


## Article

# Neutrino Charge in a Magnetized Media

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**Abstract:** In the presence of a thermal medium or an external electro-magnetic field, neutrinos can interact with photon, mediated by the corresponding charged leptons (real or virtual). The effect of a medium or an electromagnetic field is two-fold—to induce an effective  $\nu\gamma$  vertex and to modify the dispersion relations of all the particles involved to render the processes kinematically viable. It has already been noted that, in a medium, neutrinos acquire an effective charge, which in the standard model of electroweak interaction comes from the vector type vertex of weak interaction. On the other hand, in a magnetized plasma, the axial vector part also starts contributing to the effective charge of a neutrino. This contribution corresponding to the axial vector part in the interaction Lagrangian is denoted as the axial polarisation tensor. In this note, we outline the calculation of the axial polarization tensor to odd and even powers in  $eB$ . We further show its gauge invariance properties. Finally, we infer upon the zero external momentum limit of this axial polarisation tensor.

**Keywords:** neutrino; finite temperature field theory; magnetic field; Schwinger proper time representation



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## 1. Introduction

Neutrinos are of significant importance because of the observable imprints they leave on astrophysical or cosmological observables through their interactions. These imprints are related to their mass, charge, dipole moment, etc. These are generally unexplainable through a standard model of particle physics. In a cosmological context, they play a significant role in the synthesis of primordial light elements [1,2], modifying the power spectrum of cosmic microwave background (CMB) anisotropies [3] and in the large scale clustering of matter in the universe [4]. They contribute to many physical processes involving electromagnetic interactions (many of them involve astrophysical situations [5–23]), a list of them and the origin of the same can be found in [24–95]. In this note, we would rather consider the astrophysical part of it.

To understand neutrino-astrophysics, we need to understand the physics of the models that dictate neutrino behaviour. To have predictable power, such models should be re-normalizable. To have a re-normalizable model of the physics of elementary particles, the tree level  $\gamma - \nu$  interactions are protected from being realised in nature. Thus, at the tree level, neutrinos do not couple to photon in the standard model of particle physics, and this coupling can only take place at a loop level, mediated by the fermions and gauge bosons. This coupling can give birth to off-shell photons only, since, for on-shell particles, the processes like  $\nu \rightarrow \nu\gamma$  and  $\gamma \rightarrow \nu\bar{\nu}$  are restrained kinematically. Only in the presence of a medium can all the particles be on shell as there the dispersion relation of the photon changes, giving the much required phase space for the reactions. Intuitively, when a neutrino moves inside a thermal medium composed of electrons and positrons, they interact with these background particles. The background electrons and positrons themselves have interaction with the electromagnetic fields, and this fact gives rise to

an effective coupling of the neutrinos to the photons. Under these circumstances, the neutrinos may acquire an “effective electric charge” through which they interact with the ambient plasma.

Finite neutrino charge at the tree level is however possible with the introduction of gauge bosons with an extra  $U(1)$  symmetry [86–88]. As a result, neutrinos can acquire an intrinsic charge and turn out to be a Dirac particle.

Whatever may be the source of electric charge of neutrino (induced or intrinsic), neutrino–photon interactions have many far reaching consequences in this universe. To provide an example, because of in-medium neutrino–photon interaction, the supernova explosions are realized in nature. It is highly possible that the neutrinos actually dump a fraction of their energy inside an exploding star during its stellar evolution. For instance, during a type II supernovae collapse [12], neutrinos are produced deep inside the proto-neutron star surge out, carrying an effective energy whose order of magnitude estimate is  $\sim 10^{52}$  erg/s. It is conjectured that the neutrinos deposit some fraction (ranging between 1 to 10%) of its total energy during the explosion through different kinds of neutrino electromagnetic interactions, e.g.,  $\gamma\nu \rightarrow \gamma\nu$ ,  $\nu \rightarrow \nu\gamma$ ,  $\nu\bar{\nu} \rightarrow e^+e^-$ , to name a few. It is important to note here that all of these processes are of order  $G_F^2$  (where  $G_F$  is the four Fermi coupling constant). However, the amount of energy dumped by these mechanisms to the mantle of the proto-neutron star seems to be barely sufficient to blow the outer part of the same. The celebrated paper of [15] could account for  $10^{50}$  erg of energy release by hydrodynamic processes taking place in a nuclear media. The supernova shock wave was found to get stalled in respective computer simulations. Hence, neutrino-mediated processes were invoked so as to revive the shock wave, so that there can be a delayed shock revival. However, this attempt was not entirely successful [16].

However, inside a star, a nonzero magnetic field would be present owing to different hydrodynamics induced dynamo mechanisms. It is usually conjectured that this magnetic field of the supernova-progenitor—when compressed to the size of the neutron star—during the Super Nova (SN) type II explosion, its strength can reach up to  $\mathcal{B} \sim \frac{m^2}{e}$  or more in most of the regions of the nascent proto-neutron star. This follows due to the conservation of surface magnetic field of the SN precursor. The symbol  $m$  denotes the mass of electron. In addition, in the rest of this paper, we would refer to field strengths of this magnitude as critical field strength  $\mathcal{B}_c = 4.41 \times 10^{13} \text{G}$ . Given the possibility of strong  $e\mathcal{B}$  field generation in proto-neutron star, it is worth investigating their role of  $\nu - \gamma$  interaction in a magnetized media. This would help in understanding if the magnetic field induced effects can account for the release of  $10^{53}$  erg of energy in a super nova type II explosion—that is, to find out if the strong magnetic field induced contributions can exceed the same due to unmagnetized ones by the desired orders of magnitude. This happens to be one of our objectives of this work.

Apart from this, there are several high energy processes in astrophysics that are in need of some explanation for asymmetric energy transfer. This happens to be an extra motivation to this investigation. To provide an example, during gamma ray burst, a prompt gamma ray flash from the optically thick environment of the progenitor has been noticed in many GRB events and are found to be anisotropic in their distribution in space and too fast in their temporal appearance to point to the existence of a class of feebly charged weakly interacting particles at the source [17].

Other examples where explanations using asymmetric  $\gamma - \nu$  processes can be useful—which are explosions in magnetars [18] accompanied by associated jets [20] and the presence of unexplained anomalous drift-velocity that newly born pulsars are found to be accompanied with [23]. Ideally, in a symmetric SN explosion, the associated newly born neutron star should be stationary. However, they are seen to be born with some velocity—usually referred to in the literature as pulsar kicks. Since the presence of a  $\mathcal{B}$  field breaks the rotational symmetry for a system, it thus makes a good choice to look for signatures of the same while studying neutrino electromagnetic interactions in a magnetized media.

### 1.1. Intrinsic Charge of Neutrino

We have introduced the origin of neutrino charge in the Introduction. They happen to be (i) intrinsic ( $e_{int}^\nu$ ) in their origin and can be accounted for from studies beyond the standard model of elementary particle physics, (ii) medium induced neutrino effective charge ( $e_{eff}^\nu$ ) owing its existence to in-medium effects upon standard model left-handed neutrinos. The second one gets contributions to itself coming from the vector vertex of neutrino-  $W^\pm, Z$  interaction (denoted as  $e_{eff}^\nu$ ), and the axial vertex of  $\nu$  interaction (denoted as  $e_{eff}^{\nu a}$ ). The second one is realised in nature only in the presence of magnetized matter (medium).

### 1.2. Induced Charge of Neutrino

In this paper, we concentrate upon the effective neutrino photon interaction vertex coming from the axial vector part of the interaction. From there, we estimate the effective charge of the neutrino (in a magnetised medium). We name the axial contribution in the effective neutrino photon Lagrangian as the axial polarisation tensor  $\Pi_{\mu\nu}^5$ , which we will define in the next section. We discuss the physical situations where the axial polarisation tensor arises, then show how it affects the physical processes. The effective charge of the neutrino has been calculated previously by many authors, the induced charge calculated in the references [82–85] and the intrinsic charge in [86–93]. Any discussion about the origin and magnitude of a very fundamental entity—intrinsic neutrino charge  $e_{int}^\nu$ , both experimental and theoretical, deserves special attention. Since that is not the main focus of our work, we will refrain from any such attempt here, except attempts to compare their magnitude obtained in various terrestrial laboratory conditions with that of  $e_{eff}^\nu$  obtained for different background media. This particular attempt is made to keep  $e_{int}^\nu$  free from medium induced contributions. At the end, we would like to mention that any analysis on the estimates of  $e_{eff}^{\nu a}$  from experimental and astrophysical observations is yet to be performed to the best of our knowledge and understanding.

The plan of the paper is as follows. We start with Section 2 which deals with the formalism through which the physical importance of  $\Pi_{\mu\nu}^5$  is appreciated. In Section 3, the general form factor analysis of “axial polarisation tensor” on the basis of symmetry arguments is provided. In Section 4, we show the fermion propagator in a magnetised medium, and, using the same, explicitly write down  $\Pi_{\mu\nu}^5$  in the rest frame of the medium. In the next section, i.e., Section 5, we show the formal proof of gauge invariance of  $\Pi_{\mu\nu}^5$ . In Section 6, we calculate the effective electric charge from the expression of the axial polarisation tensor odd in  $e\mathcal{B}$  even in  $\mu$  as well as even in  $e\mathcal{B}$  odd in  $\mu$ . In Section 7, we discuss our results and conclude by touching upon the physical relevance of our work. Section 8 covers an outlook on future perspectives. In the Appendix A, we outline the construction of the basis tensors for  $\Pi_{\mu\nu}^5$  from 4-vectors, having appropriate CPT transformation properties. We follow the natural system of units in which  $\hbar = c = k_B = 1$ .

