

Article

Numerical Study of Bar Suppression in Galaxy Models Due to Disc Heating

Alejandro López Gómez ^{1,*} , Ruslan Gabbasov ²  and Isaura Luisa Fuentes-Carrera ¹

¹ Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, U.P. Adolfo López Mateos, Edificio 9, Zacatenco, Ciudad de México 07730, Mexico

² Departamento de Sistemas, Universidad Autónoma Metropolitana, Av. San Pablo No. 420 Col. Nueva el Rosario Alcaldía Azcapotzalco, Ciudad de México 02128, Mexico

* Correspondence: alelopez@ipn.mx

Abstract: The process of bar formation, evolution and destruction is still a controversial topic regarding galaxy dynamics. Numerical simulations show that these phenomena strongly depend on physical and numerical parameters. In this work, we study the combined influence of the softening parameter, ϵ and disc mass fraction, m_d , on the formation and evolution of bars in isolated disc-halo models via N -body simulations with different particle resolutions. Previous studies indicate that the bar strength depends on m_d as $\propto m_d^{-1}$, which is seen as a delay in bar formation. However, the distortion parameter, η , which measures the bar's momentum through time, shows that an increase in m_d does not always induce a delay in bar formation. This suggests that ϵ interact to either enhance or weaken the bar. Moreover, numerical heating dominates in models with small softening values, creating highly accelerated particles at the centre of discs, regardless of m_d or resolution. These enhanced particle accelerations produce chaotic orbits for $\epsilon \leq 5$ pc, resulting in bar suppression due to collisional dynamics in the centre. In our high-resolution models ($N \approx 10^7$), small softening values are incapable of reproducing the bar instability. The role of disc mass is as follows: increasing m_d for moderate ϵ (≥ 10 pc) reduces the amount of drift in the acceleration profile, without affecting the bar's behaviour. Models with lower m_d values, coupled with small softening values, have an excess of highly accelerated particles, introducing unwanted effects into otherwise reliable simulations. Finally, we show that the evolution of the disc's vertical acceleration profile is a reliable indicator of numerical heating introduced by ϵ and the bar.

Keywords: galactic dynamics; barred galaxies; numerical simulations



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1. Introduction

Bar formation, evolution and destruction are all long-standing subjects in astrophysics. Extensive efforts trying to understand bar phenomena include studies of theoretical [1–5], numerical [6–9] and observational [10–14] nature. Numerical simulations approach bar phenomenology from three distinctive fronts: gravitational-only [15–18], hydrodynamical including collisionless particles [19–22], and cosmological [23–26].

Studies of structure (and substructure) formation in isolated galaxies in pure N -body simulations have expanded our understanding of bar dynamics [8,27–30]. Despite lacking gas physics and all the effects that come with it (star formation, feedback, etc.), N -body experiments are able to reproduce a lot of fundamental properties involving bar-like phenomena [31–33]. For example, ref. [34] performed a series of N -body simulations composed of a dark matter (DM) halo and a stellar disc, covering different types of halos

(rotating, non-rotating, etc.) and values for Toomre’s criterion, Q , ranging from 0.05 to 1.5; with $N = 2 \times 10^5$ particles and a resolution of $\epsilon \simeq 0.7$ kpc for their Plummer softening length. They showed that bars tend to slow down, independently of the DM halo inner profile or its velocity distribution, by dynamical friction [35] between the bar and DM halo. Another important outcome of [34] is that bars persevere after periods of extreme friction, even if the slowdown is very dramatic, which means that bars are robust objects, something that was doubtful in previous works, e.g., [36]. Later on, ref. [16] evolved similar disc-halo pairs but vastly improving the number of particles, $N = 3.55 \times 10^6$ and numerical resolution with $\epsilon \simeq 0.22$ kpc (values corresponding to their model A_1), and found results that contradicted the conclusions of [34]. They demonstrated that the low particle number and choice of Q used by [34] caused a numerical artifact that led to low bar pattern speeds, a fact also verified by [37]. Ref. [16] also found differences in the lengths of their bars, especially compared to those reported by [8]. While these differences are relevant, both works agree on the fact that a recently formed bar tends to grow and does not disappear spontaneously.

Angular momentum exchange between halo and disc also seems to play a prominent role in bar growth and stabilisation [38–41]. Bars form due to resonant orbits induced by sudden fluctuations in the disc’s density distribution [41]. These resonant orbits promote angular momentum transfer between the disc and an external component that surrounds it (usually a DM halo). If the density perturbation has the right wavelength, the transfer of momentum occurs in a feedback loop that only benefits bar growth. Now, if we were able to shut down the channels¹ that allow the transfer of angular momentum, we could, in principal, *destroy* a bar [42–44]. Ref. [45] calls it bar “suicide”. In short, to destroy the orbits that comprise a bar, it is necessary to gather a huge amount of baryonic matter at its centre, i.e., $\sim 10\%$ of the total disc mass [46]. In order to investigate the existence of unbarred galaxies, ref. [47] constructed galaxy models including stellar disc, DM halo and spherical bulge. They showed that adding a sufficiently dense central bulge component (denser than the disc at the bulge half-mass radii) could prevent the formation of a bar instability. None of their models had a gas component, but, as stressed by these authors, a dissipative element would only weaken the bar; see, for example [48].

The paramount achievements listed above have one glaring flaw: the way numerical parameters are chosen. We have already mentioned some of the parameters that usually relate to bar dynamics, such as N or ϵ . In addition, gas dynamics and halo physical properties like the spin parameter λ , concentration c and virial velocity v_{200} are known to also affect the formation and growth of bars [49,50]. Nonetheless, here, we constrain our study to ϵ , N and the disc mass fraction, m_d . These parameters are involved in how bars are formed and, although the influence of m_d has already been established by N -body simulations of disc galaxies [16,37], it is not well-known how ϵ truly affects them and whether or not it interacts with other parameters. In N -body simulations it is necessary to *soften* the forces between particles to, on one hand, stay in collisionless regime and avoid *relaxation* (too small softening) and, on the other, to avoid excessive smoothing *damping* (too large softening). Moreover, simulations must include a softening kernel that maps the gravitational potential to correctly distribute it along the radial extension of each component. The most common is the Plummer softening. However, it is known that the Plummer softening behaves quite poorly when resolving the frequencies that enhance bar modes [51]. In fact, any kind of gravitational softening tends to lower the growth of $m = 2$ modes, although kernels such as the cubic spline are able to resolve them well enough [51].

Balance amid relaxation and damping is key to producing realistic N -body simulations. However, the process to achieve such balance, and therefore, to set a proper softening length for numerical models, is difficult and subject to a large number of uncertainties. Formerly, scientists adapted ϵ due to its relationship with N . For a given N , the value for ϵ

is chosen such that the error between sampling the density distribution and shot noise are minimised [52–55].

Another approach was given by [56]. He studied the effect of softening for one-component (stellar disc) and two-component (stellar and gaseous discs) two-dimensional N -body simulations, stressing that (i) for one-component systems, high softening values ($\epsilon \gtrsim 0.36$ kpc) lack physical consistency because they either have no physical equivalent (like the scale height for small softening values) or tend to quell most instabilities artificially; (ii) for both systems, when $0.20 \text{ kpc} \lesssim \epsilon \lesssim 0.36 \text{ kpc}$, discs are stable, but softening often induces artificial quenching on spiral modes, depending on the wavelength of the perturbation and (iii) models with Newtonian gravity, i.e., $\epsilon \rightarrow 0$, are unstable for a wide range of perturbations, especially those comparable with the characteristic wavelength of the stellar or gaseous components. The above assertions are tightly constrained by the “temperature”, Q , of the discs. Note that Romeo’s simulations [56] do not account for the stabilizing effect of a DM halo.

The adopted choices found in these studies assumed that the density distribution is static and the particles are of equal mass; this is not true either for simulations of cosmological or galactic scale. Specifically, N -body galaxies are, in general, multi-component and, at times, multi-phase systems, meaning that the particle distribution is not constant. Codes like GADGET [57] coped with this by adding softening values for each component. The components can have different smoothing lengths and different masses by demanding that the minimum gravitational potential has the same value for all particles. This allows one to simulate galaxies with much less halo particles without significant increase in the relaxation.

More recently, ref. [58] simulated collisionless barred galaxies with the purpose of evaluating the influence of softening in bar formation, finding virtually no difference between models using an adaptive algorithm designed to adjust ϵ when local particle concentration changes and models with a fixed value of ϵ for the entire run. The result is, at the very least, unexpected.

In Figure 1, we summarise the ranges of softening values normalised by the number of particles that have been used in galactic-scale N -body simulations². There has not been a considerable change in softening ranges after the year 2000 despite the almost exponential growth of N . One would expect a decrease of ϵ as N increases in time by following the notion that for higher N the amount of errors in force calculation minimises when ϵ decreases [53,59]. Instead, some authors use other recipes to adjust their softening values, such as stability analysis [60] or the interparticle separation [61], which may explain the scatter in Figure 1. On the other hand, the spread of ϵ in the last two decades may be attributed to the fact that some authors reported values for Plummer softening or its equivalent for other softening kernels.

Most simulations are constrained to $\epsilon < 0.2$ kpc and most simulations after the year 2000 are constrained to $\epsilon < 0.1$ kpc. All of the simulations in Figure 1 contain more than one component, meaning that they may use multiple softening lengths, one for each component. In such cases, we usually take the lowest value for ϵ , regardless of the type of component it is used on, but we restrain the set to non-gaseous components.

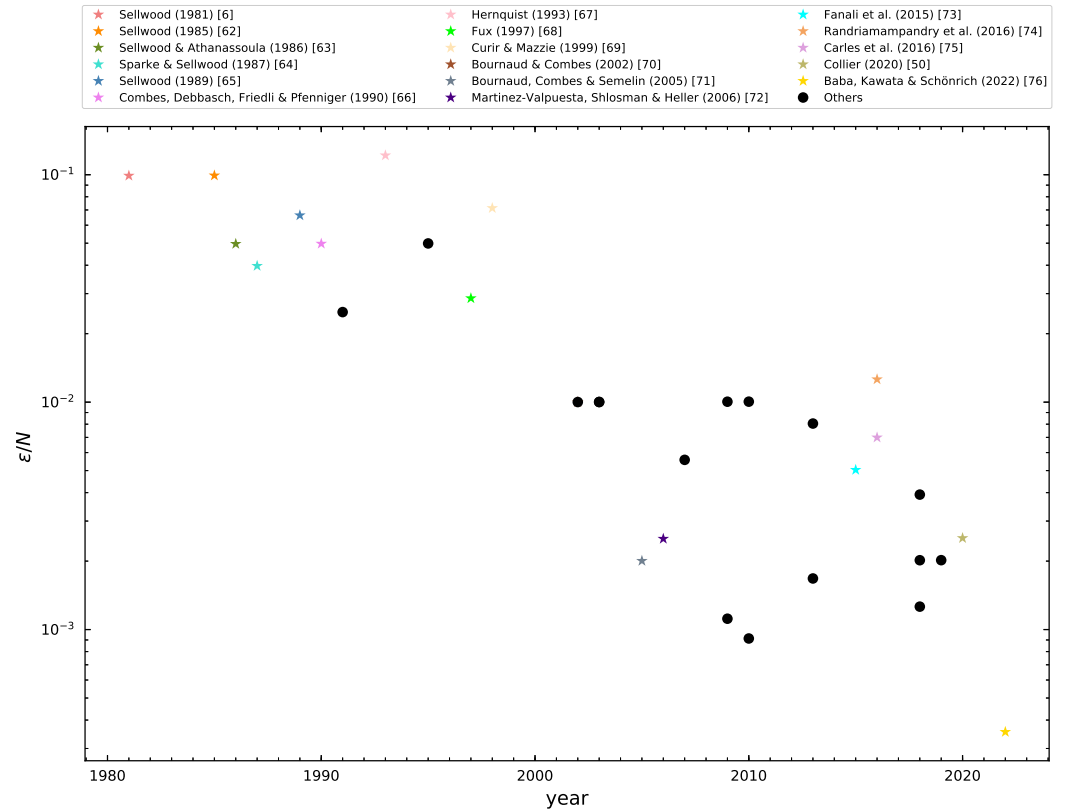


Figure 1. Values of ϵ used throughout time for galactic-scale N -body simulations normalised by N . Annotations for some of this paper’s authors are located at the right of the plot and are denoted in our plot by coloured stars; the rest of simulations are plotted with a filled black circle. The softenings in this figure are in parsecs (pc). We added the reference citation for the studies mentioned in at the top of this figure, which are [6,50,62–76].

Paper Overview

In order to establish the role of softening, disc mass and number of particles on bar formation and evolution, we made a series of simulations and measured the disc and bar properties, such as the tangential average velocity, velocity dispersions, radial accelerations and bar strength throughout 12 Gyrs.

Our paper is organised as follows: Section 2 describes the initial conditions of N -body disc galaxies, details of simulations and their analysis. Section 3 discusses and applies the parameters used to assess their disc stability. In Section 4, we revise and discuss the results obtained throughout this paper, including subsections describing the evolution of the models (Sections 4.1 and 4.2) and their numerical consistency (Section 4.3). Finally, in Section 5, we give conclusive notes about the results and implications of this work.

2. Methods

We describe the methods and techniques to achieve the simulations presented in this work. We follow the usual methodology, i.e., we propose an array of models that comply with our purposes, and then the models are evolved and analysed to obtain a qualitative (or quantitative) relation between the numerical parameters. From here on, the softening values are given in units of kpc, unless indicated otherwise.

2.1. Choice of Parameters

There are several studies that attempt to relate the numerical properties of N -body simulations to their physical behaviour [58,61]. In this work, we seek to constrain the range of two parameters: the disc mass fraction, m_d and the softening length, ϵ . Now, for this

study to be relevant in regards to reproducing actual bar phenomena, we concentrate on those sets of values that have the most influence on disc dynamics [77]. It has also been shown that bar dynamics in an isolated galaxy is dominated by the disc and DM halo components [77], and although structures like a classical bulge or a thick external gas disc indeed affect the secular evolution of bars, their role is secondary in comparison to both disc and halo.

We break down how to select the parametrical window for both m_d and ϵ down below. It is important to mention that our intent is not to find an “optimal” range of values for ϵ contrary to, for example, [52]. Our main objective is to assess the influence of both m_d and ϵ on bar formation and evolution and whether these parameters interact to benefit or hinder its development; see [78].

