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A Common Origin of the H_0 and S_8 Cosmological Tensions and a Resolution Within a Modified Λ CDM Framework

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Abstract

The two most severe cosmological tensions in the Hubble constant H_0 and the matter clustering amplitude S_8 have the same relative discrepancy of 8.3%, which suggests that they may have a common origin. Modifications of gravity and exotic dark fields with numerous free parameters introduced in the Einstein field equations often struggle to simultaneously alleviate both tensions; thus, we need to look for a common cause within the standard Λ CDM framework. At the same time, linear perturbation analyses of matter in the expanding Λ CDM universe have always neglected the impact of comoving peculiar velocities $\mathbf{v}$ (generally thought to be a second-order effect), the same velocities that, in physical space, cannot be fully accounted for in the observed late-time universe when the cosmic distance ladder is used to determine the local value of H_0 . We have reworked the linear density perturbation equations in the conformal Newtonian gauge (sub-horizon limit) by introducing an additional drag force per unit mass $-\Gamma(t)\mathbf{v}$ in the Euler equation with $\Gamma \equiv \gamma(2H)$, where $\gamma \ll 1$ is a positive dimensionless constant and $2H(t)$ is the time-dependent Hubble friction. We find that a damping parameter of $\gamma = 0.083$ is sufficient to resolve the S_8 tension by suppressing the growth of structure at low redshifts, starting at $z_* \simeq 3.5$ – 6.5 to achieve $S_8 \simeq 0.78$ – 0.76 , respectively. Furthermore, we argue that the physical source causing this additional friction (a tidal field generated by nonlinear structures in the late-time universe) is also responsible for a systematic error in the local determinations of H_0 —the inability to subtract peculiar tidal velocities along the lines of sight when determining the Hubble flow via the cosmic distance ladder. Finally, the dual action of the tidal field on the expanding background—reducing both the matter and the dark energy sources of the squared Hubble rate H^2 , thereby holding back the cosmic acceleration $\ddot{a}$ —is of fundamental importance in resolving cosmological tensions and can also substantially alleviate the density coincidence problem.

Keywords: cosmological constant; cosmology; dark energy; Friedmann equations; Hubble constant



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1. Introduction

1.1. Standard Λ CDM Model

The standard Λ CDM cosmological model has enjoyed remarkable success over the past two decades, providing a consistent framework that explains a vast array of observations, from the temperature anisotropies of the cosmic microwave background (CMB) to the

large-scale distribution of galaxies [1,2]. Based on a spatially flat universe dominated by cold dark matter (CDM) and a cosmological constant Λ [3–6], the Λ CDM model is specified by six parameters, which have been constrained with sub-percent precision by the Planck satellite mission [1]. On the other hand, the cosmological constant itself presents one of the most profound theoretical challenges in physics today, known by its components as the cosmological constant problem (CCP) and the density coincidence problem (DCP), respectively [7,8].

- (1) CCP: Quantum field theory predicts a vacuum energy density vastly greater than the observed value by more than 120 orders of magnitude [9,10].
- (2) DCP: The effective density corresponding to Λ and the density of matter are comparable at the present epoch, requiring highly contrived initial conditions to reach this universal state [11,12].

These fundamental puzzles motivate a search for cosmological mechanisms that can drive the late-time acceleration of the universe without relying on extreme fine-tuning assumptions.

1.2. Major Cosmological Tensions

As observational precision has improved over the years, some statistically significant cracks have appeared in the Λ CDM foundation. In particular, two major discrepancies have persisted and intensified, suggesting the presence of systematic errors in diverse observational datasets and the need for more physics unaccounted for in Λ CDM—or, alternatively, the necessity to modify gravity at large scales by including exotic new physics and more dark, undetected fields in cosmological theories [13–15].

The first and most prominent discrepancy is the “Hubble tension”, concerning the value of the current expansion rate H_0 . Estimates of H_0 inferred from early-universe physics (CMB) assuming Λ CDM consistently yield lower values of $H_0 \simeq 67.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$ [1]. In stark contrast, local late-universe measurements using the cosmic distance ladder (CDL) consistently favor a higher value of $H_0 \simeq 73.0 \text{ km s}^{-1} \text{ Mpc}^{-1}$ [15–17]. The statistical significance of this discrepancy has reached the 5σ level, rendering it impossible to dismiss as a mere statistical fluctuation [16].

The second major discrepancy is the “ S_8 tension”, which involves the clumpiness of baryonic matter observed in the late universe. The parameter S_8 quantifies the amplitude of density fluctuations. Predictions based on Planck CMB data project a relatively high degree of clustering ($S_8 \simeq 0.834$) [1]. Conversely, direct measurements of late-time large-scale structure (LSS) from weak gravitational lensing and galaxy clustering surveys, such as the Kilo-Degree Survey (KiDS) [18] and the Dark Energy Survey (DES) [19], consistently measure a lower amplitude ($S_8 \approx 0.76\text{--}0.78$). The growth history of the LSS provides a powerful test of the background evolution [20]. While the statistical significance of this tension ($\sim 2\text{--}3\sigma$) is lower than that of H_0 , its persistence across independent low-redshift probes suggests a genuine physical anomaly: our universe is smoother than what Λ CDM predicts.

1.3. A Remarkable Coincidence and a New Path Forward

In this work, we first highlight a striking coincidence that has largely gone unnoticed: the relative difference in the two major tensions is virtually identical.

- (a) The relative discrepancy in H_0 [1,15,16] is

$$(73.0 - 67.4) / 67.4 = 8.3\%.$$

(b) The relative discrepancy in S_8 [1,15,18] is

$$(0.834 - 0.77)/0.77 = 8.3\%.$$

We argue that this concordance in relative discrepancies is not coincidental. Rather, it points toward a single underlying physical mechanism responsible for both cosmological tensions. The missing physics ought to be capable of simultaneously (i) suppressing the predicted LSS growth rates by about 8.3% and (ii) resolving the 8.3% overestimate of H_0 that currently appears in the local CDL measurements.

While numerous extensions to Λ CDM have been proposed—ranging from early dark energy [21] to modified gravity theories [22] and models involving coupled dark sectors [23]—these solutions typically introduce complex exotic fields with multiple free parameters, a generally undesirable trade-off. Furthermore, such fields often struggle to resolve both tensions simultaneously; mechanisms that alleviate the H_0 tension frequently exacerbate the S_8 tension [24,25]. For these reasons, we choose to improve Λ CDM without invoking new exotic physics.

To this end, we revisit the standard model, searching for minimal adjustments that could simultaneously account for both of the above $\sim 8.3\%$ tensions, and we consider linear density perturbations within this framework [10,26]. Standard treatments typically assume a perfect cosmic fluid of collisionless dust, in which perturbations grow by gravitational instability [27], while gravitational attraction is counteracted to a degree by the Hubble flow. The fluid peculiar velocities affecting the linear density perturbations have traditionally been neglected in the linearized Euler equation (as a nonlinear effect [28,29]), even in the low-redshift universe when the cosmic deceleration has significantly diminished. But peculiar velocities are also sources of noise and uncertainty that complicate the determination of recession velocities in the local CDL measurements [30]. Thus, if tidal interactions from exterior nonlinear structures introduce radial velocities into the linear regime at late times [28], the resulting drag forces could systematically distort our interpretations of both key local rates—the background expansion rate H_0 and the LSS growth rate S_8 .

Based on the above reasoning, we test a modification of the linear perturbation equation in the sub-horizon limit of the conformal Newtonian gauge by introducing a first-order drag force proportional to the amplitude of the peculiar velocity itself. This drag acts as an additional friction that somewhat suppresses the growth of structure. In this paper, we demonstrate that a drag amplitude $\gamma \lesssim 8.3\%$ of $2H(t)$ at low redshifts, along with the Hubble friction $2H(t)$, is sufficient to suppress the LSS growth rate and resolve the S_8 tension. Furthermore, we argue that the same mechanism may account for the apparent excess in the locally measured H_0 values, thereby offering a unified “ $\gamma\Lambda$ CDM” resolution of the two major tensions in modern Λ CDM cosmology and other identified discrepancies [15].

1.4. Outline

The remainder of this paper is organized as follows:

- In Section 2, we introduce an additional drag term to the Euler equation and perform the linear perturbation analysis in the conformal Newtonian gauge within the FLRW background.
- In Section 3, we determine typical values of S_8 in our modified $\gamma\Lambda$ CDM framework and we examine whether the S_8 tension can be resolved by accounting for the added time-dependent friction ($\gamma \lesssim 8.3\%$ of $2H$) induced by drag forces due to an exterior tidal field (statistically isotropized, but with a nonzero mean value).
- In Section 4, we examine how this kinematic enhancement of the Hubble expansion modifies the background dynamics of the standard Λ CDM model to linear order in $\gamma \ll 1$.

- In Section 5, we develop a simplified 1D radial line-of-sight model and estimate the $\mathcal{O}(\gamma)$ net tidal acceleration acting on a target galaxy, as well as the implied magnitude of γ .
- In Section 6, we summarize our conclusions.
- In Appendix A, we show that the modifications introduced to the kinematics and the dynamics of the Hubble flow are consistent with one another.

2. Linear Perturbations with a Time-Dependent Drag Force

We formulate the linear scalar perturbation equations for an FLRW universe with dust, a cosmological constant, and a time-dependent drag force due to a peculiar velocity field. The analysis is performed in comoving spatial coordinates $\mathbf{x}$ and cosmic time t . The background dynamics follows the standard spatially flat Λ CDM model, while the perturbations obey modified Euler dynamics due to the additional friction term.

2.1. Initial Assumptions and Kinematic Framework

We consider a spatially flat FLRW spacetime with line element

$$ds^2 = -c^2 dt^2 + a^2(t) d\mathbf{x}^2, \quad (1)$$

where $a(t)$ is the scale factor and c is the speed of light in vacuum. The background expansion rate is given by the Hubble function

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}. \quad (2)$$

The matter sector is assumed to be a pressureless perfect fluid (dust) with background density $\rho(t)$ satisfying the continuity equation

$$\dot{\rho} + 3H\rho = 0. \quad (3)$$

The background evolution of a , H , and ρ is governed by the usual Friedmann equations (FEs) with a cosmological constant Λ and a gravitational constant G , viz.,

$$H^2 = \frac{\Lambda c^2}{3} + \frac{8\pi G\rho}{3}, \quad (4)$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda c^2}{3} - \frac{4\pi G\rho}{3} \quad (5)$$

Combining the FEs and using the identity $\ddot{a}/a = \dot{H} + H^2$, we obtain the Raychaudhuri–Landau equation for $H(t)$ [31], viz.,

$$2\dot{H} + 3H^2 = \Lambda c^2. \quad (6)$$

This equation relates the cosmic acceleration $\dot{H}$ with the Hubble expansion rate H and the cosmological constant Λ .

2.2. Linearized Perturbation Equations

2.2.1. Preliminaries

In the FLRW background, we introduce small perturbations to linear order. The density field is decomposed as

$$\rho(t, \mathbf{x}) = \rho(t)[1 + \delta(t, \mathbf{x})], \quad |\delta| \ll 1, \quad (7)$$

where $\delta(t, \mathbf{x})$ is the density contrast in comoving coordinates $\mathbf{x}$.

The gravitational potential $\Phi(t, \mathbf{x})$, which encodes the perturbation of the metric sourced by these density fluctuations, represents the scalar metric perturbation in the conformal Newtonian gauge. In the Newtonian (sub-horizon) limit, where time derivatives of the metric potentials are negligible and spatial gradients dominate, the linearized fluid equations reduce to their familiar comoving Newtonian forms. In this regime, Φ satisfies the standard Poisson equation relating the gravitational potential to the comoving density perturbations [10,26].

The proper velocity of the fluid relative to an observer at the origin $\mathbf{r} = \mathbf{0}$ is defined as $\mathbf{u}(t, \mathbf{x}) \equiv d\mathbf{r}/dt$. We decompose $\mathbf{u}$ into the Hubble flow and the peculiar velocity $\mathbf{v}(t, \mathbf{x})$ as

$$\mathbf{u}(t, \mathbf{x}) = H(t)\mathbf{r} + \mathbf{v}(t, \mathbf{x}), \quad (8)$$

where $H(t)$ is the Hubble rate and $|\mathbf{v}| \ll H\mathbf{r}$. The peculiar velocity measures departures from the pure Hubble flow and is related to the comoving coordinates $\mathbf{x} = \mathbf{r}/a(t)$ by

$$\mathbf{v}(t, \mathbf{x}) = a(t) \frac{d\mathbf{x}}{dt}. \quad (9)$$

The perturbed Euler equation is modified by the introduction of a time-dependent drag force per unit mass $\mathbf{a}_{\text{drag}}$. When expressed in terms of the peculiar velocity, this physical deceleration is assumed to take the form

$$\mathbf{a}_{\text{drag}}(t, \mathbf{x}) = -\Gamma(t)\mathbf{v}(t, \mathbf{x}), \quad (10)$$

where $\Gamma(t)$ is an arbitrary non-negative function of cosmic time with the dimension of frequency. This ensures that the friction enters the linear growth equation as an additive damping term.

2.2.2. First-Order Equations

To first order in the quantities δ , $\mathbf{v}$, and Φ , the continuity, Euler, and Poisson equations for a fluid of dust subject to the time-dependent drag deceleration (Equation (10)), respectively, are

$$\dot{\delta} + \frac{1}{a} \nabla \cdot \mathbf{v} = 0, \quad (11)$$

$$\dot{\mathbf{v}} + H\mathbf{v} = -\frac{1}{a} \nabla \Phi - \Gamma(t)\mathbf{v}, \quad (12)$$

$$\frac{1}{a^2} \nabla^2 \Phi = (4\pi G\rho)\delta. \quad (13)$$

This system of equations is closed and can be reduced to a single differential equation for the linear growth of density perturbations $\delta(t)$: We define the comoving divergence of peculiar velocities

$$\theta \equiv \frac{1}{a} \nabla \cdot \mathbf{v}, \quad (14)$$

which represents the rate at which the fluid is fluctuating in comoving space. The continuity Equation (11) then becomes

$$\theta = -\dot{\delta}. \quad (15)$$

Next, taking the comoving divergence of the Euler Equation (12), we obtain the equation

$$\dot{\theta} + 2H\theta = -\frac{1}{a^2} \nabla^2 \Phi - \Gamma(t)\theta, \quad (16)$$

and, substituting $\nabla^2\Phi/a^2$ from the Poisson Equation (13) and θ from Equation (15), we obtain the final form

$$\ddot{\delta} + (2H + \Gamma)\dot{\delta} - (4\pi G\rho)\delta = 0. \quad (17)$$

Equation (17) is an ordinary differential equation (ODE) that governs the cosmic evolution of a linear density Fourier mode $\delta_{\mathbf{k}}(t)$ with comoving wavenumber $\mathbf{k}$. Because the perfect fluid is assumed to be dust, the evolution of $\delta_{\mathbf{k}}(t)$ becomes independent of $\mathbf{k}$ on sub-horizon scales [32]. In this sense, this equation generalizes the standard Λ CDM growth equation by including the friction coefficient $\Gamma(t) > 0$. This term enhances the conventional Hubble friction and results in greater suppression of the growth of linear density perturbations relative to the growth in the standard Λ CDM model with $\Gamma = 0$.