## 2. Formalism

In this work, we consider neutrino momenta that are small compared to the masses of the  $W$  and  $Z$  bosons. We can, therefore, neglect the momentum dependence in the  $W$  and  $Z$  propagators, which is justified if we are performing a calculation to the leading order in the Fermi constant,  $G_F$ . In this limit, four-fermion interaction is given by the following effective Lagrangian:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{\sqrt{2}}G_F\bar{l}_\nu\gamma^\mu(1-\gamma_5)l_\nu\bar{f}\gamma_\mu(g_V+g_A\gamma_5)f \quad (1)$$

where  $l_\nu$  and  $f$  are the neutrino and the corresponding lepton fields, respectively. In Equation (1),  $g_V$  and  $g_A$  are the weak vector and axial vector coupling strengths, respectively. For electron  $Z$  boson vertex,

$$g_V = 1 - (1 - 4 \sin^2 \theta_W)/2 \quad (2)$$

$$g_A = -1 + 1/2 \quad (3)$$

where the first terms in  $g_V$  and  $g_A$  are the contributions from the  $W$  exchange diagram and the second one from the  $Z$  exchange diagram. Here,  $\theta_W$  is the Weinberg angle (where  $\theta_W = 28.7^\circ$ ) and  $G_F$  is the four Fermi coupling constant ( $G_F = (1.026 \pm 0.001) \times 10^{-5} m_p^{-2}$ , where  $m_p$  is the proton mass).

With this interaction Lagrangian, we can write down the matrix element for the Cherenkov amplitude as

$$M = -\frac{G_F}{\sqrt{2}e} Z \epsilon^\nu \bar{l}_\nu \gamma^\mu (1 - \gamma_5) l_\nu (g_V \Pi_{\mu\nu} + g_A \Pi_{\mu\nu}^5) \quad (4)$$

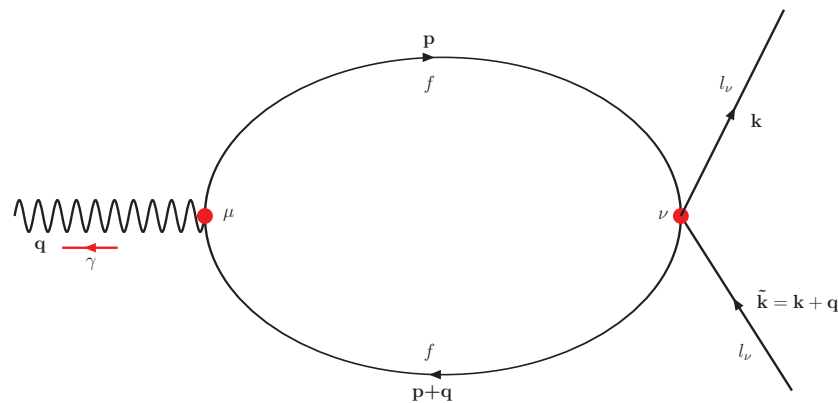
where  $\epsilon^\nu$  is the photon polarisation tensor, and  $Z$  is the wave function renormalisation factor inside a medium. The term  $\Pi_{\mu\nu}$  is defined as

$$i\Pi_{\mu\nu} = (-ie)^2 (-1) \int \frac{d^4 p}{(2\pi)^4} \text{Tr}[\gamma_\mu iS(p) \gamma_\nu iS(p')] \quad (5)$$

which looks exactly like the photon polarisation tensor, but doesn't have the same interpretation here. The symbol  $e$  in Equation (5) represents the coupling constant for the  $U(1)$  gauge field ( $e = \sqrt{4\pi/137}$ ). The momentum labels of the propagators can be understood from the corresponding Feynman diagram (Figure 1). Henceforth, we would call it the polarisation tensor.  $\Pi_{\mu\nu}^5$  is defined as

$$i\Pi_{\mu\nu}^5 = e^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr}[\gamma_\mu \gamma_5 iS(p) \gamma_\nu iS(p')] \quad (6)$$

which we call the axial polarisation tensor. Both the polarisation tensor and the axial polarisation tensor are obtained by calculating the Feynman diagram given in Figure 1 drawn in a four-Fermi-limit.



Loop Induced Neutrino Photon Vertex

**Figure 1.** One-loop diagram for the effective electromagnetic vertex of the neutrino in the limit of infinitely heavy  $W$  and  $Z$  masses.

The off-shell electromagnetic vertex function  $\Gamma_\nu$  is defined in such a way that, for on-shell neutrinos, the  $\nu\nu\gamma$  amplitude is given by:

$$\mathcal{M} = -i\bar{u}(q') \Gamma_\nu u(q) A^\nu(k) \quad (7)$$

where  $k$  is the photon momentum. Here,  $u(q)$  is the neutrino spinor and  $A^\nu$  stands for the electromagnetic vector potential. In general,  $\Gamma_\nu$  would depend on  $k$  and the characteristics of the medium. With our effective Lagrangian, the vertex  $\Gamma_\nu$  is given by

$$\Gamma_\nu = -\frac{1}{\sqrt{2}e} G_F \gamma^\mu (1 - \gamma_5) (g_V \Pi_{\mu\nu} + g_A \Pi_{\mu\nu}^5). \quad (8)$$

Charge radius of a neutrino can be obtained from the vertex function given by Equation (8). This observable in general should be gauge independent, UV finite and a renormalisation group invariant.

On the other hand, the neutrino charge radius can also be estimated using an effective field theory approach. It was performed in [95] from the coefficients of a four fermion effective field theory that was constructed from the current–current co-relators involving neutrinos and other standard model fermions, after integrating out the heavy degrees of freedom. In this approach, the energy scale of the coefficients are extrapolated to the mass of the electron to relate them to the charge radius of neutrino. The effective charge we are evaluating is related to the charge radius. Given the expression for one, it is possible to find the same for the other, from the expression of the previous one. The physical situation considered in this work is substantially different from that of [95] because of the presence of magnetic field and the medium. They belong to two different paradigms so we will not deal with it here any further.

The effective charge of the neutrinos is defined in terms of the vertex function by the following relation [82,83]:

$$e_{\text{eff}} = \frac{1}{2q_0} \bar{u}(q) \Gamma_0(k_0 = 0, \mathbf{k} \rightarrow 0) u(q). \quad (9)$$

For massless Weyl spinors, this definition can be cast into the form:

$$e_{\text{eff}} = \frac{1}{2q_0} \text{Tr} \left[ \Gamma_0(k_0 = 0, \mathbf{k} \rightarrow 0) (1 + \lambda \gamma^5) \not{q} \right], \quad (10)$$

where  $\lambda = \pm 1$  is the helicity of the fermions in the external line. Since we do not consider the right-handed neutrino, we therefore choose  $\lambda = +1$ . Before going to estimate neutrino-charge in various situations, we would like to state that this induced neutrino charge vanishes in vacuum. This statement is true in general for any number of physical dimensions 'd' and in any gauge parameter [96,97].

While discussing about  $\Pi_{\mu\nu}^5$ , it should be remembered that it has two vertices, one is axial vector vertex and the other one is a vector vertex; The current conservation relation for the electromagnetic vertex is obeyed in the form,

$$k^\nu \Pi_{\mu\nu}^5 = 0 \quad (11)$$

which also happens to be the current conservation condition.

In order to calculate the Cherenkov amplitude or the effective charge of the neutrinos inside a medium, we have to calculate  $\Pi_{\mu\nu}^5$ . The formalism thus discussed is a general one, and we extend the calculations previously done based on this formalism to the case where we have a constant background magnetic field in addition to a thermal medium. In doing so, we give the full expression of  $\Pi_{\mu\nu}^5$  in a magnetised medium and explicitly show its gauge invariance. We also comment on the effective charge contribution from the axial polarisation part.

Discussing about the effective charge of the neutrinos in a medium, the way we have done, it should be mentioned that, although it is interesting to find it theoretically, it is not the “charge” with which the neutrinos couple with a magnetic field. From the definition of the electromagnetic vertex as given in Equation (7) and the definition of charge in Equation (9), it is clear that we are interested to find the coupling of the photon field

with  $\bar{u}\Gamma^0 u$  and not  $\bar{u}\Gamma^i u$ . The magnetic interaction will come from the term  $\bar{u}(q)\Gamma^i u(q)A_i$ , but we will see at the end of the calculation that no  $\Gamma^i$  exists in the limit  $k_0 \rightarrow 0, \vec{k} \rightarrow 0$ , whereby we cannot speak of any possible interaction of the neutrinos with the external static uniform magnetic field.

### 3. General Analysis

We start this section with a discussion on the possible tensor structure and form factor analysis of  $\Pi_{\mu\nu}^5(k)$ , based on the symmetry of the interaction. To begin with, we note that  $\Pi_{\mu\nu}^5(k)$  in the vacuum should vanish. This follows from the following arguments. In vacuum, the available vectors and tensors at hand are the following:

$$k_\mu, g_{\mu\nu} \text{ and } \epsilon_{\mu\nu\lambda\sigma}. \quad (12)$$

The two point axial-vector correlation function  $\Pi_{\mu\nu}^5$  can be expanded in a basis, constructed out of tensors  $g_{\mu\nu}$ ,  $\epsilon_{\mu\nu\lambda\sigma}$ , and vector  $k_\lambda$ . Given the parity structure of the theory, it is impossible to construct a tensor of rank two using  $g_{\mu\nu}$  and  $k_\mu$ . Thus, the only available tensor (with the right parity structure) we have at hand is  $\epsilon_{\mu\nu\lambda\sigma}$ . The other vector needed to make it a tensor of rank two is  $k_\lambda$ . As we contract  $\epsilon_{\mu\nu\lambda\sigma}$  with  $k_\lambda, k_\sigma$ , since  $\epsilon_{\mu\nu\lambda\sigma}$  is completely antisymmetric tensor of rank four, the corresponding term vanishes.

