

## Article

# Constraints on Space-Time-Matter Theory in the Framework of the Standard-Model Extension

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**Abstract:** We use experimental limits on Lorentz violation within the framework of the Standard-Model Extension to derive quantitative constraints on Space-Time-Matter theory, a version of Kaluza–Klein theory in which the cylinder condition is relaxed so that four-dimensional physics can in principle depend on the extra coordinates. The extra dimensions are not necessarily compact or length-like. We find that the associated variation in fundamental quantities such as rest mass must occur slowly, on cosmological scales.

**Keywords:** Kaluza–Klein theory; large extra dimensions; Lorentz violation; Robertson–Mansouri–Sexl theory; Standard-Model Extension



**Citation:** Overduin, J.; Ali, H.; Walz, F. Constraints on Space-Time-Matter Theory in the Framework of the Standard-Model Extension. *Galaxies* **2021**, *9*, 26. <https://doi.org/10.3390/galaxies9020026>

Academic Editor: Marco Schreck

Received: 1 February 2021

Accepted: 22 April 2021

Published: 26 April 2021

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## 1. Space-Time-Matter Theory

Current approaches to unification of fundamental interactions based on extra dimensions assume that those dimensions are compact and therefore undetectable at experimentally accessible energies (as in string theories), or that they are large but “off limits” to standard-model fields (as in brane theories), or that they are mathematical artifacts only (as in projective theories). Here we explore the alternative idea that extra dimensions may be large, but may not share the length-like character of the three macroscopic spatial dimensions. The original prototype is of course Minkowski’s fourth dimension of time, which was originally proposed in 1908 as a geometrical way of interpreting Einstein’s theory of special relativity [1]. In retrospect, we now appreciate that the existence of this large fourth dimension was already implicit in the Lorentz symmetry that underlies Maxwell’s laws of electromagnetism, dating from 1865. (Indeed, historians of science have conclusively established that it was the Lorentz invariance of electromagnetism that led Einstein to the discovery of relativity, not the lack of experimental evidence for an electromagnetic ether [2].)

Kaluza’s 1919 discovery that Einstein’s general relativity could be unified with Maxwell’s electromagnetism by means of a fifth dimension may well have been inspired by Minkowski’s example. Unrestricted extra dimensions spoil four-dimensional (4D) Lorentz symmetry, so Kaluza was obliged to treat his fifth coordinate differently from the others, assuming that it did not enter into observable four-dimensional physics. Klein provided a physical justification for this cylinder condition in 1926 by postulating that additional spatial dimension(s) might be compact in scale. The resulting Kaluza–Klein theory, though relatively dormant for some decades, eventually bore fruit in the 1980s and 1990s as the foundation for modern string theories (see [3] for review).

Space-Time-Matter or STM theory (also referred to by some authors as Kaluza–Klein gravity or induced-matter theory) is an alternative approach in which cylindricity is not imposed from the outset, and extra coordinates are not assumed to be closed or compact in scale, or even to have the physical dimension of length [4–6]. Minkowski’s time coordinate again serves as an inspiration. The theory originally grew out of a scale-invariant version

of 4D general relativity with variable rest mass, and indeed one possible implementation of the idea is to “geometrize” mass as a fifth dimension [7–9]. The approach is general: to take the new coordinates seriously, whatever their physical dimension, and not to unnecessarily hobble the theory by imposing cylindricity from the outset. Or, to put it in colloquial terms: not to throw out the baby of possible new physics with the bathwater of Lorentz violation. It should be noted that there are other approaches to classical Kaluza theory, including some in which cylindricity is preserved and no Lorentz violation is expected [10–12].