2.3. Scaling the Friction Coefficient $\Gamma(t)$

In the subsequent analysis, we scale $\Gamma(t)$ to the expansion-induced Hubble friction with coefficient $2H(t)$. We adopt the convenient form

$$\Gamma = \gamma(2H), \quad (18)$$

where $\gamma > 0$ is a dimensionless constant parameterizing the strength of the drag forces relative to the usual Hubble damping term. This choice ensures that the added friction is naturally scaled to the universal expansion, thereby simplifying further investigations. It is, however, expected that $\gamma \ll 1$.

Substituting Equation (18) into Equation (17), we find that the evolution of the density contrast $\delta(t)$ satisfies the linear differential equation

$$\ddot{\delta} + 2H(1 + \gamma)\dot{\delta} - (4\pi G\rho)\delta = 0. \quad (19)$$

Naturally, for $\gamma = 0$, we recover the standard Λ CDM linear growth Equation [10,26].

The scaling constant γ in Equation (19) naturally provides a framework for addressing the observed S_8 tension, and it may also potentially alleviate the H_0 tension. In the following section, we determine the range of γ values that reconcile the predicted linear growth with current observational constraints on S_8 (≈ 0.76 – 0.78) [15].

3. Clustering Amplitudes and Growth Rates in Λ CDM with a Drag Force

3.1. Preliminaries

The linear growth factor (D) of structure is commonly defined in terms of the matter density contrast $\delta(a)$ [10,26], viz.,

$$D(a) \equiv \frac{\delta(a)}{\delta(a_i)}, \quad (20)$$

where a_i is usually taken to be the scale factor at recombination (redshift $z_{\text{rec}} \approx 1100$). For this reason, we introduce the scale factor as the independent variable in the ODE (19) for the matter density contrast, as shown in Sections 3.2 and 3.3 below. For the calculations, we also adopt

$$\ln a_i = -7, \quad (21)$$

corresponding to $z_{\text{rec}} = 1096$.

The RMS fluctuation amplitude $\sigma_8(\gamma = 0)$, linearly extrapolated to the present epoch and corresponding to the Planck-2018 Λ CDM normalization [1], is defined in terms of

$D(a = 1; \gamma = 0)$ [10,26]. In our modified Λ CDM model, we correspondingly write $\sigma_8(\gamma)$, which we determine from the scaling relation

$$\sigma_8(\gamma) = \sigma_8(0) \left[\frac{D(a = 1; \gamma)}{D(a = 1; 0)} \right], \quad (22)$$

where $\sigma_8(0) = 0.811 \pm 0.006$ [1].

The observational clustering parameter S_8 [1,33] is then determined by the relation

$$S_8(\gamma) = \sigma_8(\gamma) \sqrt{\frac{\Omega_{m0}}{0.3}}, \quad (23)$$

where Ω_{m0} is the matter density fraction at present time. Our modified model is essentially Λ CDM with enhanced friction, so we adopt the Planck-2018 value of $\Omega_{m0} = 0.315 \pm 0.007$, for which Equation (23) reduces to

$$S_8(\gamma) = \frac{\sigma_8(\gamma)}{0.975900}, \quad (24)$$

precise to six significant digits.

In general, the matter density fraction varies with the scale factor and is defined as [10,26]

$$\Omega_m(a) \equiv \frac{\rho(a)}{\rho_c(a)}, \quad (25)$$

where $\rho(a) = \rho_0 a^{-3}$ is the background matter density, $\rho_0 = \rho(a = 1)$ at present ($a = 1$), and

$$\rho_c(a) = \frac{3H(a)^2}{8\pi G}, \quad (26)$$

is the critical density in FLRW cosmology. We note that $\rho(a)$ refers to the total (CDM plus baryonic) matter component; on the linear scales of interest, the growth of structure $\delta(a)$ is dominated by dark matter with baryons acting as passive tracers of the overall gravitational potential wells.

It is convenient to introduce the dimensionless Hubble function $E(a) \equiv H(a)/H_0$, in terms of which the fraction $\Omega_m(a)$ may be written as

$$\Omega_m(a) = \frac{\Omega_{m0} a^{-3}}{E(a)^2}, \quad (27)$$

which satisfies $\Omega_m(1) = \Omega_{m0}$. The dimensionless Hubble function $E(a)$ is given by the well-known relation

$$E(a) = \sqrt{\Omega_{m0} a^{-3} + \Omega_{\Lambda 0}}, \quad (28)$$

where $\Omega_{\Lambda 0} = 1 - \Omega_{m0}$ is the vacuum energy fraction at present [1,10,26]. These quantities naturally appear in the gravitational source term of the linear growth equation derived below.

3.2. The ODE of the Matter Density Contrast $\delta(a)$

We change the independent variable in Equation (19) from cosmic time t to scale factor a [34] by using the identities

$$\frac{d}{dt} = aH \frac{d}{da} \quad \text{and} \quad \frac{d^2}{dt^2} = a^2 H^2 \frac{d^2}{da^2} + aH^2 \left(1 + \frac{d \ln H}{d \ln a} \right) \frac{d}{da}. \quad (29)$$

Using primes to denote the derivatives with respect to a and replacing δ from Equation (20), we find that the linear growth equation is transformed into

$$D''(a) + \frac{1}{a} \left(2\gamma + 3 + \frac{d \ln H}{d \ln a} \right) D'(a) - \frac{3}{2} \frac{\Omega_m(a)}{a^2} D(a) = 0. \quad (30)$$

In this ODE, a factor of 2γ effectively enhances the coefficient of Hubble friction ($3 + d \ln H / d \ln a$), and the gravitational source term depends on the mean universal density of matter at each instance.

For computational efficiency, it is convenient to also express the coefficient of friction in Equation (30) in terms of $\Omega_m(a)$. The procedure simplifies the ODE considerably, viz.,

$$D''(a) + \frac{1}{a} \left(2\gamma + 3 - \frac{3}{2} \Omega_m(a) \right) D'(a) - \frac{3}{2} \frac{\Omega_m(a)}{a^2} D(a) = 0, \quad (31)$$

where $\Omega_m(a)$ can be computed from the compact expression

$$\Omega_m(a) = \left(1 + \frac{\Omega_{\Lambda 0}}{\Omega_{m0}} a^3 \right)^{-1}, \quad (32)$$

which is derived by combining Equations (27) and (28). Here, the fiducial ratio $\Omega_{\Lambda 0} / \Omega_{m0} = 2.174603$, precise to seven significant digits [1].

3.3. Analytic Approximation for the Matter-Dominated Epoch

To gain analytic insight into the effect of the enhanced friction parameter γ , we consider an approximate solution of the growth Equation (31) in the matter-dominated regime. At early times ($a \ll 1$), Equation (32) implies that $\Omega_m(a) \simeq 1$, so that Equation (31) reduces to a Cauchy–Euler ODE [35,36], viz.,

$$D''(a) + \frac{1}{a} \left(2\gamma + \frac{3}{2} \right) D'(a) - \frac{3}{2a^2} D(a) = 0, \quad (33)$$

which admits power-law solutions of the form $D \propto a^p$, where p is a constant to be determined.

Substituting the power-law form into Equation (33), we find a quadratic equation for p , viz.,

$$p^2 + \left(2\gamma + \frac{1}{2} \right) p - \frac{3}{2} = 0. \quad (34)$$

The power p_+ of the growing mode, expanded to linear order in $\gamma \ll 1$, is $p_+ = 1 - 4\gamma/5$. Since $D(a_i) = 1$ (Equation (20)), the growing solution of the matter-dominated ODE (33) is

$$D(a) = \left(\frac{a}{a_i} \right)^{1-4\gamma/5}. \quad (35)$$

In standard Λ CDM where $\gamma = 0$, $p_+ = 1$ and the growing mode is simply $D(a) = a/a_i$.

Extrapolating this solution to the present epoch ($a = 1$), we apply the results to Equation (22) for $\sigma_8(\gamma)$; using Equation (21), we find, to linear order, that

$$\mathcal{D}_1(\gamma) \equiv \frac{D(a=1; \gamma)}{D(a=1; 0)} = (a_i)^{4\gamma/5} \simeq 1 - 5.6 \gamma, \quad (36)$$

and that

$$\sigma_8(\gamma) \simeq \sigma_8(0) (1 - 5.6 \gamma). \quad (37)$$

The steep slope of -5.6 in $\sigma_8(\gamma)$ reflects the cumulative effect of a slightly enhanced Hubble friction (by $\gamma \ll 1$) over the long timescale from recombination ($a_i \simeq 10^{-3}$) to today. In a realistic Λ CDM background, this cumulative suppression of growth is expected to be mitigated because the drag forces are expected to become significant only at late times, when the causal nonlinear structures have been formed in the universe and $H(t)$ has significantly decreased with time.

Finally, a qualitative comparison with observations may be carried out using Equation (23) for $S_8(\gamma)$, with the important caveat that the analytic approximations above assume $\Omega_m \simeq 1$ and $\ln a_i = -7$. Within this framework, small values of the enhanced friction parameter of $\gamma \simeq 0.01$ – 0.015 reduce the Λ CDM-predicted clustering amplitude and alleviate the S_8 tension because the drag forces have been active at all times since recombination. In Section 3.4, we refine this rough estimate by numerically integrating the full growth Equation (31) for a range of γ values and a range of initial conditions in a realistic Λ CDM background. In these computations, the critical redshift z_* at which drag forces first appear becomes a free parameter.

3.4. Numerical Solutions of the Growth Equation

To obtain precise estimates of the impact of the enhanced Hubble friction parameter γ on the growth of linear density perturbations for redshifts $z \leq z_*$, we integrate numerically the growth Equation (31). The integration of the growing solution begins in the early matter-dominated regime (standard Λ CDM with $\gamma = 0$ and $D = a/a_i$) at a scale factor corresponding to recombination, viz. $a_i = \exp(-7)$, where

$$D(a_i) = 1 \quad \text{and} \quad D'(a_i) = (a_i)^{-1} = \exp(+7). \quad (38)$$

The $\gamma = 0$ integration is carried out from $a = a_i$ to an intermediate value of $a = a_*$ (a free parameter corresponding to $z_* = 1/a_* - 1$), where the 2γ term, the other free parameter, is activated smoothly and persists up to the present epoch ($a = 1$, $z = 0$). The numerical algorithm uses the adaptive-step Runge–Kutta solver `ode45`, which has been widely employed in computational physics [37,38]. This solver automatically adjusts the step size to maintain accuracy, making it suitable for integrating the rapidly varying growth function over many orders of magnitude in scale factor a .

The resulting growth factors at $a = 1$ are then used to compute the corresponding linear clustering amplitude $S_8(\gamma)$ via

$$S_8(\gamma) = S_8(0) \left[\frac{D(a=1; \gamma)}{D(a=1; 0)} \right] = S_8(0) \mathcal{D}_1(\gamma), \quad (39)$$

where $S_8(0) = 0.8310$ is obtained from Equation (24) for $\gamma = 0$ and $\sigma_8(0) = 0.811$ [1], thereby allowing for a direct comparison with observational results.

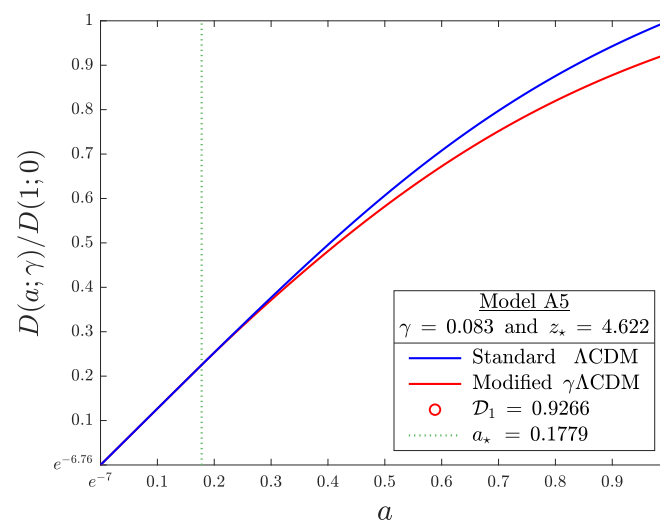
The numerical solutions are controlled by the two free parameters γ and z_* (or a_*) and are grouped into two sets (A and B) according to the chosen typical values of the parameter γ .

3.4.1. Parameter $\gamma \simeq 8\%$ of the Hubble Friction $2H(a)$

In the first set of solutions listed in Table 1, we choose $\gamma = 0.083$ and we determine the critical values of z_* and a_* that result in reduced growth at the present time relative to the standard Λ CDM solution. The reduced growth results in the listed values of $S_8 \in [0.75, 0.79]$, where one-half of the values determined by observations lie [15] (a total of $N = 38$ published measurements of $S_8 \in [0.7, 0.8]$, excluding four outliers, with mean $\bar{x} = 0.756$ and standard deviation $s = 0.026$). The complete solution with $\gamma = 0.083$ and $z \leq z_* = 4.622$ that results in $S_8 = 0.77$ (Model A5 in Table 1) is illustrated in Figure 1.

Table 1. Numerical solutions of the growth Equation (31) with $\gamma = 0.083$ (i.e., 8.3% of the Hubble friction $2H$) at all redshifts $z \leq z_*$.