On the other hand, in a medium, we have an additional vector  $u^\mu$ , i.e., the velocity of the centre of mass of the medium. Therefore, the polarisation tensor can be expanded in terms of form factors along with the new tensors constructed out of  $u^\mu$  and the ones we already had in the absence of a medium as

$$\Pi_{\mu\nu}(k) = \Pi_T T_{\mu\nu} + \Pi_L L_{\mu\nu}. \quad (13)$$

Here,

$$T_{\mu\nu} = \tilde{g}_{\mu\nu} - L_{\mu\nu} \quad (14)$$

$$L_{\mu\nu} = \frac{\tilde{u}_\mu \tilde{u}_\nu}{\tilde{u}^2} \quad (15)$$

with

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \quad (16)$$

$$\tilde{u}_\mu = \tilde{g}_{\mu\rho} u^\rho. \quad (17)$$

In the rest frame of the medium, the four velocity is given by  $u^\mu = (1, 0, 0, 0)$ . It is easy to see that the longitudinal projector  $L_{\mu\nu}$  is not zero in the limit  $k_0 = 0, \vec{k} \rightarrow 0$  and  $\Pi_L$  is also not zero in the above-mentioned limit. This fact is responsible for giving nonzero contribution to the effective charge of neutrinos.

It has already been mentioned that, in a medium, we have another extra four vector  $u^\mu$  and hence it is possible to construct the axial polarisation tensor of rank two, out of  $\epsilon_{\mu\nu\alpha\beta}, u_\mu, k_\mu$ , i.e.,  $\epsilon_{\mu\nu\alpha\beta} u^\alpha k^\beta$  that would verify the Ward identity for the two point function. An explicit calculation of  $\Pi_{\mu\nu}^5(k)$  verifies the tensor structure of it as predicted here. It is worth noting that this contributes to the Cherenkov amplitude, but not to the effective electric charge of the neutrinos since, for charge calculation, we have to put the index  $\nu = 0$ . In the rest frame, only  $u^0$  exists that forces the totally antisymmetric tensor to vanish.

In a constant background magnetic field in addition to the ones mentioned in Equation (12), one has the freedom of having other extra vectors and tensors (to first order in field strength), such as

$$F_{\mu\nu}, \tilde{F}_{\mu\nu} \quad (18)$$



along with

$$\bar{E}_\mu^{(1)} = k^\lambda F_{\lambda\mu}, \quad \bar{E}_\mu^{(2)} = k^\lambda \tilde{F}_{\lambda\mu}. \quad (19)$$

Explicit evaluation of the axial polarisation tensor, in a constant background magnetic field, is (however, the metric used by the authors in references mentioned is different from that of us) [98,99],

$$\Pi_{\mu\nu}^5(k) = \frac{e^3}{(4\pi)^2 m^2} \left[ -C_\parallel k_{\nu\parallel} (\tilde{F}k)_\mu + C_\perp \left\{ k_{\nu\perp} (k\tilde{F})_\mu + k_{\mu\perp} (k\tilde{F})_\nu - k_\perp^2 \tilde{F}_{\mu\nu} \right\} \right], \quad (20)$$

where  $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ ,  $F_{12} = -F_{21} = \mathcal{B}$  and  $(k\tilde{F})_\nu = \tilde{F}^{\mu\nu}k_\mu$ . According to the notation used in Equation (20),  $k_\parallel = (k_0, 0, 0, k_3)$  and  $k_\perp = (0, k_x, k_y, 0)$ . Lastly,  $C_\parallel$  and  $C_\perp$  are functions of  $\mathcal{B}, k_\parallel^2, k_\perp^2$ . It is easy to note that, in consonance with the general parity structure of the theory, the basis tensors for this case are  $\tilde{F}_{\mu\nu}$ ,  $\tilde{E}_\mu^{(2)}k_{\nu\perp}$  and  $\tilde{E}_\nu^{(2)}k_{\mu\perp}$ . The most general tensorial structure including medium and field effects can be obtained using the four-vectors provided in the Appendix A of this text.

We would like to digress a little and point out that the polarization operator  $\Pi_{\mu\nu}^5$  given in Equation (20) can be obtained from a fermion triangle diagram having one axial current and two vector currents when one of the vector current carries the momentum that is ultra soft. It has been shown in Ref. [100] that this diagram won't admit any radiative corrections due to various non-renormalisation theorems.

#### 4. One Loop Calculation of the Axial Polarisation Tensor

Since we investigate the case with a background magnetic field, without any loss of generality, it can be taken to be in the  $z$ -direction. We denote the magnitude of this field by  $\mathcal{B}$ . Ignoring first the presence of the medium, the electron propagator in such a field can be written down following Schwinger's approach [101–103]:

$$iS_B^V(p) = \int_0^\infty ds e^{\Phi(p,s)} G(p,s) \quad (21)$$

where  $\Phi$  and  $G$  are as given below:

$$\Phi(p,s) \equiv is \left( p_\parallel^2 - \frac{\tan(e\mathcal{B}s)}{e\mathcal{B}s} p_\perp^2 - m^2 \right) - \epsilon|s| \quad (22)$$

$$\begin{aligned} G(p,s) &\equiv \frac{e^{ie\mathcal{B}s\sigma_z}}{\cos(e\mathcal{B}s)} \left( \not{p}_\parallel + \frac{e^{-ie\mathcal{B}s\sigma_z}}{\cos(e\mathcal{B}s)} \not{p}_\perp + m \right) \\ &= \left[ (1 + i\sigma_z \tan(e\mathcal{B}s))(\not{p}_\parallel + m) + \sec^2(e\mathcal{B}s) \not{p}_\perp \right], \end{aligned} \quad (23)$$

where

$$\sigma_z = i\gamma_1\gamma_2 = -\gamma_0\gamma_3\gamma_5 \quad (24)$$

and we have used

$$e^{ie\mathcal{B}s\sigma_z} = \cos(e\mathcal{B}s) + i\sigma_z \sin(e\mathcal{B}s) \quad (25)$$

and  $m$  is electron mass. To make the expressions transparent, we specify our convention in the following way:

$$\begin{aligned}\not{p}_{\parallel} &= \gamma_0 p^0 + \gamma_3 p^3 \\ \not{p}_{\perp} &= \gamma_1 p^1 + \gamma_2 p^2 \\ p_{\parallel}^2 &= p_0^2 - p_3^2 \\ p_{\perp}^2 &= p_1^2 + p_2^2.\end{aligned}$$

Of course, in the range of integration indicated in Equation (21),  $s$  is never negative and hence  $|s|$  equals  $s$ . In the presence of a background medium, the above propagator is now modified to [104]:

$$iS(p) = iS_B^V(p) + S_B^{\eta}(p) \quad (26)$$

where

$$S_B^{\eta}(p) \equiv -\eta_F(p) \left[ iS_B^V(p) - i\bar{S}_B^V(p) \right] \quad (27)$$

and

$$\bar{S}_B^V(p) \equiv \gamma_0 S_B^{V\dagger}(p) \gamma_0 \quad (28)$$

for a fermion propagator, such that

$$S_B^{\eta}(p) = -\eta_F(p) \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} G(p,s). \quad (29)$$

Here,  $\eta_F(p)$  contains the distribution function for the fermions and the anti-fermions:

$$\eta_F(p) = \Theta(p \cdot u) f_F(p, \mu, \beta) + \Theta(-p \cdot u) f_F(-p, -\mu, \beta), \quad (30)$$

$f_F$  denotes the Fermi–Dirac distribution function:

$$f_F(p, \mu, \beta) = \frac{1}{e^{\beta(p \cdot u - \mu)} + 1} \quad (31)$$

and  $\Theta$  is the step function given by:

$$\Theta(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x < 0. \end{cases}$$

Here, the centre of mass four velocity of the medium is  $u$ , in the rest frame, it looks like (in units  $\hbar = c = k_B = 1$ .)  $u^{\mu} = (1, 0, 0, 0)$ ,  $\beta = \frac{1}{T}$ , when  $T$  is the temperature and  $\mu$  is the chemical potential.

#### 4.1. The Expression for $\Pi_{\mu\nu}^5$ in Thermal Medium and in the Presence of a Background Uniform Magnetic Field

The axial polarisation tensor  $\Pi_{\mu\nu}^5$  is expressed as

$$i\Pi_{\mu\nu}^5 = (-ie)^2 (-1) \int \frac{d^4 p}{(2\pi)^4} \text{Tr}[\gamma_{\mu} \gamma_5 iS(p) \gamma_{\nu} iS(p')]. \quad (32)$$



Leaving out the vacuum contribution (the contribution devoid of any thermal effects) and the contributions with two thermal factors, we are left with

$$i\Pi_{\mu\nu}^5(k) = (-ie)^2(-1) \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ \gamma_\mu \gamma_5 S_B^\eta(p) \gamma_\nu iS_B^V(p') + \gamma_\mu \gamma_5 iS_B^V(p) \gamma_\nu S_B^\eta(p') \right]. \quad (33)$$

The vacuum part has already been done in [98] and the thermal part is related with pure absorption effects in the medium, which we are leaving out for the time being.

Using the form of the fermion propagator in a magnetic field in the presence of a thermal medium, as given by expressions (21) and (29), we get

$$\begin{aligned} i\Pi_{\mu\nu}^5(k) &= -(-ie)^2(-1) \int \frac{d^4p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \\ &\quad \times \int_0^{\infty} ds' e^{\Phi(p',s')} [\text{Tr} [\gamma_\mu \gamma_5 G(p,s) \gamma_\nu G(p',s')] \eta_F(p) \\ &\quad + \text{Tr} [\gamma_\mu \gamma_5 G(-p',s') \gamma_\nu G(-p,s)] \eta_F(-p)] \\ &= -(-ie)^2(-1) \int \frac{d^4p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} R_{\mu\nu}(p, p', s, s'), \end{aligned} \quad (34)$$

where  $R_{\mu\nu}(p, p', s, s')$  contains the trace part.