In five dimensions, the Einstein equations read  $G_{AB} = T_{AB}$ , where uppercase Latin indices  $A, B$  range over  $0 - 4$  (for convenience, we adopt units where  $c = 8\pi G = 1$  in this section only). The 4D subset of these equations is  $G_{\alpha\beta}(x^\gamma, \ell) = T_{\alpha\beta}(x^\gamma, \ell)$ , where Greek indices  $\alpha, \beta, \gamma$  range over  $0 - 3$  as usual. These equations now contain new, potentially Lorentz-violating terms that depend on the fifth coordinate  $x^4 = \ell$ . The approach in STM theory is to move all these terms to the right-hand side of the 4D field equations; that is, to regard them as sources for 4D physics. 4D matter and energy can, in this way, be “induced” from higher-dimensional geometry. In fact, it was proposed early on that *all* of what we regard as matter/energy in 4D might arise in this way from a higher-dimensional world that is intrinsically empty. Then, the appropriate generalization of Einstein’s equation is not  $G_{AB} = T_{AB}$  but  $G_{AB} = 0$  or, equivalently,  $R_{AB} = 0$  [13].

STM theory is a realization of what is often referred to as “Einstein’s dream” of unification, which he expressed perhaps most clearly in a paper with Jakob Grommer in 1923 [14,15]:

We presently have in mind as ultimate goal a pure field theory, in which the field variables produce the field of ‘empty space’ as well as the ... elementary particles that constitute ‘matter.’

As Einstein himself recognized, this goal is only partially realized in 4D general relativity, where an equals sign relates “gravity” on the left-hand side of the field equations with “matter” on the right [16,17]. In the 5D generalization of this theory, it can be realized completely: the field and its source are logically fused in the single quantity  $R_{AB}$ .

As it stands, however, 5D STM theory is also incomplete, in that it gives an account only of gravity, electromagnetism, and a scalar field [6]. The induced 4D matter sources are describable mostly in terms of macroscopic properties such as energy density, pressure, and bulk equation of state. The nature and role of the scalar field are not yet clear. A comprehensive account of fermionic matter and elementary particle spin remains elusive. (This latter issue has also proved to be a challenge for other geometrical approaches to unification, including those based on geons [18], kinks [19], torsion [20,21], Clifford algebras [22], and sigma models [23].) Progress may come from the discovery of new exact solutions of the field equations in higher dimensions with appropriate properties on 4D hypersurfaces. In general, rich scope remains for further inquiry on the theoretical side.

On the experimental side, the theory is consistent with observation, but as yet lacks a “smoking gun” that could distinguish it unambiguously from 4D general relativity (GR). A degree of ambiguity necessarily arises here because the same 4D physics can be embedded differently in different solutions of the 5D field equations. Thus, the classical tests of GR, for example, were originally applied to a 5D generalization of the Schwarzschild metric known as the soliton metric, leading to strong constraints on a primary dimensionless free parameter of the theory, of order parts in  $10^4$  [24,25], or even  $10^8$  or more [26,27]. Later work has established that a different 5D metric known as the canonical metric provides a more unambiguous basis for comparison with 4D physics, and constraints in this case are weaker [28–30]. For a comprehensive review of tests of alternative theories of gravity, see Will [31].

## 2. Physical Interpretation

Any 5D-covariant theory must necessarily violate 4D Lorentz invariance at some level, and must therefore be testable in principle. Here, we consider what constraints may be placed on STM theory in the “special relativity limit” where fields are weak.

In 4D relativity, experimental effects are typically expressed in terms of the Lorentz factor  $\gamma$ , which arises from the definition of proper time  $d\tau^2 = -ds^2/c^2$  and the line element of 4D Minkowski space,  $ds^2 = -c^2dt^2 + |d\vec{x}|^2$ :

$$\gamma(v) \equiv \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad (1)$$

where  $v = |d\vec{x}|/dt$ . A fifth coordinate  $x^5 = \ell$  introduces an additional term in this factor, as follows:

$$\gamma_5(v) = \left(1 - \frac{v^2}{c^2} \mp \frac{v_5^2}{c^2}\right)^{-1/2}, \quad (2)$$

where  $v_5 = d\ell/dt$  and we maintain an open mind with regard to the metric signature of the new term. To go further, it is necessary to interpret the extra coordinate physically. In classical Kaluza–Klein theory, with  $v^5$  proportional to electric charge, it was originally hoped that the compactification of  $\ell$  might explain charge quantization (see Ref. [32] for a review).