γ	Model	z_*	a_*	$\mathcal{D}_1$	Growth Suppression ($1 - \mathcal{D}_1$) (%)	S_8
0.083	A1	7.551	0.1170	0.9025	9.75	0.75
	A2	7.000	0.1250	0.9063	9.37	0.753
	A3	6.536	0.1327	0.9097	9.03	0.756
	A4	5.930	0.1443	0.9145	8.55	0.76
	A5	4.622	0.1779	0.9266	7.34	0.77
	A6	3.560	0.2193	0.9386	6.14	0.78
	A7	2.690	0.2710	0.9506	4.94	0.79
	A8	1.132	0.4690	0.9795	2.05	0.814

**Figure 1.** The numerical solution $D(a; \gamma)$ with $\gamma = 0.083$ and $z_* = 4.622$ (red curve) that results in $S_8 = 0.77$ (Model A5 in Table 1). The $\gamma = 0$ standard Λ CDM solution is also shown for comparison (blue curve). Both solutions are normalized to the $a = 1$ value of the standard Λ CDM solution.

Model A8 is included in Table 1 to accommodate the latest KiDS–Legacy survey that has produced a rather surprising result: $\sigma_8 = 0.802^{+0.022}_{-0.018}$ and $S_8 = 0.814^{+0.011}_{-0.012}$ [39,40]. Naturally, a value of $\gamma \simeq 0.08$ is too large for such a small tension versus S_8 from Λ CDM, and this is why only a very recent activation redshift of $z_* \simeq 1.13$ is required in the $\gamma\Lambda$ CDM model to resolve the remaining small discrepancy.^{1,2}

3.4.2. Parameter $\gamma \simeq 4\%$ of the Hubble Friction $2H(a)$

In the second set of solutions listed in Table 2, we choose $\gamma = 0.039$ and we analyze critical values of $z_* < 10$ (perhaps a palatable upper limit [28]) that result in reduced growth at the present time. This smaller value of γ characterizes the smaller H_0 tension between standard Λ CDM [1] and a subset of the SNIa–TRGB measurements that have resulted in $H_0 \lesssim 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$ [15,17]. Evidently, drag forces of such a lower magnitude (only $\sim 4\%$ of the Hubble friction $2H$) are capable of resolving the S_8 tension only if S_8 truly lies at the high end (~ 0.79) of the observational range, and, even then, z_* has to be ≥ 7 , a relatively high threshold. The complete solution with $\gamma = 0.039$ and $z_* = 7$ that results in $S_8 = 0.793$ (Model B3 in Table 2) is illustrated in Figure 2.

Table 2. Numerical solutions of the growth Equation (31) with $\gamma = 0.039$ (i.e., 3.9% of the Hubble friction $2H$) at all redshifts $z \leq z_*$.

γ	Model	z_*	a_*	$\mathcal{D}_1$	Growth Suppression ($1 - \mathcal{D}_1$) (%)	S_8
0.039	B1	9	0.1000	0.9475	5.25	0.787
	B2	8	0.1111	0.9506	4.94	0.790
	B3	7	0.1250	0.9540	4.60	0.793
	B4	6	0.1429	0.9578	4.22	0.796
	B5	4	0.2000	0.9675	3.25	0.804
	B6	3	0.2500	0.9738	2.62	0.809
	B7	2.258	0.3069	0.9795	2.05	0.814

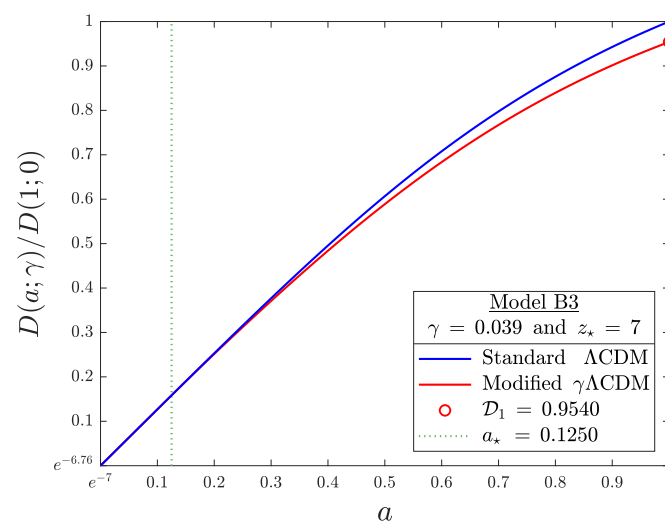


Figure 2. The numerical solution $D(a; \gamma)$ with $\gamma = 0.039$ and $z_* = 7$ (red curve) that results in $S_8 = 0.793$ (Model B3 in Table 2). The $\gamma = 0$ standard Λ CDM solution is also shown for comparison (blue curve). Both solutions are normalized to the $a = 1$ value of the standard Λ CDM solution.

Models B6 and B7 are included in Table 2 to accommodate the latest KiDS–Legacy survey that has produced a large best-fit value of $S_8 = 0.814^{+0.011}_{-0.012}$ [39,40]. A small value of $\gamma \simeq 0.04$ appears to be in line with the small tension ($\lesssim 1\sigma$ or 2.4%) versus S_8 from Λ CDM, and an activation redshift of $z_* \simeq 2.26$ is quite reasonable as well.³

3.5. Growth Rate $f(a)$ from the Nonlinear First-Order Growth ODE

Further characterization of the linear growth of structure is provided by the dimensionless growth rate

$$f(a) \equiv \frac{d \ln D}{d \ln a}, \quad (40)$$

where the initial value problem (IVP) for $D(a)$ is described by Equations (31), (32), and (38) [26,28].

Recast in terms of $f(a)$, the IVP is described by a nonlinear first-order ODE of the Riccati type, viz.,

$$\frac{df(a)}{d \ln a} + (f(a))^2 + \left(2(1 + \gamma) - \frac{3}{2} \Omega_m(a)\right) f(a) = \frac{3}{2} \Omega_m(a), \quad (41)$$

along with the initial condition

$$f(a_i) = 1, \quad (42)$$

where $a_i = \exp(-7)$. Table 3 summarizes the present-day growth rates $f(1; \gamma)$ obtained for Λ CDM and the $\gamma\Lambda$ CDM Models A1–A8 of Table 1. In general terms, the growth rates of the $\gamma > 0$ models are reduced by $\simeq 8\%$, in contrast to the $\mathcal{D}_1$ ratios that are reduced by 5–10%.

Table 3. Present-day growth rates $f(1; \gamma)$ obtained from numerical integrations of the IVP (41) and (42) for the models listed in Table 1. The first row shows the standard Λ CDM model. The last column shows the normalized ratio $\mathcal{F}_1 \equiv f(1; \gamma)/f(1; 0)$.

γ	Model	z_*	a_*	$\mathcal{D}_1$	$f(1; \gamma)$	$\mathcal{F}_1$
0	Λ CDM	...	...	1	0.52710	1
0.083	A1	7.551	0.1170	0.9025	0.48385	0.91795
	A2	7.000	0.1250	0.9063	0.48390	0.91803
	A3	6.536	0.1327	0.9097	0.48394	0.91812
	A4	5.930	0.1443	0.9145	0.48402	0.91826
	A5	4.622	0.1779	0.9266	0.48430	0.91880
	A6	3.560	0.2193	0.9386	0.48478	0.91971
	A7	2.690	0.2710	0.9506	0.48561	0.92128
	A8	1.132	0.4690	0.9795	0.49145	0.93236

The two diagnostics in Table 3, i.e., $\mathcal{D}_1$ copied from Table 1 and the new ratio $\mathcal{F}_1$, quantify different aspects of the same growth history.

- Ratio $\mathcal{D}_1$ measures an integrated amplitude effect: once γ is activated at $a = a_*$, the enhanced friction suppresses growth over the entire interval $a \in [a_*, 1]$, and the accumulated suppression increases substantially when the activation redshift z_* is set farther into the past.
- By contrast, ratio $\mathcal{F}_1$ probes only a local logarithmic slope at the present epoch. Because this slope is determined mainly by the late-time background evolution (near Ω_{m0}) and the local friction term, varying z_* leaves the late-time slope $f(1; \gamma)$ largely unaffected. Consequently, the spread in $\mathcal{F}_1$ is much smaller than that in $\mathcal{D}_1$ across the $\gamma = 0.083$ models in Table 3.

3.5.1. Application: Models of the Growth Rate $f(a) \propto (\Omega_m(a))^n$

Table 4 summarizes two power-law representations of the numerically integrated linear growth rate $f(a)$ over the interval $a \in [a_*, 1]$ for the models shown in Tables 1 and 3. The standard Λ CDM model [20] is also listed for comparison.

For each choice of a_* , we solve the IVP (41) and (42) numerically across the interval $\mathcal{I}(a) = [a_i, 1]$ and evaluate $f(a; \gamma)$ on a dense sampling in $(\ln a)$ space. Then, we perform two least-squares fits of the resulting curve in terms of the matter fraction $\Omega_m(a)$: (a) a two-parameter $\{A, n\}$ fit of the standard form $f(a) = A(\Omega_m(a))^n$ and (b) a one-parameter $\{\ell\}$ fit of the form $f(a) = p_+(\Omega_m(a))^\ell$, in which the amplitude is fixed to the value $p_+ = 1 - 4\gamma/5$ predicted by the analytic model in Section 3.3 and only the exponent ℓ is optimized. Table 4 displays the best-fit parameters for each ansatz, along with a goodness-of-fit metric based on the RMS residuals over the interval $\mathcal{I}(a)$. A comparison between $\{A, n\}$ and $\{p_+, \ell\}$ isolates the deviation from a pure $(\Omega_m)^\ell$ power law that is absorbed by the floating normalization $A(a_*)$ versus the pure power law in which the amplitude is constrained to be fixed ($p_+ = 0.9336$ for $\gamma = 0.083$).

Table 4. Power-law fits of growth rate $f(a)$ in $a \in [a_*, 1]$: (a) two-parameter $\{A, n\}$ fit $f(a) = A(\Omega_m(a))^n$; (b) one-parameter $\{\ell\}$ fit $f(a) = p_+(\Omega_m(a))^\ell$, where $p_+ = 1 - 4\gamma/5$ (Section 3.3) and $p_+ = 0.9336$ for $\gamma = 0.083$.

γ	Model	z_*	a_*	A	n	RMS Residual Error (%)	ℓ	RMS Residual Error (%)
0	Λ CDM	...	...	1.00011	0.55173	0.0423	0.55156	0.0436
0.083	A1	7.551	0.1170	0.95450	0.59635	1.31	0.56122	2.14
	A2	7.00	0.1250	0.95531	0.59727	1.30	0.56082	2.18
	A3	6.536	0.1327	0.95610	0.59814	1.29	0.56041	2.21
	A4	5.930	0.1443	0.95730	0.59940	1.27	0.55974	2.25
	A5	4.622	0.1779	0.96089	0.60281	1.21	0.55746	2.38
	A6	3.560	0.2193	0.96555	0.60656	1.12	0.55404	2.53
	A7	2.690	0.2710	0.97181	0.61065	1.00	0.54899	2.70
	A8	1.132	0.4690	1.00063	0.62242	0.51	0.52632	2.98

In Models A1–A8, the best-fit exponents are systematically higher than those determined for the Λ CDM model and listed in row 1 of Table 4 with $\gamma = 0$ and fitted parameters

$$\{A, n\} \simeq \{1.000, 0.552\} \simeq \{p_+, \ell\} \quad (\text{Standard } \Lambda\text{CDM}). \quad (43)$$

This is the expected model behavior in $\gamma\Lambda$ CDM: for fixed $\Omega_m(a)$, a higher power makes $f(a; \gamma)$ smaller at late times (when $\Omega_m < 1$), reflecting the modest additional suppression of growth relative to standard Λ CDM. In this sense, the growth rates listed in Table 4 are consistent with the suppressed clustering amplitudes reported in Section 3.4.1 above.

3.5.2. Application: Models of the Observable Growth Rate $(f\sigma_8)(a)$

Using the numerical solution $f(a)$ of the IVP (41) and (42), we can determine the evolution of the primary observable—the weighted growth rate $(f\sigma_8)(a)$ with conditions $f(a_i) = 1$ (Equation (42)) and $\sigma_8(a_i) = \exp(-6.97)$. The calibrated value $\sigma_8(a_i)$ is such that the $\gamma = 0$ Λ CDM model recovers the Planck-2018 value of $\sigma_8(0) = 0.811$ at $a = 1$. The $\gamma\Lambda$ CDM models then evolve from this shared initial state, allowing for the cumulative effect of the γ -modified growth to manifest as the physical suppression of the present-day weighted growth rate $(f\sigma_8)(a = 1)$.

Figure 3 shows a typical model with $\gamma = 0.083$ and $z_* = 4.622$ (Model A5), for which the present-day weighted growth rate $(f\sigma_8)(a = 1) = 0.3639$ is suppressed by about 15% relative to 0.4275 in the standard Λ CDM.

The growth index ℓ , defined here by the parameterization $f(1) = (\Omega_{m0})^\ell$ for both models in Figure 3, provides another commonly used quantitative measure of growth suppression. For the standard Λ CDM model, we calculate the present-day index as $\ell = \ln(f(1))/\ln(\Omega_{m0}) = 0.554$, which aligns with the expectation from General Relativity ($\ell_{\text{GR}} = 6/11$ [20]) and with the global index $\ell = 0.552$ (Equation (43)) produced by fitting the entire history of the model's past recombination (Section 3.5.1).

On the other hand, the present-day index shifts significantly to $\ell = 0.628$ in Model A5, where $f(1) = 0.48430$ (Table 3). This elevated index is the signature of the 15% γ -suppressed growth rate at $a = 1$. However, when we consider the entire evolution of Model A5, the best-fit global index in the interval $\mathcal{I}(a) = [a_i, 1]$ is $\ell = 0.557$ (Table 4).