#### 4.2. $R_{\mu\nu}$ to Even and Odd Orders in the Magnetic Field

We calculate  $R_{\mu\nu}(p, p', s, s')$  to even and odd orders in the external magnetic field and call them  $R_{\mu\nu}^{(e)}$  and  $R_{\mu\nu}^{(o)}$ . The reason for doing this is that the two contributions have different properties as far as their dependence on medium is concerned, a topic which will be discussed in the concluding section. Calculating the traces, we obtain

$$\begin{aligned} R_{\mu\nu}^{(e)} &= 4i\eta_-(p) \left[ \varepsilon_{\mu\nu\alpha\parallel\beta\parallel} p^{\alpha\parallel} p'^{\beta\parallel} (1 + \tan(eBs) \tan(eBs')) \right. \\ &\quad + \varepsilon_{\mu\nu\alpha\parallel\beta\perp} p^{\alpha\parallel} p'^{\beta\perp} \sec^2(eBs') + \varepsilon_{\mu\nu\alpha\perp\beta\parallel} p^{\alpha\perp} p'^{\beta\parallel} \sec^2(eBs) \\ &\quad \left. + \varepsilon_{\mu\nu\alpha\perp\beta\perp} p^{\alpha\perp} p'^{\beta\perp} \sec^2(eBs) \sec^2(eBs') \right] \end{aligned} \quad (35)$$

and

$$\begin{aligned} R_{\mu\nu}^{(o)} &= 4i\eta_+(p) \left[ m^2 \varepsilon_{\mu\nu 12} (\tan(eBs) + \tan(eBs')) \right. \\ &\quad + \left\{ (g_{\mu\alpha\parallel} p^{\widetilde{\alpha\parallel}} p'_{\nu\parallel} - g_{\mu\nu} p'_{\alpha\parallel} p^{\widetilde{\alpha\parallel}} + g_{\nu\alpha\parallel} p^{\widetilde{\alpha\parallel}} p'_{\mu\parallel}) \right. \\ &\quad + \left. (g_{\mu\alpha\parallel} p^{\widetilde{\alpha\parallel}} p'_{\nu\perp} + g_{\nu\alpha\parallel} p^{\widetilde{\alpha\parallel}} p'_{\mu\perp}) \sec^2(eBs') \right\} \tan(eBs) \\ &\quad + \left\{ (g_{\mu\alpha\parallel} p'^{\widetilde{\alpha\parallel}} p_{\nu\parallel} - g_{\mu\nu} p_{\alpha\parallel} p'^{\widetilde{\alpha\parallel}} + g_{\nu\alpha\parallel} p'^{\widetilde{\alpha\parallel}} p_{\mu\parallel}) \right. \\ &\quad + \left. (g_{\mu\alpha\parallel} p'^{\widetilde{\alpha\parallel}} p_{\nu\perp} + g_{\nu\alpha\parallel} p'^{\widetilde{\alpha\parallel}} p_{\mu\perp}) \sec^2(eBs) \right\} \tan(eBs') \left. \right]. \end{aligned} \quad (36)$$

Here,

$$\eta_+(p) = \eta_F(p) + \eta_F(-p) \quad (37)$$

$$\eta_-(p) = \eta_F(p) - \eta_F(-p) \quad (38)$$

which contain the information about the distribution functions. In addition, it should be noted that, in our convention,

$$a_\mu b^{\widetilde{\mu\parallel}} = a_0 b^3 + a_3 b^0.$$

If we concentrate on the rest frame of the medium, then  $p \cdot u = p_0$ . Thus, the distribution function does not depend on the spatial components of  $p$ . In this case, we can write the expressions of  $R_{\mu\nu}^{(e)}$  and  $R_{\mu\nu}^{(o)}$  using the relations derived earlier [105] inside the integral sign, as

$$p^{\beta\perp} \stackrel{\circ}{=} -\frac{\tan(eBs')}{\tan(eBs) + \tan(eBs')} k^{\beta\perp} \quad (39)$$

$$p'^{\beta\perp} \stackrel{\circ}{=} \frac{\tan(eBs)}{\tan(eBs) + \tan(eBs')} k^{\beta\perp} \quad (40)$$

$$p_{\perp}^2 \stackrel{\circ}{=} \frac{1}{\tan(eBs) + \tan(eBs')} \left[ -ieB + \frac{\tan(eBs')^2}{\tan(eBs) + \tan(eBs')} k_{\perp}^2 \right], \quad (41)$$

$$p_{\perp}'^2 \stackrel{\circ}{=} \frac{1}{\tan(eBs) + \tan(eBs')} \left[ -ieB + \frac{\tan(eBs')^2}{\tan(eBs) + \tan(eBs')} k_{\perp}^2 \right], \quad (42)$$

$$m^2 \stackrel{\circ}{=} \left( i \frac{d}{ds} + (p_{\parallel}^2 - \sec^2(eBs) p_{\perp}^2) \right) \quad (43)$$

and get

$$R_{\mu\nu}^{(e)} \stackrel{\circ}{=} 4i\eta_-(p_0) \left[ \varepsilon_{\mu\nu\alpha\parallel\beta\parallel} p^{\alpha\parallel} p'^{\beta\parallel} (1 + \tan(eBs) \tan(eBs')) \right. \\ \left. + \varepsilon_{\mu\nu\alpha\parallel\beta\perp} p^{\alpha\parallel} p'^{\beta\perp} \sec^2(eBs') + \varepsilon_{\mu\nu\alpha\perp\beta\parallel} p^{\alpha\perp} p'^{\beta\parallel} \sec^2(eBs) \right] \quad (44)$$

and

$$R_{\mu\nu}^{(o)} \stackrel{\circ}{=} 4i\eta_+(p_0) \left[ -\varepsilon_{\mu\nu 12} \left\{ \frac{\sec^2(eBs) \tan^2(eBs')}{\tan(eBs) + \tan(eBs')} k_{\perp}^2 + (k \cdot p)_{\parallel} (\tan(eBs) + \tan(eBs')) \right\} \right. \\ + 2\varepsilon_{\mu 12\alpha\parallel} (p'_{\nu\parallel} p^{\alpha\parallel} \tan(eBs) + p_{\nu\parallel} p'^{\alpha\parallel} \tan(eBs')) \\ + g_{\mu\alpha\parallel} k_{\nu\perp} \left\{ p^{\tilde{\alpha}\parallel} (\tan(eBs) - \tan(eBs')) - k^{\tilde{\alpha}\parallel} \frac{\sec^2(eBs) \tan^2(eBs')}{\tan(eBs) + \tan(eBs')} \right\} \\ + \{g_{\mu\nu} (p \cdot \tilde{k})_{\parallel} + g_{\nu\alpha\parallel} p^{\tilde{\alpha}\parallel} k_{\mu\perp}\} (\tan(eBs) - \tan(eBs')) \\ \left. + g_{\nu\alpha\parallel} k^{\tilde{\alpha}\parallel} p_{\mu\perp} \sec^2(eBs) \tan(eBs') \right]. \quad (45)$$

The  $\stackrel{\circ}{=}$  symbol signifies that the above relations are not proper equations, and the equality holds only inside the momentum integrals in Equation (34).

## 5. Gauge Invariance

### 5.1. Gauge Invariance for $\Pi_{\mu\nu}^5$ to Even Orders in the External Field

The axial polarisation tensor even in the external field is given by

$$\Pi_{\mu\nu}^{5(e)} = -(-ie)^2(-1) \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} R_{\mu\nu}^{(e)}(p, p', s, s'). \quad (46)$$

Using Equation (44) in the rest frame of the medium, we have

$$R_{\mu\nu}^{(e)} \stackrel{\circ}{=} 4i\eta_-(p_0) \left[ \varepsilon_{\mu\nu\alpha\parallel\beta\parallel} p^{\alpha\parallel} p'^{\beta\parallel} (1 + \tan(eBs) \tan(eBs')) \right. \\ \left. + \varepsilon_{\mu\nu\alpha\parallel\beta\perp} p^{\alpha\parallel} p'^{\beta\perp} \sec^2(eBs') + \varepsilon_{\mu\nu\alpha\perp\beta\parallel} p^{\alpha\perp} p'^{\beta\parallel} \sec^2(eBs) \right]. \quad (47)$$

Noting that it is possible to write

$$q^{\alpha} p_{\alpha} = q^{\alpha\parallel} p_{\alpha\parallel} + q^{\alpha\perp} p_{\alpha\perp}.$$

Equation (47) can be written as

$$R_{\mu\nu}^{(e)} \stackrel{\circ}{=} 4i\eta_{-}(p_0) \left[ (\varepsilon_{\mu\nu\alpha\beta} p^{\alpha} p'^{\beta} - \varepsilon_{\mu\nu\alpha\beta_{\perp}} p^{\alpha} p'^{\beta_{\perp}} - \varepsilon_{\mu\nu\alpha_{\perp}\beta} p^{\alpha_{\perp}} p'^{\beta}) (1 + \tan(e\mathcal{B}s) \tan(e\mathcal{B}s')) \right. \\ \left. + \varepsilon_{\mu\nu\alpha\beta_{\perp}} p^{\alpha} p'^{\beta_{\perp}} \sec^2(e\mathcal{B}s') + \varepsilon_{\mu\nu\alpha_{\perp}\beta} p^{\alpha_{\perp}} p'^{\beta} \sec^2(e\mathcal{B}s) \right]. \quad (48)$$

Here, throughout, we have omitted terms such as  $\varepsilon_{\mu\nu\alpha_{\perp}\beta_{\perp}} p^{\alpha_{\perp}} p'^{\beta_{\perp}}$ , since, by the application of Equation (39), we have

$$\varepsilon_{\mu\nu\alpha_{\perp}\beta_{\perp}} p^{\alpha_{\perp}} p'^{\beta_{\perp}} = \varepsilon_{\mu\nu\alpha_{\perp}\beta_{\perp}} p^{\alpha_{\perp}} p^{\beta_{\perp}} + \varepsilon_{\mu\nu\alpha_{\perp}\beta_{\perp}} p^{\alpha_{\perp}} k^{\beta_{\perp}} \\ \stackrel{\circ}{=} - \frac{\tan(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} \varepsilon_{\mu\nu\alpha_{\perp}\beta_{\perp}} k^{\alpha_{\perp}} k^{\beta_{\perp}}$$

which is zero. After rearranging the terms appearing in Equation (48), and by the application of Equations (39) and (40), we arrive at the expression

$$R_{\mu\nu}^{(e)} \stackrel{\circ}{=} 4i\eta_{-}(p_0) \left[ \varepsilon_{\mu\nu\alpha\beta} p^{\alpha} k^{\beta} (1 + \tan(e\mathcal{B}s) \tan(e\mathcal{B}s')) \right. \\ \left. + \varepsilon_{\mu\nu\alpha\beta_{\perp}} k^{\alpha} k^{\beta_{\perp}} \tan(e\mathcal{B}s) \tan(e\mathcal{B}s') \frac{\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} \right]. \quad (49)$$

Because of the presence of terms like  $\varepsilon_{\mu\nu\alpha\beta} k^{\beta}$  and  $\varepsilon_{\mu\nu\alpha\beta_{\perp}} k^{\alpha}$  if we contract  $R_{\mu\nu}^{(e)}$  by  $k^{\nu}$ , it vanishes.