The process to choose a proper array of values for ϵ is, in some regards, arbitrary. Softening is a particularly difficult parameter to constrain, especially for simulations of galactic scale, because there is no direct analogue with observational parameters. Despite numerous attempts to find an appropriate softening value, e.g., [54,60,79], there is still no way to determine an ideal numerical value for ϵ . The fact that N -body simulations are subjected to a large number of free parameters is to blame. Numerical parameters such as ϵ appear to be unrelated to other physical parameters, despite having a tremendous impact on the dynamical properties of N -body galaxies. However, parameters like m_d and ϵ are clearly related to each other through the equations of motion, which in turn are involved in force calculation. This is because the force for each particle within the density profile is proportional to the mass of the particle, m , and inversely proportional to ϵ^2 , and such mass changes due to m_d as $m = m_d M_{200}/N_d$, for the disc, and $m = M_{200}(1 - m_d)/N_h$, for the halo. This means that constraining ϵ should not be reduced to studying its relation to the particle resolution but also include its interaction with other relevant quantities.

Another important factor is the softening kernel introduced in the modification of the Newtonian gravity. GADGET-2 uses a cubic spline kernel [80], which is an improvement from the commonly used Plummer softening kernel because the spline kernel is exactly Newtonian at several softening lengths. For historical reasons, the spline softening length h is related to the Plummer softening length as $h = 2.8\epsilon$.

So, how to pick the right softening length for our models if there is no “correct” choice yet? The simplest approach is to choose values from some of the studies already conducted by other authors, e.g., [58,61,81] that bring the best results, and then expand our study to limits where calculated forces could become artificial. For example, [58] used $\epsilon = 0.006$ as lower limit in their adaptative softening code. They obtain these values by finding the number of neighbours in the density field for a given particle and mapping the result to the softening kernel (in this case, the cubic spline); the more distance there is between neighbours, the larger the softening length is. Something that stands out from these simulations is that the softening length distribution does not appear to change throughout their simulations, despite them clearly displaying a bar at the end (~ 10 Gyrs). They also demonstrate that changes in the fixed softening value do not affect the behaviour of the simulations³. Large softening values ($\epsilon > 0.1$) normally bring spurious outcomes [56,60,82], but there is no agreement to what is the upper limit for the appearance of serious discretisation effects in force calculation. We propose to use an upper limit of $\epsilon = 0.1$. We select $\epsilon = 0.001$ as our lower limit. We justify the latter selection given that several numerical experiments already use gravitational softening lengths close to 0.001, e.g., [39,50]. Another reason is that we are interested in studying the behaviour of N -body discs when particles come close to being collisional. We use the same fixed softening length for both the disc and DM halo. This is because our initial conditions use the same softening value to perform force calculation (see Section 2.2). It is worth noting that one may use different softening lengths for multicom-

ponent systems, e.g., ref. [83], choosing their values so that the force between all particles is the same regardless of mass. We consider such a setting in Appendix A by comparing the evolution of three models: a low-resolution model ($N \sim 10^6$), a high-resolution model ($N \sim 3 \times 10^6$) and a low-resolution model where the halo softening length, ϵ_h , is given by

$$\epsilon_h = \frac{m_h}{m_d} \epsilon_d, \quad (1)$$

where m_d is the mass of a disc particle, m_h is the mass of a halo particle and ϵ_d is the disc softening.

The amount of disc mass assigned to N -body simulations is roughly constrained by cosmological numerical experiments between $0.02M$ and $0.2M$ [84], where M is the total mass of the galaxy. For example, ref. [85] studied over two thousand disc galaxies of the Illustris-1 [86] and TNG100 [87] sets of cosmological simulations. They find that stellar discs have bars in $\sim 30\%$ of them when stellar masses are $M_* \gtrsim 10^{11.25} M_\odot$ on the Illustris-1 catalogue and that, for the TNG100 catalogue, there is a fraction of $\sim 50\%$ of barred discs when stellar masses are $M_* = 10^{10.66-11.25} M_\odot$. In both catalogues, the barred disc fraction decreases when the stellar mass decreases. Galaxies with lower stellar masses ($3.3 \times 10^{10} M_\odot < M_* < 8.3 \times 10^{10} M_\odot$) rarely have bars; such results are redshift-dependent. This range can also be found via observational constraints. Using the Tully–Fisher relation [88] for a sample of 81 disc galaxies and taking into account adiabatic contraction, ref. [89] find the average disc mass fraction, $\tilde{m}_d$, that best fits each object. All the fitted galaxies fall within the range $0.02 \leq \tilde{m}_d \leq 0.1$, which is in agreement with other studies of the same nature, e.g., [90,91]. Therefore, assuming a lower limit of $3.3 \times 10^{10} M_\odot$ and an upper limit of $10^{11.25} M_\odot$ for the stellar mass of galaxies, and using a virial velocity for the DM halo⁴ of $v_{200} = 160 \text{ km s}^{-1}$, it is possible to use $0.033 \leq m_d \leq 0.17$ as a plausible range for the disc mass fraction, which is not far from the values estimated by theoretical considerations.

Taking the above into account, we now impose a range of values for m_d . Since we only have a disc and a DM halo, the total mass of our systems is given by $M = M_d + M_h$. Disc and DM halo masses are $M_d = m_d M$ and $M_h = (1 - m_d)M$, respectively. As mentioned before, realistic discs form when the disc mass fraction falls within the range $0.02 \leq m_d \leq 0.2$; this range is in agreement with the ranges given by cosmological simulations and observations. However, not all discs in this range are susceptible to bar formations. Along with the models presented in this work, we also performed simulations with a wider range of values for the disc mass fraction ($0.02 \leq m_d \leq 0.2$)⁵. These experiments indicate that bars form roughly within the range $0.02 \leq m_d \leq 0.08$, and most of the bars with m_d close to 0.08 are considerably weak. So, with this information in mind, we chose $0.035 \leq m_d \leq 0.05$, an interval that allows stellar discs to be stable in a global sense but at the same time still susceptible to local instabilities, like bars.

The configuration of the galaxy models is completed by specifying the *spin* parameter λ , and the fraction of angular momentum of the disc with respect to the halo’s angular momentum, j_d . In order to avoid any unphysical values for scale length and velocity structure on the disc particles, we adopt the fiducial convention of $m_d = \lambda = j_d$ [81,84,92]. In this way, we ensure that disc and halo are embedded properly, at least according to the standard galaxy formation theory. This configuration also means that $m_d/\lambda = 1$, which does not necessarily benefit the development of bars in our discs, but leaves the door open for other parameters to disrupt their stability [49].

2.2. Initial Conditions

To generate proper initial conditions (ICs), we use GALIC [93], a code created with the specific purpose of generating disc galaxies that, ideally, are in perfect equilibrium.

GALIC applies Rodionov's method to find iterations for Poisson's equation that comply with conditions suited to a certain equilibrium state using the so-called *merit function* [94]. GALIC defines two merit functions, one to map the desired density distribution and another for the velocity structure of said distribution. The latter is needed because it is not enough to use one merit function to approximate the density distribution of a system. In turn, one must also minimise the velocity structure so that it resembles the correct conditions set by the specified profile. This method allows GALIC to find precise values for the distribution function of well-known density profiles (see below). The iterative method in GALIC has the disadvantage of being time- and resource-consuming in the long run, so we limit the number of iterations to 10, which is the same number of iterations recommended by those authors.

We must also mention that GALIC calculates the forces for each component in the ICs using the same ϵ in the multipole expansion, so we must apply the same softening for all particles in our models, regardless of the species. We use the same fixed softening length in both ICs and simulations.

2.2.1. Dark Matter Halo

The halo distribution is the same as in [92] and is equivalent to the profile proposed by [95]. The particle distribution inside a radius r is defined by

$$\rho_h(r) = \frac{M_h}{2\pi} \frac{a_h}{r(r + a_h)^3}, \quad (2)$$

where a_h and M_h are the scale factor and mass of the DM halo, respectively. This distribution can be associated with a NFW halo [96] through the concentration parameter, c , which is given by

$$a_h = \frac{r_{200}}{c} \sqrt{2[\ln(1+c) - c/(1+c)]}, \quad (3)$$

where r_{200} is the virial radius of the NFW halo and is related to its virial mass, M_{200} . It can be assumed that $M_{200} = M_h$ [84]. In this case, the tangential velocity is essentially one of a *cuspy* DM halo and is depicted in Figure 2.

2.2.2. Stellar Disc

The disc distribution is an exponential radial profile with a $\text{sech}^2(z)$ vertical profile, which is given by

$$\rho_d(R, z) = \frac{M_d}{4\pi z_0 R_d^2} e^{-R/R_d} \text{sech}^2(z/z_0). \quad (4)$$

Here, the disc scale length, R_d , is calculated from the prescription given in [84], which contains a direct dependency on the disc mass fraction. The scale height z_0 is assumed constant and in all our models, $z_0 = 0.2R_d$.

The model shown in Figure 2 resembles model D1 in [93], which was tested by the authors with $N_h = 10^6$ and $N_d = 10^6$, $\epsilon = 0.05$ and $m_d = 0.035$. This model has a stability threshold of $\min(Q_{t=0}) \equiv Q_{\min} \approx 1$ and the authors report that it should be susceptible to experience axisymmetric perturbations⁶.

The halo profile is symmetric and isotropic, with $\sigma_r/\sigma_z = 1$. The disc profile has a distribution function defined by $f(E, L_z)$ and its velocity structure is that of an isotropic rotator, i.e.,

$$\sigma_r = \sigma_z = \sigma_\phi = \langle v_\phi^2 \rangle - \langle v_\phi \rangle^2, \quad (5)$$

where $\langle v_\phi^2 \rangle$ is the second moment of the azimuthal motion and $\langle v_\phi \rangle$ is the streaming in the azimuthal direction.

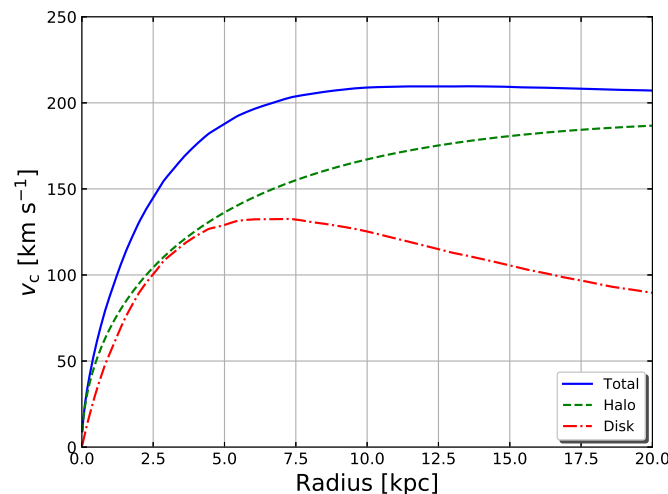


Figure 2. Circular velocities for the DM halo (green dashed line), stellar disc (red dashed-dotted line) and the contribution of both (blue solid line), given by Equations (2) and (4). These curves correspond to model SGS2 in Table 1.

In addition to the very nature of the collisionless systems we analyse, the models studied here, if compared to real galaxies, should resemble early-type systems due to their lack of a gas component.

2.3. Simulations

All of our simulations were run with the N -body/SPH code GADGET-2 [57,81]. Other relevant parameters such as the number of particles were assigned values that are commonly used in similar works, $N_d = 240,000$ and $N_h = 800,000$, which are the number of particles for disc and halo, respectively. Previous studies indicate that in order to avoid statistical errors in force calculation, the number of particles should exceed $N \approx 10^6$ [39,97]. For generality, we set $c = 10$ and $v_{200} = 160 \text{ km s}^{-1}$ [92,98] which gives a mass of a Milky Way-sized galaxy of $M \approx 10^{12} M_\odot$, with $M_d \approx 3.5 \times 10^{10} M_\odot$, $M_h \approx 96.5 \times 10^{10} M_\odot$ for $m_d = 0.035$, and $M_d \approx 5 \times 10^{10} M_\odot$, $M_h \approx 95 \times 10^{10} M_\odot$ for $m_d = 0.05$. We also performed additional simulations, tripling N_d and N_h of the original set to verify the results obtained with the low-resolution models (see Section 4.3).

Table 1 summarises all the simulations conducted here, including the model’s parameters and some of the measured quantities. Figure 3 shows several snapshots of models that display clear bar formation and Figure 4 shows models where bar formation is either transient or non-existent (see Section 4 for details). We discuss their properties later on.

3. Analysis Framework

To assess and quantify the models, we carry out kinematic and stability analyses on the stellar discs. We also follow the formation and evolution of bars via the so-called *distortion parameter*, η [99], which can be used as a measure of any non-axisymmetric perturbations, and the magnitude of Fourier’s second harmonic, A_2 , of the disc surface density specifically captures bar strength.

Table 1. Simulations with different values of m_d and ϵ . Column 1: name of model. Column 2: disc mass fraction with respect to halo mass parameter. Column 3: softening length parameter in kpc. Column 4: minimum value of Toomre’s parameter for the IC ($t = 0$). Column 5: average deviation at $t = 1.8$ Gyrs of the simulation from ICs radial dispersion. Column 6: average deviation at $t = 1.8$ Gyrs of the simulation from ICs tangential dispersion. Column 7: average deviation at $t = 1.8$ Gyrs of the simulation from ICs vertical dispersion. Column 8: distortion parameter once the simulation has finished (last snapshot). Column 9: amount of steps necessary to complete the simulation. Column 10: timestep between each step in Gyrs. Column 11: radial scale length of the stellar disc’s IC in kpc. Column 12: percentage of angular momentum lost by the disc. Every simulation lasts approximately 12 Gyrs.