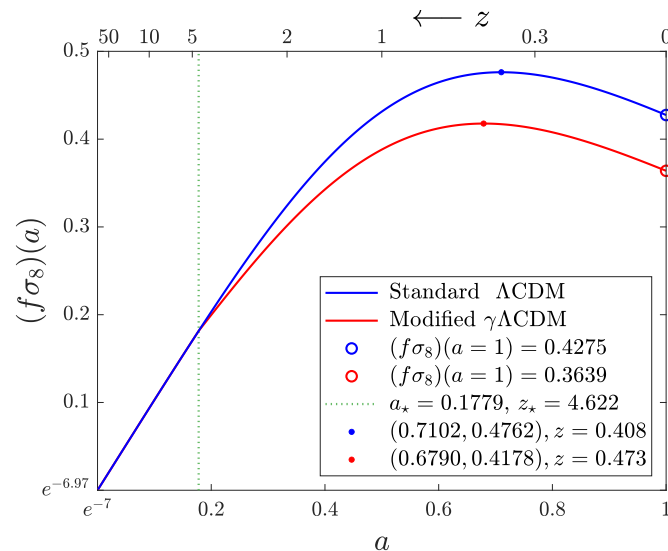


Figure 3. The numerical solution $(f\sigma_8)(a)$ with $\gamma = 0.083$ and $z_* = 4.622$ (red curve) that results in $S_8 = 0.77$ and $f(1; \gamma) = 0.4843$ (Model A5 in Tables 1 and 3, respectively). The $\gamma = 0$ standard Λ CDM solution is also shown for comparison (blue curve). The blue and red dots indicate the locations of $\max(f\sigma_8)$ on the two curves. The redshift (z) axis is also shown at the top of the diagram. Both of the IVP solutions are anchored at the recombination scale factor of $a_i = \exp(-7)$, where $(f\sigma_8)(a_i) = \exp(-6.97)$, a value obtained by evolving the Planck-2018 result $\sigma_8(0) = 0.811$ [1] backward to a_i using the standard Λ CDM growth history.

The difference between these two results is a direct consequence of the activation of γ beyond $a = a_*$. Since Model A5 is identical to Λ CDM for $a < a_*$ (Figure 1), the global fit in Section 3.5.1 is dominated by the standard growth regime of small scale factors ($a < 0.3$ in Figure 1), resulting in a global index $\ell = 0.557$, close to the Λ CDM prediction. Thus, the sharp increase in the local index to $\ell = 0.628$ at $a = 1$ confirms that growth suppression is a late-time phenomenon that manifests most strongly as the universe approaches the present epoch.

The other models in set A (Tables 1 and 3) show similar behaviors. This is reflected in the tight range of $f(1; \gamma)$ values shown in Table 3, which also produce a tight range of local growth indices

$$\ell \in [0.615, 0.629]. \quad (44)$$

These results are consistent with the recent analysis of Nguyen et al. [43], who obtained a best-fit local index $\ell = 0.633^{+0.025}_{-0.024}$ (a tension of $\sim 3.5\sigma$ with Λ CDM) by combining the Planck-2018 CMB data and the LSS data from weak lensing, galaxy clustering, and cosmic velocities. The largest deviation occurs in Model A8, in which the local index $\ell = 0.615$ is lower by 0.75σ .^{4,5}

4. Modified Background Dynamics in $\gamma\Lambda$ CDM

The drag force per unit mass introduced to the perturbed Euler Equation (12) in Section 2 slightly enhances the Hubble flow by a factor of $(1 + \gamma)$, where $\gamma \ll 1$. Accounting for this in the kinematics of the flow, Equation (8) takes the form

$$\mathbf{u}(t, \mathbf{x}) = (1 + \gamma)H(t)\mathbf{r} + \mathbf{v}(t, \mathbf{x}), \quad (45)$$

and the modified Hubble rate affects the dynamics of the background Hubble expansion to $\mathcal{O}(\gamma)$ in the $\gamma\Lambda$ CDM model. This can be implemented in the dynamical Equations (3), (4), and (6) of Section 2.1 by substituting $H \rightarrow (1 + \gamma)H$ and expanding the terms to linear order in γ .

Below, we analyze the diagnostics of the $\gamma\Lambda$ CDM expansion, all related to the evolution of the Hubble rate $\dot{H}(t)$, which is discussed first.

4.1. Hubble Rate Evolution

In the $\mathcal{O}(\gamma)$ regime of $\gamma\Lambda$ CDM, Equation (6) with $H \rightarrow (1 + \gamma)H$ reads

$$2\dot{H} = (\Lambda c^2 - 3H^2) - \gamma(\Lambda c^2 + 3H^2), \quad (46)$$

or, in compact form,

$$2\dot{H} = -3H^2 \left[(1 - \Omega_\Lambda) + \gamma(1 + \Omega_\Lambda) \right], \quad (47)$$

where

$$\Omega_\Lambda(t) \equiv \frac{\Lambda c^2}{3H^2}. \quad (48)$$

In the matter-dominated regime where $\Omega_\Lambda(t) \ll 1$, we see that $\dot{H} = -3(1 + \gamma)H^2/2 < 0$, and $\gamma > 0$ always renders $\dot{H}$ more negative relative to its Λ CDM value. At any fixed H , the decrease in $H(t)$ is slightly faster in $\gamma\Lambda$ CDM, and the location of the late-time attractor fixed point H_{fp} also changes slightly compared to the de Sitter value of $H_{\text{dS}} = c\sqrt{\Lambda/3}$ (see below).

4.2. Hubble Rate Fixed Point

Setting $\dot{H} = 0$ in the above equations for $2\dot{H}$, we obtain the location of the fixed point of the late-time attractor to $\mathcal{O}(\gamma)$, viz.,

$$H_{\text{fp}} = c\sqrt{\frac{\Lambda}{3}}(1 - \gamma) = H_{\text{dS}}(1 - \gamma), \quad (49)$$

or, equivalently,

$$(\Omega_\Lambda)_{\text{fp}} = \frac{1 + \gamma}{1 - \gamma} \simeq 1 + 2\gamma + \mathcal{O}(\gamma^2). \quad (50)$$

The value of $(\Omega_\Lambda)_{\text{fp}} > 1$ does not signal an inconsistency; rather, it reflects that the standard closure relation $\Omega_m + \Omega_\Lambda = 1$ is modified in $\gamma\Lambda$ CDM when the standard definitions of the Λ CDM parameters are retained (see Section 4.3): Equation (49) shows that $H_{\text{fp}} < H_{\text{dS}}$ since $\gamma > 0$, and Equation (50) follows directly from the standard definition of the parameter $\Omega_\Lambda \propto H^{-2}$ (Equation (48)). The result is that the fixed point of the late-time attractor moves higher by $+2\gamma$ (see also Section 4.4).

At the FLRW background level, the stability of the fixed point is not changed by γ to linear order. Considering Equation (46) in the form $\dot{H} = F(H)$ —viz., $\dot{H}$ as a function of H —we find

$$\frac{dF}{dH}(H = H_{\text{fp}}) = -3H_{\text{dS}} + \mathcal{O}(\gamma^2) < 0.$$

Therefore, small perturbations $\delta H(t)$ about H_{fp} decay exponentially in time as $\delta H \propto \exp(-3H_{\text{dS}}t)$.

4.3. Vacuum and Matter Scaling

To interpret the γ modifications to the background dynamics, we recast the FEs in a form that makes the effective changes in the vacuum and matter couplings explicit. In $\gamma\Lambda$ CDM, the first Friedmann Equation (4) reads

$$\left((1 + \gamma)H \right)^2 = \frac{\Lambda c^2}{3} + \frac{8\pi G\rho}{3}, \quad (51)$$

and, to linear order in γ , the modified expression of H^2 is

$$H^2 = (1 - 2\gamma) \left(\frac{\Lambda c^2}{3} + \frac{8\pi G\rho}{3} \right) + \mathcal{O}(\gamma^2). \quad (52)$$

Thus, the contributions of the dark and matter fields to H^2 are lowered by the same factor, and this is where our interpretation of an exterior tidal field comes in: a radially outward tidal force effectively counteracts the source of gravity ($\propto G\rho$) and also holds back the acceleration of the background—because of the $(1 - 2\gamma)$ scaling of Λ and ρ , the dark energy pushes forward slightly less, and gravity pulls inward slightly less (for more details, see Section 4.8 below).

We quantify the outward field in the following manner. After straightforward manipulations, we recast Equations (51) and (52) to the convenient form

$$\Omega_m(t) + \Omega_\Lambda(t) = (1 + \gamma)^2 = 1 + 2\gamma + \mathcal{O}(\gamma^2), \quad (53)$$

where $\Omega_m(t) \equiv 8\pi G\rho/(3H^2)$, and $\Omega_\Lambda(t)$ is defined in Equation (48). Naturally, at the asymptotic limit $\Omega_m \rightarrow 0$ and to linear order in γ , Equation (53) reduces to the $\mathcal{O}(\gamma)$ form of Equation (50) for the fixed point of the late-time attractor.

4.4. Cosmic Acceleration

Here, we consider the onset of the accelerated expansion in $\gamma\Lambda$ CDM. We begin with the kinematic identity $\ddot{a}/a = \dot{H} + H^2$, which cannot be rescaled by hand, just as the kinematic definition of H (Equation (2)) cannot be rescaled. To build this identity in the $\mathcal{O}(\gamma)$ regime, we rearrange the various terms in Equation (47), which becomes

$$2 \left(\dot{H} + H^2 \right) = H^2 \left[(3\Omega_\Lambda - 1) - 3(1 + \Omega_\Lambda)\gamma \right] = 2 \frac{\ddot{a}}{a}. \quad (54)$$

Then, the condition $\ddot{a} \geq 0$ is satisfied at all times for which

$$\Omega_\Lambda(t) \geq \frac{1 + 3\gamma}{3(1 - \gamma)} = \frac{1 + 4\gamma}{3} + \mathcal{O}(\gamma^2). \quad (55)$$

Next, we determine the onset of acceleration in redshift space. For convenience, we calculate the derivative

$$\frac{d\Omega_\Lambda}{d(\ln a)} = \frac{\dot{\Omega}_\Lambda}{H} = -2\Omega_\Lambda \frac{\dot{H}}{H^2}; \quad (56)$$

using Equation (47) and the definition of redshift $z \equiv 1/a - 1$, we find a logistic-type Riccati ODE, viz.,

$$\frac{d\Omega_\Lambda}{d(\ln a)} = -(1 + z) \frac{d\Omega_\Lambda}{dz} = 3\Omega_\Lambda \left[(1 + \gamma) - (1 - \gamma)\Omega_\Lambda \right]. \quad (57)$$

We do not display the closed-form solution $\Omega_\Lambda(z)$ of this first-order ODE, as it is cumbersome and allows little physical intuition. Instead, we integrate the ODE for various initial conditions $(\Omega_\Lambda)_{\text{rec}}$ set at recombination ($\ln a_i = -7$, $z_{\text{rec}} = 1096$) to construct the phase portraits shown in Figure 4 for $\gamma = 0$ (Λ CDM) and $\gamma = 0.083$ ($\gamma\Lambda$ CDM). In each portrait, the particular solution that matches the present-day condition $\Omega_{\Lambda 0} = 0.685$ [1] is highlighted by a thick red curve, and the fixed point $(\Omega_\Lambda)_{\text{fp}}$ is indicated by a horizontal dashed line.

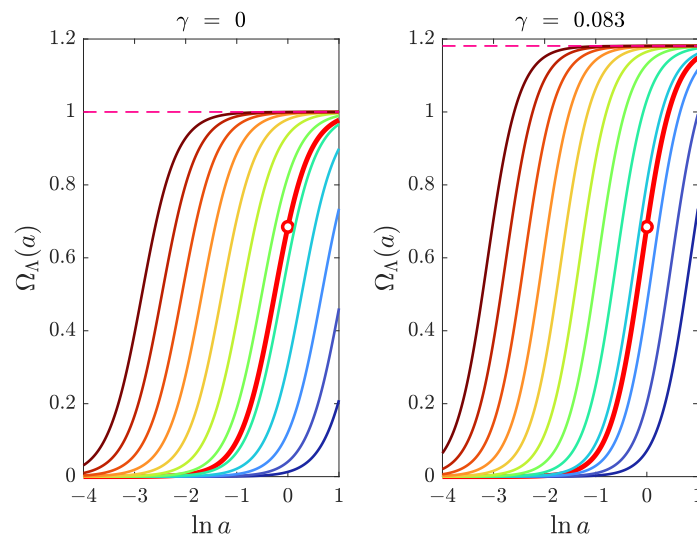


Figure 4. Phase portraits of the Riccati ODE (57) for $\gamma = 0$ (Λ CDM) and $\gamma = 0.083$ ($\gamma\Lambda$ CDM). The initial conditions set at recombination ($\ln a_i = -7$) are logarithmically spaced in the interval $\log(\Omega_\Lambda)_{\text{rec}} \in [-11, -5.4]$. In each panel, the thick red curve denotes the particular solution that passes through $\Omega_{\Lambda 0} = 0.685$ (red circle), and the horizontal dashed line indicates the location of the late-time attractor fixed point (Equation (50)).

Next, we formulate the evolution of $\Omega_\Lambda(z)$ as a Cauchy problem implied by Definition (48) together with the $\mathcal{O}(\gamma)$ modified Hubble dynamics and integrate Equation (57) subject to the present-day condition

$$\Omega_\Lambda(z=0) = \Omega_{\Lambda 0}. \quad (58)$$

The solutions are strictly monotonic: for any choice of $\gamma > 0$, parameter $\Omega_\Lambda(z)$ rises from negligible values at high redshifts to $\Omega_{\Lambda 0}$ at present and tends to the attractor (50) at much later times. In this way, γ generates a one-parameter family of past and future histories, $\Omega_\Lambda(z; \gamma)$, including the standard Λ CDM evolution with $\gamma = 0$. Figure 5 illustrates the future trajectories of three family members with $\gamma \in [0, 0.039, 0.083]$.

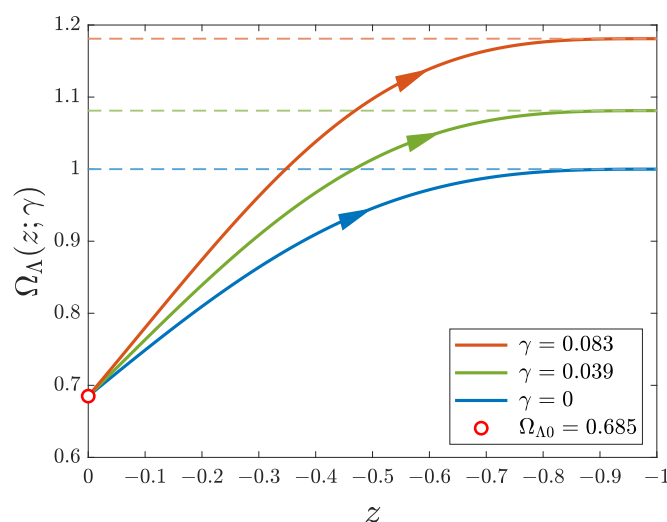


Figure 5. Future trajectories of the Cauchy problem (57) and (58) with $\gamma \in [0, 0.039, 0.083]$ are plotted versus redshift $z \leq 0$. The red circle represents the initial condition $\Omega_{\Lambda 0} = 0.685$. The dashed lines indicate the corresponding locations of the late-time attractor fixed points (Equation (50)).