Large extra dimensions open up the possibility of significant violations of well-tested conservation laws in 4D. Thus, braneworld or membrane theories assume from the outset that the extra-dimensional “bulk” is inaccessible to all standard-model fields except gravity. This attitude can be justified to the extent that gravitation is fundamentally different from the other interactions (e.g., insofar as standard-model fields are seen as propagating *in* spacetime, while gravitation *is* spacetime). Therefore, it is not necessarily ad hoc to allow gravitons to propagate off the brane, while all other fields are restricted to it (an assumption that would otherwise be little more than another version of Kaluza’s cylinder condition). Of course, such a picture also helps address the hierarchy problem: the weakness of gravitational interactions relative to those of the standard model of particle physics is explained by the fact that they are “diluted” over a greater volume of space.

While they have been influential, however, the extra dimensions of the brane and Kaluza–Klein theories are less radical than Minkowski’s, in that they share the length-like character of ordinary space. Philosophically, this represents a return to the Pythagorean prejudice that geometry is only for quantities that can be measured with a meter-stick [1]. Given the long history of connections between unification and extra dimensions, it is surprising that more physicists have not followed Minkowski’s lead and attempted to expand the domain of geometry farther beyond space and time.

Additional time-like dimensions are associated with tachyons (imaginary-mass states), causality violation via closed time-like curves, and the wrong sign for the Maxwell action [33]. Nevertheless, they have occasionally been considered. Sakharov allowed the possibility of even numbers of additional compact time dimensions and argued that causality could be preserved for macroscopic processes if the radius of compactification were sufficiently small [34]. This work was further developed by Aref’eva and Volovich [35]. Other “two-time” theories have been propounded by Burakovsky and Horwitz [36], Bars and Kounnas [37], Wesson [38], and others.

Among the first truly “post-Minkowskian coordinates” were those investigated by William Band and Osamu Hara (1959) relating  $x^5$  to particle spin [39,40], and that of Yurii Borisovich Rumer, who proposed in 1949 a fifth coordinate based on action via  $x^5 = S/mc$  [41]. Rumer applied this idea to what he termed “5-optics” and imposed a restriction (called the “requirement of physical admissibility”) similar to Kaluza’s cylinder condition [42]. More recently, Fukui studied the possibility of extra coordinates proportional to the fundamental constants  $e$  and  $h$  [43,44]. Carmeli explored a theory in which  $x^5$  was associated with the cosmological expansion speed [45]. These and other suggestions have been briefly reviewed in Ref. [46].

At its most fundamental, physics deals with dimensions of length [L], time [T], and mass [M], so the most natural physical interpretation for a new coordinate is arguably one related to mass via either  $x^5 = Gm/c^2$  or  $x^5 = h/mc$ . Newton’s gravitational constant  $G$  (or alternatively Planck’s constant  $h$ ) is then promoted to the same dimension-transposing

role as  $c$  in 4D special relativity. Aspects of this idea were discussed as early as 1967 by de Vos and Hilgevoord [47] and 1974 by Edmonds [48,49], but it is most widely associated with STM theory and its precursors beginning in 1984 [4–8]. In the context of this theory, the identification of  $x^5$  with rest mass follows naturally from several lines of argument including the fact that the 4D relativistic energy–momentum relation  $p^\alpha p_\alpha = E^2 - c^2 p^2 = m^2 c^4$  reduces simply to  $p^A p_A = 0$  in 5D; and that the 4D free-particle action principle  $\delta(\int m ds) = 0$  is contained in the simpler 5D one  $\delta(dS) = 0$ , where  $ds^2 = g_{\alpha\beta} g^{\alpha\beta}$  and  $dS^2 = g_{AB} g^{AB}$ , respectively.