Model (1)	m_d (2)	ϵ (3)	$Q_{\min}$ (4)	$\bar{\sigma}_r$ (5)	$\bar{\sigma}_t$ (6)	$\bar{\sigma}_z$ (7)	η_e (8)	Steps (9)	Δt (10)	R_d (11)	% $\Delta L_{d,f}$ (12)
SGS1	0.035	0.1	1.13	0.386	0.086	0.088	0.248	32,768	3.66×10^{-4}	2.87	27.63
SGS2	-	0.05	1.13	0.383	0.089	0.100	0.229	56,518	1.83×10^{-4}	-	24.29
SGS3	-	0.025	1.11	0.389	0.092	0.098	0.217	65,536	1.83×10^{-4}	-	23.93
SGS4	-	0.010	1.07	0.421	0.117	0.112	0.217	131,115	4.58×10^{-5}	-	22.39
SGS5	-	0.005	1.07	0.434	0.113	0.108	0.126	375,840	2.29×10^{-5}	-	23.22
SGS6	-	0.001	1.08	0.489	0.177	0.165	0.008	3,433,757	1.43×10^{-6}	-	6.23
SGS7	0.038	0.1	1.14	0.335	0.054	0.068	0.234	32,768	3.66×10^{-4}	3.08	23.26
SGS8	-	0.05	1.14	0.358	0.066	0.065	0.213	53,683	1.83×10^{-4}	-	23.03
SGS9	-	0.025	1.13	0.373	0.078	0.076	0.211	65,536	1.83×10^{-4}	-	22.23
SGS10	-	0.010	1.13	0.372	0.089	0.101	0.221	131,109	4.58×10^{-5}	-	22.96
SGS11	-	0.005	1.12	0.396	0.082	0.083	0.036	348,741	2.29×10^{-5}	-	10.51
SGS12	-	0.001	1.05	0.433	0.128	0.110	0.015	2,748,011	1.43×10^{-6}	-	6.10
SGS13	0.041	0.1	1.14	0.316	0.044	0.068	0.179	32,768	3.66×10^{-4}	3.29	17.73
SGS14	-	0.05	1.15	0.336	0.074	0.089	0.248	53,687	1.83×10^{-4}	-	24.68
SGS15	-	0.025	1.14	0.342	0.066	0.082	0.185	65,536	1.83×10^{-4}	-	21.00
SGS16	-	0.010	1.14	0.347	0.070	0.074	0.159	131,093	4.58×10^{-5}	-	19.05
SGS17	-	0.005	1.11	0.349	0.065	0.089	0.006	345,694	2.29×10^{-5}	-	5.98
SGS18	-	0.001	1.11	0.395	0.103	0.099	0.010	2,752,160	1.43×10^{-6}	-	4.87
SGS19	0.044	0.1	1.16	0.297	0.040	0.041	0.187	32,768	3.66×10^{-4}	3.50	17.17
SGS20	-	0.05	1.15	0.300	0.039	0.056	0.102	34,016	1.83×10^{-4}	-	12.13
SGS21	-	0.025	1.15	0.331	0.042	0.051	0.166	65,536	1.83×10^{-4}	-	16.58
SGS22	-	0.010	1.14	0.335	0.060	0.078	0.152	131,093	4.58×10^{-5}	-	16.16
SGS23	-	0.005	1.14	0.319	0.069	0.064	0.017	322,834	2.29×10^{-5}	-	8.16
SGS24	-	0.001	1.14	0.363	0.081	0.086	0.006	2,489,617	1.43×10^{-6}	-	5.14
SGS25	0.047	0.1	1.17	0.280	0.026	0.045	0.184	32,768	3.66×10^{-4}	3.71	19.98
SGS26	-	0.05	1.17	0.291	0.037	0.065	0.211	44,219	1.83×10^{-4}	-	21.34
SGS27	-	0.025	1.16	0.300	0.044	0.043	0.127	65,536	1.83×10^{-4}	-	13.39
SGS28	-	0.010	1.13	0.297	0.048	0.064	0.116	131,093	4.58×10^{-5}	-	13.38
SGS29	-	0.005	1.10	0.298	0.041	0.055	0.005	325,608	2.29×10^{-5}	-	4.48
SGS30	-	0.001	1.10	0.334	0.067	0.069	0.014	2,845,525	1.43×10^{-6}	-	4.49
SGS31	0.05	0.1	1.18	0.268	0.022	0.031	0.112	32,768	3.66×10^{-4}	3.92	12.11
SGS32	-	0.05	1.18	0.275	0.031	0.050	0.144	32,949	1.83×10^{-4}	-	14.67
SGS33	-	0.025	1.18	0.289	0.033	0.049	0.164	65,536	1.83×10^{-4}	-	17.65
SGS34	-	0.010	1.15	0.298	0.040	0.044	0.129	131,078	4.58×10^{-5}	-	16.70
SGS35	-	0.005	1.14	0.287	0.041	0.044	0.029	315,762	2.29×10^{-5}	-	9.19
SGS36	-	0.001	1.14	0.312	0.053	0.061	0.005	2,481,852	1.43×10^{-6}	-	3.57

3.1. Kinematics

Our kinematic study includes the analysis of the rotational velocity curves and their time evolution for all of our models (see Figure 5). Velocities in an annulus, k , of the disc are simply the average tangential velocities of all the particles (n_k) belonging to that annulus,

$$\bar{v}_t = \frac{1}{n_k} \sum_{i=1}^{n_k} v_t^i, \quad (6)$$

where v_t^i is the tangential velocity of particle i .

Other quantities, such as the velocity dispersions, are used to measure disc heating. Their behaviour gives us clues about the velocity structure once the simulation goes further in time. Figure 6 shows the velocity dispersions for some of the models at different times. Dispersions are trivially calculated from Equation (6), i.e.,

$$\sigma_i^2 = \overline{(v_i^j - \bar{v}_i)^2} \quad \text{with} \quad i = r, t, z, \quad (7)$$

where v_i^j is the velocity of component i for particle j . σ_r represents the radial dispersion, σ_t is the tangential dispersion and σ_z is the vertical dispersion. Every plot in Figures 5 and 6 is measured along the disc's plane, both for simplicity and to correctly assess Toomre's parameter, given that the original conception of this criterion was first applied to infinitely thin discs.

Additionally, in order to quantify the heating, we measure the deviations of the velocity dispersions with respect to the initial average values in the radial, tangential and vertical directions:

$$\bar{\sigma}_\alpha^2 = \frac{1}{N_a} \sum_{i=0}^{N_a} \left(\frac{\sigma_{\alpha,i}^t - \sigma_{\alpha,i}^{\text{ini}}}{\sigma_{\alpha,i}^{\text{ini}}} \right)^2 \quad \text{with} \quad \alpha = (r, t, z), \quad (8)$$

where, for example, $\bar{\sigma}_r$ is the deviation from radial dispersion with respect to the IC, and $\sigma_{r,i}^{\text{ini}}$ and $\sigma_{r,i}^t$ are the dispersion for every ring in the initial disc and the dispersion for every ring at time t of the simulation, respectively. N_a is the number of rings from a particular snapshot.

3.2. Stability of Galactic Discs

The stability of a collisionless disc against non-axisymmetric perturbations is determined by the Toomre's stability criterion, Q [3], defined by

$$Q = \frac{\sigma_r \kappa}{3.36 G \Sigma(r)}, \quad (9)$$

where σ_r is the radial velocity dispersion, $\Sigma(r)$ is the surface density and κ is the epicyclic frequency

$$\kappa = \frac{\partial \Omega^2}{\partial r} + 4\Omega, \quad (10)$$

with Ω being the angular velocity in the disc. Calculating σ_r and κ is relatively easy using Equations (7) and (10). Note that this criterion was originally derived for razor-thin discs. Because $z_0 \ll R_d$, we can restrict ourselves to compute Q in the disc's plane [100]. Also, $z_0 \geq 10\epsilon$ for all of our models, which means that we are able to correctly resolve the vertical structure of the discs. We show the Q profiles and their time evolution for four different types of perturbed discs in Figure 7.

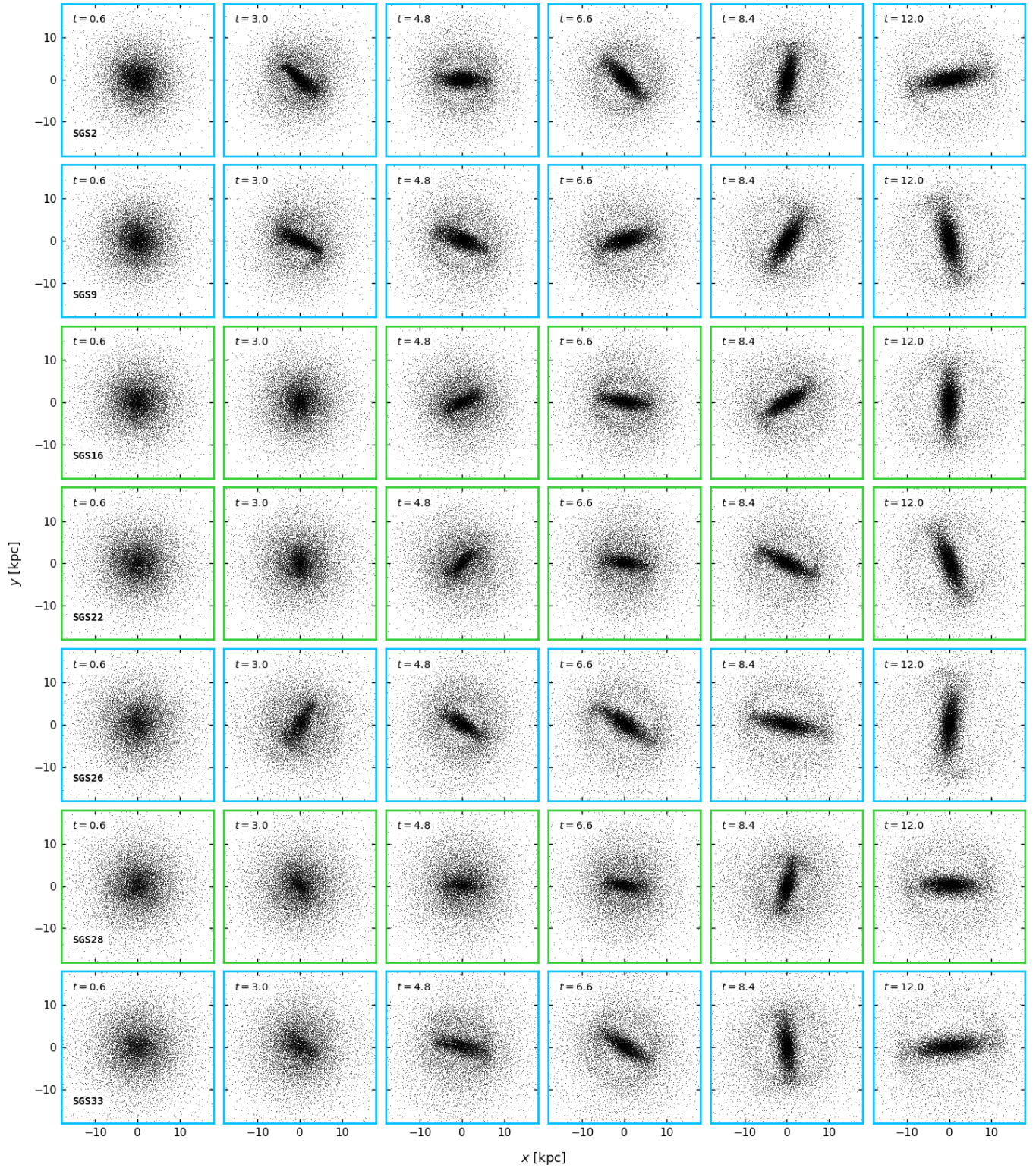


Figure 3. Disc face-on particle projection for some of simulations from the Table 1 that produce strong bars through time. Model name is indicated on the left bottom corner of the first column. Timestamps shown at the top of every snapshot are in units of Gyrs. The frame colour of each row indicates common bar formation **blue** and delayed bar formation **green**. See text for details.

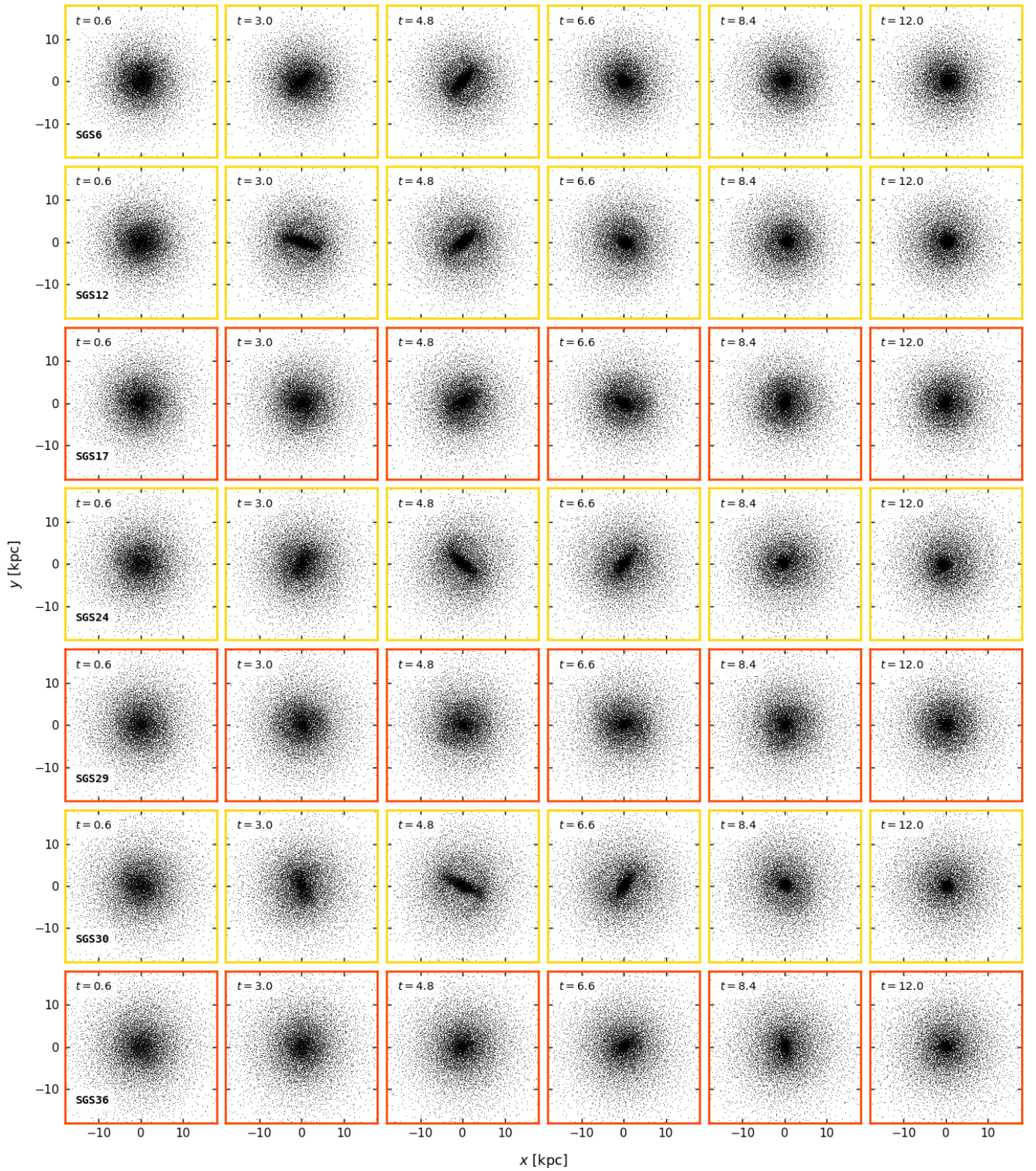


Figure 4. Same as in Figure 3 but this time the simulations show bar stable discs or, in some cases, bar destruction (e.g., SGS12 or SGS30). The frame colours of each row are **yellow**, transient bar formation and **red**, no bar formation. See text for details.