Application: The $\ddot{a} = 0$ Transition Redshift

In the solutions of the Cauchy problem in Equations (57) and (58), the inequality $\ddot{a} \geq 0$ is satisfied for redshifts $z \leq z_T(\gamma)$. To continue, we choose $\gamma = 0.083$ and we solve the Cauchy problem to determine $z_T(\gamma = 8.3\%)$. Using the Planck-2018 condition that $\Omega_{\Lambda 0} = 0.685$ [1], we find that

$$z_T(\gamma = 8.3\%) = 0.2767, \quad (59)$$

a threshold to be compared to the Λ CDM value of $z_T(\gamma = 0) = (2\Omega_{\Lambda 0}/\Omega_{m0})^{1/3} - 1 = 0.6323$.

The dramatic decrease in z_T in $\gamma\Lambda$ CDM is a nonlinear $[\Omega_{\Lambda}(z)]^2$ effect that can be traced to the Riccati ODE (57), although the γ modification also pushes z_T lower by raising the threshold value of $\Omega_{\Lambda}(z_T)$. The γ -modified nonlinear Riccati term displaces the late-time attractor (Equation (50)) to a higher value, and the higher fixed point leads to the selection of a different family of trajectories as compared to the Λ CDM fixed point of $(\Omega_{\Lambda\text{CDM}})_{\text{fp}} = 1$. Together, these two effects serve to reduce $z_T(\gamma = 8.3\%)$ relative to $z_T(\gamma = 0)$ by more than 50%.

4.5. Matter–Vacuum Equality Threshold

Here, we turn to the “matter–vacuum equality” (MVE) threshold underlying the DCP in standard Λ CDM (Section 1.1). This “recent” threshold, $z_{\text{eq}}(\gamma)$, is also modified in $\gamma\Lambda$ CDM relative to the standard Λ CDM value of $z_{\text{eq}}(\gamma = 0) = (\Omega_{\Lambda 0}/\Omega_{m0})^{1/3} - 1 = 0.2956$.

For $\gamma > 0$, the same tidal field that modifies the background dynamics also modifies the effective dilution of matter. This dual action is the key reason for which $\gamma\Lambda$ CDM is a successful model, and it will be discussed in more detail in Sections 4.8 and 5. In the matter sector of $\gamma\Lambda$ CDM, the solution $\rho(z)$ of the $(1 + \gamma)$ -modified continuity Equation (3) is

$$\rho(z) = \rho_0(1+z)^{3(1+\gamma)}, \quad \rho_0 \equiv \rho(z=0), \quad (60)$$

so that the threshold of matter–vacuum equality $\rho(z_{\text{eq}}) = \rho_{\Lambda}$ (where $\rho_{\Lambda} \equiv \Lambda c^2/(8\pi G) = \text{const.}$) is reached at a different redshift $z_{\text{eq}}(\gamma)$ than in the $\gamma = 0$ Λ CDM case.

Equating $\rho(z_{\text{eq}})$ to the constant vacuum density ρ_{Λ} , replacing the densities with their Ω parameters, and solving for $z_{\text{eq}}(\gamma)$, we find that

$$z_{\text{eq}}(\gamma) = \left(\frac{\Omega_{\Lambda 0}}{\Omega_{m0}} \right)^{\frac{1}{3(1+\gamma)}} - 1. \quad (61)$$

Using the Planck-2018 values [1] and $\gamma = 0.083$, we obtain

$$z_{\text{eq}}(\gamma = 8.3\%) = 0.2701, \quad (62)$$

a value that is lower than $z_{\text{eq}}(\gamma = 0) = 0.2956$ by 8.6%.

Application: No Density Coincidence Problem in $\gamma\Lambda$ CDM

At this point, we can compare the $\gamma = 0.083$ redshift thresholds of Equations (59) and (62). We find that

$$\frac{z_T - z_{\text{eq}}}{z_{\text{eq}}} \simeq 2.4\%, \quad (63)$$

i.e., the acceleration threshold was crossed only 65 Myr before the MVE threshold was reached. Furthermore, each of these redshift thresholds differs from the mean $\bar{z} = 0.2734$

by only $\pm 1.2\%$, where $\bar{z}$ corresponds to a lookback time of 3.28 Gyr in the standard Λ CDM universe of age 13.8 Gyr [1].

Based on these numerical results, we can argue that the DCP is effectively absent in $\gamma\Lambda$ CDM: the $\mathcal{O}(\gamma)$ onset of the accelerated epoch nearly coincides with the MVE threshold, with corresponding densities (relative to the present-day density ρ_0) of

$$\rho(z_T = 0.2767) = 2.211\rho_0, \quad (64)$$

and

$$\rho(z_{\text{eq}} = 0.2701) = 2.175\rho_0, \quad (65)$$

respectively, showing a relative difference of merely 1.7%. Figure 6 summarizes the matter density profiles and the corresponding thresholds for both Λ CDM models ($\gamma = 0, 0.083$) in the recent epoch with $z \in [0, 0.7]$.

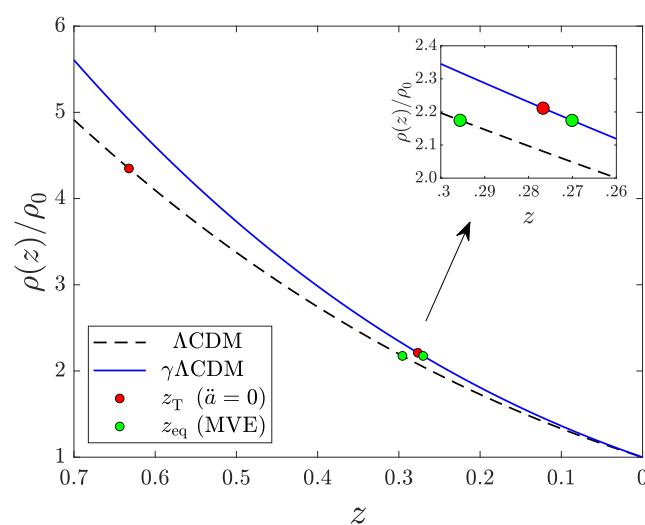


Figure 6. Matter density profiles in Λ CDM and $\gamma\Lambda$ CDM ($\gamma = 0.083$) for $z \in [0, 0.7]$ plotted in reverse. The corresponding thresholds z_T and z_{eq} are marked by red and green dots, respectively.

4.6. Deceleration Parameter

The deceleration parameter is defined kinematically by

$$q \equiv -\frac{\ddot{a}/a}{H^2} = -1 - \frac{\dot{H}}{H^2}. \quad (66)$$

Using Equation (47), the $\mathcal{O}(\gamma)$ deceleration parameter reads

$$q = \frac{1 - 3\Omega_\Lambda}{2} + \frac{3(1 + \Omega_\Lambda)}{2} \gamma. \quad (67)$$

The γ term increases q in the matter-dominated epoch, and the deceleration $\ddot{a} = -qaH^2$ is stronger at fixed H . Thus, the transition point $\ddot{a}(z_T) = 0$ occurs at a lower redshift z_T or, equivalently, at a higher value of $\Omega_\Lambda(t)$, as seen in Equation (55).

Equation (67) highlights some interesting properties of the deceleration parameter in $\gamma\Lambda$ CDM. We adopt the Λ CDM form of $q(\gamma = 0) \equiv q^{(0)}$, viz.,

$$q^{(0)} = \frac{1 - 3\Omega_\Lambda}{2}, \quad (68)$$

and we use it to eliminate Ω_Λ from Equation (67), which then takes the compact $\mathcal{O}(\gamma)$ form

$$q = q^{(0)} + \gamma(2 - q^{(0)}). \quad (69)$$

Beyond its simplicity, Equation (69) is also physically important for the following reasons.

- Since $q^{(0)} \leq 1$ in Λ CDM and $\gamma > 0$, the $\mathcal{O}(\gamma)$ term increases q , and, since $\ddot{a} \propto -q$, the difference

$$\Delta q \equiv q - q^{(0)} = \gamma(2 - q^{(0)}), \quad (70)$$

renders $\ddot{a}/a$ more negative at fixed H and enhances deceleration (or holds back acceleration in the corresponding regime), as noted above in the discussion of Equation (67).

- The $\mathcal{O}(\gamma)$ difference Δq is modulated by the Λ CDM baseline kinematics through the factor $(2 - q^{(0)})$. In particular, $\Delta q \rightarrow \frac{3}{2}\gamma$ (matter domination) and $\Delta q \rightarrow 3\gamma$ (Λ domination). Thus, the same constant γ produces an $\mathcal{O}(1)$ larger correction in the acceleration era than in the deceleration era, with the magnitude of Δq remaining bounded across all intermediate epochs.
- The acceleration transition point $\ddot{a} = 0$ occurs when $q = 0$, and it is shifted in $\gamma\Lambda$ CDM to

$$q^{(0)} = -\frac{2\gamma}{1-\gamma} = -2\gamma + \mathcal{O}(\gamma^2). \quad (71)$$

This is a kinematic statement that the $\gamma > 0$ modification delays the onset of acceleration until redshift $z_T(\gamma)$ relative to the higher Λ CDM threshold of $z_T(\gamma = 0)$.

- The $\gamma\Lambda$ CDM model does not permit an arbitrary z -dependent departure of the deceleration history $q(z)$. To $\mathcal{O}(\gamma)$, the deviation Δq is proportional to the constant parameter γ (Equation (70)). Hence, varying γ only rescales a definite, model-prescribed departure of $q(z)$ (relative to $\gamma = 0$ Λ CDM), rather than introducing a free correction function of z . Consequently, observational reconstructions of $q(z)$ —from distance indicators or via the transition redshift z_T —can constrain the parameter γ directly.

Application: A Present-Day Deceleration Parameter

Applying Equation (67) with $\gamma = 0.083$ to the present time ($\Omega_{\Lambda 0} = 0.685$ [1]), we find that

$$q(z = 0) = -0.318. \quad (72)$$

Compared to the corresponding Λ CDM value of -0.528 [44], the magnitude of $q(z = 0)$ is reduced by about 40%. This implies that the present-day acceleration $\ddot{a}$ is nearly 40% smaller in $\gamma\Lambda$ CDM.

4.7. An Effective Equation of State

In the standard FLRW framework (cf. Ref. [44]), the deceleration diagnostic may be written as

$$q = \frac{1}{2}(1 + 3w_{\text{tot}}), \quad (73)$$

where the total EOS index $w_{\text{tot}} = P_{\text{tot}}/(\rho_{\text{tot}}c^2)$ is determined by the total pressure $P_{\text{tot}} = P + P_\Lambda$ and density $\rho_{\text{tot}} = \rho + \rho_\Lambda$. This relation remains valid in our treatment because, in the Friedmann Equation (52), both contributions to H^2 are modified by the same constant factor $(1 - 2\gamma)$. Then, the matter pressure P appears to be nonzero, but it does not come from the dust; the solution (60) of the $(1 + \gamma)$ -modified continuity equation shows that, effectively,

$$P(\rho; \gamma) = \gamma \rho c^2, \quad (74)$$

an EOS that corresponds to an effective index

$$w_\gamma = \gamma. \quad (75)$$

So, in $\gamma\Lambda$ CDM, we have two contributions to w_{tot} : the effective index w_γ and the index $w_\Lambda = -1$ of the cosmological dark fluid with negative pressure $P_\Lambda = -\rho_\Lambda c^2$.

On the other hand, the modified Friedmann Equation (52) alters the standard sum rule $\Omega_{\text{tot}} = 1$ to the $\gamma\Lambda$ CDM rule

$$\Omega_{\text{tot}} \equiv \Omega_{\text{m}} + \Omega_\Lambda = 1 + 2\gamma + \mathcal{O}(\gamma^2), \quad (76)$$

as is also seen in Equation (53). Then, the total index w_{tot} is determined from the weighted average

$$w_{\text{tot}} = \frac{-\Omega_\Lambda + \gamma\Omega_{\text{m}}}{\Omega_{\text{tot}}}, \quad (77)$$

or, equivalently, from the $\mathcal{O}(\gamma)$ equation

$$w_{\text{tot}} = -\Omega_\Lambda + \gamma(1 + \Omega_\Lambda). \quad (78)$$

Substituting Equation (78) into (73) reproduces Equation (67), which indicates that the $\mathcal{O}(\gamma)$ modifications of the FEs can be described within the relativistic FLRW framework by an effective pressure term, $P(\rho; \gamma) = \gamma\rho c^2$, assigned to the matter sector, along with the modified sum rule (76).

Application: A Present-Day Effective EOS Index

To form a connection with the standard two-component general relativistic phenomenology (pressureless matter plus a single dark fluid), we introduce an effective dark energy index w_{eff} , defined as the constant EOS index that reproduces the same deceleration $q_0 \equiv q(z=0)$ in a universe of dust. According to this definition, the effective index reads [44]

$$w_{\text{eff}}(z=0) = \frac{2q_0 - 1}{3\Omega_{\Lambda 0}}. \quad (79)$$

Using $\Omega_{\Lambda 0} = 0.685$ [1] and Equation (72), we obtain

$$w_{\text{eff}}(z=0) = -0.796, \quad (80)$$

corresponding to a shift of $\Delta w_{\text{eff}} \simeq +0.20$ relative to the Λ CDM value of $w_\Lambda = -1$. Here, w_{eff} is introduced as a purely phenomenological mapping to the usual “dust plus dark energy” fit and should not be confused with the EOS indices w_γ and w_{tot} evaluated above.

4.8. Global Energy Conservation

The $\mathcal{O}(\gamma)$ tidal field modifications discussed in Section 4.3 can alternatively be considered as arising from an isotropic exotic field described by a stress–energy tensor $T_{\mu\nu}^{(\text{exo})}$ in the Einstein field equations, viz.,

$$G_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu}^{(\text{mat})} + T_{\mu\nu}^{(\text{vac})} + T_{\mu\nu}^{(\text{exo})} \right), \quad (81)$$

where superscripts also label the matter sector and the vacuum sector. Each component is taken to be a perfect fluid comoving with the background four-velocity u^μ . The matter and vacuum sectors obey the usual Λ CDM properties, viz., $\{\rho > 0, P = 0\}$ for dust and

$$\left\{ \rho_\Lambda = \text{const.} > 0, \quad P_\Lambda = -\rho_\Lambda c^2 < 0 \right\}, \quad (82)$$

for vacuum, respectively.