In this interpretation, the fifth component of velocity manifests itself as a variable rest mass. On dimensional grounds,  $v_4 = G\dot{m}/c^2$  where  $\dot{m} \equiv dm/dt$ , so Equation (2) becomes

$$\gamma_5(v) = \left[ 1 - \left(\frac{v}{c}\right)^2 \mp \left(\frac{G\dot{m}}{c^3}\right)^2 \right]^{-1/2}, \quad (3)$$

where  $\dot{m} \equiv dm/dt$ . To estimate the difference between  $\gamma$  and  $\gamma_5$  in a practical situation, consider for example an electron in the hydrogen ground state. It moves in the space direction at 1/137th of its speed in the time direction,  $v/c = 0.0073$ . What about the mass direction? The success of observational cosmology implies that its mass has not changed appreciably since the time of cosmic nucleosynthesis, about 10 min after the big bang. To get an upper limit, suppose that its entire rest mass was generated by some unknown process during those first 10 min; then  $G\dot{m}/c^3 = 4 \times 10^{-69}$ . No wonder that a mass-like fifth dimension, if real, has not been detected experimentally to date. While such a proposition may be hard to test in practice, its importance in principle can be suggested by a hypothetical paraphrase of Minkowski’s famous pronouncement in 1908:

Space, time *and* mass in themselves are doomed to fade away into mere shadows, and only a kind of union of the *three* will preserve an independent reality.

Less often quoted, however, is the preceding line in Minkowski’s speech: “The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength” [50]. Are there any hints in experimental physics to support the notion of mass as a fifth coordinate along with space and time?

### 3. Lorentz Violation in the Flat-Space Limit

Existing experiments tell us that any new term in Equation (3) must be small [51], so we can Taylor-expand:

$$\gamma_5(v) = 1 + \frac{v^2}{2c^2} \left[ 1 \pm \left(\frac{G\dot{m}}{c^2 v}\right)^2 \right]. \quad (4)$$

The term in square brackets violates Lorentz symmetry. Violations of this kind appear in a kinematical generalization of special relativity known as Robertson–Mansouri–Sexl (RMS) theory [52,53]. In RMS theory, the existence of a preferred frame modifies the standard Lorentz transforms such that

$$\begin{aligned} t &= a(v)T + e(v)x & , & & x &= b(v)(X - vT) , \\ y &= d(v)Y & , & & z &= d(v)Z , \end{aligned} \quad (5)$$

where  $T, X, Y, Z$  are coordinates in the preferred frame, and the functions  $a(v), b(v), d(v)$ , and  $e(v)$  describe time dilation, length contraction, transverse length contraction, and clock synchronization, respectively. The function  $b(v)$  is the RMS generalization of the Lorentz factor  $\gamma(v)$  of special relativity. Mansouri and Sexl showed on general grounds that

$$a(v) \approx 1 + \alpha \left(\frac{v}{c}\right)^2, \quad b(v) \approx 1 + \beta \left(\frac{v}{c}\right)^2, \quad (6)$$

where  $\alpha, \beta$  are constants whose values go over to 0,0 in the limit of Galilean relativity and  $-\frac{1}{2}, \frac{1}{2}$  in the limit of special relativity. Comparing Equations (4) and (6), we identify

$$\beta - \frac{1}{2} = \pm \frac{1}{2} \left( \frac{G\dot{m}}{c^2 v} \right)^2. \quad (7)$$

Experimental constraints on the RMS parameters  $\alpha, \beta$  have come from a test of the relativistic Doppler effect known as the Ives–Stilwell experiment (IS), and from the Kennedy–Thorndike experiment (KT), a modified form of the original Michelson–Morley experiment with arms of different lengths. Recent limits are [54,55]:

$$\begin{aligned} |\alpha + \frac{1}{2}| &\leq 8.4 \times 10^{-8} \text{ (IS)}, \\ \alpha - \beta + 1 &= 0.0_{-3.7}^{+4.8} \times 10^{-8} \text{ (KT)}. \end{aligned} \quad (8)$$

Combining these expressions, we find that  $|\beta - \frac{1}{2}| \leq 1 \times 10^{-7}$ . From Equation (7) it then follows that  $|\dot{m}| \leq 2 \times 10^{32} \text{ kg/s}$  (assuming  $v \leq c$ ). The weakness of this constraint is due to the large value of the dimension-transposing constant  $c^3/G$  ( $= 4 \times 10^{35} \text{ kg/s}$  in SI units), relative to the mass currents typically encountered in experimental physics. That is, the mass-like fifth dimension has observational consequences only at “speeds”  $\dot{m}$  close to  $c^3/G$ —just as Minkowski’s fourth dimension becomes detectable only at speeds  $\dot{x}$  close to  $c$  [56].