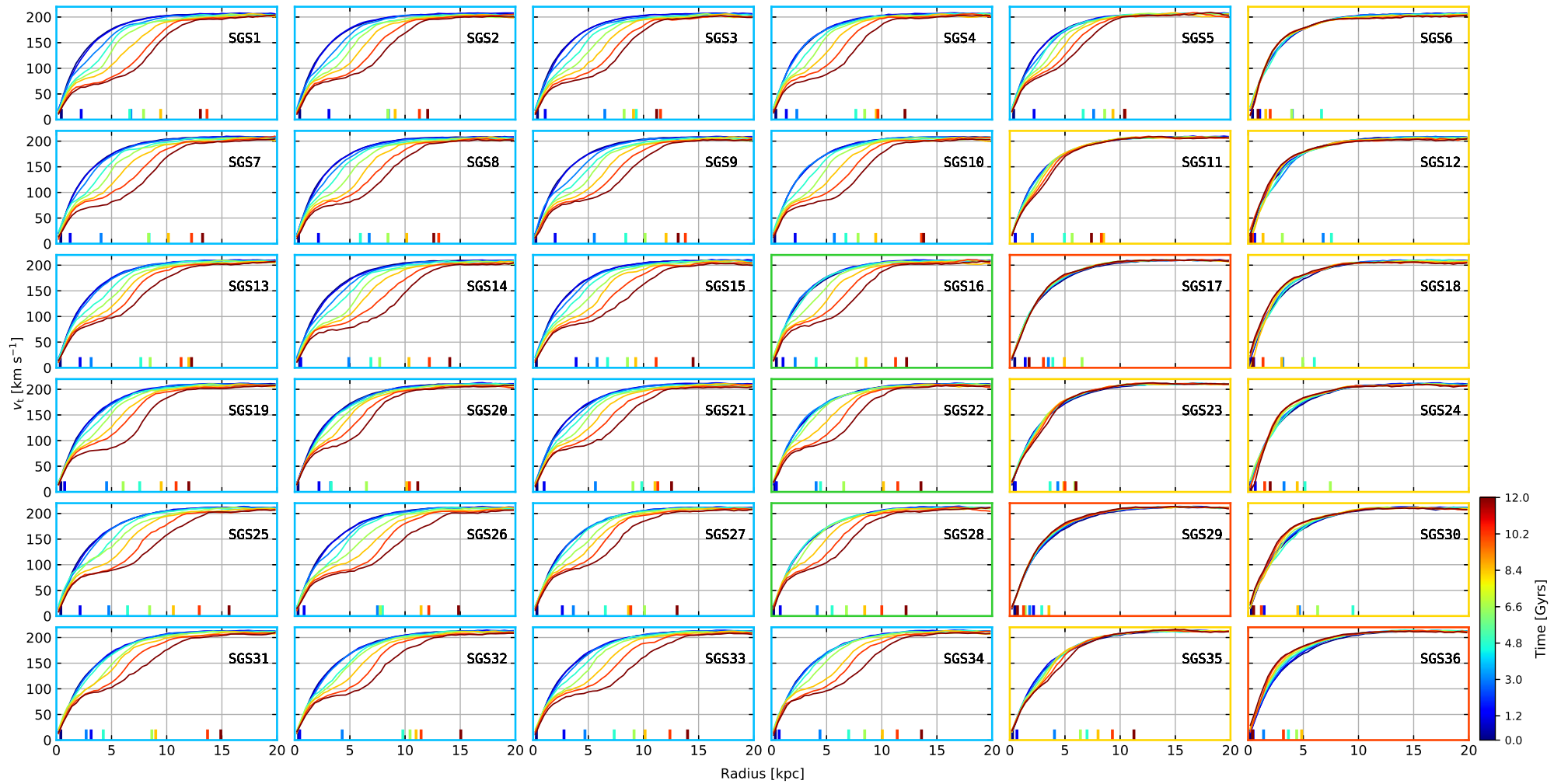


Figure 5. Tangential velocities for all models in Table 1. Different colours show the temporal evolution of the tangential velocities indicated by colourbar. The apparent length of the bar is indicated by the coloured ticks over the abscissa and the colour of each tick matches the colour of the corresponding tangential velocity. The colour code for each frame represents the same as in Figures 3 and 4. See text for details.

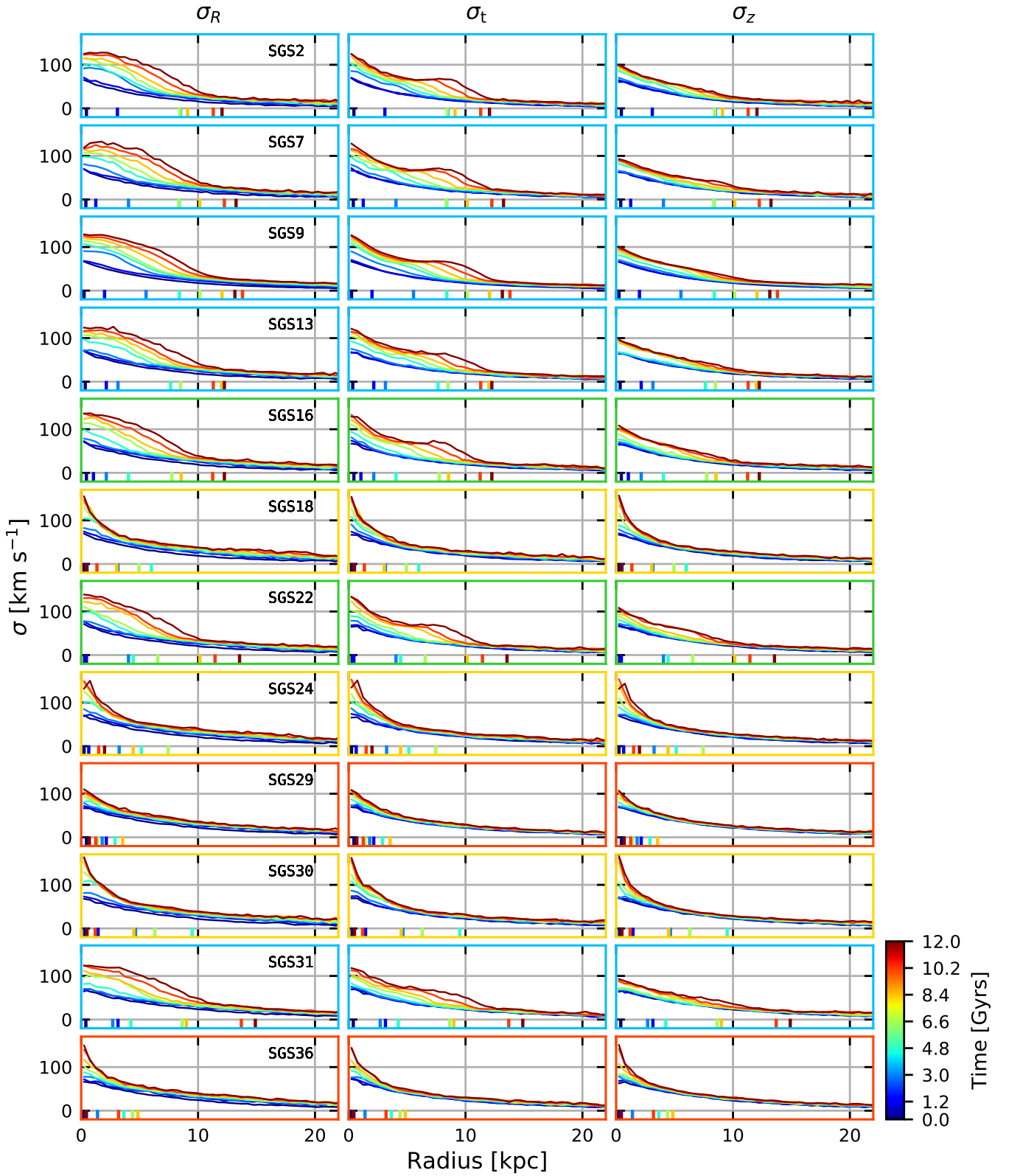


Figure 6. Velocity dispersion curves for some models from the Table 1. Columns represent, from left to right, the radial, tangential and vertical dispersions of the models in each row. Different colours show the temporal evolution of each simulation according to colourbar. The length of the bar is indicated by the coloured ticks over the abscissa and the colour of each tick matches the colour of the corresponding tangential velocity. The colour code for the frames represents the same as in Figures 3 and 4. See text for details.

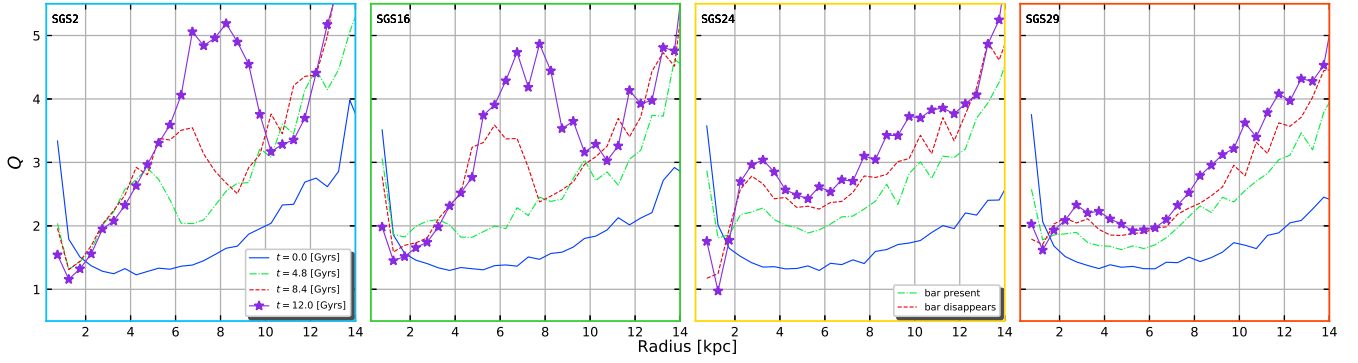


Figure 7. Evolution of Toomre’s parameter for four different models, each with different degrees of disc distortion, η_e . The models with more mass and less softening (SGS24 and SGS29) show the least perturbed discs as can be noted by absence of humps at $R = 6 - 8$ kpc. In the third panel from left to right, we indicate with a legend on the lower right, which line corresponds to a bar being present in SGS24 (green line) and which one corresponds a time where there is no longer a bar in it (red line). The colour code for the frame on each plot represents the same as in Figures 3 and 4.

3.3. Distortion Parameter

Another way to assess the stability of pure stellar discs is by computing the distortion parameter, η , which was originally used to measure high density concentrations of matter in differentially rotating stars [99]. This concept can be applied in a similar manner to a bulk of particles. Obtaining the moments of inertia for the disc, we can estimate subtle changes in its density, thus allowing us to know if the distribution exhibits any non-axisymmetric structure. [61] already applied this method to simulations of isolated galaxy models. They determined that there is a threshold in η when simulated discs tend to display signs of bar-like perturbations. In the present work, we are not only aiming to replicate and confirm the results of [61], but also to expand the usage of η to describe properties kindled by the bar, if any.

The distortion parameter η is defined as

$$\eta = \sqrt{\eta_+^2 + \eta_\times^2}, \quad (11)$$

where

$$\eta_+ = \frac{I_{xx} - I_{yy}}{I_{xx} + I_{yy}} \quad \text{and} \quad \eta_\times = \frac{2I_{xy}}{I_{xx} + I_{yy}}, \quad (12)$$

and the moments of inertia are

$$I_{ij} = \sum_{k=1}^{N_d} m_k x_k^i x_k^j \quad \text{with} \quad i, j = (x, y). \quad (13)$$

Here, m_k is the mass of each particle in GADGET units and x_k are the positions of every particle in the disc. An important property of η is that it allows one to track the average movement of every particle. So, in regions where particle density loses symmetry and starts to concentrate in specific parts of the disc, the parameter η should increase, thus reflecting the overall motion of the bar component. In Section 4, we demonstrate the different applications of η . The final value of η (called η_e) for all simulations is shown in Table 1. The threshold η_e only shows us whether or not the model holds a bar at the end of the run. Time evolution of η will give a full picture of the bar’s evolution, as is shown in Figure 8. In Section 4, we delve into the radial and secular evolution of η .

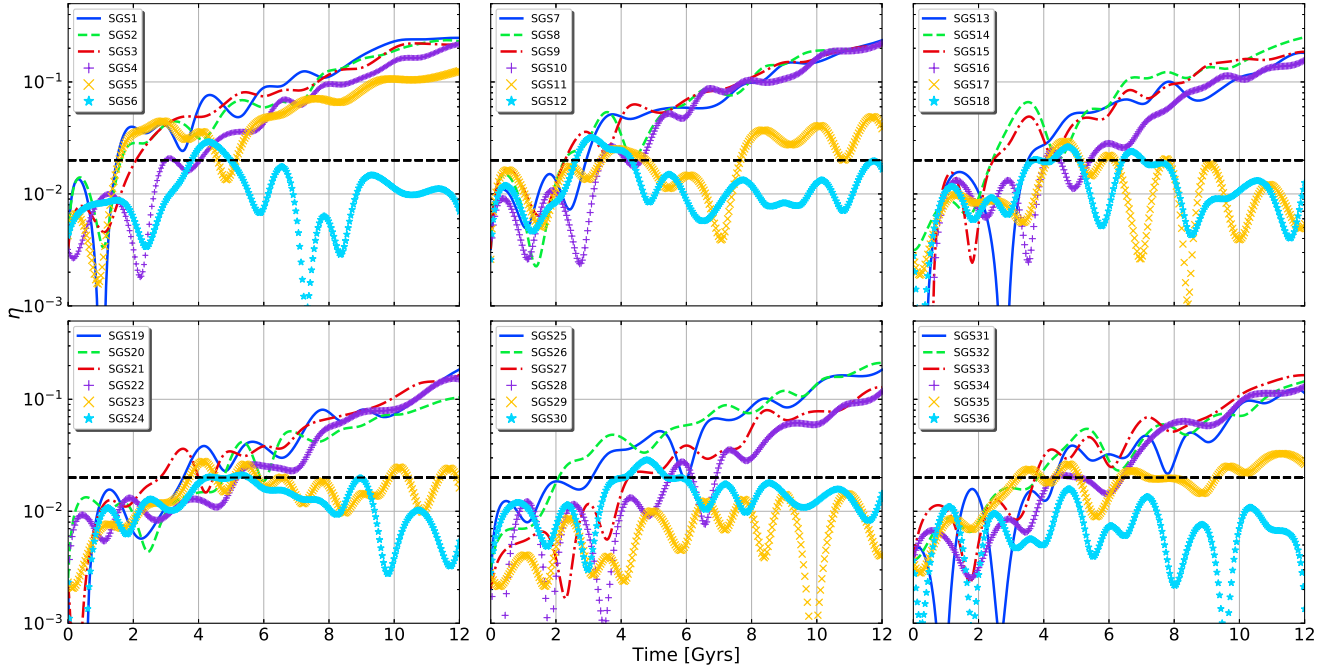


Figure 8. Time evolution of η . The dashed black line represents $\eta_{\text{th}} = 0.02$, which is the threshold for definite bar formation.