## 5.2. Gauge Invariance for $\Pi_{\mu\nu}^5$ to Odd Orders in the External Field

The axial polarisation tensor odd in the external field is given by

$$\Pi_{\mu\nu}^{5(o)} = -(-ie)^2(-1) \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} R_{\mu\nu}^{(o)}(p, p', s, s'), \quad (50)$$

where  $R_{\mu\nu}^{(o)}(p, p', s, s')$  is given by Equation (45). The general gauge invariance condition in this case

$$k^{\nu} \Pi_{\mu\nu}^{5(o)} = 0 \quad (51)$$

can always be written down in terms of the following two equations:

$$k^{\nu} \Pi_{\mu_{\parallel}\nu}^{5(o)} = 0 \quad (52)$$

$$k^{\nu} \Pi_{\mu_{\perp}\nu}^{5(o)} = 0 \quad (53)$$

where  $\Pi_{\mu_{\parallel}\nu}^{5(o)}$  is that part of  $\Pi_{\mu\nu}^{5(o)}$  where the index  $\mu$  can take the values 0 and 3 only. Similarly,  $\Pi_{\mu_{\perp}\nu}^{5(o)}$  stands for the part of  $\Pi_{\mu\nu}^{5(o)}$ , where  $\mu$  can take the values 1 and 2 only.  $\Pi_{\mu_{\parallel}\nu}^{5(o)}$  contains  $R_{\mu_{\parallel}\nu}^{(o)}(p, p', s, s')$ , which, from Equation (45), is as follows:

$$R_{\mu_{\parallel}\nu}^{(o)} \stackrel{\circ}{=} 4i\eta_{+}(p_0) \left[ -\varepsilon_{\mu_{\parallel}\nu 12} \left\{ \frac{\sec^2(e\mathcal{B}s) \tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} k_{\perp}^2 + (k \cdot p)_{\parallel} (\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')) \right\} \right. \\ + 2\varepsilon_{\mu_{\parallel} 12\alpha_{\parallel}} (p'_{\nu_{\parallel}} p^{\alpha_{\parallel}} \tan(e\mathcal{B}s) + p_{\nu_{\parallel}} p'^{\alpha_{\parallel}} \tan(e\mathcal{B}s')) \\ + g_{\mu_{\parallel}\alpha_{\parallel}} k_{\nu_{\perp}} \left\{ p^{\tilde{\alpha}_{\parallel}} (\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s')) - k^{\tilde{\alpha}_{\parallel}} \frac{\sec^2(e\mathcal{B}s) \tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} \right\} \\ \left. + g_{\mu_{\parallel}\nu} (p \cdot \tilde{k})_{\parallel} (\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s')) \right] \quad (54)$$

and  $\Pi_{\mu_{\perp}\nu}^{5(o)}$  contains  $R_{\mu_{\perp}\nu}^{(o)}(p, p', s, s')$ , which is

$$R_{\mu_{\perp}\nu}^{(o)} \stackrel{\circ}{=} 4i\eta_+(p_0) \left[ \{g_{\mu_{\perp}\nu}(p \cdot \tilde{k})_{\parallel} + g_{\nu\alpha_{\parallel}} p^{\tilde{\alpha}_{\parallel}} k_{\mu_{\perp}}\} (\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s')) + g_{\nu\alpha_{\parallel}} k^{\tilde{\alpha}_{\parallel}} p_{\mu_{\perp}} \sec^2(e\mathcal{B}s) \tan(e\mathcal{B}s') \right]. \quad (55)$$

Equations (52) and (53) imply that one should have the following relations satisfied,

$$k^{\nu} \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} R_{\mu_{\perp}\nu}^{(o)} = 0 \quad (56)$$

and

$$k^{\nu} \cdot \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} R_{\mu_{\parallel}\nu}^{(o)} = 0. \quad (57)$$

Out of the two above equations, Equation (56) can be verified easily since

$$k^{\nu} R_{\mu_{\perp}\nu} = 0. \quad (58)$$

Now, we look at Equation (57). We explicitly consider the case  $\mu_{\parallel} = 3$  (the  $\mu_{\parallel} = 0$  case leads to similar results). For  $\mu_{\parallel} = 3$ ,

$$k^{\nu} R_{3\nu}^{(o)} \stackrel{\circ}{=} -p_0 [(p'_{\parallel}{}^2 - p_{\parallel}^2)(\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')) - k_{\perp}^2 (\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s'))] (4i\eta_+(p_0)). \quad (59)$$

Apart from the small convergence factors,

$$\begin{aligned} \frac{i}{e\mathcal{B}} (\Phi(p, s) + \Phi(p', s')) &= (p'_{\parallel}{}^2 + p_{\parallel}^2 - 2m^2) \xi - (p'_{\parallel}{}^2 - p_{\parallel}^2) \zeta \\ &\quad - p_{\perp}^2 \tan(\xi - \zeta) - p_{\perp}^2 \tan(\xi + \zeta) \end{aligned} \quad (60)$$

where we have defined the parameters

$$\begin{aligned} \xi &= \frac{1}{2} e\mathcal{B}(s + s') \\ \zeta &= \frac{1}{2} e\mathcal{B}(s - s'). \end{aligned} \quad (61)$$

From the last two equations, we can write

$$ie\mathcal{B} \frac{d}{d\zeta} e^{\Phi(p,s) + \Phi(p',s')} = e^{\Phi(p,s) + \Phi(p',s')} (p'_{\parallel}{}^2 - p_{\parallel}^2 - p_{\perp}^2 \sec^2(\xi - \zeta) + p_{\perp}^2 \sec^2(\xi + \zeta)) \quad (62)$$

which implies

$$p'_{\parallel}{}^2 - p_{\parallel}^2 = ie\mathcal{B} \frac{d}{d\zeta} + [p_{\perp}^2 \sec^2(e\mathcal{B}s') - p_{\perp}^2 \sec^2(e\mathcal{B}s)]. \quad (63)$$

The equation above is valid in the sense that both sides of it actually act upon  $e^{\tilde{\Phi}(p,s,p',s')}$ , where

$$\tilde{\Phi}(p, p', s, s') = \Phi(p, s) + \Phi(p', s'). \quad (64)$$

From Equations (59) and (63), we have

$$k^\nu R_{3\nu} e^{\tilde{\Phi}} \stackrel{\circ}{=} -4i\eta_+(p_0)p_0 \left[ (p_\perp'^2 \sec^2(eBs) - p_\perp^2 \sec^2(eBs))(\tan(eBs) + \tan(eBs')) - k_\perp^2 (\tan(eBs) - \tan(eBs')) + ieBp_0(\tan(eBs) + \tan(eBs')) \frac{d}{d\xi} \right] e^{\tilde{\Phi}}. \quad (65)$$

Now, using the the expressions for  $p_\perp^2$  and  $p_\perp'^2$  from Equations (41) and (42), we can write

$$k^\nu R_{3\nu} e^{\tilde{\Phi}} \stackrel{\circ}{=} 4eB\eta_+(p_0)p_0 \left[ (\sec^2(eBs) - \sec^2(eBs')) + (\tan(eBs) + \tan(eBs')) \frac{d}{d\xi} \right] e^{\tilde{\Phi}}. \quad (66)$$

The above equation can also be written as

$$k^\nu \times R_{3\nu} e^{\tilde{\Phi}} \stackrel{\circ}{=} 4eB\eta_+(p_0)p_0 \frac{d}{d\xi} \left[ e^{\tilde{\Phi}} (\tan(eBs) + \tan(eBs')) \right]. \quad (67)$$

Transforming to  $\xi, \zeta$  variables and, using the above equation, we can write the parametric integrations (integrations over  $s$  and  $s'$ ) on the left-hand side of Equation (57) as

$$\int_{-\infty}^{\infty} ds \int_0^{\infty} ds' k^\nu R_{3\nu} e^{\tilde{\Phi}} = \frac{8\eta_+(p_0)p_0}{eB} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \Theta(\xi - \zeta) \frac{d}{d\xi} \mathcal{F}(\xi, \zeta) \quad (68)$$

where

$$\mathcal{F}(\xi, \zeta) = e^{\tilde{\Phi}} (\tan(eBs) + \tan(eBs')).$$

The integration over the  $\xi$  and  $\zeta$  variables in Equation (68) can be represented as

$$\begin{aligned} & \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \Theta(\xi - \zeta) \frac{d}{d\xi} \mathcal{F}(\xi, \zeta) \\ &= \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \left[ \frac{d}{d\xi} \{ \Theta(\xi - \zeta) \mathcal{F}(\xi, \zeta) \} - \delta(\xi - \zeta) \mathcal{F}(\xi, \zeta) \right] \\ &= - \int_{-\infty}^{\infty} d\zeta \mathcal{F}(\xi, \zeta) \end{aligned} \quad (69)$$

Here, the second step follows from the first one as the first integrand containing the  $\Theta$  function vanishes at both limits of the integration. The remaining integral is now only a function of  $\zeta$  and is even in  $p_0$ . However, in Equation (68), we have  $\eta_+(p_0)p_0$  sitting, which makes the the integrand odd under  $p_0$  integration in the left-hand side of Equation (57), as  $\eta_+(p_0)$  is an even function in  $p_0$ . Thus, the  $p_0$  integral as it occurs in the left-hand side of Equation (57) vanishes as expected, yielding the required result shown in Equation (52).