As a purely kinematical model of Lorentz violation, RMS theory has significant limitations. An alternative, fully dynamical field theory framework that contains general relativity and the Standard Model of particle physics along with additional fields that allow for all possible violations of Lorentz symmetry is the Standard-Model Extension or SME [57,58]. RMS-type models are parameterized within the SME in terms of “rods” and “clocks,” observers whose length and time scales can disagree with those of photons when moving at a velocity  $\vec{v}$  relative to the preferred frame. In the simplest possible model [59,60], Lorentz violation is controlled by two new constants,  $(c_{\text{clock}})_{TT}$  and  $(c_{\text{rod}})_{TT}$ . To leading order in  $v$ ,

$$\alpha = -\frac{1}{2} - \frac{5}{12}(c_{\text{clock}})_{TT}, \quad \beta = \frac{1}{2} + \frac{7}{12}(c_{\text{rod}})_{TT}. \quad (9)$$

Thus, in this case, constraints on the RMS parameter  $\beta - \frac{1}{2}$  in Equation (7) translate directly into constraints on the SME parameter  $(c_{\text{rod}})_{TT}$ .

To obtain numerical bounds, we identify our “rods” with particles such as electrons, protons, neutrons, or neutrinos, and associate “ $v$ ” with the speed of the standard Sun-centered inertial frame relative to the average rest frame of photons in the cosmic microwave background,  $v \approx 370 \text{ km/s}$ . Experimental limits on  $(c_{\text{rod}})_{TT}$  are summarized in the SME Data Tables [61]. For electrons, they come from synchrotron energy losses in accelerators [62], atomic interferometry [63], spectroscopy [64], nuclear binding energy [65], observations of the Crab Nebula flare [66], and clock frequencies [67], implying that  $(c_e)_{TT} \geq -5 \times 10^{-21}$  and  $|(c_e)_{TT}| < 4 \times 10^{-15}$ . For protons and neutrons, they come from atom interferometry [63], nuclear binding energy [65], and atomic fountain clocks [68]. For neutrinos, they come from oscillation experiments like MiniBooNE [69] and Super-Kamiokande [70].

The strongest such limit currently comes from the world’s first dual atomic fountain clock and implies that  $|(c_p)_{TT}| < 9 \times 10^{-16}$  for protons [71]. Taking this last number as an example, we find from Equations (7) and (9) that:

$$|\dot{m}_p| < \sqrt{\frac{7}{6}|(c_p)_{TT}|} \frac{c^3}{G} \leq 2 \times 10^{25} \text{ kg/s}, \quad (10)$$

where we have taken  $v < c$  to get an upper bound. Such a result might be consistent with a slow variation in particle rest mass on cosmological timescales. Indeed, in this picture mass could *originate* from a displacement along  $x^5$ , in what might be seen as a geometrical

analog of the Higgs mechanism [56]. Related ideas have been explored by Bekenstein [72], Liu and Wesson [73], and Wetterich [74].

In this interpretation, we would expect on dimensional grounds that  $|\dot{m}_p|/M_{\text{bar}}$  is not too far from the Hubble expansion rate  $H_0$ , where  $M_{\text{bar}}$  is the mass of baryonic matter within the causal or particle horizon of the Universe,  $L_p \approx ct_0$ , and  $t_0$  is the elapsed time since the big bang. We can test this expectation using Equation (10). For a spatially flat universe,  $M_{\text{bar}} \approx \rho_{\text{bar}} L_p^3$  where  $\rho_{\text{bar}} = 0.05\rho_{\text{crit}}$  from observation and primordial big-bang nucleosynthesis,  $\rho_{\text{crit}} \equiv 3H_0^2/8\pi G$  is the critical density, and  $H_0 t_0 = 1$  for the now standard and well-tested  $\Lambda$ CDM cosmological model [75]. Combining these expressions, we find:

$$\frac{|\dot{m}_p|}{M_{\text{bar}}} < \sqrt{\frac{7}{6}|(c_p)_{TT}|} \frac{160\pi}{3} H_0 \leq 5 \times 10^{-6} H_0. \quad (11)$$

The corresponding constraint for electrons is four times weaker. The strength of this limit reflects the remarkable progress in sensitivity of fundamental physics experiments over the past few decades.