In this framework, the vacuum–matter coupling Equation (52) takes the form

$$\rho_c = (1 - 2\gamma)(\rho + \rho_\Lambda) + \mathcal{O}(\gamma^2), \quad (83)$$

where the critical FLRW density $\rho_c(a)$ is given by Equation (26). This equation implies that the $\mathcal{O}(\gamma)$ effective exotic density is

$$\rho_{\text{exo}} = -2\gamma(\rho + \rho_\Lambda). \quad (84)$$

Thus, at the background level, the exotic field provides a screening contribution that produces the particular $(1 - 2\gamma)$ rescaling of both the matter and vacuum sectors in Equations (52) and (83).

The continuity equation is derived from the contracted Bianchi identity of the Einstein tensor, viz., $\nabla^\mu G_{\mu\nu} = 0$, which enforces the conservation of the total stress–energy tensor in Equation (81). In the flat FLRW background, the vacuum terms do not appear in the continuity equation, and global energy conservation is expressed by a pair of coupled equations, viz.,

$$\dot{\rho} + 3H\rho = +Q, \quad (85)$$

$$\dot{\rho}_{\text{exo}} + 3H(\rho_{\text{exo}} + P_{\text{exo}}/c^2) = -Q, \quad (86)$$

where the $\gamma\Lambda$ CDM rate

$$Q(t) = -3\gamma H\rho, \quad (87)$$

encodes the energy transfer between dust and the effective exotic reservoir (whereas ρ_Λ and P_Λ remain strictly constant in time).

Equation (85) is the $(1 + \gamma)$ -modified FLRW continuity equation for dust (as compared to the Λ CDM Equation (3)). Equation (86) helps to determine the effective exotic pressure P_{exo} as follows: differentiating Equation (84) with respect to cosmic time and setting $\dot{\rho}_\Lambda = 0$, we find that $\dot{\rho}_{\text{exo}} = -2\gamma\dot{\rho}$ and, to $\mathcal{O}(\gamma)$, that $\dot{\rho}_{\text{exo}} = 6\gamma H\rho$. Then, Equation (86), solved for P_{exo} , is reduced to the form

$$P_{\text{exo}} = \gamma(\rho + 2\rho_\Lambda)c^2 + \mathcal{O}(\gamma^2). \quad (88)$$

Equations (84)–(88) demonstrate that the $\gamma\Lambda$ CDM background evolution, as described by Equations (52) and (83), admits a fully conservative interpretation in which energy–momentum is redistributed between the dust sector and the isotropic exotic sector, while the global relativistic conservation law $\nabla^\mu G_{\mu\nu} = 0$ is exactly preserved to linear order in the parameter γ . In particular, when the above relations are recast into the conventional two-component form (dust plus a single dark fluid), they reproduce to $\mathcal{O}(\gamma)$ the effective EOS index $w_{\text{eff}}(z = 0) = -0.796$ of Section Application: A Present-Day Effective EOS Index.

Equations (84) and (88) show that the exotic sector is characterized by negative energy density $\rho_{\text{exo}} < 0$ and positive pressure $P_{\text{exo}} > 0$, with both quantities evolving in time. The signs are opposite to those of the density and pressure of the vacuum sector (Equation (82)). More generally, any time-dependent dark component (or exotic field) with the same sign structure—entering the Einstein field Equations (81) in the same manner as the stress–energy tensor $T_{\mu\nu}^{(\text{exo})}(\rho_{\text{exo}}, P_{\text{exo}})$ —could provide an alternative source/origin of the modified kinematics and dynamics produced by $\gamma\Lambda$ CDM. The essential requirement is that such

a dark component or field, say $T_{\mu\nu}(\gamma)$, must simultaneously rescale the $\mathcal{O}(\gamma)$ effective contributions of both the matter and vacuum sectors to the expansion history as follows:

$$\begin{aligned}\rho_{\text{tot}} &= +(1 - 2\gamma)(\rho + \rho_\Lambda), \\ P_{\text{tot}} &= -(1 - 2\gamma)\rho_\Lambda c^2 + \gamma\rho c^2.\end{aligned}\quad (89)$$

In this fundamental set of $\gamma\Lambda$ CDM equations, notice, in particular, the dual action of the reduction factor $(1 - 2\gamma)$ on both ρ and ρ_Λ (cf. Equation (52)), as well as the remarkable appearance of the “dust pressure” term given by Equation (74)—which, of course, does not come from the dust, but it is traced to the $(1 + \gamma)$ -modified matter continuity Equation (85).

We do note, however, that the specific $T_{\mu\nu}^{(\text{exo})}$ modification of Equations (81)—or any other sources of $T_{\mu\nu}(\gamma)$ modifications adhering to Equations (89)—effectively rules out exotic dark fields that require effective EOS indices $w_{\text{eff}}(z = 0) < -1$ or total EOS indices $w_{\text{tot}} < -1$ (see Section 4.7).

Application: A Tidal Sector EOS Index

In $\gamma\Lambda$ CDM, the tidal sector itself admits a well-defined, albeit time-dependent, EOS index, viz.,

$$w_{\text{tid}}(a) \equiv \frac{P_{\text{tid}}}{\rho_{\text{tid}} c^2} = -\frac{1}{2} \left(1 + \frac{\rho_\Lambda}{\rho(a) + \rho_\Lambda} \right) = -1 + \frac{1}{2} \Omega_m(a), \quad (90)$$

where $\Omega_m(a)$ is given by Equation (32). At the present epoch, $\Omega_{m0} = 0.315$ [1] and

$$w_{\text{tid}}(a = 1) = -0.8425. \quad (91)$$

In the Λ -dominated de Sitter future $a \rightarrow \infty$, $\Omega_m \rightarrow 0$ and $w_{\text{tid}} \rightarrow -1$. In this asymptotic limit, the tidal density and pressure saturate to $\rho_{\text{tid}} \rightarrow -2\gamma\rho_\Lambda < 0$ and $P_{\text{tid}} \rightarrow +2\gamma\rho_\Lambda c^2 > 0$, respectively, where they continue to modify to $\mathcal{O}(\gamma)$ the Hubble flow $H(t)$ (Equations (52) and (83)) and partially counteract the background acceleration $\ddot{a}(t)$. However, note that the sign structure of the tidal field is markedly opposite to that of the constant Λ field. This generally means that any introduced exterior field replacing the tidal field must also have a negative energy density and positive pressure.

4.9. Λ CDM- $\gamma\Lambda$ CDM Comparison

For comparison purposes, Table 5 lists several widely known parameter values for standard Λ CDM and the corresponding values obtained in this work from the benchmark $\gamma\Lambda$ CDM model with $\gamma = 0.083$.

The most remarkable difference appears in row 6, which shows the transition redshift z_T of the inflection point $\ddot{a} = 0$. The benchmark $\gamma\Lambda$ CDM value of z_T has shifted down to the level of the MVE redshift (row 8). The nonlinearity responsible for this large shift was discussed in Section Application: The $\ddot{a} = 0$ Transition Redshift. Besides the dramatic decrease in the z_T threshold, (a) the LSS parameter S_8 has clearly decreased to the serviceable range required by observations, resolving thus the S_8 tension (row 3); and (b) the effective Λ -EOS index $w_{\text{eff}}(z = 0)$ is now $\simeq -0.80$ in $\gamma\Lambda$ CDM, rather than -1 as in standard Λ CDM (row 11).

Table 5. Comparison of results between standard Λ CDM ($\gamma = 0$) and benchmark $\gamma\Lambda$ CDM ($\gamma = 0.083$).

Row #	Parameter	Symbol	Λ CDM Value ($\gamma = 0$)	$\gamma\Lambda$ CDM Value ($\gamma = 0.083$)
1	Hubble Constant	H_0	— 67.4 km s ^{−1} Mpc ^{−1} —	
2	Dark Energy Fraction	$\Omega_{\Lambda 0}$	— 0.685 —	
3	LSS Parameter	S_8	$\simeq 0.83$	$\simeq 0.77$
4	Hubble Friction	$2(1 + \gamma)H(t)$	$2H(t)$	$2.166H(t)$
5	Fixed Point	$(\Omega_{\Lambda})_{fp}$	1	1.166
6	Transition Redshift	z_T	0.6323	0.2767
7	Zero-Acceleration Ω_{Λ}	$\Omega_{\Lambda}(z_T)$	$\frac{1}{3}$	0.454
8	MVE Redshift	z_{eq}	0.2956	0.2701
9	MVE Ω_{Λ}	$\Omega_{\Lambda}(z_{eq})$	$\frac{1}{2}$	$\simeq 0.6$
10	Deceleration Parameter	$q(z = 0)$	−0.528	−0.318
11	Effective Λ -EOS Index	$w_{eff}(z = 0)$	−1	−0.796
12	Total EOS Index	$w_{tot}(z = 0)$	−0.685	−0.545
13	Tidal Sector EOS Index	$w_{tid}(z = 0)$	...	−0.8425
14	Dust EOS Index	w_{γ}	...	0.083

4.10. Hubble Tension Unraveled

Rows 1 and 2 in Table 5 indicate that the values of H_0 and $\Omega_{\Lambda 0}$ remain unchanged in $\gamma\Lambda$ CDM. On the other hand, Equation (45) with $\gamma \ll 1$ implies that the local CDL measurements infer an enlarged value of the Hubble constant

$$H_0(\gamma) = H_0(1 + \gamma), \quad (92)$$

relative to the CMB-inferred value of $H_0 = (67.4 \pm 0.5) \text{ km s}^{-1} \text{ Mpc}^{-1}$.

For the γ values adopted in this work ($\gamma \simeq 4\%–8\%$), the Hubble tension is thereby resolved. In Table 6, we summarize the $\gamma\Lambda$ CDM values of $H_0(\gamma)$ in comparison to the results of the two leading sets of local CDL measurements [15–17].

Table 6. $\gamma\Lambda$ CDM values of $H_0(\gamma)$ (Equation (92) with $H_0 = 67.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$) compared to the H_0 [CDL] results from local CDL measurements.

γ	$\longrightarrow$	$H_0(\gamma)$ (km s ^{−1} Mpc ^{−1})	H_0 [CDL] (km s ^{−1} Mpc ^{−1})	$\longleftarrow$	Data Set	Reference–Review
0.083		73.0	73.0 ± 1.0		Cepheids–SnIa	[16]–[15]
0.076		72.5	72.5 ± 1.0		Cepheids–SnIa (+TRGB)	
0.039		70.0	69.8 ± 1.7		SnIa–TRGB	[17]–[15]
0.035		69.8				

5. The Dual Action of the Tidal Field in $\gamma\Lambda$ CDM

We noted in Section 4 that the dual action of the radial tidal field (Equation (52)) distinguishes $\gamma\Lambda$ CDM from other models that modify gravity/drag or the background dark energy but not both at the same time [13–15]. This dual action can also be seen in Table 5 by comparing the results listed in rows 3–4 and 11–12, respectively.

Here, we analyze a simplified model of tidal forces that can systematically bias CDL inferences of H_0 toward higher values relative to the Λ CDM-CMB outcome. The main idea is that, along an observer’s line of sight (LOS) leading to a low-redshift target at $r = r_0$, there are always more nearby nonlinear perturbors at distances $r > r_0$ than on the side of the observer at $r = 0$. This is independent of the observer’s location lying at the origin $r = 0$.

5.1. Preliminaries

We assume a spherical isotropic distribution of perturbors that can be reduced to an effective 1D radial model along the observer’s LOS. But we wish to preserve the shell geometry of the 3D distribution, so we adopt a “coherence” cone with a fixed solid angle $\Delta\Omega \ll 4\pi$ anchored at $r = 0$ and extending out to the Hubble radius $r_H \gg r_0$. However, the integrations will not extend this far; they will be limited by the assumption that nonlinear perturbors are formed at late times and their mean tidal field becomes active only for scale factors $a \geq a_*$ (Section 3.4).

The target galaxy is assumed to have a physical radius $s_0 \ll r_0$, so there are no perturbors in the interval $|r - r_0| < s_0$ at a given epoch $a(t)$. In comoving coordinates $x = r/a$, the mean density of the perturbors is assumed to be

$$\bar{\rho}(a) = m \bar{n}(a), \quad (93)$$

where m is a typical perturber mass and $\bar{n}$ is the comoving mean number density. Then, the section of a shell of width dx at distance x within the coherence cone contains mass $dM = \bar{\rho} \Delta\Omega x^2 dx$, and the comoving density per unit length is

$$\mu(x) = \frac{dM}{dx} = (\bar{\rho} \Delta\Omega) x^2, \quad (94)$$

excluding the interval $|x - x_0| < \sigma_0$, where $x_0 = r_0/a$ and $\sigma_0 = s_0/a \ll x_0$. Then, the radial profile (94) has the domain

$$x \in [\max(0, x_0 - x_*), x_0^-] \cup [x_0^+, x_0 + x_*], \quad (95)$$

where $x_0^\pm = x_0 \pm \sigma_0 > 0$, and the cutoff scale x_* is determined from the requirement that $a \geq a_*$ and the epoch of the observation $a = 1$ (Section 3.4). Specifically,

$$x_* = c \int_{a_*}^1 \frac{da}{a^2 H(a)}, \quad (96)$$

and this specifies the conformal distance that a signal propagating at the speed of light c can traverse between $a = a_*$ and $a = 1$ in a flat FLRW background.

5.2. Tidal Acceleration Components

First, we derive the differential tidal acceleration along the observer’s LOS produced by a perturbing mass element dM located at comoving radius x onto the target at x_0 . The relevant Newtonian separation between the two galaxies is their physical separation

$$\Delta r = |r - r_0| = a \Delta x, \quad (97)$$

where $\Delta x = |x - x_0|$ is the comoving separation. In the far-field regime (where $\Delta r \gg s_0$), the magnitude of the Newtonian tidal acceleration da generated by an exterior mass dM is thus

$$da = \frac{2Gs_0dM}{(\Delta r)^3} = \frac{2G\sigma_0dM}{a^2(\Delta x)^3}, \quad (98)$$

where $da > 0$ when it points away from the observer.

