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Review

Spin Equilibrium of Rapidly Spinning Neutron Stars via Transient Accretion

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Abstract: The concept of spin equilibrium due to an interaction between the stellar magnetosphere and a thin, Keplerian accretion disk, and a well-known formula of the corresponding equilibrium spin frequency, provide a key understanding of spin evolution and the distribution of rapidly spinning neutron stars, viz., millisecond pulsars. However, this concept and formula are for stable accretion, but the mass transfer to most accreting millisecond pulsars is transient and the accretion rate evolves by orders of magnitude during an outburst. In this short and focussed review, we briefly discuss a relatively new concept of the spin equilibrium condition and a new formula for the equilibrium spin frequency for transiently accreting millisecond pulsars. We also review a new method to estimate this equilibrium spin frequency for observed transiently accreting millisecond pulsars, even when a pulsar has not yet attained the spin equilibrium. These will be crucial to probe the spin evolution and distribution of millisecond pulsars, and should also be applicable to all magnetic stars transiently accreting via a thin, Keplerian accretion disk.

Keywords: accretion; neutron star; pulsar; rotation; spin equilibrium; X-ray binary

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1. Introduction

Rapidly spinning neutron stars, viz., millisecond pulsars (MSPs; spin period $P < 30$ ms), manifest in various incarnations [1–3]. The largest known population ($\sim$500) is of radio MSPs, which show regular brightness variation at the neutron star spin period in radio wavelengths [1]. There are also more than a hundred known γ-ray MSPs, almost all of which are a subset of radio MSPs [3]. While most of these MSPs are in binary systems [1–3], they, with very few exceptions, do not accrete matter from the companion star [4]. These MSPs (RMSPs) are powered by neutron star spin or rotational kinetic energy [5]. Another kind of MSP accretes matter from a low-mass companion star (mass $\lesssim 1\ M_\odot$) in a neutron star low-mass X-ray binary (LMXB) system [6]. As the spinning neutron star's magnetic field channels this matter to the stellar magnetic poles, X-ray intensity variation at the stellar spin frequency (ν) is observed. These are accretion-powered MSPs (AMSPs), also known as X-ray MSPs, and more than 20 such sources are known [2,4]. There is also another set of accreting X-ray MSPs in LMXB systems, which show nuclear-powered X-ray pulsations during thermonuclear X-ray bursts [5]. These bursts occur due to intermittent, unstable burning of accumulated matter on the accreting neutron star surface [7], and an asymmetric brightness pattern on this spinning star during such a burst causes these pulsations [5]. There are about 20 known nuclear-powered MSPs (NMSPs), and some of them are also AMSPs [5].

Thus, there are different types of MSPs, and the ν-values of these neutron stars are measured by multiple methods using observations in various wavelengths (e.g., radio, X-ray, γ-ray). However, how do MSPs attain high spin rates? Neutron stars are typically born with relatively high spin frequencies (tens of Hz) and large magnetic field values ($B \sim 10^{12}$ G), and then slow down due to angular momentum loss via the radiation and polar outflow of particles [8]. If a neutron star is in a binary system with a low-mass companion star, then this companion may eventually fill its Roche lobe, and the neutron star may accrete matter through an accretion disk [6]. The high specific angular momentum of the disk matter can spin up the star and give rise to MSPs in the LMXB phase [9,10]. Discoveries of AMSPs and transitional MSPs, which swing between RMSP and AMSP phases, have confirmed this [11–16]. The neutron star magnetic field decreases to much lower values ($B \sim 10^{7-9}$ G [17]), possibly due to accretion-induced Ohmic dissipation and/or diamagnetic screening, during the initial part of the spin-up in the LMXB phase [6,18–20]. Such a field decay allows the accretion disk to approach the neutron star and hence to spin it up more efficiently.