3.4. Fourier Magnitude

The Fourier magnitude or *strength* is another relevant quantity that describes different stages of bar evolution; see [16]. The magnitudes of different modes for the disc surface density allows one to observe how the perturbation grows, whether this growth is rapid or slow, and if other instabilities arise that cohabit with the bar. We measure the strength of the bar as the second harmonic of the Fourier spectra of the disc's particle distribution, that is,

$$A_2 = \frac{\sqrt{a_2^2 + b_2^2}}{N_d}. \quad (14)$$

The Fourier components for the second mode are defined as

$$a_2 = \sum_{i=1}^{N_d} \cos(2\theta_i) \quad b_2 = \sum_{i=1}^{N_d} \sin(2\theta_i), \quad (15)$$

where θ_i is the polar angle of particle i . In order to obtain representative values of the strength, we take the maximum of A_2 at each time, which is a valid description of bar amplitude [30].

4. Results and Discussion

We now describe and analyse the results obtained throughout this work. Table 1 summarises some of the quantities extracted from all the simulations.

From these data, we notice that $Q_{\min}$ (column 4) is generally higher as m_d increases. This is true when comparing models with the same ϵ (e.g., SGS1 and SGS31). We also note that it is unclear how disc stability depends on ϵ . When $\epsilon \leq 0.005$ and m_d remain the same, $Q_{\min}$ reaches its lowest values. However, for the rest of the models (which have $\epsilon \geq 0.01$) $Q_{\min}$ is practically the same.

Now, $Q_{\min}$ tells us that, generally, simulations with greater disc mass tend to be more stable, which is not a surprising result [8,41], given that disc dominated models are less prone to manifesting structures such as bars. In our models, the increase in $Q_{\min}$ is

mostly due to an increase in the radial dispersion for higher values of m_d . The models also increase in R_d , but this quantity does not affect the behaviour of $Q_{\min}$ because both surface density and epicyclic frequency, and their effects, more or less cancel out when R_d increases. Such an increase in the radial dispersions may be explained by enhanced kinematic pressure in the disc due to the fact that increasing m_d actually reduces the central surface density of discs, which lowers the value of $Q_{\min}$. The latter requires a thorough analysis that is out of the scope of this work. The dependence on ϵ is much more noticeable: for smaller values of ϵ , $Q_{\min}$ decreases, making the disc theoretically more responsive to local instabilities. However, we must note that the dependency between m_d , ϵ and $Q_{\min}$ is small since basically all of the models have $Q_{\min} \approx 1.1$. Thus, we may regard the models as being marginally stable against bar formation.

The four models in Figure 7, SGS2, SGS16, SGS24 and SGS29 have different masses and softenings. It is evident that SGS2 and SGS16 are very similar, structurally speaking and, as shown in Figures 3 and 5, they effectively exhibit a bar. Model SGS24 seems to have a bump at $t = 4.8$ Gyrs in Q and, if we look carefully in Figure 4, it could host a bar in an early stage around the same time, but is nowhere to be found in later snapshots (see also model SGS30 in Figure 4). Model SGS29 exhibits a similar bump at $t = 8.4$ Gyrs, but there is no bar in either its particle snapshots or velocity curves. This bump in SGS29 is caused by the gathering of particles at its centre, something that also occurs in models that have transient bars, like SGS24. We offer an explanation for this phenomenon in Section 4.4.

Columns 5, 6 and 7 in Table 1 are the deviations from average dispersion velocities, $\bar{\sigma}_r$, $\bar{\sigma}_t$ and $\bar{\sigma}_z$ in the radial, tangential and vertical directions, respectively. The deviations are evaluated at $t = 1.8$ Gyrs, the time where the first bar just started to form⁷ (model SGS5). The values given in the table are interpreted as follows. If, for instance, $\bar{\sigma}_\alpha = 1$, it means that, for coordinate α , the average dispersion velocity at the end of the simulation is 2 times higher compared to that of the IC. Conversely, if $\bar{\sigma}_\alpha = 0$, the average dispersion has remained the same throughout the simulation. In this manner, we can measure the disc heating due to selected attributes rather than caused by the bar instability.

Column 8 is the distortion parameter η_e and captures the disc symmetry at the end of the simulation and columns 9 and 10 give the average timestep and the total number of steps per simulation. The tendency for the timestep to decrease when ϵ decreases is expected due to the presence of particles with higher accelerations.

Figures 3 and 4 exemplify disc evolution for models that exhibit barred discs and transient/non-barred discs, respectively. Only 10% of disc particles are shown in order to correctly distinguish the bars. We identify, from both Figures 3 and 4, and the distortion parameter in Figure 8, four scenarios of bar formation and colour-coded plot frames according to the following: (i) **blue** : common bar formation, where the bar is clearly formed and there is no comparative delay in its growth; (ii) **green**: delayed bar formation, at least in comparison to the common scenario; (iii) **yellow**: transient bar formation, where at some point the bar forms but then it is destroyed; (iv) **red**: no bar formation, for cases where no clear distortion was detected at any time. Models in Figure 3 display strong persistent bars, where it can be noted that the bar triggering varies in time. On the contrary, models in Figure 4 show either stable discs or weak bars and their destruction. Visual comparison of bar growth may already give some clues on the influence of the model parameters. For example, model SGS26 produces the bar more rapidly than model SGS28 and, given that both models have the same properties except ϵ , we might conclude that the amount of softening could have affected the bar growth rate. Models with the lowest ϵ show little signs of perturbation, at least visually. The cases of SGS24 and SGS30 especially stand out because there is a barred structure forming at $t \geq 3.0$ Gyrs, but it then disappears

at $t \geq 7.2$ Gyrs. These two models only differ in disc mass by $\sim 7\%$. We further confirm these results through the measurement of distortion parameter in Section 4.2.

Snapshots of models SGS30 and, in a fainter fashion, SGS24, in Figure 4 show traces of bar creation and destruction (or perhaps suppression) in a period of about 3 Gyrs. The self-consistent vanishment of bars in these models is not due to matter concentration or the accumulation of eccentric orbits, e.g., [45,101,102] because our models do not have gas to gather and the dispersion curves do not reveal any strange behaviour, such as extreme vertical heating. It is in fact possible that we are dealing with a numerical artifact. In such cases, we could use statistical tools designed to unmask spurious effects, e.g., [53,61]. The most useful is perhaps the estimation of errors in force calculation [54]. Unfortunately, the complex nature of disc-halo dynamics imposes serious challenges for them to be applied appropriately. This is because quantities such as the Average Square Error (ASE) and Integrated Square Error (ISE) are commonly calculated for density distributions that are spherically symmetric [52,53], which is not the case for our simulations since the density of our DM halos has been affected by the presence of the bar. Nonetheless, we performed several realisations of a model with an isolated DM halo (no disc) to compute ASE in order to determine their optimal softening length. The base model has $N_h = 800,000$, a total mass of $M_{200} \approx 10^{12} M_\odot$, scale length $a = 27.06$ kpc and $c = 10$. We created 24 realisations of this model with a softening length window of $0.001 \leq \epsilon \leq 3$. The forces associated with the halo realisations are calculated using the cubic spline and are then compared to the actual forces produced by the Hernquist profile through the ASE measurement. The value of the resulting errors is normalised with a factor defined by $C = 1/F_{\max}^2$, where $F_{\max}$ is the maximum force exerted by the theoretical halo density distribution [103,104]. We determined an optimal Plummer softening length of $\epsilon_{\text{opt}} \approx 0.3$, which falls outside of our original softening window. However, we are interested in the behaviour of the disc subjected to changes in particle and space resolution, and the disc density is higher than that of the halo. Furthermore, the halo in our models is affected by adiabatic contraction due to its interaction with the disc, so we expect it to have a denser core than in the isolated case. All of this means that the optimal softening for the models should be considerably smaller than the one found for the realisations.

We estimated the relaxation time via two methods described in [105]:

$$T_{\text{relax}} = \frac{N}{8 \ln(R/\epsilon)} T_{\text{cross}}, \quad (16)$$

where R is the whole halo extension and T_{cross} is the crossing time of the system, and through the derived fitting equation that depends on the number of particles

$$\log_{10}(T_{\text{relax}}) = 0.63 + 0.78 \log_{10}(N) \quad (17)$$

Using Equation (16), we find that all of our models have a relaxation time that exceeds $\sim 10^{12}$ years. However, if we calculate the relaxation times using Equation (17) assuming a model with only a halo component (with a $\gamma = 0$ Dehnen profile and $\epsilon = 0.27$ kpc), we find that, for $N = 8 \times 10^5$ particles, $T_{\text{relax}} \sim 10^5$ years; for $N = 2.4 \times 10^6$ particles we also find relaxation times of $\sim 10^5$ years. So, Equation (16) seems to overestimate the relaxation times and Equation (17) seems to underestimate them. We are not entirely sure about the applicability of the fitting equation to a broader family of models (e.g., Hernquist or NFW profiles), but there are several examples of numerical studies of galaxy models that already use softening lengths in the range we are examining, with similar number of particles and mass resolutions [30,39,50], that appear to be physically realistic.

4.1. Bar's Impact on the Tangential Velocity Curves

The model depicted in Figure 2 and described in Section 2.2 has a velocity structure that mirrors a rigid body in its central regions (high dispersions) and a differentially rotating body in its outer regions (low dispersions). After just a few rotation periods, the models in Figure 5 already show effects induced by a bar-like perturbation. The shapes of our barred tangential velocities do resemble some of the observed tangential velocities for barred galaxies. For example, Ref. [106] measured, among other things, the velocity curves of 10 late-type barred galaxies. The tangential velocities of their stronger bars—for instance, NGC 7741 or NGC 7479—are quite similar to what we obtained in our simulations. When a bar appears, the velocity profile in such a region should be similar to a rigid body; as the bar grows, this effect strengthens. In our simulations, the bar seems to occupy almost all of the disc, which is shown by the coloured bar length ticks in Figure 5. The lengths are determined by following the phase of the bar, which can be found using the Fourier magnitudes described in Section 3.4, i.e., $\phi_b = \arctan(b_2/a_2)$. The bar is found in regions where ϕ_b remains constant; the length of the bar is defined as the annulus where ϕ_b deviates $\pm \arcsin(0.3)$ from its constant value (see [8] for a detailed description of this method). The drawback of this method is that it is reliable only for strong bars where their phase can be uniquely determined.

From the tangential velocity curves and velocity dispersions in Figures 5 and 6, respectively, we can notice some characteristic features ignited by a bar-like perturbation. For all the models showing a bar, the tangential velocities exhibit a decrease in velocity for central regions of the disc. This is because bars considerably change the density distribution of discs, in such extent that they are able to produce eccentric orbits (commonly known as the x_2 family orbit) in the centre of the disc, which are the main constituents of bulge components [45,101]. Knee-shaped curves at later stages of evolution indicate the existence of four different regions: a central solid body region, a differentially rotating plateau, an external solid-body region, and an external differentially rotating disc. It is important to note that the shape of the tangential velocities in the central and external regions barely change throughout the simulation.

Comparing the velocity dispersion curves, it is evident that radial and tangential dispersions reflect the same correlation with the bar strength as seen in the tangential velocities, forming a bump in the centre for models with a bar present. At the same time, vertical dispersions—commonly used to measure the degree of the disc heating—show little change when subjected to the bar. For comparison, the models with vanishing or no apparent bar (SGS24, SGS29, SGS30 and SGS36) show an increase only in the centre of the disc. We expect that models with small softening values to experience disc heating (in all directions), since these models tend to become collisional. We see no clear evidence of such an effect, other than in the central region. We may conclude that the main factor responsible for the disc heating in our models is the bar instability.

Looking at Figure 6, we can notice a pattern for σ_z : as ϵ decreases, discs tend to become hotter only in their innermost region (models SGS24, SGS30 and SGS36). SGS24 shows no trace of a bar in the finishing stages of its evolution, but Figures 5, 6 and 8 show, in earlier stages ($t \simeq 4$ Gyrs), signs of bar formation. This means that the perturbation was suppressed spontaneously without the need of an external instigator. There is also a noticeable discrepancy between the dispersion curves of SGS29 and SGS36. Both models never formed a bar; however, SGS36 displays considerable amount of central heating in all velocity components while SGS29 does not; this effect could be related to changes in either m_d or ϵ .

4.2. Evolution of Parameter η

Figure 8 shows η values evolving through time for all of our models, giving a more complete view of disc and bar behaviour. It is important to mention that all of the η curves in Figure 8 were smoothed with a cubic kernel interpolation, simply to reduce the excess of noise in the curve caused by oscillations in the position of the disc's centre of mass.

According to [61], when $\eta \geq \eta_{\text{th}} = 0.02$, the disc is set to form perturbations similar to bars. Our experiments confirm this threshold, even for transient bars. Most of the models do have clear bar presence (compare Figures 3 and 8) and all of these models surpass the prescribed threshold of $\eta_{\text{th}} \approx 0.02$. Moreover, all the models with undisputed bar formation have $\epsilon \geq 0.01$. On the other hand, the effect of the disc mass is indeed small due to the narrow range used for this parameter, although it is still noticeable. When m_d increases, the time the simulation takes to form its bar is slightly longer (compare upper-left and lower-right panes in Figure 8), from ~ 2 to ~ 4 Grys. Once a simulation exceeds the value of η_{th} , the bar recently created keeps on gaining momentum, accumulating more mass (particles), and thus increasing its overall strength. Results for $\epsilon \leq 0.005$ are puzzling. For the lowest values of softening, the bar instability suffers to keep itself alive; in fact, the only model with $\epsilon \leq 0.005$ that maintains its bar growing is SGS5, which also has the smallest possible value of disc mass in our set ($m_d = 0.035$). Models SGS29 and SGS36 are the only ones that never exhibit signs of clear distortion. Model SGS17 surpasses η_{th} at $t \sim 4$ Grys, but there is no clear visual confirmation, so we classified it as an unbarred model. The rest of the models with these softening values (SGS6, SGS11, SGS12, SGS18, SGS23, SGS24, SGS30 and SGS35) show transient or unclear bar formation. For example, SGS24 gathers enough particles to elevate η beyond the threshold of 0.02 in ~ 4 Grys and then, ~ 1 Gyr later, loses such mass and is never capable to recover its (faint) bar. The mechanism that describes such bar suppression ought to be different from the ones proposed by [47,101,107]. An earlier explanation for this effect was first exposed by [6], pointing that ϵ could play a paramount role in the suppression of bar-like instabilities. Later on, [56] concluded that, for simulations of one-component galaxy, i.e., stars only, and extremely low values of softening lengths, the bar instability was considerably suppressed. The degree of suppression depended on how small the softening was. [56] also find that bar suppression correlates to the size of the perturbation, λ ; if $\lambda < \epsilon$, then the disc will hardly react to it.