Using Equation (94), subtracting the interior from the exterior contribution, and integrating, we find that the net tidal acceleration of the target is

$$a_{\text{net}}(a) = \frac{2G\sigma_0}{a^2} (\bar{\rho} \Delta\Omega) \left\{ \int_{x_0+\sigma_0}^{x_0+x_*} \frac{x^2 dx}{(x-x_0)^3} - \int_{\max(0, x_0-x_*)}^{x_0-\sigma_0} \frac{x^2 dx}{(x_0-x)^3} \right\}, \quad (99)$$

where x_* is given by Equation (96).

Using new variables $u = x - x_0$ for the exterior integral and $u = x_0 - x$ for the interior integral, we transform Equation (99) to the form

$$a_{\text{net}}(a) = \frac{2G\sigma_0}{a^2} (\bar{\rho} \Delta\Omega) \left\{ \int_{\sigma_0}^{x_*} \frac{(x_0+u)^2 du}{u^3} - \int_{\sigma_0}^{\min(x_0, x_*)} \frac{(x_0-u)^2 du}{u^3} \right\}. \quad (100)$$

Performing the integrations in the regime of interest $x_0 < x_*$, we find that

$$a_{\text{net}}(a) = \frac{8Gx_0}{a^2} (\bar{\rho} \Delta\Omega) \left\{ 1 - \frac{1}{8} \frac{\sigma_0}{x_0} \left(3 + 2 \ln \varepsilon + 4\varepsilon + \varepsilon^2 \right) \right\}, \quad (101)$$

where $\sigma_0/x_0 \ll 1$ and

$$\varepsilon \equiv \frac{x_0}{x_*} < 1. \quad (102)$$

The zeroth-order term is clearly the most significant term in $a_{\text{net}}(a)$. Then, evaluating the zeroth-order tidal acceleration for the present time, we set $a = 1$ and $x_0 = r_0$, and we determine that

$$a_{\text{net}}^{(0)}(a=1) = 32\pi G \bar{\rho}(a=1) r_0 \Delta\omega + \mathcal{O}(s_0/r_0), \quad (103)$$

where $\bar{\rho}(a=1)$ is given by Equation (93), and $\Delta\omega = \Delta\Omega/(4\pi) \ll 1$ is the normalized solid angle of the coherence cone that we adopted for the 1D radial model.

It is worth pointing out that $a_{\text{net}}^{(0)}$ is also obtained (much more easily) by replacing the interior upper limit $\min(x_0, x_*)$ with x_* in Equation (100). However, the interior integration is then extended past $x = 0$ and onto the opposite ray of the observer's LOS. In the 1D radial model analyzed here, such a two-ray extension may be acceptable because some nonlinear perturbers "behind" the observer are within the same activation-limited causal range and contribute to the long-range tidal environment of the target galaxy.

5.3. Characteristic Estimates

We return to the comoving form of the zeroth-order tidal acceleration of Equation (101), viz.,

$$a_{\text{net}}^{(0)}(a) = 32\pi G \frac{\bar{\rho}(a)}{a^2} x_0 \Delta\omega, \quad (104)$$

and we approximate $\bar{\rho}(a)$ by $\bar{\rho}(1) = f_0 \rho_0$, where f_0 is the present-day participation fraction of the perturber population, and

$$\rho_0 \equiv \frac{3\Omega_{\text{m}0} H_0^2}{8\pi G}. \quad (105)$$

The approximation is justified in practice as we are working in the $z < 10$ domain, where the perturber galaxies are already well formed [45–47] and exert tidal forces throughout the domain of $a \in [a_*, 1]$. Then, Equation (104) becomes

$$a_{\text{net}}^{(0)}(a) = 12 \Omega_{\text{m}0} H_0^2 \frac{x_0}{a^2} (f_0 \Delta\omega), \quad (106)$$

and it reduces at the present epoch to

$$a_{\text{net}}^{(0)}(a = 1) = 12 \Omega_{\text{m}0} H_0^2 r_0 (f_0 \Delta\omega). \quad (107)$$

The parentheses highlight the geometric factors that are sources of uncertainty in evaluations of the net tidal acceleration.

The recession speed due to the acceleration (106) accumulated up to the present is

$$v_{\text{rec}}(a = 1) = \int_{a_*}^1 \frac{a_{\text{net}}^{(0)}(a)}{H(a)} \frac{da}{a}. \quad (108)$$

In the matter-dominated epoch of Λ CDM, the Hubble parameter is written as $H(a) = H_0 E(a)$, where the dimensionless function $E(a)$ is given by Equation (28). Then, Equation (108) takes the form

$$v_{\text{rec}}(a = 1) = 12 \Omega_{\text{m}0} H_0 r_0 (f_0 \Delta\omega) \mathcal{J}(a_*), \quad (109)$$

where the tidal activation integral $\mathcal{J}(a_*)$ is given by

$$\mathcal{J}(a_*) = \int_{a_*}^1 \frac{da}{a^3 E(a)}. \quad (110)$$

We can now form the dimensionless ratio $a_{\text{net}}^{(0)}/(H_0 v_{\text{rec}})$ at $a = 1$, which, according to our heuristic drag model in Section 2, is a measure of the coefficient of friction 2γ . An added benefit is that the uncertain geometric factors and the distance to the target galaxy cancel out when Equation (107) is divided by Equation (109). Then, we find that

$$\gamma = \frac{1}{2 \mathcal{J}(a_*)}. \quad (111)$$

Thus, in this approximation, the zeroth-order 1D radial model shows that γ is determined primarily by the tidal activation epoch a_* within the standard Λ CDM expanding background.

For $\Omega_{\Lambda 0} \neq 0$, the integral $\mathcal{J}(a_*)$ does not admit a closed-form expression in terms of elementary functions, so, in our estimates, we revert to numerical evaluations and a first-order approximation in the small parameter $(\Omega_{\Lambda 0}/\Omega_{\text{m}0})a^3$. Expanding the integrand of $\mathcal{J}(a_*)$ and the right-hand side of Equation (111) to first order, we find that

$$\gamma^{(1)} \simeq \frac{1}{4} \frac{\sqrt{\Omega_{\text{m}0} a_*}}{1 - \sqrt{a_*}} \left\{ 1 - \frac{1}{10} \left(\frac{\Omega_{\Lambda 0}}{\Omega_{\text{m}0}} \right) \frac{\sqrt{a_*} - a_*^3}{\sqrt{a_*} - 1} \right\}. \quad (112)$$

Some representative results for which the resulting exact values of $\gamma \in [0.039, 0.083]$ are listed in Table 7 for the corresponding a_* and $z_* = 1/a_* - 1$ values and for $\Omega_{\text{m}0} = 0.315$ and $\Omega_{\Lambda 0} = 0.685$ [1]. The zeroth-order approximation,

$$\gamma^{(0)} \simeq \frac{1}{4} \frac{\sqrt{\Omega_{\text{m}0} a_*}}{1 - \sqrt{a_*}}, \quad (113)$$

shows deviations from the exact numerical γ values by about 4–8%, and the first-order approximation (112) shows deviations of about 2–4%. The exact and approximate results are plotted in Figure 7.

For the tidal activation scales of $a_* \in [0.045, 0.125]$ in Table 7, the exact γ values are bracketed by the approximate analytic values, with $\gamma^{(0)}$ underpredicting and $\gamma^{(1)}$ overpredicting the exact results (Figure 7). For this reason, analytic values of higher accuracy can be obtained via the midpoint estimator

$$\gamma^{(\text{mid})} = \frac{1}{2}(\gamma^{(0)} + \gamma^{(1)}), \quad (114)$$

without having to calculate the next higher-order approximation. This amounts to simply halving the coefficient $\frac{1}{10}$ of the correction term in Equation (112) and results in deviations of $-\Delta\gamma^{(\text{mid})} \simeq 1\text{--}2\%$. The $\gamma^{(\text{mid})}$ estimates are also listed in Table 7 and plotted in Figure 7.

Table 7. Determined γ values in the 1D radial model in this section. Exact values of γ were determined numerically from Equation (111) for the chosen a_* values. Approximate values $\gamma^{(0)}$ and $\gamma^{(1)}$ were determined from Equations (113) and (112), respectively, for the chosen a_* values. Percent deviations $\Delta\gamma^{(0)}$ and $\Delta\gamma^{(1)}$ from the exact values are also listed for comparison. Finally, midpoint estimates of $\gamma^{(\text{mid})}$ (Equation (114)) and their percent deviations are listed in the last two columns.

Model	a_*	z_*	S_8	γ	$\gamma^{(0)}$	$\Delta\gamma^{(0)}$ (%)	$\gamma^{(1)}$	$\Delta\gamma^{(1)}$ (%)	$\gamma^{(\text{mid})}$	$\Delta\gamma^{(\text{mid})}$ (%)
C1	0.045	21.22	0.768	0.0392	0.0378	−3.7	0.0400	+1.9	0.0389	−0.9
C2	0.075	12.33	0.760	0.0558	0.0529	−5.2	0.0573	+2.6	0.0551	−1.3
C3	0.100	9.00	0.756	0.0693	0.0649	−6.4	0.0714	+3.0	0.0681	−1.7
C4	0.125	7.00	0.753	0.0830	0.0767	−7.5	0.0858	+3.5	0.0813	−2.0

Notes. (a) The z_* values were calculated from $z_* = 1/a_* - 1$. (b) The S_8 values were computed by numerical integration (see Section 3.4) for the listed pairs of (a_*, γ) . (c) There is no S_8 tension in the 1D radial model because the tidal field is activated too early in redshift space ($z_* \geq 7$). (d) Model C4, a palatable model, is also listed in Table 1 as Model A2.

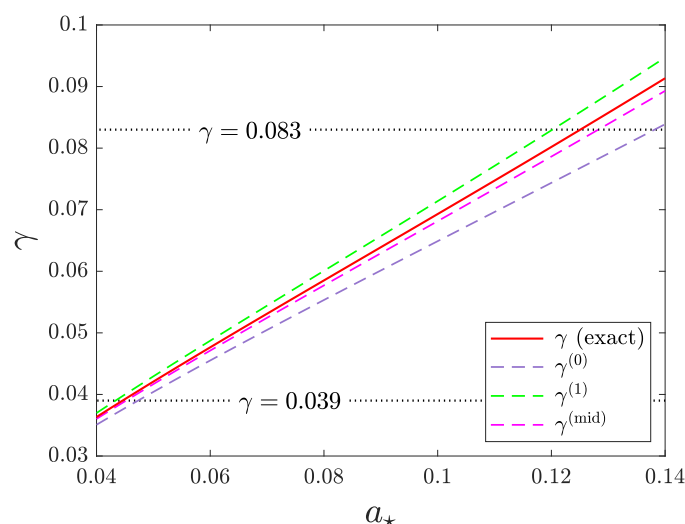


Figure 7. Exact and approximate values of γ versus a_* in the 1D radial model in Section 5. The exact values were computed numerically from Equation (111), and the approximate values were calculated from Equations (112)–(114).

5.4. Comparisons Between Tables

We single out the results of the 1D radial model with $\gamma = 0.083$ and 0.039 (Table 7), and we compare them with the S_8 models listed in Tables 1 and 2. There is a general consensus in the following aspects.

- The $\gamma = 0.039$ models indicate that the S_8 tension can only be resolved if the tidal activation redshift is enormous (typically $z_* \approx 30\text{--}12$ for $S_8 \approx 0.76\text{--}0.78$). Unless S_8 turns out to be ≥ 0.79 (see Table 2), this γ amplitude is inadequate for $\gamma\Lambda\text{CDM}$.

- The $\gamma = 0.083$ models consistently demonstrate that an exterior drag force resolves the tensions in S_8 and H_0 , provided that the tidal activation redshift is $z_* \leq 7$ (see also the 1D radial Model C4 with $z_* = 7$ in Table 7). Given the approximate nature of the 1D radial model and the exact models listed in Table 1, this threshold may have to be revised to be lower [45–47], notably to $z_* \simeq 3.5$ – 6.5 (see Models A3–A6 in Table 1), a range also indicated by recent LSS simulations [48–52]. Regardless, the adopted estimate of $\gamma = 0.083$ is clearly the best choice for the benchmark $\gamma\Lambda$ CDM model.

The agreement between the $\gamma = 0.083$ Models A2 and C4 (Tables 1 and 7, respectively) raises an important issue: in this work, ad hoc modifications (γ terms) were introduced in both the kinematic description of the Hubble flow and the dynamical equations that determine the growth of linear density perturbations. However, it is not entirely clear why these two seemingly independent, arbitrary modifications produced very similar outcomes. We address this issue in Appendix A, where we show that the two independent modifications of the kinematics and the dynamics are consistent with one another.

6. Summary and Conclusions

6.1. Summary

This cosmological study can be summarized as follows.

- In Section 1, we reviewed the two foundational Λ CDM problems (CCP and DCP) and the two major observational tensions (H_0 and S_8), emphasizing the empirical $\gamma = 8.3\%$ difference in observations that motivates a unified late-time explanation within the new $\gamma\Lambda$ CDM model.
- In Section 2, we introduced a late-time drag term in the Euler equation and derived the corresponding linear perturbation equations in the conformal Newtonian gauge (sub-horizon limit) on a flat FLRW Λ CDM background.
- In Section 3, we solved the modified growth equation and showed that the added friction can suppress late-time structure growth and lower the predicted clustering amplitude to the observed low-redshift range, thereby addressing the S_8 tension.
- In Section 4, we analyzed the associated $\mathcal{O}(\gamma)$ background modifications implied by the same kinematical prescription, including its impacts on standard late-time and present-day expansion diagnostics.
- In Section 5, we developed a simplified 1D radial LOS model for the late-time tidal environment of nonlinear structures and used it to estimate both the accumulated LOS recession contribution and the implied drag amplitude $\gamma(a_*)$, where a_* is the scale factor of the tidal activation epoch.