Why is the study of neutron star spin evolution important? The MSPs have shown a wide range of spin frequencies (up to 716 Hz [21]). Apart from these ν-values, several other properties, such as the rate of change of ν (i.e., $\dot{\nu}$), nature of the companion star, binary orbital period, etc., of many radio MSPs are known and could be accessed, e.g., from the ATNF pulsar catalogue[1]. Moreover, the upcoming radio observatory *Square Kilometre Array* (*SKA*) could increase the number of such radio MSPs by about an order of magnitude. The earlier evolution of these MSPs through their LMXB phases determined their parameter values. Such an evolution depends on several fundamental aspects, e.g., accretion through the stellar magnetosphere and strong gravity region, the stellar structure that depends on the equation of state (EoS) models, gravitational wave emission and binary evolution, etc., of neutron stars. Thus, one could probe these aspects of physics and astrophysics of neutron stars by computing the stellar and binary evolution, including the ν-evolution, through the LMXB phase and comparing the calculated parameter values at each evolutionary stage with the known values. This is why a realistic estimation of ν-evolution is essential.

We see that the neutron star spin frequency attains or approaches an equilibrium value due to accretion through the stellar magnetosphere. For a neutron in an LMXB, this equilibrium frequency ($\nu_{\rm eq}$) is high, and this is why MSPs attain high ν-values. The traditional expression of $\nu_{\rm eq}$ does not consider transient or episodic accretion in the LMXB phase (Section 2). However, in Section 3, we show the crucial effects of transient accretion. We mention the implications of such accretion in Section 4.

2. Spin Equilibrium without Transient Accretion

Let us consider a neutron star accreting matter through a thin, Keplerian disk. If the stellar magnetic field is relatively weak, the disk can extend up to the stellar surface, or up to the innermost stable circular orbit (ISCO), the radius of whichever is larger. In such cases, the accreted matter could spin up the neutron star till the ν-value almost reaches the Keplerian value or the so-called breakup spin rate, and no spin equilibrium via disk–magnetosphere interaction is attained. Note that all measured ν-values [22,23] are substantially less than the expected breakup spin rates of neutron stars [24,25].

If the neutron star magnetic field is relatively strong, and the energy density of this magnetic field balances the kinetic energy density of the inflowing matter at a certain radius $r_{\rm m}$, which is greater than both ISCO and stellar radii, then the disk stops at $r_{\rm m}$ and the accreted matter is channelled by the field onto the stellar magnetic polar caps [2,6,26]. Equating the magnetic and material stresses, we obtain [20,27]:

$$r_{\rm m} = \xi \left(\frac{\mu^4}{2GM\dot{M}^2} \right)^{1/7}. \tag{1}$$

Here, μ (=BR^3) is the neutron star magnetic dipole moment, B is the stellar surface dipole magnetic field, R and M are the stellar radius and mass, respectively, $\dot{M}$ is the accretion rate, and ξ is a constant, which depends on the disk–magnetosphere interaction and the magnetic pitch at the disk inner edge [27]. The value of ξ could be ~0.3–0.5 [2].

Another characteristic radius of the system is the corotation radius ($r_{\rm co}$), where the disk Keplerian angular velocity is the same as the stellar angular velocity. This condition gives

$$r_{\rm co} = \left(\frac{GM}{4\pi^2 \nu^2} \right)^{1/3}. \tag{2}$$

A third important radius is the speed-of-light cylinder radius $r_{\rm lc} = c/2\pi\nu$. Typically, $r_{\rm lc} > r_{\rm co}$.

For a thin, Keplerian disk, there can be the following three regimes of accretion.
(i) $r_{\rm m} < r_{\rm co}$: In this accretion phase a steady accretion occurs, and the neutron star spins up because the matter at the disk inner edge rotates faster than the star, and hence, applies a net positive torque on the star [28,29]. If $r_{\rm m} > R$ and $r_{\rm ISCO}$ ($r_{\rm ISCO}$: ISCO radius), the accreted matter is channelled onto the neutron star by a magnetic field, and X-ray pulsations occur.
(ii) $r_{\rm co} < r_{\rm m} < r_{\rm lc}$: In this so-called "propeller" phase, the neutron star magnetosphere rotates faster than the accreted matter at the disk inner edge, and hence, a centrifugal barrier could shut off the accretion [30,31]. A part of the accreted matter could be driven away from the system in this phase at the expense of the neutron star's angular momentum, and hence, the star slows down [20,32].
(iii) $r_{\rm lc} < r_{\rm m}$: In this phase, the accreted matter does not enter the magnetosphere, and the radio pulsar mechanism could turn on [33]. This happens when $\dot{M}$ is relatively low.