#### 4. Conclusions and Discussion

We have tested Space-Time-Matter theory in the flat-space limit for the case of a mass-like fifth dimension  $x^5 = Gm/c^2$ , expressing our constraints in terms of  $(c_{\text{rod}})$ , a free parameter within a Lorentz-violating generalization of the standard model known as the Standard-Model Extension.

Experimental constraints on this parameter restrict any cosmological variation in rest masses of fundamental particles to less than  $\sim 10^{-6}$  times the cosmic expansion rate. This is a small number, but not *unnaturally* small in the context of fundamental physics. It is comparable, for example, to existing upper limits imposed by tests of the Equivalence Principle on the coupling strength between standard-model fields and new scalar fields of the kind suggested by string and other unified theories [76]. Further progress in those experiments (carrying them out in space at cryogenic temperatures, for example) will drive that bound down to the  $\sim 10^{-9}$  level, which *would* be judged unnatural by most particle theorists. For comparison, one need only think of the strong CP problem in nuclear physics; here experimental limits on the electric dipole moment of the neutron force a dimensionless free parameter in quantum chromodynamics to take values of  $\sim 10^{-9}$  or less. This is considered so unnatural that a new particle, the axion, has been postulated to drive the CP-violating parameter to zero dynamically. Although it has never been observed, the axion is believed by many to make up the dark matter that accounts for 25% of the universe by mass, and experiments around the world are underway to detect it. In the same way, if experimental limits on Lorentz violation were to strengthen by another six orders of magnitude or so, then the rate of variation in rest masses of fundamental particles implied by Equation (11) would be so much smaller than the Hubble expansion rate that the motivation for a mass-like fifth dimension would come into question.

There are caveats to the conclusions above. It is possible to parameterize a mass-like fifth dimension differently using Planck's constant  $h$  in place of Newton's constant  $G$  via  $x^5 = h/mc$ . In this case, Equation (7) takes the form

$$\beta - \frac{1}{2} = \pm \frac{1}{2} \left( \frac{hm}{cm^2} \right)^2. \quad (12)$$

Carrying through the same analysis as above, one gets in place of Equations (10) and (11) a constraint that can be expressed as

$$\left| \frac{\dot{m}}{m} \right| < \sqrt{\frac{7}{6}|(c_{\text{rod}})_{TT}|} \frac{mc^2}{h}. \quad (13)$$

The meaning of this expression is harder to interpret. Applied to protons with  $m = m_p$  and  $c_{\text{rod}} = c_p$ , it would imply that  $|m/m| < 7 \times 10^{15} \text{ s}^{-1}$ , which is so weak that it would not seem to constrain the theory in any useful way. Further study may yield a way to extract a “local” or microscopic limit in this way, as opposed to the “global” or macroscopic one above. Alternatively, it may be possible to interpret extra dimensions in terms of fundamental quantities other than rest mass, like intrinsic spin (again using Planck’s constant) or even entropy (using Boltzmann’s constant). These are subjects for future investigation.

The important point is that Lorentz symmetry must be violated at some level within any 5D-covariant theory, and that a promising way to test such theories in general is to situate them systematically within an established framework of experimental constraints like that offered by the Standard-Model Extension.

**Author Contributions:** All the authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research was supported in part by the Maryland Space Grant Consortium and the Fisher College of Science and Mathematics at Towson University.

**Acknowledgments:** This paper is dedicated to the memory of three founding members of the 5D Space-Time-Matter consortium ([5dstm.org](http://5dstm.org)): Paul Wesson, Hongya Liu, and Jaime Ponce de Leon.

**Conflicts of Interest:** The authors declare no conflict of interest.

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