Figure 8 confirms that the model showing the least perturbed disc also has the most softened potential (SGS36). Unfortunately, this conclusion is not definitive, since models with the lowest ϵ are not always the least perturbed (see η curve in Figure 8 for models SGS29 and SGS30, respectively).

These results may lead to the assumption that if one decides to increase m_d or decrease ϵ independently, the simulated discs would follow a trend: increasing m_d or decreasing ϵ heats up the disc and increases its stability; decreasing m_d or increasing ϵ does the opposite. In this case, distinctive pairs of values for m_d and ϵ seem to resonate with each other, i.e., either strengthen or weaken the bar. Furthermore, models SGS29 and SGS35 both have the same softening length but SGS35 has a bit more disc mass, so we would expect to witness less distortion in SGS35, which is not true (see their respective η curves). Another example is model SGS26. Comparing this model with others of the same softening but less disc mass (e.g., SGS20), we find that SGS26 starts forming its bar earlier. A clear-cut explanation for this non-linear behaviour is tough to find. It makes sense that m_d and ϵ interact with each other: an increase in disc mass also increases the mass of individual particles, which alters force computation. In consequence, we must take into account these changes in the force field in order to choose an appropriate value for ϵ .

4.3. Particle Resolution

An important factor on the reliability of N -body simulations is the particle resolution. We have already established in Section 1 that ϵ and N are intimately related, e.g., [52,54], no matter the scale [108,109]. Improper balance between these parameters may result in undesired outcomes. For a given N , small softening values might cause undesired relaxation (heating), which could damp the generation of local instabilities, either barred or spiral [109]. Conversely, high softening values may introduce large biases that could render simulations unrealistic.

This means that, for the range of softening lengths used here, it may be expected that some of the results presented above show erratic behaviour. Following our previous analysis, we will consider the next relationships between η and the numerical parameters m_d and ϵ : $\eta \propto m_d^{-1}$ and $\eta \propto \epsilon$, and the models that conform to the above are labelled as “well-behaved” or normal. These relations are not followed by all of our models. For instance, models SGS14 and SGS26 have longer and stronger bars than models with larger softening values, e.g., SGS13 and SGS25 (see Figures 5 and 8), despite all having the same amount of mass. On the contrary, models SGS8 and SGS20 with $\epsilon = 0.05$ have shorter and weaker bars than SGS9 and SGS21 with $\epsilon = 0.025$. These results may imply a connection between parameters ϵ and m_d , given that they do not present normal behaviour.

To verify that this is not a coincidence or randomness within our simulations due to the low number of particles or range of softenings, we ran some of our models again, this time tripling the number of particles for each component, which means that $N_d = 0.72 \times 10^6$ and $N_h = 2.4 \times 10^6$, leaving all the other quantities untouched. We added the label ‘p’ to these models to distinguish them from our original set. The resulting properties of the discs in these experiments are summarised in Table 2. Comparing $Q_{\min}$ of this table and the one in Table 1, we immediately notice that the models with more particles have hotter discs than the fiducial ones, and hence, they are more stable. This fact is reflected by η_e , which is lower for almost all of the models in Table 2 compared to those in Table 1. In our low-resolution models, the strength of bars is not necessarily connected to Toomre’s stability criterion, which is self-evident when comparing the values of $Q_{\min}$ and η_e for models with $\epsilon \leq 0.005$, as discussed in the beginning of Section 4. Furthermore, ϵ and m_d still show a non-linear influence in regards to bar formation and growth, e.g., SGS2p reaches the same bar strength ($\eta_e \approx 0.12$) as SGS29p, despite having less disc mass and a lower softening length (see Table 2).

As mentioned earlier, ϵ and N are closely related. This was already studied for some simple configurations with analytically known density-potential pairs, e.g., [52,53]. So, for a Plummer sphere and a fixed value for ϵ , the total error decreases with N , independently from the choice of ϵ [55,109]. Consequently, our extra set of simulations should be physically more faithful than our fiducial one, regardless of our parameter choices. By increasing N , we are essentially moving the softening length window so that force calculation for particles found at short distances (e.g. $\epsilon \approx 0.005$) is closer to the ‘real’ interacting force, instead of using a lower force magnitude to avoid divergence in the accelerations [81]. A similar effect may be achieved by setting the softening length of halo particles so that the force between these and disc particles is the same. In Appendix A, we study the latter scenario on model SGS5 more closely.

Further evidence that there is an interplay between ϵ and m_d can be found in Figures 9 and 10. For each panel in Figure 9, a fixed value of softening is chosen and $\max(A_2)$ is plotted against disc mass fraction for both the low-resolution (LR) and high-resolution (HR) models. For the largest softening value considered in this work ($\epsilon = 0.1$; top left panel), the value of the bar strength remains approximately constant, $\max(A_2) \sim 0.675 \pm 0.025$, with the exception being $m_d = 0.044$. For this m_d value, the

LR model has $\max(A_2) \sim 0.7$ while for the HR model, $\max(A_2) \sim 0.5$. For a softening value of 0.025 (top right panel), the bar strength remains approximately constant, $\max(A_2) \sim 0.6 \pm 0.05$, regardless of disc mass fraction or numerical resolution. For a softening value of 0.005 (bottom left panel), the LR and HR models show important discrepancies in the bar strength (larger than 0.43 between HR and LR models) except for the lowest value of disc mass fraction. This might be pointing out a limiting (ϵ, m_d) value for which the models can be considered as physically truthful using a low resolution, $N \sim 10^6$, meaning that, for $m_d \geq 0.038$, LR models with $\epsilon = 0.005$ cannot be trusted. For the lowest softening value ($\epsilon = 0.001$), $\max(A_2)$ values show no tendency with disc mass fraction regardless of the resolution, and may be an indication that a softening value of $\epsilon = 0.001$ holds no physical validity, regardless of the m_d or N values.

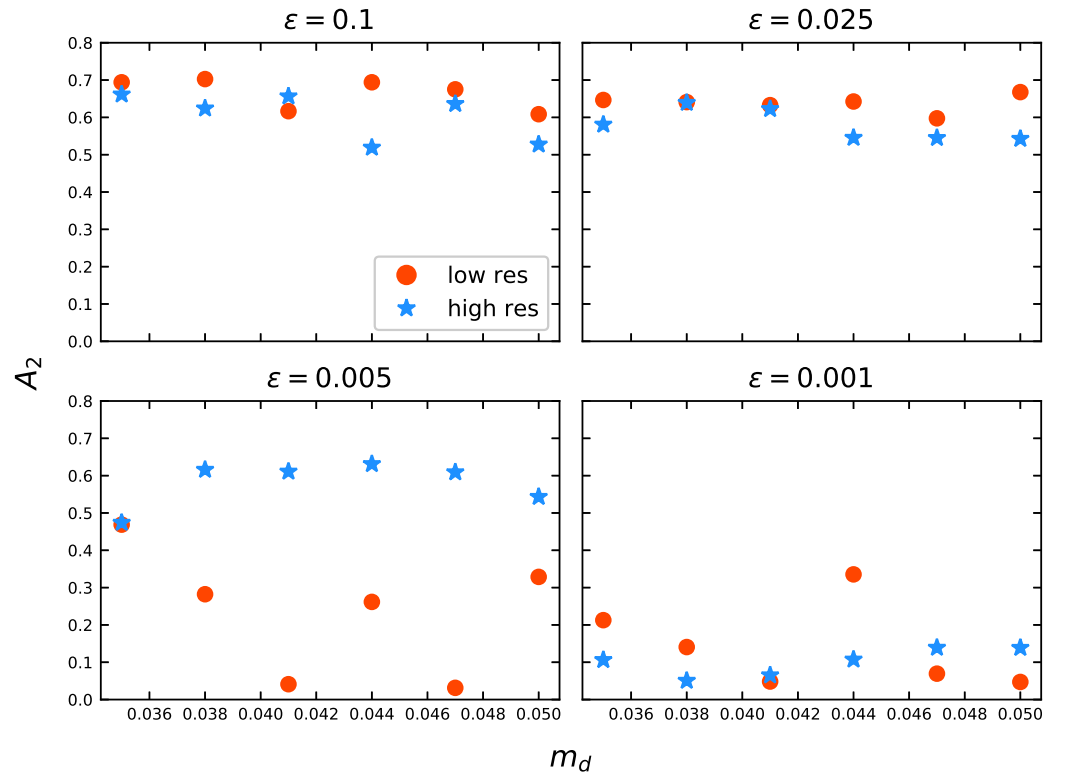


Figure 9. $\max(A_2)$ dependence on m_d for constant values of ϵ at $t = 12$ Gyrs. The models marked with a red bullet (●) are of high resolution, while those with a filled blue star (★) are of low resolution.

On the other hand, for each panel in Figure 10, $\max(A_2)$ is plotted against softening for a fixed value of m_d . When $\epsilon \geq 0.025$, all models form a bar with a strength of $\max(A_2) \sim 0.6 \pm 0.1$. For $\epsilon = 0.01$, the strengths only differ when $m_d = 0.041$. For $\epsilon = 0.005$, the differences become much more significant, having LR and HR pair models with a clear bar ($m_d = 0.035$; $\max(A_2) \sim 0.5$), another pair with a strong bar in the HR model ($m_d = 0.047$; $\max(A_2) \sim 0.6$) and no bar at all in the LR model ($m_d = 0.047$; $\max(A_2) \sim 0$). In fact, all HR models with $\epsilon = 0.005$ develop a bar. These results suggest a threshold in terms of ϵ for bar formation in LR models. Such a threshold is also affected by the value of m_d . When $\epsilon = 0.001$, none of our models develop a bar, regardless of the resolution. Considering that LR models appear to have a bar threshold, it is also fair to suggest that the same happens for HR models, but this threshold is moved by increasing N . We resume this analysis in Section 4.4.

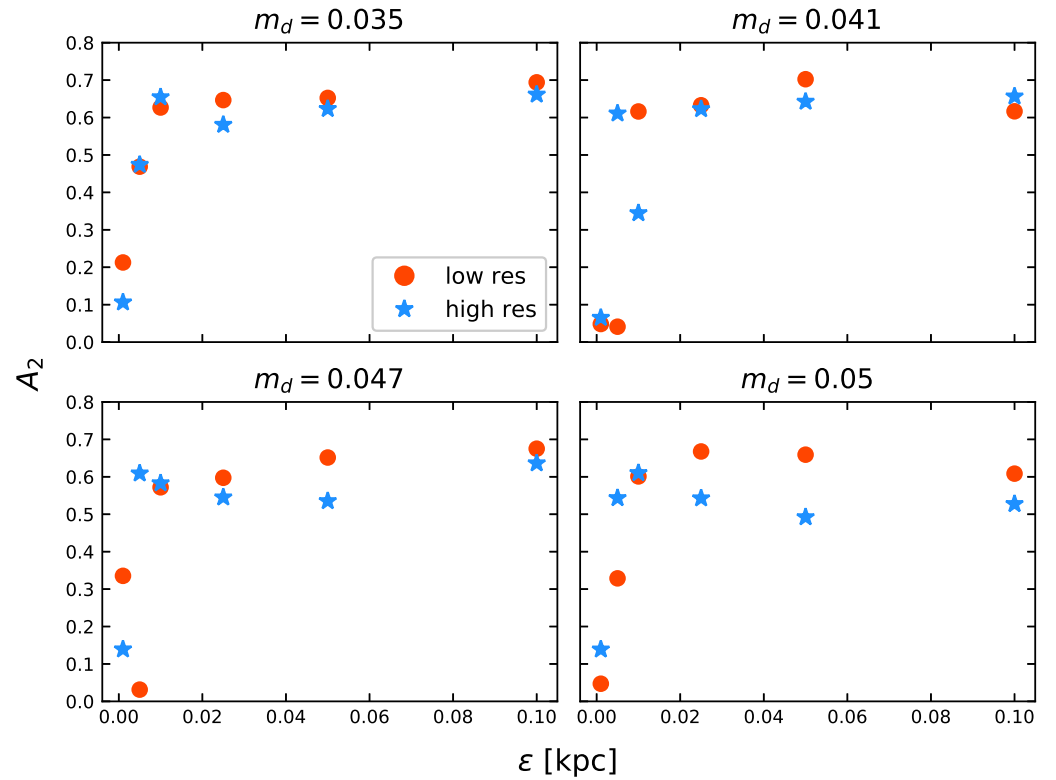


Figure 10. $\max(A_2)$ dependence on ϵ for constant values of m_d at $t = 12$ Gyrs. The symbols represent the same as in Figure 9.

Figure 11 shows the temporal evolution of η (top row) and the A_2 Fourier component (bottom row) for models SGS31p, SGS34p and SGS36p. We chose these models because they are the most stable in terms of their $Q_{\min}$. Also, their stability criteria are basically the same, so any differences among them would be related to m_d and/or ϵ . Both SGS31p and SGS34p have, in essence, the same behaviour: they develop a bar at around 6 Gyrs (as shown by η) and, according to the strength, reach their peak at 8 Gyrs and evolve secularly after the buckling phase. The buckling is a well-known vertical instability that arises due to the ongoing bar instability and is related to the vertical to radial velocity dispersion ratio, σ_z/σ_r [72,110]. Some authors point out that violent buckling, that is, a sudden increase of σ_z/σ_r during the simulation, greatly benefits the growth of bars after the buckling phase, e.g., [50]. Here, we consider an increase of 100% in the dispersion ratio from $t = 0$ to the moment the buckling is ignited as a strong buckling, e.g., [111]. From Figure 6, we notice that σ_r is always greater than σ_z for models with a strong bar; hence, the dispersion ratio stays the same for most of the simulation. This implies that our models do not have strong bucklings, which means that this is not the deciding factor in models that possess strong bars. However, only the models that reach the buckling instability stage, such as SGS31p or SGS34p, are able to sustain their bars, as shown by Figure 11. On the contrary, models with $\epsilon = 0.001$, like SGS36p, develop a bar but are unable to maintain it. This indicates that the buckling may be relevant for bars to hit their secular phase, but such a study is out of the scope of this work. We aim to explain why models with small softenings cannot reach the secular phase in the next section.