From the above, we see that for $r_{\rm m} < r_{\rm co}$, ν increases due to a positive torque applied on the neutron star, and hence, $r_{\rm co}$ decreases (Equation (2)). On the other hand, for $r_{\rm co} < r_{\rm m} < r_{\rm lc}$, ν decreases because of a negative torque applied on the neutron star, and hence, $r_{\rm co}$ increases (Equation (2)). Thus, due to this self-regulated mechanism, $r_{\rm co}$ tends to become $r_{\rm m}$, which is the spin equilibrium condition. For this condition, using Equations (1) and (2), one obtains the following expression of the equilibrium spin frequency [6,20]:

$$\nu_{\rm eq} = \frac{1}{2\pi} \sqrt{\frac{GM}{r_{\rm m}^3}} = \frac{1}{2^{11/14} \pi \xi^{3/2}} \left(\frac{G^5 M^5 \dot{M}^3}{\mu^6} \right)^{1/7}. \tag{3}$$

This is also the maximum ν-value a spinning-up neutron star could attain by the disk–magnetosphere interaction [34]. Moreover, this expression could be applicable to any spinning, magnetic star (e.g., a neutron star in a high-mass X-ray binary (HMXB), protostar) accreting through a thin, Keplerian disk. Note that the stellar magnetic dipole moment μ, and hence the magnetic field B, are related to the equilibrium spin frequency by $\nu_{\rm eq} \propto \mu^{-6/7} \propto B^{-6/7}$. This is why the low-magnetic field ($\sim 10^{7-9}$ G) neutron stars in LMXBs can attain orders of magnitude higher ν-values compared to those of the high-magnetic field ($\sim 10^{12}$ G) neutron stars in HMXBs.

Realistic expressions of disk–magnetosphere interaction torque formulae are [20,35,36]:

$$N_{\rm acc} = \dot{M}\sqrt{GMr_{\rm m}} + \frac{\mu^2}{9r_{\rm m}^3} \left[2\left(\frac{r_{\rm m}}{r_{\rm co}}\right)^3 - 6\left(\frac{r_{\rm m}}{r_{\rm co}}\right)^{3/2} + 3 \right] \tag{4}$$

for the accretion phase, and

$$N_{\rm prop} = -\eta\dot{M}\sqrt{GMr_{\rm m}} - \frac{\mu^2}{9r_{\rm m}^3} \left[3 - 2\left(\frac{r_{\rm co}}{r_{\rm m}}\right)^{3/2} \right] \tag{5}$$

for the propeller phase. In each of Equations (4) and (5), the first term is the contribution due to accretion disk matter, and the second term is due to the interaction of the stellar magnetic field with the entire disk. In the propeller phase, an unknown fraction of disk matter could be ejected, which is taken care of by an order of unity positive constant η (Equation (5) [20,32]).

The equilibrium spin frequency provides the main concept to understand the spin evolution and distribution of MSPs. However, apart from the spin-down in the propeller phase, a neutron star in an LMXB could also spin down due to electromagnetic (e.g., magnetic dipole) radiation, the polar outflow of particles, the ejection of accreted matter via jets powered by the stellar spin kinetic energy, and the emission of gravitational waves (GW). For example, the magnetic dipole radiation torque is given by [8]:

$$N_{\rm EM} = -\frac{2\mu^2}{3r_{\rm lc}^3} = -\frac{16\pi^3\mu^2\nu^3}{3c^3}. \tag{6}$$

Moreover, the gravitational wave torque is given by [37]:

$$N_{\rm GW} = -\frac{32GQ^2}{5}\left(\frac{2\pi\nu}{c}\right)^5, \tag{7}$$

where Q is the stellar rotating misaligned mass quadrupole moment. One or more of such negative torques acting on the neutron star should affect the spin equilibrium condition. For example, if negative $N_{\rm GW}$ balances positive $N_{\rm acc}$, then accretion continues, and hence, $r_{\rm m} < r_{\rm co}$. This higher $r_{\rm co}$ value implies a lower ν-value for the spin equilibrium, and hence, a lower equilibrium spin value. However, we primarily consider only the disk–magnetosphere interaction, and not these extra negative torques, in this review.

3. Spin Equilibrium with Transient Accretion

We did not consider the effects of transient accretion in Section 2. It has recently been shown that these effects are crucial when computing neutron star spin evolution [20,34,36].