## 6. Effective Charge

Now, we concentrate on the neutrino effective charge. From the outset, it is to be made clear that we are only calculating the axial contribution to the effective charge (in a forthcoming publication, we will comment on the vector contribution to the effective charge of the neutrino [106,107]). We can now write the full expression of the axial polarisation tensor as

$$i\Pi_{\mu\nu}^5(k) = -(-ie)^2(-1) \int \frac{d^4p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} \left[ R_{\mu\nu}^{(o)} + R_{\mu\nu}^{(e)} \right], \quad (70)$$

where  $R_{\mu\nu}^{(o)}$  and  $R_{\mu\nu}^{(e)}$  are given by Equations (45) and (44) in the rest frame of the medium.

### 6.1. Effective Charge to Odd Orders in External Field

In the limit when the external momentum tends to zero, only two terms survive from  $\Pi_{\mu\nu}^5(k)$ . Denoting  $\Pi_{\mu\nu}^5(k_0 = 0, \vec{k} \rightarrow 0) = \Pi_{\mu\nu}^5$ , we obtain

$$\begin{aligned} \Pi_{\mu 0}^5 = & \lim_{k_0=0, \vec{k} \rightarrow 0} 4e^2 \int \frac{d^4 p}{(2\pi)^4} \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} (\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')) \\ & \times \eta_+(p_0) [2p_0^2 - (k \cdot p)_{\parallel}] \varepsilon_{\mu 012} \end{aligned} \quad (71)$$

The other terms turn out to be zero in this limit. The above equation shows that, except the exponential functions, the integrand is free of the perpendicular components of momenta. This implies that we can integrate out the perpendicular component of the loop momentum. Upon performing the Gaussian integration over the perpendicular components and taking the limit  $k_{\perp} \rightarrow 0$ , we obtain

$$\begin{aligned} \Pi_{\mu 0}^5 = & \lim_{k_0=0, \vec{k} \rightarrow 0} \frac{(-4ie^3 B)}{4\pi} \int \frac{d^2 p_{\parallel}}{(2\pi)^2} \int_{-\infty}^{\infty} ds e^{is(p_{\parallel}^2 - m^2) - \varepsilon|s|} \int_0^{\infty} ds' e^{is'(p_{\parallel}'^2 - m^2) - \varepsilon|s'|} \\ & \times \eta_+(p_0) [2p_0^2 - (k \cdot p)_{\parallel}] \varepsilon_{\mu 012}. \end{aligned} \quad (72)$$

It is worth noting that the  $s$  integral gives

$$\int_{-\infty}^{\infty} ds e^{is(p_{\parallel}^2 - m^2) - \varepsilon|s|} = 2\pi \delta(p_{\parallel}^2 - m^2) \quad (73)$$

and the  $s'$  integral gives

$$\int_0^{\infty} ds' e^{is'(p_{\parallel}'^2 - m^2) - \varepsilon|s'|} = \frac{i}{(p_{\parallel}'^2 - m^2) + i\varepsilon}. \quad (74)$$

Using the above results in Equation (72) and using the delta function constraint, we arrive at

$$\Pi_{\mu 0}^5 = \lim_{k_0=0, \vec{k} \rightarrow 0} 2(e^3 B) \int \frac{d^2 p_{\parallel}}{(2\pi)^2} \delta(p_{\parallel}^2 - m^2) \eta_+(p_0) \left[ \frac{2p_0^2}{(k_{\parallel}^2 + 2(p \cdot k)_{\parallel})} - \frac{1}{2} \right] \varepsilon_{\mu 012}. \quad (75)$$

In deriving Equation (75), pieces proportional to  $k_{\parallel}^2$  in the numerator were neglected. Now, if one makes the substitution,  $p'_{\parallel} \rightarrow (p_{\parallel} + k_{\parallel}/2)$  and sets  $k_0 = 0$ , one arrives at

$$\Pi_{\mu 0}^5 = - \lim_{k_0=0, \vec{k} \rightarrow 0} 2(e^3 B) \int \frac{dp_3}{(2\pi)^2} (n_+(E'_p) + n_-(E'_p)) \left[ \frac{E'_p}{p_3 k_3} + \frac{1}{2E'_p} \right] \varepsilon_{\mu 012}. \quad (76)$$

Here,  $n_{\pm}(E'_p)$  are the functions  $f_F(E'_p, -\mu, \beta)$ , and  $f_F(E'_p, \mu, \beta)$ , as given in Equation (31), which are nothing but the Fermi–Dirac distribution functions of the particles and the antiparticles in the medium. The new term  $E'_p$  is defined as follows:

$$E_p'^2 = [(p_3 - k_3/2)]^2 + m^2$$

and it can be expanded for small external momenta in the following way:

$$E_p'^2 \simeq p_3^2 + m^2 - p_3 k_3 = E_p^2 - p_3 k_3$$

where  $E_p^2 = p_3^2 + m^2$ . Noting that

$$E'_p = E_p - \frac{p_3 k_3}{2E_p} + O(k_3^2) \quad (77)$$

one can use this expansion in Equation (76), to arrive at:

$$\Pi_{\mu 0}^5 = - \lim_{k_0=0, \vec{k} \rightarrow 0} 2(e^3 \mathcal{B}) \int \frac{dp_3}{(2\pi)^2} \left( n_+(E'_p) + n_-(E'_p) \right) \left[ \frac{E_p}{p_3 k_3} \right] \varepsilon_{\mu 012}. \quad (78)$$

The expression for  $\eta_+(E'_p) = n_+(E'_p) + n_-(E'_p)$  when expanded in the powers of the external momentum  $k_3$  is given by

$$\eta_+(E'_p) = \left( 1 + \frac{1}{2} \frac{\beta p_3 k_3}{E_p} \right) \eta_+(E_p) \quad (79)$$

up to first order terms in the external momentum  $k_3$ . In the expression above,  $\beta = \frac{1}{T}$  in units where  $\hbar = c = k_B = 1$ .

#### 6.1.1. Effective Charge for $\mu \ll m$

In the limit, when  $\mu \ll m$ , one can use the following expansion,

$$\begin{aligned} \eta_+(E'_p) &= \left[ n_+(E'_p) + n_-(E'_p) \right] \\ &= 2 \sum_{n=0}^{\infty} (-1)^n \cosh([n+1]\beta\mu) e^{-(n+1)\beta E_p} \left( 1 + \frac{\beta p_3 k_3}{2E_p} + O(k_3^2) + \dots \right). \end{aligned} \quad (80)$$

Now, using Equation (80) in Equation (78), we get

$$\begin{aligned} \Pi_{\mu 0}^5 &= -\varepsilon_{\mu 012} \lim_{k_0=0, \vec{k} \rightarrow 0} (4e^3 \mathcal{B}) \sum_{n=0}^{\infty} (-1)^n \cosh([n+1]\beta\mu) \\ &\quad \times \int \frac{dp_3}{(2\pi)^2} e^{-(n+1)\beta E_p} \left[ \frac{E_p}{(p_3 k_3)} + \frac{\beta}{2} \right]. \end{aligned} \quad (81)$$

The first term vanishes by symmetry of the integral, but the second term is finite and thus we get:

$$\Pi_{\mu 0}^5 = -\beta \varepsilon_{\mu 012} \lim_{k_0=0, \vec{k} \rightarrow 0} \frac{(e^3 \mathcal{B})}{2\pi^2} \sum_{n=0}^{\infty} (-1)^n \cosh([n+1]\beta\mu) \int dp_3 e^{-(n+1)\beta E_p}. \quad (82)$$

To perform the momentum integration, use of the following integral transform turns out to be extremely convenient

$$e^{-\alpha\sqrt{s}} = \frac{\alpha}{2\sqrt{\pi}} \int_0^{\infty} du e^{-us - \frac{\alpha^2}{4u}} u^{-3/2}. \quad (83)$$

Identifying  $\sqrt{s}$  with  $E_p$  and  $[(n+1)\beta]$  as  $\alpha$  (since the square root opens up), one can easily perform the Gaussian  $p_3$  integration without any difficulty. The result is:

$$\begin{aligned} \Pi_{\mu 0}^5 &= -\beta \varepsilon_{\mu 012} \frac{(e^3 \mathcal{B})}{2\pi^2} \sum_{n=0}^{\infty} (-1)^n \cosh([n+1]\beta\mu) \\ &\quad \times (\beta(n+1)/2) \int du e^{-m^2 u - \frac{((n+1)\beta/2)^2}{u}} u^{-2}. \end{aligned} \quad (84)$$



Performing the  $u$  integration, the axial part of the effective charge of neutrino, being called  $e_{eff}^{v_a}$  (the subscript  $a$  in  $v_a$  is to denote the axial vector vertex origin of the same)—in the limit of  $m > \mu$ —turns out to be

$$e_{eff}^{v_a} = -\sqrt{2}g_A m \beta G_F \frac{e^2 \mathcal{B}}{\pi^2} (1-\lambda) \cos(\theta) \sum_{n=0}^{\infty} (-1)^n \cosh((n+1)\beta\mu) K_{-1}(m\beta(n+1)). \quad (85)$$

Here,  $\theta$  is the angle between the neutrino three momentum and the background magnetic field  $e\mathcal{B}$ . The superscript  $v_a$  on  $e_{eff}^{v_a}$  denotes that we are calculating the axial contribution of the effective charge.  $K_{-1}(m\beta(n+1))$  is the modified Bessel function (of the second kind) of order one (for this function,  $K_{-1}(x) = K_1(x)$ ), which sharply falls off as we move away from the origin in the positive direction. Recall that  $g_A$  is the axial coupling constant of the electron in the loop as already mentioned in Section 2. However, as the temperature tends to zero, Equation (85) seems to blow up because of the presence of  $m\beta$ , but  $K_{-1}(m\beta(n+1))$  would damp its growth as  $e^{-m\beta}$ , hence the result remains finite.