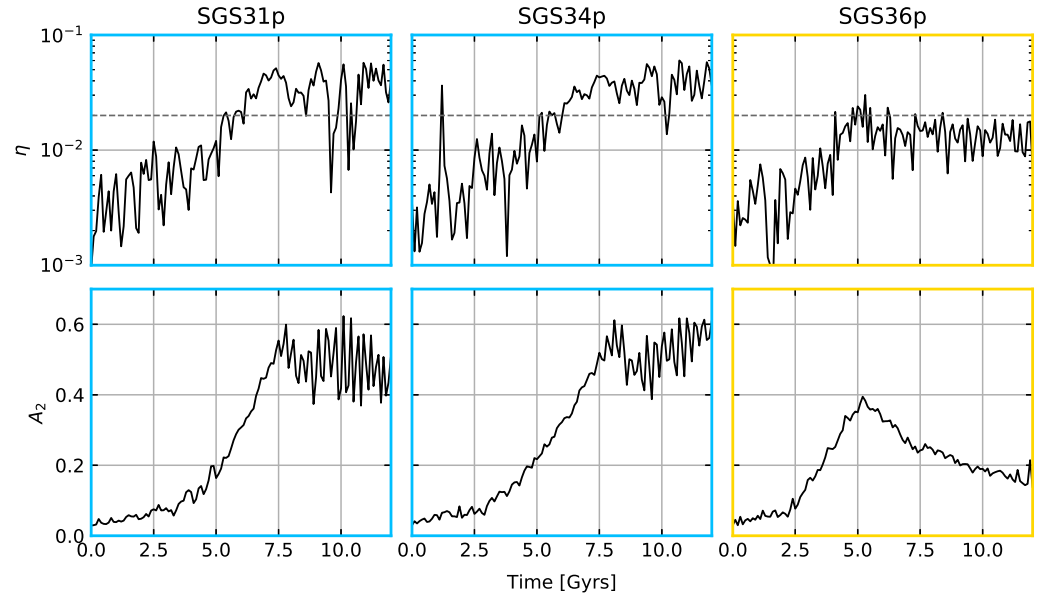


Figure 11. Time evolution of η (top) and the strength A_2 (bottom) for the discs of models SGS31p (left), SGS34p (centre) and SGS36p (right). The horizontal dashed line in the η plot is our bar threshold, $\eta_{\text{th}} = 0.02$. The colour code for the frame in each plot represents the same as in Figures 3 and 4.

Table 2. Structural properties of the discs in runs with extra particles. Column 1 : model name. Column 2: Toomre’s criterion at $t = 0$. Column 3: radial dispersion deviation at $t = 2.2$ Gyrs. Column 4: tangential dispersion deviation at $t = 2.2$ Gyrs. Column 5: vertical dispersion deviation at $t = 2.2$ Gyrs. Column 6: distortion parameter at $t = 12$ Gyrs. Column 7: bar strength at $t = 12$ Gyrs.

Model (1)	Q_{min} (2)	$\bar{\sigma}_r$ (3)	$\bar{\sigma}_t$ (4)	$\bar{\sigma}_z$ (5)	η_e (6)	A_2 (7)
SGS1p	1.24	0.320	0.039	0.063	0.152	0.661
SGS2p	1.24	0.313	0.044	0.055	0.122	0.623
SGS3p	1.22	0.330	0.038	0.056	0.158	0.580
SGS4p	1.24	0.355	0.058	0.055	0.143	0.654
SGS5p	1.24	0.334	0.041	0.056	0.095	0.473
SGS6p	1.24	0.368	0.078	0.095	0.008	0.106
SGS7p	1.26	0.275	0.025	0.043	0.130	0.624
SGS8p	1.24	0.288	0.035	0.039	0.141	0.685
SGS9p	1.24	0.303	0.036	0.043	0.112	0.639
SGS10p	1.24	0.319	0.044	0.047	0.140	0.665
SGS11p	1.25	0.314	0.040	0.072	0.149	0.615
SGS12p	1.25	0.390	0.094	0.063	0.004	0.050
SGS13p	1.27	0.272	0.029	0.056	0.145	0.656
SGS14p	1.26	0.303	0.037	0.037	0.115	0.642
SGS15p	1.27	0.284	0.030	0.071	0.120	0.622
SGS16p	1.28	0.304	0.034	0.046	0.004	0.344
SGS17p	1.27	0.296	0.027	0.051	0.136	0.610
SGS18p	1.28	0.316	0.050	0.043	0.008	0.065
SGS19p	1.28	0.268	0.033	0.032	0.051	0.519
SGS20p	1.30	0.284	0.031	0.033	0.112	0.606
SGS21p	1.28	0.274	0.026	0.036	0.065	0.545
SGS22p	1.29	0.289	0.028	0.041	0.094	0.662
SGS23p	1.29	0.288	0.026	0.036	0.115	0.631
SGS24p	1.30	0.298	0.042	0.045	0.023	0.107
SGS25p	1.31	0.255	0.032	0.039	0.057	0.636
SGS26p	1.28	0.264	0.032	0.035	0.068	0.535

Table 2. Cont.

Model (1)	$Q_{\min}$ (2)	$\bar{\sigma}_r$ (3)	$\bar{\sigma}_t$ (4)	$\bar{\sigma}_z$ (5)	η_e (6)	A_2 (7)
SGS27p	1.31	0.260	0.035	0.042	0.067	0.545
SGS28p	1.30	0.272	0.032	0.032	0.081	0.582
SGS29p	1.30	0.271	0.035	0.042	0.122	0.609
SGS30p	1.31	0.274	0.050	0.033	0.007	0.139
SGS31p	1.32	0.256	0.038	0.028	0.020	0.527
SGS32p	1.31	0.261	0.033	0.027	0.022	0.492
SGS33p	1.31	0.260	0.030	0.026	0.057	0.542
SGS34p	1.31	0.262	0.033	0.039	0.036	0.610
SGS35p	1.30	0.256	0.033	0.038	0.056	0.543
SGS36p	1.31	0.267	0.047	0.040	0.007	0.138

4.4. Vertical Acceleration Profiles

Models with $\epsilon = 0.001$, for either low or high resolution, display an odd behaviour when compared to other models with higher softening lengths. This scenario is depicted in Figure 11, where SGS36p has a bar peaking at around 5 Gyrs but then it dilutes at 12 Gyrs; this phenomenon occurs in several other models with the same amount of softening, and even at a higher resolution⁸. SGS31p ($\epsilon = 0.1$) and SGS34p ($\epsilon = 0.01$) develop more or less in the same manner. In fact, all high-resolution models in Table 2 with $\epsilon \geq 0.005$ have similar dynamics.

To shed some light on the disparity between the evolution of these models and how it could relate to ϵ , we track the vertical acceleration profiles, $|a_z|$, of every particle in our high-resolution set and map the results for different times (Figure 12). Each plot in the grid shows the evolution of $|a_z|$ against r for every model in Table 2. It is clear that vertical accelerations in the centre of discs increase with time, independently of the bar's properties. Accelerations increase at a lesser rate for models with higher m_d . Decreasing ϵ appears to have a positive impact on the central acceleration rate; both these effects are tightly linked to the bar strength. For instance, SGS19 is less accelerated than SGS25 despite having less disc mass. Conversely, SGS20 is more accelerated than SGS21 despite having a higher softening. Both SGS19 and SGS21 have noticeably weaker bars than most of the models in Table 2, at least according to their η_e values. Nevertheless, if bar strength remains comparable, which happens for the first two rows in Figure 12, we see that accelerations increase when ϵ decreases. An increase in vertical acceleration is not always correlated to bar strength. For example, SGS5 has a comparably weaker bar than SGS3 or SGS4 (following both η_e and A_2), despite being highly accelerated. This is where the noise introduced by an excess in softening starts to dominate the acceleration profile, overriding any effect induced by the bar; such an event occurs for models with $\epsilon \leq 0.005$, although it diminishes when m_d increases. Simulations with $\epsilon = 0.001$ appear to be physically unreliable since their vertical acceleration profiles reach values that do not correspond to a disc-like system. For simulations with this softening length, the particles with the highest accelerations are concentrated in the centre, which could explain their dynamics, particularly why bars are suppressed in these models.

We now summarise the main results emerging from Figure 12. Any trend involving accelerations depends on two effects: numerical noise and bar instability. At first glance, it is hard to distinguish which effect is the one meddling with the accelerations. For example, the strength of bars shown by η_e in Table 2 is comparatively similar between models SGS1p, SGS2p, SGS3p, SGS4p and SGS5p, but the $|a_z|$ peak is definitely greater for SGS4p and SGS5p, which have lower softening values. Conversely, models with the weakest bars at $t = 12$ Gyrs, e.g., SGS19p, SGS21p or SGS31p, are significantly less accelerated than models

with stronger bars. However, we are able to say that lower softenings accelerate the central regions of phase-space, which may contribute to the bar's suppression or destruction; such effect is much less prevalent as one increases ϵ .

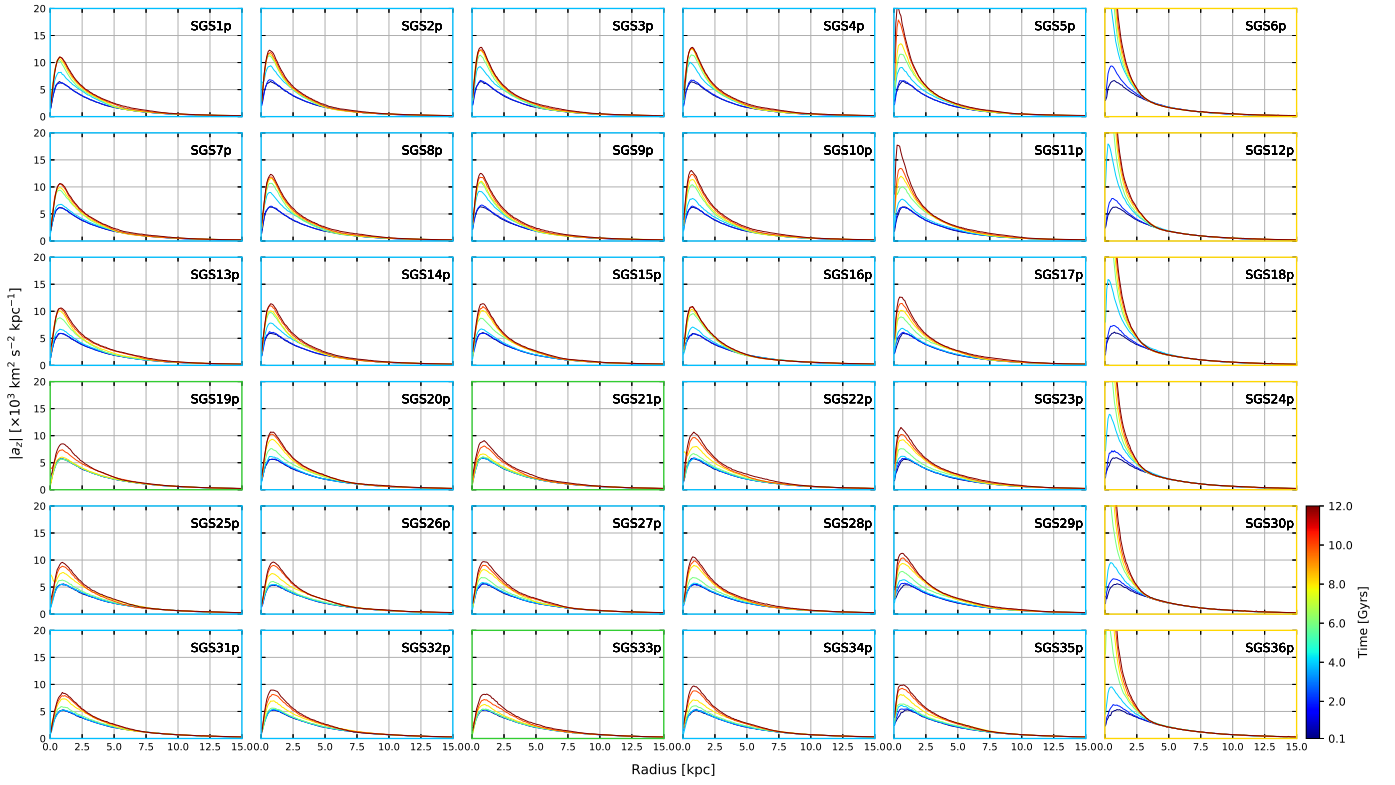


Figure 12. Vertical acceleration profile, $|a_z|$, of every disc in our high-resolution models. Different colours show the temporal evolution of each simulation. The colour code for the frame on each plot represents the same as in Figures 3 and 4.

Although our intent is to make our study as comprehensive as possible, there are still many scenarios left to explore. Our range for m_d is limited and although it is assumed that higher m_d values should generate more bar-stable models, it would be interesting to study its interaction with ϵ . The models are also relatively simple, containing only two dynamical components and one profile for each component; gas components and bulges not only intervene in bar dynamics, they also must be modelled using some softening recipe and it has been shown that the mass of a gaseous component and its softening length value may affect how a bar is triggered and formed [78].

We also used the same fixed ϵ for both components regardless of N , but there are other strategies available (see Appendix A) to set an ϵ for each component. One may use an adaptive approach to set the softening lengths, but such an approach still requires setting upper and lower limits for each species.

5. Conclusions

We ran a series of numerical models to follow the influence of two parameters (ϵ and m_d) on bar formation and secular evolution, also accounting for the effect of N in particle dynamics. We choose $0.001 \leq \epsilon \leq 0.1$ as our softening window, which sets it around the values opted by the majority of numerical studies, including ones that use an adaptive algorithm. The disc mass range is adjusted using the standard disc formation theory and is set to $0.035 \leq m_d \leq 0.05$, which properly fits observed disc galaxies. The results of these experiments are summarised as follows.