All AMSPs and most neutron star LMXBs accrete matter transiently [2]. This means the source passes through alternate outburst and quiescent periods. An outburst period typically lasts days to weeks, and the usual durations of quiescent periods are months to years. The instantaneous accretion rate $\dot{M}$ increases by several orders of magnitude during an outburst [38]. Two instabilities in the accretion disk—thermal and viscous—cause such outbursts when the long-term average accretion rate $\dot{M}_{\rm av}$ is less than a critical $\dot{M}_{\rm av,crit}$ value [39,40]. The matter from the companion star accumulates during a long quiescent period, and then falls onto the neutron star causing a relatively short outburst. Note that, while such a transient behaviour could have various observational implications, e.g., for repeating fast radio bursts according to a model [41], here we focus on its effects on the neutron star spin evolution.

Note that Equation (3) tacitly assumes a constant accretion rate, i.e., $\dot{M} = \dot{M}_{\rm av}$. Thus, in this case, $r_{\rm m}$ does not change much, and $r_{\rm co}$ evolves to ultimately satisfy the spin equilibrium condition ($r_{\rm m} = r_{\rm co}$). However, for a transient accretion, $r_{\rm m}$ drastically changes, as $\dot{M}$ ($\propto r_{\rm m}^{-7/2}$) evolves by several orders of magnitude in an outburst cycle. On the other hand, ν, and hence $r_{\rm co}$, do not evolve appreciably during one outburst. Therefore, for one $\dot{M}$-value of the outburst, $r_{\rm m}$ equals $r_{\rm co}$, but otherwise, the standard equilibrium condition ($r_{\rm m} = r_{\rm co}$) is not satisfied during an outburst cycle. Then, what is the condition of spin equilibrium for transient accretion?

3.1. Spin Equilibrium Condition

During the rise of an outburst, when the $\dot{M}$ value increases by several orders of magnitude, the disk approaches the star, and $r_{\rm m}$ decreases. Thus, the source could pass through $r_{\rm m} > r_{\rm co}$, $r_{\rm m} = r_{\rm co}$ and $r_{\rm m} < r_{\rm co}$ conditions. These conditions should occur in the opposite order during the outburst decay. The $r_{\rm m} = r_{\rm co}$ condition (dashed horizontal line in Figure 1a–d) separates the accretion phase from the propeller phase. The former phase ($r_{\rm m} < r_{\rm co}$) is for $\dot{M}$ values above the dashed horizontal line, while the latter phase ($r_{\rm m} > r_{\rm co}$) is for $\dot{M}$ values below this dashed horizontal line. Since $r_{\rm co}$ is a function of ν, whether for a given $\dot{M}$ value the source will be in the accretion phase or in the propeller phase depends on ν.