The factor of  $e^2 \mathcal{B}$  is the contribution of the magnetic field and  $m\beta$  along with  $\cosh((n+1)\beta\mu) K_{-1}(m\beta(n+1))$  are the contributions from the medium effects of the background magnetized media. The constant  $G_F = \frac{\sqrt{2}}{8} \left( \frac{e}{\sin\theta_W} \frac{1}{M_W} \right)^2$  is related to the mass of  $W$  and  $Z$  bosons and  $\lambda = \pm 1$  is the helicity of the neutrino spinors.

#### 6.1.2. Effective Charge for $\mu \gg m$

Here, we would try to estimate neutrino effective charge when  $\mu \gg m$  and  $\beta \neq \infty$ . We would like to emphasise that the last condition should be strictly followed, i.e., temperature  $T \neq 0$ . Using Equations (78) and (79), we would obtain

$$\Pi_{30}^5 = \frac{e^3 \mathcal{B}}{2\pi} \beta \int \frac{dp}{2\pi} \eta_+(E_p). \quad (86)$$

Neglecting  $m$  in the expression in  $E_p$ , we would obtain

$$\Pi_{30}^5 = \frac{e^3 \mathcal{B}}{2\pi^2} \ln[(1 + e^{\beta\mu})(1 + e^{-\beta\mu})]. \quad (87)$$

The same can also be written as

$$\Pi_{30}^5 = \frac{e^3 \mathcal{B}}{\pi^2} \ln\left(2 \cosh\left(\frac{\beta\mu}{2}\right)\right). \quad (88)$$

The expression for the effective charge then turns out to be

$$e_{eff}^{v_a} = -\sqrt{2}g_A G_F \frac{e^2 \mathcal{B}}{\pi^2} (1-\lambda) \cos(\theta) \ln\left(2 \cosh\left(\frac{\beta\mu}{2}\right)\right). \quad (89)$$

It can be seen from the expression above that only left-handed Weyl neutrinos acquire induced charge. The expressions for  $e_{eff}^{v_a}$  in the limit of  $\mu \ll m$  and  $\mu \gg m$  can further be cast, after expressing  $e^2 \mathcal{B} = \frac{e\mathcal{B}}{m^2/e} m^2 = \frac{e\mathcal{B}}{B_c} m^2$  and substituting the values of the physical constant  $G_F$ , in the following forms, for  $\mu \ll m$ :

$$\begin{aligned} e_{eff}^{v_a} &= -(3.036 \times 10^{-12}) e \left[ g_A \left( \frac{\mathcal{B}}{B_c} \right) \right] \left( \frac{\sqrt{2}}{\pi^2} \right) (1-\lambda) \cos(\theta) \\ &\times \sum_{n=0}^{\infty} (-1)^n \cosh((n+1)\beta\mu) [(m\beta) K_{-1}(m\beta(n+1))] \end{aligned} \quad (90)$$

and the same for  $\mu \gg m$ :

$$e_{\text{eff}}^{\nu_a} = -(3.036 \times 10^{-12})e \left[ g_A \left( \frac{\mathcal{B}}{B_c} \right) \right] \left( \frac{\sqrt{2}}{\pi^2} \right) (1-\lambda) \cos(\theta) \ln \left( 2 \cosh \left( \frac{\beta\mu}{2} \right) \right). \quad (91)$$

It should be noted that Equation (91) goes to zero as  $\mu \rightarrow 0$ . For a sufficiently dense medium, when  $\mu \gg m$  and low temperature,  $e_{\text{eff}}^{\nu_a}$  can be as close to unity. For example, when  $\mu \sim 10$  MeV,  $T \sim 10^{-4}$  and  $\mathcal{B}/B_c \sim 10^2$  and  $\theta = \pi/3$ , the  $e_{\text{eff}}^{\nu_a}$  may turn out to be the order of electric charge  $e$ .

## 6.2. $\nu_e$ Effective Charge at Even Order in the External Field and Their Coupling with the Magnetic Fields

From the part of  $\Pi_{\mu\nu}$ , which is even in the external fields, we see from Equation (49) that

$$\begin{aligned} R_{\mu 0}^{(e)} = 4i\eta_-(p_0) & \left[ \varepsilon_{\mu 0 \alpha \beta} p^\alpha k^\beta (1 + \tan(e\mathcal{B}s) \tan(e\mathcal{B}s')) \right. \\ & \left. + \varepsilon_{\mu 0 \alpha \beta} k^\alpha k^\beta \tan(e\mathcal{B}s) \tan(e\mathcal{B}s') \frac{\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} \right]. \quad (92) \end{aligned}$$

which shows that  $\Pi_{\mu\nu}^5(k)$  to even orders in the external field will vanish when  $k_0 \rightarrow 0$ ,  $\vec{k} \rightarrow 0$ . This implies that there will be no contribution to the effective neutrino charge from the sector which is even in the powers of  $\mathcal{B}$ . Can the neutrinos that are propagating in a magnetised plasma couple with the classical magnetic field? The situation is a little bit subtle here, as the vertex of the neutrinos with the dynamical photons do get changed here due to the presence of the magnetic field, but this change cannot induce any electromagnetic form factor responsible for coupling of the neutrinos with any magnetic field. In order to find the effective charge of the neutrinos which couples them with time independent magnetic field, one should look for (as given in Equation (7)) the  $\Gamma^i$ 's, where  $i = 1, 2$ . A magnetic field in the  $z$ -direction is given by a gauge where  $A_1, A_2$  are both non zero, or one of them is nonzero. Thus, to calculate the charge that is essential for the neutrino current to couple with a magnetic field, one has to put the index  $\nu = 1, 2$  in Equation (70) and take the limit  $k_0 \rightarrow 0, \vec{k} \rightarrow 0$  and see which component of  $\Pi_{\mu 1}^5(k)$  exists in the pre-mentioned momentum limit. For the odd  $\mathcal{B}$  part, we see from Equation (45) that

$$\begin{aligned} R_{\mu 1}^{(o)} = 4i\eta_+(p_0) & \left[ g_{\mu \alpha} k_1 \left\{ p^{\alpha} (\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s')) - k^{\alpha} \frac{\sec^2(e\mathcal{B}s) \tan^2(e\mathcal{B}s')}{\tan(e\mathcal{B}s) + \tan(e\mathcal{B}s')} \right\} \right. \\ & \left. + g_{\mu 1} (p \cdot \vec{k}) (\tan(e\mathcal{B}s) - \tan(e\mathcal{B}s')) \right] \quad (93) \end{aligned}$$

which goes to zero as the photon momentum tends to zero. By the same argument, it follows that, for  $\nu = 2$ , there is vanishing contribution. Thus, it shows that there is no effective magnetic coupling from the  $\mathcal{B}$  odd part. For the  $\mathcal{B}$  even part, it is seen from Equation (44) that only  $R_{12}^{(e)}$  survives, and is given by

$$R_{12}^{(e)} = 4i\eta_-(p_0) \left[ (p \cdot \vec{k}) (1 + \tan(e\mathcal{B}s) \tan(e\mathcal{B}s')) \right], \quad (94)$$

which vanishes in the limit when the external momentum goes to zero. Thus, from this, we can say that  $\Pi_{\mu\nu}^5$  has no contribution for any charge of the neutrinos which can couple them with the magnetic field.

## 7. Conclusions

In our analysis, we have calculated the contributions to  $\Pi_{\mu\nu}^{5(o)}(k)$  to odd and even orders in the external constant magnetic field. The main reason for doing so is the fact that  $\Pi_{\mu\nu}^{5(o)}(k)$  and  $\Pi_{\mu\nu}^{5(e)}(k)$ , the axial polarisation tensors to odd and even powers in  $e\mathcal{B}$ , have different dependence on the background matter. Pieces proportional to even powers in  $\mathcal{B}$  are proportional to  $\eta_-(p_0)$ , an odd function of the chemical potential. On the other hand, pieces proportional to odd powers in  $\mathcal{B}$  depend on  $\eta_+(p_0)$ , and are even in  $\mu$  and, as a result, it survives in the limit  $\mu \rightarrow 0$ . Equations (90) and (91) corroborate this observation. The underlying theory, being a theory of weak interaction, violates parity but preserves CP and CPT. The same can be verified from the C, P and T transformation properties of Equations (35) and (36). We have estimated the field dependent part of the induced neutrino effective charge  $e_{eff}^{\nu_a}$  from odd  $\mathcal{B}$  and even  $\mu$  contribution of  $\Pi_{\mu\nu}^5$ .

The leading order estimate of  $e_{eff}^{\nu_a}$  that follows from our work can be grouped into two categories: (i) non degenerate  $\mu < m$  and (ii) highly degenerate  $\mu \gg m$ . The complete estimate of  $e_{eff}^{\nu_a}$  belonging to the first class can be approximated by keeping the leading order effects in  $(m/T)$  for  $T \ll m$ , and it turns out to be

$$e_{eff}^{\nu_a} \sim -(3.036 \times 10^{-12}) \left[ g_A \left( \frac{\mathcal{B}}{B_c} \right) \frac{1}{\pi^{3/2}} \right] \left( \sqrt{\frac{m}{T}} \right) e^{-m/T} \cosh(\mu/T) (1 - \lambda) \cos \theta e. \quad (95)$$

The same belonging to the second class can similarly be estimated for  $\mu \gg m \gg T$  as

$$e_{eff}^{\nu_a} \sim -(3.036 \times 10^{-12}) \left[ g_A \left( \frac{\mathcal{B}}{B_c} \right) \frac{\sqrt{2}}{\pi^2} \right] \left( \frac{\beta\mu}{2} \right) (1 - \lambda) \cos \theta e. \quad (96)$$

For Weyl neutrinos, one can notice that only left-handed neutrinos contribute to the effective charge provided the angle between neutrino momentum and the magnetic field  $\mathcal{B}$  is less than  $\pi/2$ . We would like to point out that the neutrino charge for  $\mathcal{B}$  dependent part has turned out to be direction-dependent—a result that is expected because of the presence of magnetic field  $\mathcal{B}$  that breaks the isotropy of the system.