6.2. Conclusions

- ① **S_8 Tension**—After activation, the drag force adds more friction to the linear growth equation of perturbations, suppressing late-time growth and lowering the predicted clustering amplitude [43,53–55]. The solutions presented in Section 3 demonstrate that the parameter range consistent with the kinematic H_0 interpretation simultaneously yields the required reduction in parameter S_8 in a $\gamma\Lambda$ CDM background with a benchmark value of $\gamma = 8.3\%$.
- ② **Hubble Tension**—The $\gamma\Lambda$ CDM model provides a late-time mechanism that biases local CDL determinations of the Hubble flow through an additional LOS contribution sourced by a statistically isotropized tidal environment with a nonzero mean value (Section 5) [55–57]. In this picture, the inferred expansion rate is shifted relative to the Λ CDM background in a manner controlled by the same parameter γ that enters the modified kinematics (cf. Equation (45)).

- ③ **Density Coincidence Problem**—In $\gamma\Lambda$ CDM, parameter γ modifies both the recent acceleration threshold and the matter–vacuum equality threshold [26,44]. For the benchmark case analyzed in Section 4.5, these thresholds occur at nearly the same redshift (cf. Equations (59) and (62)), so the “recent” coincidence underlying the DCP in standard Λ CDM is effectively absent in $\gamma\Lambda$ CDM (Section Application: No Density Coincidence Problem in $\gamma\Lambda$ CDM).
- ④ **Cosmological Constant Problem**—The present framework does not address the magnitude of the vacuum energy predicted by quantum field theory (cf. Refs. [58,59]). As in the standard Λ CDM, the Λ field is treated phenomenologically as an input to the background dynamics (Section 4). We do, however, find a scaled-down value $(1 - 2\gamma)\Lambda$ in $\gamma\Lambda$ CDM, with the same scaling also modifying the dust–matter density field (cf. Equations (52) and (89)).

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Abbreviations

The following abbreviations are used in this manuscript:

CCP	Cosmological Constant Problem
CDL	Cosmic Distance Ladder
CDM	Cold Dark Matter
CMB	Cosmic Microwave Background
DCP	Density Coincidence Problem
DES	Dark Energy Survey
EOS	Equation of State
FEs	Friedmann Equations
FLRW	Friedman–Lemaître–Robertson–Walker
IVP	Initial Value Problem
KiDS	Kilo-Degree Survey
LOS	Line of Sight
LSS	Large-Scale Structure
MVE	Matter–Vacuum Equality
ODE	Ordinary Differential Equation
RMS	Root Mean Square
SnIa	Supernovae of Type Ia
TRGB	Tip of Red Giant Branch
1D	One-Dimensional
3D	Three-Dimensional

Appendix A. Reconciling Hubble Flow Kinematics and Dynamics in $\gamma\Lambda$ CDM

Appendix A.1. Modified Hubble Flow Kinematics

In a purely kinematic model, the recession velocity of a target moving in an expanding FLRW background assumes the modified form

$$\mathbf{u}(t) = (1 + \epsilon(t))H(t) \mathbf{r}(t), \quad (\text{A1})$$

where the drag parameter $\epsilon(t) \ll 1$.

Differentiating with respect to cosmic time and using dot notation for the operator $\partial/\partial t$, we find, to order $\mathcal{O}(\epsilon)$, that

$$\dot{\mathbf{u}} = [H\dot{\epsilon} + (1 + \epsilon)\dot{H} + (1 + 2\epsilon)H^2] \mathbf{r}. \quad (\text{A2})$$

Subtracting the conventional $\epsilon = 0$ terms $\dot{\mathbf{u}}_0 = (\dot{H} + H^2) \mathbf{r}$, we find, to order $\mathcal{O}(\epsilon)$, that the tidal part of the acceleration is

$$\dot{\mathbf{u}}^{(\text{tid})} = [H\dot{\epsilon} + \epsilon(\dot{H} + 2H^2)] \mathbf{r}. \quad (\text{A3})$$

Appendix A.2. Modified Hubble Flow Dynamics

The recession velocity of a target in $\gamma\Lambda$ CDM was assumed to take the modified form

$$\mathbf{u}(t, \mathbf{x}) = (1 + \gamma)H(t) \mathbf{r}(t, \mathbf{x}) + \mathbf{v}(t, \mathbf{x}), \quad (\text{A4})$$

where the drag parameter $\gamma \ll 1$ is a constant and the peculiar velocity $|\mathbf{v}| \ll H\mathbf{r}$.

The peculiar velocity field $\mathbf{v}$ is described by the linearized Euler equation modified to include a drag acceleration term of $\mathcal{O}(\gamma)$, viz.,

$$\dot{\mathbf{v}} + (1 + 2\gamma)H\mathbf{v} = -\frac{1}{a}\nabla\Phi, \quad (\text{A5})$$

where the del operator is taken in comoving coordinates $\mathbf{x}$, as in Section 2.2.2 of the main text.

Differentiating Equation (A4) with respect to cosmic time, we find that

$$\dot{\mathbf{u}} = (1 + \gamma)\dot{H}\mathbf{r} + (1 + \gamma)H\mathbf{u} + \dot{\mathbf{v}}, \quad (\text{A6})$$

or, after some algebraic manipulations preserving only $\mathcal{O}(\gamma)$ terms, that

$$\dot{\mathbf{u}} = (\dot{H} + H^2) \mathbf{r} + (H\mathbf{v} + \dot{\mathbf{v}}) + \gamma(\dot{H}\mathbf{r} + 2H^2\mathbf{r} + H\mathbf{v}). \quad (\text{A7})$$

Next, using Equation (A5) to replace the term $(H\mathbf{v} + \dot{\mathbf{v}})$ and finally subtracting the conventional $\gamma = 0$ terms $\dot{\mathbf{u}}_0 = (\dot{H} + H^2) \mathbf{r} - (\nabla\Phi)/a$, we find, to order $\mathcal{O}(\gamma)$, that the tidal part of the acceleration is

$$\dot{\mathbf{u}}^{(\text{tid})} = \gamma(\dot{H} + 2H^2) \mathbf{r} - \gamma H\mathbf{v}. \quad (\text{A8})$$

The term $-\gamma H \mathbf{v}$ represents the drag exerted on the peculiar velocity field $\mathbf{v}$. Since $|\mathbf{v}| \ll H \mathbf{r}$, this term is of mixed higher order, viz., $\mathcal{O}(\gamma |\mathbf{v}| / (H \mathbf{r}))$, and can subsequently be ignored. So, the tidal acceleration added to the Hubble expansion formally becomes

$$\dot{\mathbf{u}}^{(\text{tid})} = \gamma (\dot{H} + 2H^2) \mathbf{r} + \mathcal{O}(\gamma |\mathbf{v}|). \quad (\text{A9})$$

Appendix A.3. Linear First-Order ODE for the Drag Parameter $\epsilon(t)$

Comparing Equations (A3) and (A9), we find a linear first-order ODE for the time-dependent drag parameter $\epsilon(t)$ to $\mathcal{O}(\gamma)$, viz.,

$$\dot{\epsilon} + \frac{\dot{H} + 2H^2}{H} (\epsilon - \gamma) = 0. \quad (\text{A10})$$

The fraction can be further simplified by introducing the deceleration parameter $q(t)$ defined by Equation (66) and a new independent variable,

$$\zeta(t) \equiv \epsilon(t) - \gamma. \quad (\text{A11})$$

Then, Equation (A10) takes the homogeneous form

$$\dot{\zeta} + (1 - q)H \zeta = 0. \quad (\text{A12})$$

The trivial solution of this ODE is $\zeta = 0$, which implies that

$$\epsilon(t) = \gamma. \quad (\text{A13})$$

Thus, the kinematic drag parameter ϵ turns out to be independent of time and equal to constant γ . This is the desired mathematical result that we set out to derive initially (see Section 5.4), but it may not be the most interesting result in the realm of universal expansion. We discuss the general solution of Equation (A10) below.

Appendix A.4. General Solution of the Drag Parameter $\epsilon(t)$

Remarkably, the nontrivial solution of the ODE (A12) implies a time-dependent general solution $\epsilon(t)$ of the ODE (A10), viz.,

$$\epsilon(t) = \gamma + \frac{CH_0}{a(t)^2 H(t)}, \quad (\text{A14})$$

where CH_0 is the integration constant and $\mathcal{C}$ is a dimensionless constant.

We investigate the general case in which $\mathcal{C} \neq 0$. The time-dependent part of $\epsilon(t)$, viz.,

$$\Delta\epsilon(t) \equiv \frac{CH_0}{a(t)^2 H(t)}, \quad (\text{A15})$$

although set to zero in $\gamma\Lambda\text{CDM}$, is physically very interesting for two reasons:

- (1) In the matter-dominated ΛCDM , the product $a^2 H(a)$ increases with cosmic time as $\sqrt{a \Omega_{m0}}$, so $\Delta\epsilon(t)$ decays in time and $\epsilon(t) \rightarrow \gamma$ at late times. For example, considering the set of redshifts $z \in \{7, 10, 45, 100\}$, $\Delta\epsilon(t)$ is diluted by factors of $\{5, 6, 12, 18\}$ between these epochs and the present, respectively. This dilution becomes much faster in the Λ -dominated future.
- (2) The time-dependent term $\Delta\epsilon(t)$ can be written most transparently in conformal rather than cosmic time. In what follows, we recast Equation (A15) in conformal time η , thereby making its conformal properties explicit.

Appendix A.5. The Conformal Nature of $\Delta\epsilon(t)$

Conformal time η [26] is defined (up to an additive constant that we set to zero) by

$$\eta(t) \equiv \int_0^t \frac{d\tau}{a(\tau)}, \quad (\text{A16})$$

so that

$$\frac{dt}{d\eta} = a(t). \quad (\text{A17})$$

In this convention, cosmic time $t \in [0, \infty)$ is mapped monotonically to $\eta \in [0, \eta_\infty]$, where η_∞ is the finite late-time limit in an eternally expanding matter+ Λ universe.

Using the definition of the Hubble parameter $H(t) \equiv \dot{a}/a$, as well as prime notation for the operator $d/d\eta$, we recast Equation (A15) to its conformal form, viz.,

$$\Delta\epsilon(\eta) = \frac{\mathcal{C}H_0}{a'(\eta)}. \quad (\text{A18})$$

Thus, the product $a'(\eta)\Delta\epsilon(\eta)$ is an invariant of the homogeneous ODE (A12) (that is, the ODE admits a first integral fixed by the initial conditions). Furthermore, Equation (A18) shows that $\Delta\epsilon$ is naturally expressed in conformal time, since it is directly tied to the conformal-time expansion $a'(\eta) = a(\eta)\mathcal{H}(\eta)$ (with the conformal Hubble parameter defined by $\mathcal{H}(\eta) \equiv a'/a$), rather than to the cosmic-time rate $\dot{a}(t) = a(t)H(t)$.

Then, in the matter-dominated and Λ -dominated epochs, we find that

$$a(\eta) \propto \eta^2 \quad \implies \quad \Delta\epsilon(\eta) \propto \eta^{-1}, \quad (\text{A19})$$

and

$$a(\eta) \propto (\eta_\infty - \eta)^{-1} \quad \implies \quad \Delta\epsilon(\eta) \propto (\eta_\infty - \eta)^2, \quad (\text{A20})$$

respectively. Thus, during matter domination, $\Delta\epsilon(\eta)$ decreases as a mild power law in conformal time, whereas, in the Λ -dominated regime, it is driven rapidly to zero as $\eta \rightarrow \eta_\infty$, reflecting the asymptotic de Sitter scaling of the conformal expansion $\mathcal{H}(\eta) \propto (\eta_\infty - \eta)^{-1}$. (Here, $\mathcal{H}(\eta) \rightarrow \infty$ as $\eta \rightarrow \eta_\infty$, but the physical Hubble rate $H = \mathcal{H}/a$ tends to a finite constant, as $a \rightarrow \infty$ too.)

On the other hand, our present time lies in the transition regime of η , where the above limiting behaviors are not representative. For $a \rightarrow a_0 = 1$, it is easier to work with cosmic time $t \rightarrow t_0$ and Equation (A15), which takes the form

$$\Delta\epsilon(t_0) = \mathcal{C}, \quad (\text{A21})$$

which provides an initial condition that fixes the integration constant $\mathcal{C}$ of the solution of ODE (A12). Thus, in $\gamma\Lambda$ CDM, we have effectively chosen to set $\Delta\epsilon(t_0) = 0$, in which case $\epsilon(t) = \gamma$ is constant at all times.

Notes

- ¹ The latest S_8 result from the KiDS–Legacy data set [39,40] is an outlier according to the list of 42 independent values given in Table II of Ref. [15]. Of these 42 values, only 13 rely on combined data sets that also include KiDS data. Excluding these 13 values, there remain 26 values in $S_8 \in [0.7, 0.8]$ with mean $\bar{x} = 0.757$ and standard deviation $s = 0.028$. This mean value lies about 5σ below $S_8 = 0.814$ [39]. Conversely, the new KiDS–Legacy lies 2σ above the mean $\bar{x} = 0.757$.
- ² This new, unexpectedly large S_8 value will need to be verified by other independent methods, which so far have produced much smaller S_8 values, and the process will have to also determine the reasons for these older small contemporary values. Otherwise, the latest KiDS–Legacy value will remain as one of the most obvious outliers, together with the two $S_8 = 0.65$ values on the opposite side of the potential S_8 interval [41,42].

- 3 Nevertheless, the latest KiDS–Legacy result of $S_8 = 0.814^{+0.011}_{-0.012}$ [39,40] is presently a marked outlier relative to the many independent S_8 measurements listed in Table II of Ref. [15] with a mean value of $\bar{x} = 0.757 \pm 0.028(1\sigma)$. More details are given in Note 1 above.
- 4 We note that Model A8 (Tables 1–4) was included to accommodate the recent KiDS–Legacy survey results [39,40] that, perhaps surprisingly, indicate a high value of $S_8 = 0.814^{+0.011}_{-0.012}$. Although not in tension with standard Λ CDM, this value is a marked outlier among a variety of S_8 determinations, most of which do not rely on KiDS data (see Note 1 and Table II in Ref. [15]).
- 5 On the other hand, the $\gamma\Lambda$ CDM model with two free parameters (γ and z_*) has no difficulty reproducing the KiDS–Legacy result [39,40] (Tables 1 and 2), and we find it interesting that the S_8 tension is eliminated for very low values of the γ -activation redshift $z_* \simeq 1$ –2 for $\gamma \simeq 8\%$ –4%, respectively. These low z_* values effectively support the arguments of Akarsu et al. [32] that the redshift dependence of Λ CDM parameters should be studied in various bins within the interval $z \in [0, 2]$ in order to localize systematic errors and physics missing from the standard model.

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