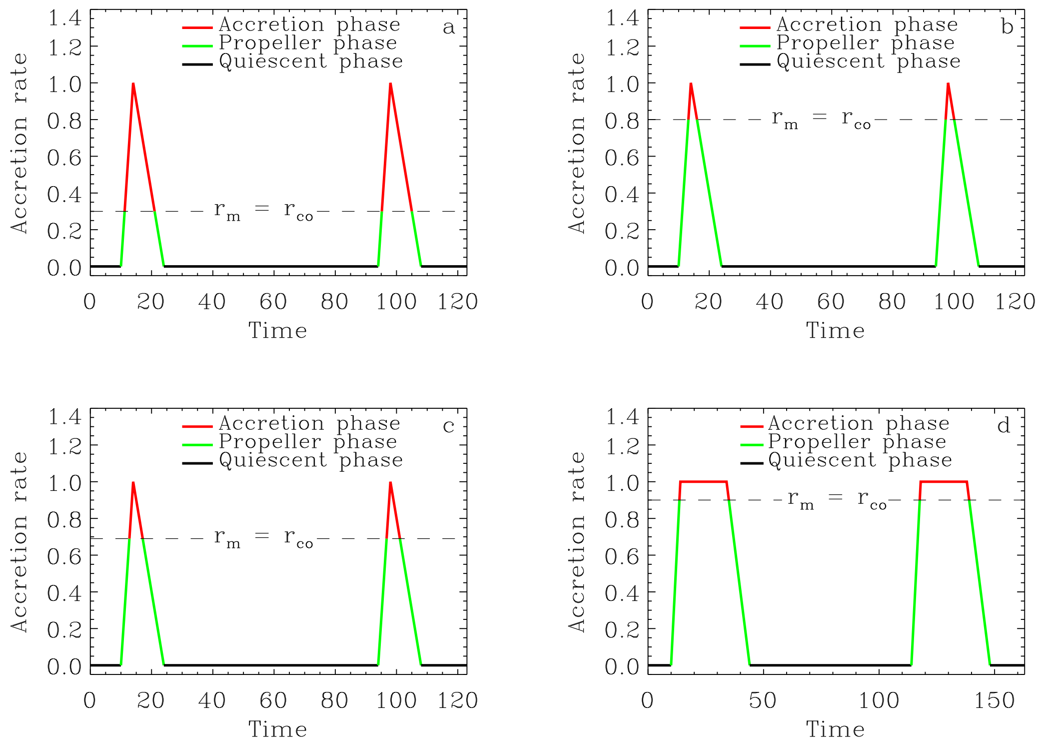

Figure 1. Schematic illustrations of outburst and quiescent periods of a neutron star LMXB [20,34,36]. The evolution of the accretion rate, normalized by the peak accretion rate, through accretion, propeller and quiescent phases is shown. Here, triangular outburst profiles (linear rise and decay) are displayed in panels (**a**–**c**) and profiles with flat tops are shown in panel (**d**). The time is in an arbitrary unit. Panel (**a**): The spin equilibrium is not yet reached and the neutron star is overall spinning up. Panel (**b**): The spin equilibrium is not yet reached and the neutron star is overall spinning down. Panel (**c**): The spin equilibrium is reached. Panel (**d**): Since the outburst profiles have flat tops, the transition between the accretion phase and the propeller phase happens at a higher rate compared to panel (**c**), the normalized accretion rate in spin equilibrium (see Section 3).

To understand the spin equilibrium condition for transient accretion, let us now consider a relatively low ν-value, which implies a comparatively high $r_{\rm co}$ value. Thus, the $r_{\rm m} = r_{\rm co}$ condition occurs at a high $r_{\rm m}$ value, and hence, for a relatively low $\dot{M}$ value. This means that most of the time during an outburst the source remains in the accretion phase (see Figure 1a). Moreover, since $\dot{M}$ is larger in the accretion phase, the total positive angular momentum transfer to the star in the accretion phase is higher than the total negative angular momentum transfer in the propeller phase during an outburst. Therefore, a net spin-up of the star happens.

On the other hand, if the ν-value is relatively high, $r_{\rm co}$ is low. Thus, the $r_{\rm m} = r_{\rm co}$ condition occurs at a low $r_{\rm m}$ value, and hence, for a relatively high $\dot{M}$ value (see Figure 1b). This means the total positive angular momentum transfer to the star in the accretion phase is lower than the total negative angular momentum transfer in the propeller phase during an outburst. Therefore, a net spin-down of the star happens.

Thus, the neutron star spins up when the ν-value is low and spins down when this value is high. Thus, a self-regulated mechanism works even for transient accretion and the star approaches a spin equilibrium (see Figure 1c). The condition for this equilibrium is that the total angular momentum transferred to the neutron star is zero during an outburst cycle. Considering no angular momentum is transferred in the quiescent phase, and accretion torque (e.g., $N_{\rm acc}$) and propeller torque (e.g., $N_{\rm prop}$) are the only torques applied on the star, the equilibrium condition implies that the total positive angular momentum transfer to the star in the accretion phase balances the total negative angular momentum transfer in the propeller phase during an outburst.

Note that during such an equilibrium, the ν-value decreases and increases during an outburst. However, the magnitude of this ν-change is very small during days to weeks of an outburst because the spin-up timescale is typically about a billion years for a neutron star in an LMXB. As a result, one can ignore such ν-value fluctuations during an outburst for the long-term spin evolution. Nevertheless, we call the spin frequency corresponding to the spin equilibrium for transient accretion the effective equilibrium spin frequency ($\nu_{\rm eq,eff}$). This is the maximum frequency for a transiently accreting neutron star.

3.2. Effective Equilibrium Spin Frequency: A Simple Formula

A primary aim of this short review is to discuss simple formulae for equilibrium spin frequency due to disk–magnetosphere interaction during transient accretion. The formula (Equation (3)) for persistent accretion ($\dot{M} = \text{constant}$) has been popularly used for decades. However, this persistent accretion formula is not applicable to most sources because most neutron star LMXBs accrete transiently.