For the sake of completeness, we next provide the estimate of the same originating from the unmagnetized medium; the same is

$$e_{eff}^{\nu} = -0.68 \times 10^{-31} e \left( \frac{1cm}{r_D} \right)^2, \quad (97)$$

where  $r_D^2 = \frac{T}{n_e e^2}$  is the Debye radius. This result was originally obtained in [82,83,85].

A magnitude wise comparison of neutrino charge owing to the vector vertex contribution from unmagnetized medium and that owing to axial vector vertex contribution from magnetized medium would reveal why, in some environments, energy transfer mechanism is isotropic and in some of the same is anisotropic. For instance, if we make a comparison of the induced charge that neutrinos acquire in SNII, white dwarves and red-giants with the same that neutrinos acquire in the environment of GRB, Magnetar or astrophysical objects having jets associated with strong magnetic fields, we shall see that the size of  $e_{eff}^{\nu_a}$  in the latter dominates over  $e_{eff}^{\nu}$  of the former.

Having the limiting expressions of neutrino charge for different physical situations, it makes sense to compare the contributions and determine the dominant one. Using the limiting expressions, the ratio of the neutrino axial charge in magnetized medium to that in unmagnetized medium for  $T \ll m$  and  $\mu \ll T$  turns out to be

$$\frac{e_{eff}^{\nu_a}}{e_{eff}^{\nu}} \sim \sqrt{2} \frac{g_A \pi^2}{(1 + 2 \sin^2 \theta_W)} \left( \frac{\mathcal{B}}{B_c} \right) \left( \frac{m}{T} \right) \cos \theta, \quad (98)$$

and, in the degenerate limit, it is

$$\frac{e_{eff}^{\nu_a}}{e_{eff}^{\nu}} \sim \frac{3\pi^2}{2} \frac{g_A}{(1 + 2\sin^2\theta_W)} \left(\frac{B}{B_c}\right) \left(\frac{m}{T}\right) \cos\theta. \quad (99)$$

Notable points in these two expressions are the appearance of  $g_A$  in the ratio and the appearance of the factor  $\left(\frac{B}{B_c}\right) \left(\frac{m}{T}\right) \cos\theta$ . Thus, for  $\cos\theta \sim 1$  and  $\left(\frac{B}{B_c}\right) \left(\frac{m}{T}\right) \gg 1$ , the axial induced neutrino charge  $e_{eff}^{\nu_a}$  can dominate over that due to unmagnetized or weakly magnetized media. Thus, in a strongly magnetized media, neutrino induced high energy processes are likely to introduce anisotropy in the physical processes taking place in astrophysical situations e.g., GRB, magnetar and anisotropic distribution of isolated moving pulsars with initial kick. This is what was one of the motivating issues behind this investigation, laid out in the Introduction section of this work.

In view of the investigations being carried out to find the intrinsic charge of neutrino, the induced charge of the same due to an unmagnetized medium acquires some importance—the reason being the presence of an ambient plasma of varying density all over the universe that can contribute to the intrinsic charge as the terrestrial and celestial experiments are being performed. For that purpose, we have considered three astrophysical objects (i) type-II supernovae ( $\mu = 350$  MeV,  $T = 30 - 60$  MeV), red giant ( $\mu \sim 400$  keV and  $T \sim 10$  keV) and white dwarf ( $\mu \sim 500$  keV and  $T \sim 0.1 - 1$  keV) [108]. The estimates of the induced charges are:

$$e_{eff}^{\nu}|_{SN-II} = -2.1 \times 10^{-7}e, \quad (100)$$

$$e_{eff}^{\nu}|_{RG} = -9.6 \times 10^{-13}e, \quad (101)$$

$$e_{eff}^{\nu}|_{WD} = -1.8 \times 10^{-10}e. \quad (102)$$

To complete the comparison of the induced charges, we provide the estimate of induced neutrino electric charge coming from axial vertex contribution next. The limiting expression of the same is found in Equation (96). Assuming an old pulsar (age  $\sim 10^9$  years,  $T \sim 100^\circ$  K and fermi momentum ( $P_F$ )  $\sim 100$  MeV) [109], the  $e_{eff}^{\nu_a}$  for this case for left handed neutrinos turns out to be

$$e_{eff}^{\nu_a} \sim 0.21 \times 10^{-2} \left(\frac{B}{B_c}\right) e, \quad (103)$$

when the angle ( $\theta$ ) between the neutrino momentum and the ambient magnetic field tends to zero. For a very conservative estimate of  $\left(\frac{B}{B_c}\right) \sim 10^{-6}$ , the induced charge would be of the order of  $\sim 10^{-8}e$ .

There are several laboratory based studies available in the literature that tried to put a bound on neutrino effective charge. This charge can be intrinsic ( $e_{int}^{\nu}$ ) or induced ( $e_{ind}^{\nu}$ ). Out of these,  $e_{int}^{\nu} < 2 \times 10^{-15}e$  for an intergalactic field  $B \sim 1nG$  and  $e_{int}^{\nu} < -2 \times 10^{-17}e$  for  $B \sim 1 \mu G$  were obtained from  $\nu$  time of flight estimate from SN-1987A in [110],  $e_{int}^{\nu} < 3 \times 10^{-4}e$  was obtained in [111] from a SLAC electron beam dump experiment, and  $e_{int}^{\nu} < 10^{-35}e$  was obtained in [112] demanding charge neutrality of the universe. The limits in terrestrial laboratory based experiments were found to be significantly larger in size—for instance, estimates of the same in  $\bar{\nu} - e$  scattering experiments performed in reactor based experiments dedicated for measuring neutrino effective charge as was explored in [38,81] and in the TEXONO reactor based experiment [113–115]. The reactor limit on neutrino charge as was reported in [113] happened to be  $1.0 \times 10^{-12}$  with 90 % C.L. (considering the Debye length for Ge detector to be  $0.68 \mu m$ ).

On the other hand, the induced charge  $e_{int}^{\nu} < 10^{-14}e$  reported in [85,116] is due to medium effects in astrophysical situations. This induced charge is not due to any intrinsic

property of neutrinos and hence can vary from situation to situation. Their estimates can be best verified through consistency checks of various medium dependent processes.

However, the medium induced charge has an important role in establishing the intrinsic charge of neutrino. Because the terrestrial experiments aiming to find  $e_{int}^\nu$  are contaminated with medium effects, the induced charge would add on to the intrinsic charge, thus affecting the conclusions. For instance, many body effects of Ge used in determining  $e_{int}^\nu$  in TEXONO experiments [113–115] enter in the estimates of  $e_{int}^\nu$  through the Debye radius. On the other hand, even in the absence of  $e_{int}^\nu$ , there will be a contributing factor due to loop and medium effects (entering through Debye length). This factor should be sub-dominant. Our estimates show that, for charge-neutral neutrinos, the contribution to the effective neutrino charge is of the order of  $10^{-24}e$ , which is twelve orders less than reported in [113–115] from these experiments. If we consider the effect of atmospheric plasma at standard temperature and pressure, it turns out to be  $10^{-20}e$ , which is also way beyond the recognisable contribution compared to the results reported in [113–115].

## 8. Outlook

In this work, we have explored the physical implication of medium induced neutrino effective charge for celestial as well as terrestrial events. We have found many situations where their effect can be substantial. However, in addition to what we have discussed, there are other issues of beyond the standard model physics that can be connected to the same. For example, existence of non-zero mass for neutrino [117,118] indicates a beyond the standard model issue involved in the neutrino sector. Similarly, the existence of various electromagnetic properties of neutrinos like charge, etc., if confirmed, can be considered as a consequence of the physics beyond the standard model. However, in order to indicate the corrections appearing from beyond the standard model physics, one needs to quantify the same appearing from medium induced effects. In this paper, a small step was taken in that direction. More about it will be communicated elsewhere.

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## Appendix A. Structure of Axial Polarization Tensor

The axial polarization tensor  $\Pi_{\mu\nu}^5$  given by Equation (70) can be expressed in terms of the form factors and basis tensors constructed out of:

$$\begin{aligned}\hat{b}^{(1)\nu} &= N_1 k_\mu \bar{F}^{\mu\nu}, \quad \hat{I}^\nu = N_2 \left( b^{(2)\nu} - \frac{(\tilde{u}^\mu b_\mu^{(2)})}{\tilde{u}^2} \tilde{u}^\nu \right), \quad \hat{u}^\nu = N_L \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) u_\mu, \\ b^{(2)\nu} &= k_\mu \tilde{F}^{\mu\nu}, \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} \bar{F}_{\lambda\rho}.\end{aligned}\quad (\text{A1})$$

following [119,120], where  $N_1$ ,  $N_2$ ,  $N_L$  and  $N_k$  are normalization constants given by

$$N_1 = \frac{1}{\sqrt{-b_\mu^{(1)} b^{(1)\mu}}} = \frac{1}{K_\perp B}, \quad N_2 = \frac{1}{\sqrt{-I_\mu I^\mu}} = \frac{K}{\omega K_\perp B},$$

$$N_L = \frac{1}{\sqrt{-\tilde{u}_\mu \tilde{u}^\mu}} = \frac{k^2}{|K|}, \quad \text{and } N_k = \frac{1}{\sqrt{-k^2}}. \quad (\text{A2})$$

Here, the variable,  $K_\perp = (k_1^2 + k_2^2)^{\frac{1}{2}}$ . The same for vacuum part of  $\Pi_{\mu\nu}^5$  was done in [106]. The medium induced part can be constructed keeping the CP symmetry of  $\Pi_{\mu\nu}^5$  and the ward identity of  $\Pi_{\mu\nu}^5$  at the electromagnetic vertex  $k^\nu \Pi_{\nu\mu}^5 = 0$ . More about it will be conveyed in a separate communication.

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