- ◊ The set of galaxy models considered here vary in mass and disc extension but have, by construction, nearly the same initial value of $Q_{\min}$ (≈ 1.1). In this sense, the models are marginally stable against bar formation.
- ◊ Further evolution of Toomre's criterion does not reflect bar instability for all of our models, particularly for models with the lowest softening values, i.e., $\epsilon \leq 0.005$ kpc (see Section 2.3). This behaviour still occurs after increasing the number of particles of the models.
- ◊ We find that η yields an accurate picture of the bar's formation and evolution (see Figure 8). We confirm that a threshold of $\eta_{\text{th}} = 0.02$ effectively indicates recent bar formation. Additionally, this parameter allows us to observe the formation, destruction and resurgence of the bar, which is not possible with other indicators such as the bar strength, A_2 . The distortion parameter is more apt at measuring when the bar has formed and A_2 is more apt at identifying the different phases of bar evolution (rapid growth, buckling, secular growth). In this case, we are inclined to use the distortion parameter because we are interested in when the bar appears instead of determining how strong it might become.
- ◊ We find a close relationship between ϵ and m_d that affects how fast the bar forms, its strength and its length. We notice that the empirical linear dependencies, that is, $\eta \propto m_d^{-1}$ and $\eta \propto \epsilon$, between our simulations and (ϵ, m_d) pairs do not hold when these two parameters interact. Thus, the process of choosing an appropriate softening value seems to be more complicated than previously thought.
- ◊ The vertical acceleration profile is a better estimator of the disc heating than the velocity dispersions. Our models with small softening values ($\epsilon \leq 0.005$ kpc) tend to accumulate and accelerate particles at the centre of discs, driving them toward chaotic motion. Such an effect normally results in the destruction of the bar.
- ◊ While dispersions do increase at the centre of unbarred models, such an increase is not that extreme and may be explained by natural causes (e.g., disc mass redistribution, bar residue, non-axisymmetric distortions, etc.). Given that the central disc density increases with time in our models, regardless of parameter values, it is also reasonable to assume that the near-circular particle orbits have simply shrunk, dimly increasing the dispersions. However, none of these effects necessarily explain the bar dilution. Because accelerations are not affected by these natural causes, they are better tracers of these unphysical/odd behaviours.
- ◊ We are also able to conclude that, for particle resolutions close to $N \approx 10^7$, softening values lower than 0.005 kpc are not well-suited to reproduce the bar instability. However, this depends on the disc mass fraction. If $m_d \approx 0.05$, the bar appears to cohabit with numerical noise introduced by small softening values, specifically, $\epsilon = 0.005$ kpc. Models with lower m_d values ($\lesssim 0.040$) are not able to overcome the interference of small softening values, affecting the behaviour of the bar (see Figure 12).

Applying a comprehensive set of numerical tools to study the bars in collisionless disc models, we conclude that such an analysis is necessary to understand the effects induced by ϵ and its interplay with m_d and N . Our analysis indicates that the vertical acceleration profile, $|a_z|$, adequately describes the behaviour of our disc, and particularly, why there is bar suppression in some of our models. However, since both the bar and numerical noise related to ϵ heat up the vertical acceleration profile, it is not trivial to tell which one is affecting the acceleration at any particular region. Numerical noise usually gathers at the innermost regions of the disc, but it is unclear at what point in phase-space the bar starts to affect these regions too. This means that there is a close relationship between the bar distortion and ϵ , one that is much more intricate than previously thought cf. [58]. Balance amid these effects may result in more faithful simulations of galactic discs.

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Appendix A. The Effect of Unequal Softening Lengths

In order to avoid using the same particle masses for halo and disc components, a usual practice is to assign different softening lengths to each one. Usually, the softening length of one component is chosen in terms of the other to guarantee the same gravitational force between particles of different species [83]. This way, one avoids undesired effects such as two-body scattering and gravitational drag force.

Here, we repeat simulation SGS5 in Table 1, this time applying the softening recipe described above. So, the softening length of halo particles is set to $\epsilon_h = 0.041$, while keeping $\epsilon_d = 0.005$ for the disc particles. The initial conditions were created with $\epsilon = 0.041$. As mentioned in Section 2.2, GALIC uses a single value for ϵ for all components while producing its equilibrium galaxy models. For comparison, an equivalent NFW halo with $N = 10^6$ that minimises the acceleration errors will have $\epsilon = 0.288$ [59].

The outcome of this run, named SGS5eq, is now compared to the original low-resolution SGS5 model and the high-resolution SGS5p rerun. Figure A1 shows the angular momentum transfer (top), bar amplitude (middle) and distortion parameter (bottom) for these three models. The upper panel shows that all three runs behave similarly until their respective bars emerge at $t \sim 2.5$ Gyrs. After that, the discs lose their momentum proportionally to the bar strength. From the middle panel, we notice that SGS5 and SGS5eq start in a similar manner (up to ~ 2.5 Gyrs), but there is a clear departure from this point where the bar of SGS5eq rapidly grows until the end of the simulation. The evolution of the bar in SGS5p is similar to SGS5eq, but there is a small delay in its growth. The buckling phase in SGS5p (~ 5 Gyrs) is visibly stronger than both SGS5 and SGS5eq, but the bar growth in SGS5p stagnates after this phase, contrary to SGS5eq, where the bar continues to grow after the end of the buckling. The behaviour of the bar strength may also be observed in the transfer of angular momentum, where the strongest bar induces a higher transfer rate. The distortion parameter graphs reflect the delays suffered by the bars in SGS5p and, most notably, SGS5.

Furthermore, by comparing the velocity dispersions profiles and the tangential velocities in Figure A2, it can be seen that in the original SGS5 run there is an excess in the velocity dispersion at the centre of the disc at late stages of bar formation, and more intense suppression of the tangential velocity in SGS5eq due to a stronger bar. In this sense, the tangential velocity and vertical velocity dispersion of model SGS5p lies between the SGS5

and SGS5eq. One key difference in SGS5eq is that the vertical heating (σ_z) is not confined to the region occupied by the bar, having enhanced dispersions for the whole disc. This fact would not be expected as heating should be inhibited by setting different ϵ values for each galaxy component. We suggest that it is rather due to the bar dynamics itself, which favours the transfer of orbital momentum into the vertical motions (see top panel in Figure A1).

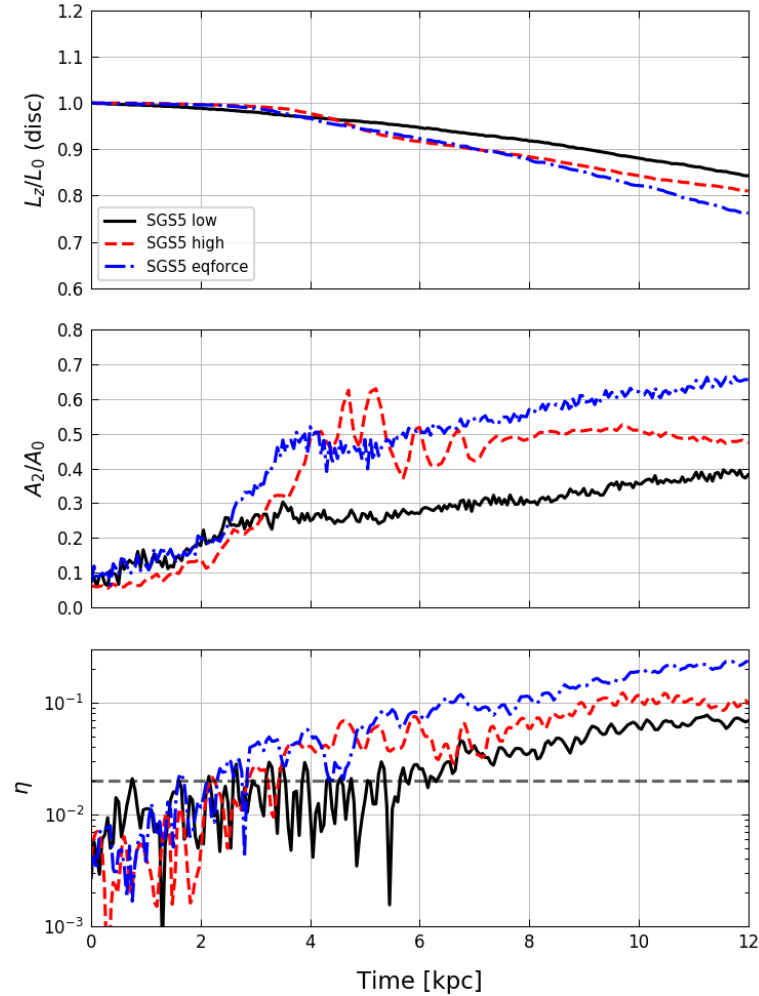


Figure A1. Properties of discs for SGS5 (black full line), SGS5p (red dashed line) and SGS5eq (blue dash-dotted line). *Top*: normalised angular momentum transfer, L_z/L_0 . *Middle*: normalised $m = 2$ Fourier amplitude of the surface density. *Bottom*: distortion parameter, η . The horizontal dashed line is the bar threshold given by $\eta_{th} = 0.02$.

Finally, we address the evolution of the disc particle vertical accelerations shown in the bottom of Figure A2. Up to $t = 0.1$ Gyrs, the discs in all three models conserve the same dynamical equilibrium despite the value of ϵ . After $t \sim 2$ Gyrs the original SGS5 model rapidly heats up in the centre albeit having a weaker bar. We attribute this difference to the choice of the single ϵ value. In fact, the disparity between SGS5 and SGS5eq in terms of bar strength also happens at $t \sim 2$, confirming that enhanced accelerations are, in the experiments shown here, the culprit for bar suppression. Again, the high-resolution model shows an intermediate behaviour.

The intermediate behaviour of SGS5p falls right between SGS5 and SGS5eq due to its smaller mean interparticle distance to ϵ ratio, which puts it closer to the ratio of SGS5eq, given that the latter has a larger ϵ for the halo component.

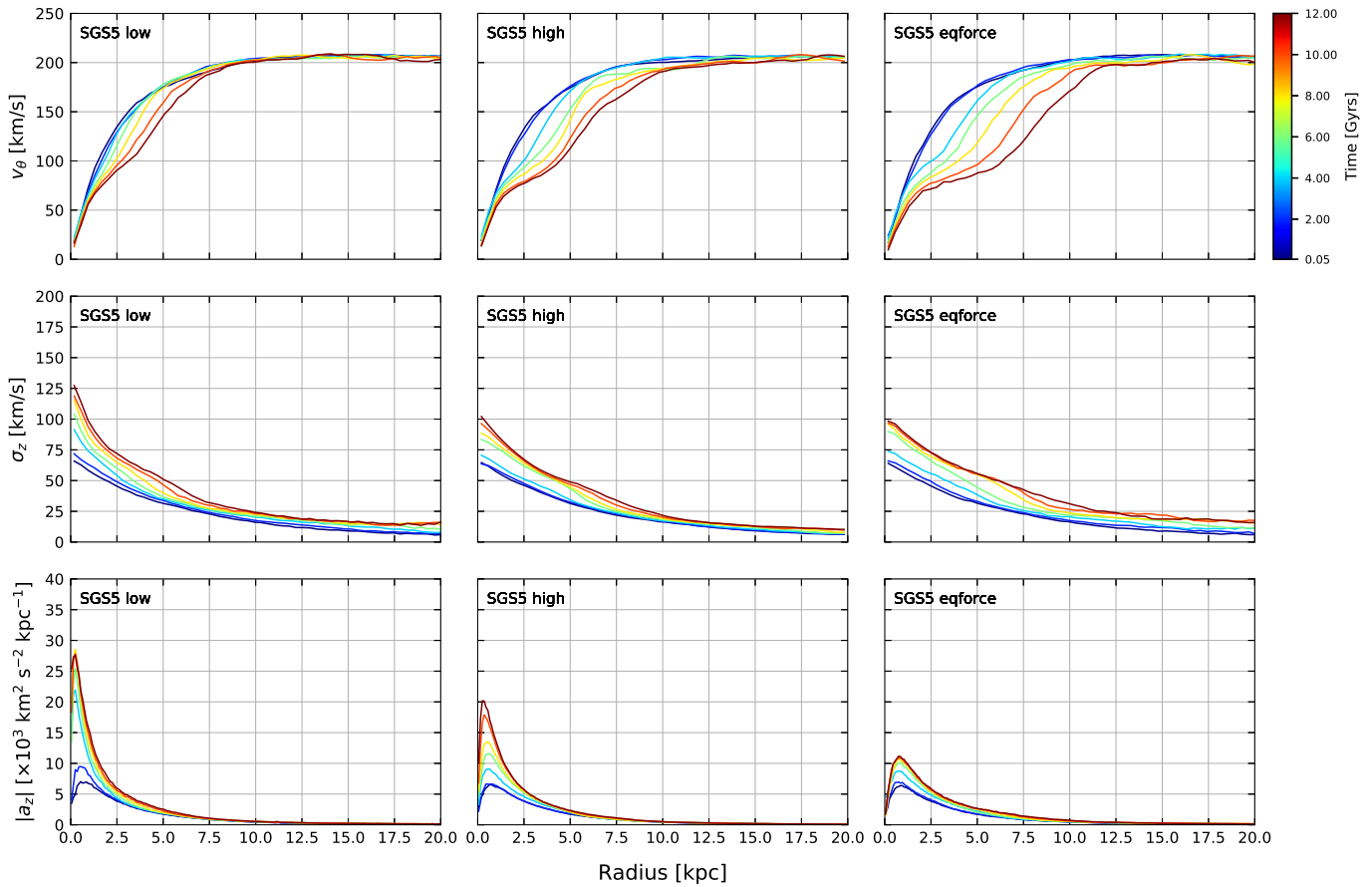


Figure A2. Evolution of the tangential velocities (**top row**), vertical velocity dispersion (**center row**) and vertical acceleration profile (**bottom row**) of models SGS5, SGS5p and SGSeq. The colour code for the frame on each plot represents the same as in Figures 3 and 4.

The results in this section imply that the choice of the softening value has an impact on long term bar evolution, rather than any sudden change in dynamics at the beginning due to the difference in softening values between the ICs and the actual run.

Notes

- ¹ In this case, resonant orbits, mostly of the x_1 family.
- ² We do not distinguish between Plummer or kernel-based softenings. Some of these simulations may also include hydrodynamical effects.
- ³ [58] use a fiducial value of $\epsilon = 0.05$ in their fixed-softening simulations. They also ran simulations with $\epsilon = 0.025, 0.1$
- ⁴ This value roughly corresponds to the mass of Milky Way-size galaxies, given that $M_{200} \propto v_{200}^3$.
- ⁵ We ran a couple of experiments using $m_d = \{0.02, 0.3\}$ that confirm our results.
- ⁶ Whether or not this model developed a bar is unknown, [93] only showed the first gigayear of the simulation.
- ⁷ We use this time as a reference point to assess the involvement that bar ignition has on any increase in the velocity dispersions (“heating”).
- ⁸ We ran SGS36 with $N \approx 6 \times 10^6$ and the bar is still suppressed, although it should be more physically consistent, since the errors in force calculations have smaller impact than for models with less particles.

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