Since our aim is to find simple analytical formulae, we ignore the second terms of the torque formulae (Equations (4) and (5)), and assume $\eta = 1$. Numerical computations have shown that this does not introduce an error more than $\sim$10% [20]. From this and Equation (1), we can write the torque on the neutron star:

$$N = \frac{\mathrm{d}J}{\mathrm{d}t} = \pm C\dot{M}^{6/7}, \tag{8}$$

where J is the stellar angular momentum, C is a constant independent of $\dot{M}$, and positive and negative signs are for the accretion phase and propeller phase, respectively.

Now, let us assume triangular outburst profiles with the same peak accretion rate ($\dot{M}_{\rm peak}$) for all outbursts (see Figure 1c). Following the spin equilibrium condition mentioned in Section 3.1, we equate the total angular momentum transfer to the star during an outburst to zero [20]:

$$+[C_1 \int \dot{M}^{6/7}\mathrm{d}\dot{M}]_{\rm acc} - [C_1 \int \dot{M}^{6/7}\mathrm{d}\dot{M}]_{\rm prop} = 0, \tag{9}$$

where $C_1 = C/(\mathrm{d}\dot{M}/\mathrm{d}t)$ is independent of $\dot{M}$ for linear outburst rise and decay. For the accretion phase, one needs to integrate from $\dot{M}_{\rm eff}$ to $\dot{M}_{\rm peak}$, where $\dot{M}_{\rm eff}$ is the accretion rate that separates the accretion regime from the propeller regime, and thus corresponds to $r_{\rm m} = r_{\rm co}$, at the spin equilibrium for transient accretion (see Figure 1c). Similarly, for the propeller phase, one needs to integrate from 0 to $\dot{M}_{\rm eff}$, assuming the quiescent phase $\dot{M} = 0$. Thus, one can write:

$$\dot{M}_{\rm peak}^{13/7} - \dot{M}_{\rm eff}^{13/7} = \dot{M}_{\rm eff}^{13/7}. \tag{10}$$

Here, the L.H.S. gives the positive angular momentum transfer in the accretion phase, while the R.H.S. gives the negative angular momentum transfer in the propeller phase. This gives ([20,34]; see also Figure 1c):

$$\frac{\dot{M}_{\rm eff}}{\dot{M}_{\rm peak}} = 2^{-7/13} \approx 0.69. \tag{11}$$

Since $\dot{M}_{\rm peak}$ could be inferred from the observed burst peak intensity, one could estimate $\dot{M}_{\rm eff}$ from Equation (11).

The effective equilibrium spin frequency ($\nu_{\rm eq,eff}$) is the ν at spin equilibrium for transient accretion (Section 3.1). This ν-value corresponds to a $r_{\rm co}$ value (Equation (2)), which equals the $r_{\rm m}$ value corresponding to $\dot{M}_{\rm eff}$ (Figure 1c). Thus, using Equation (3), one obtains $\nu_{\rm eq,eff} \propto \dot{M}_{\rm eff}^{3/7}$. For the same values of other source parameters (ξ, μ, M), if $\nu_{\rm eq,peak}$ is the equilibrium spin frequency corresponding to $\dot{M}_{\rm peak}$, then from Equation (3), one obtains $\nu_{\rm eq,peak} \propto \dot{M}_{\rm peak}^{3/7}$. Therefore [20,34],

$$\frac{\nu_{\rm eq,eff}}{\nu_{\rm eq,peak}} = \left[\frac{\dot{M}_{\rm eff}}{\dot{M}_{\rm peak}}\right]^{3/7} = 2^{-3/13} \approx 0.85. \tag{12}$$

Thus, the equilibrium spin frequency for transient accretion for linear outburst rise and decay could be expressed by the following simple formula [36]:

$$\nu_{\rm eq,eff} = j\nu_{\rm eq,peak} = \frac{j}{2^{11/14}\pi\xi^{3/2}}\left(\frac{G^5 M^5 \dot{M}_{\rm peak}^3}{\mu^6}\right)^{1/7}, \tag{13}$$

where $j \approx 0.85$.

Suppose we consider a more general formula of torques in accretion and propeller regimes, viz.,

$$N = \frac{{\rm d}J}{{\rm d}t} = \pm C\dot{M}^n, \tag{14}$$

which is similar to Equation (8) but $6/7$ in the power of $\dot{M}$ is replaced with n [20]. In this case,

$$\frac{\dot{M}_{\rm eff}}{\dot{M}_{\rm peak}} = 2^{-1/(n+1)}, \tag{15}$$

and $j = 2^{-3/[7(n+1)]}$.

How significant is the effect of transient accretion on the equilibrium spin frequency? This can be seen from the ratio $\nu_{\rm eq,eff}/\nu_{\rm eq,per}$, if all source parameter values (ξ, μ, M, $\dot{M}_{\rm av}$) are the same for both transient and persistent accretion. Note that one can obtain the equilibrium spin frequency $\nu_{\rm eq,per}$ for persistent accretion from Equation (3) assuming $\dot{M} = \dot{M}_{\rm av}$. Thus, one can write [20,36]

$$\frac{\nu_{\rm eq,eff}}{\nu_{\rm eq,per}} = j\left[\frac{\dot{M}_{\rm peak}}{\dot{M}_{\rm av}}\right]^{3/7}. \tag{16}$$

A typical range of $\dot{M}_{\rm peak}/\dot{M}_{\rm av}$ is 10–100 [42], which implies $\nu_{\rm eq,eff}/\nu_{\rm eq,per} \sim 2$–6 using Equation (12). These values have been confirmed by numerical computation. This result shows that transient accretion can spin up neutron stars to frequencies much higher than those by persistent accretion, which is crucial for the spin evolution and equilibrium of MSPs [20].

3.3. *A Way to Find If a Transiently Accreting MSP Has Reached Spin Equilibrium*

Most AMSPs and NMSPs accrete transiently and have a wide range of ν-values. Some of these MSPs may have reached the spin equilibrium, while others have not. Moreover, the ν-values of some of them could have resulted from the disk–magnetosphere interaction alone, while other torques (e.g., $N_{\rm GW}$) might have played a significant role for some other MSPs. Thus, to understand the spin evolution and distribution of accreting MSPs, it can be very useful to know if a given transiently accreting MSP has reached the spin equilibrium by disk–magnetosphere interaction alone, and if not, what this equilibrium spin frequency could be. A recent work suggested the following method to achieve this [34].

Let us consider a transiently accreting MSP. Suppose $\dot{M}_{\rm tran}$ is the accretion rate that corresponds to $r_{\rm m} = r_{\rm co}$ and thus separates the accretion phase from the propeller phase during an outburst (Figure 1a–c). Note that the $\dot{M}_{\rm tran}$ value could be inferred from specific evolutions of the outburst light curve and spectrum, e.g., [43,44]. Moreover, $\dot{M}_{\rm peak}$ could be estimated from the observed outburst peak intensity. If one finds that $\dot{M}_{\rm tran}/\dot{M}_{\rm peak}$ is consistent with 0.69 for linear outburst rise and decay, then the source is likely to be found in the spin equilibrium by disk–magnetosphere interaction (see Equation (11); Figure 1c). However, if $\dot{M}_{\rm tran}/\dot{M}_{\rm peak}$ is significantly less than 0.69, one may conclude that the MSP has not yet reached the spin equilibrium (e.g., Figure 1a).

How can one estimate the $\nu_{\rm eq,eff}$ value of this MSP despite it currently being $\nu < \nu_{\rm eq,eff}$?

Note that the $r_{\rm co}$ value corresponding to current ν equals the $r_{\rm m}$ value corresponding to $\dot{M}_{\rm tran}$. Therefore, using Equation (3), $\nu \propto \dot{M}_{\rm tran}^{3/7}$. This and Equation (11) give

$$\nu_{\rm eq,eff} \approx \nu \left[0.69\left(\frac{\dot{M}_{\rm peak}}{\dot{M}_{\rm tran}}\right)\right]^{3/7}, \tag{17}$$

which is for linear outburst rise and decay, and could be estimated from the known pulsar spin frequency and inferred accretion rates. If the estimated $\nu_{\rm eq,eff}$ value is very high, e.g., above the observed range, or even above the breakup frequency, then that could suggest the presence of an additional spin-down torque, such as the one due to gravitational wave emission. The above-mentioned method has been successfully applied to the source Aql X-1 [34].

3.4. Effective Equilibrium Spin Frequency for Various Outburst Profiles

We have so far considered only triangular outburst profiles. A realistic profile could be more complex, e.g., [38]. For example, while the rise might be near-linear, there could be a somewhat flat top portion and then a relatively slow decay part that is typically close to exponential. Moreover, there could be significant fluctuations, and even multiple peaks and dips. Nevertheless, while the $\nu_{\rm eq,eff}$ value depends on the exact nature of the outburst profile, our conclusions do not change for a complex profile.

To begin with, the spin equilibrium condition would remain the same. Therefore, whatever the profile's shape, one may find an accretion rate ($\dot{M}_{\rm eff}$) above which (accretion phase) the positive angular momentum transfer to the neutron star and below which (propeller phase) the negative angular momentum transfer to the star balance each other. The $\dot{M}_{\rm eff}/\dot{M}_{\rm peak}$ value depends on the outburst shape, but typically, $\nu_{\rm eq,eff}/\nu_{\rm eq,per}$ remains significantly greater than 1.

As an example, let us consider the exponential decay of an outburst ($\dot{M} \propto \exp[-t/T]$, T: constant [20]). For such a profile (compare with Equations (11) and (12)),

$$\frac{\dot{M}_{\rm eff}}{\dot{M}_{\rm peak}} = 2^{-7/6} \approx 0.45, \tag{18}$$

and

$$\frac{\nu_{\rm eq,eff}}{\nu_{\rm eq,peak}} = \left[\frac{\dot{M}_{\rm eff}}{\dot{M}_{\rm peak}}\right]^{3/7} = 2^{-1/2} \approx 0.71. \tag{19}$$

Therefore, for a commonly seen linear rise and exponential decay profile, j (see Equation (13)) should be between ≈0.71 and ≈0.85 [20].

Why is j smaller for an exponential decay than for a linear decay? This is because, for the former, $\dot{M}$ is lower for a larger fraction of the outburst period. Similarly, for a profile with a longer flat top, the source spends a larger fraction of the outburst period with a higher $\dot{M}$ value, and hence, $\dot{M}_{\rm eff}/\dot{M}_{\rm peak}$ and j should be larger (see Figure 1d). For this figure, the spin equilibrium condition gives (compare with Equation (10)):

$$\dot{M}_{\rm peak}^{13/7} - \dot{M}_{\rm eff}^{13/7} + C_2\Delta t = \dot{M}_{\rm eff}^{13/7}, \tag{20}$$

where C_2 is a constant and a function of $\dot{M}_{\rm peak}$, and Δt is the duration of the flat top (see Figure 1d). This implies a larger $\dot{M}_{\rm eff}/\dot{M}_{\rm peak}$ value compared with that from Equation (11), as expected.

4. Final Remarks

In this short review, we discussed the spin equilibrium condition and formulae of the equilibrium spin frequency for transiently accreting MSPs. Note that these should also be applicable to all magnetic stars accreting via a thin, Keplerian accretion disk. We explained that transiently accreting neutron stars can attain much higher spin rates than those of persistently accreting stars. The time scale to reach the spin equilibrium depends on the $\dot{M}_{\rm av}$ value, but since $\nu_{\rm eq,eff} > \nu_{\rm eq,per}$, a transient source would take more time to reach the spin equilibrium than a persistent source for the same parameter values for both cases [20].

Here, we note that the disk–magnetosphere interaction could be more complex, e.g., [45–47], than what is assumed in this review. However, the spin equilibrium condition for a transiently accreting MSP requires zero angular momentum transfer to the MSP during an outburst cycle and $\nu_{\rm eq,eff} > \nu_{\rm eq,per}$ depends on $\dot{M}_{\rm peak} > \dot{M}_{\rm av}$. Both these spin equilibrium conditions and $\dot{M}_{\rm peak} > \dot{M}_{\rm av}$ would be valid for a different disk–magnetosphere interaction, and hence, the conclusions of this review should also be qualitatively valid for a more complex interaction.

We further elaborated on a way to find out if an MSP has already attained the spin equilibrium due to disk–magnetosphere interaction alone and mentioned a method to estimate this equilibrium spin frequency for those sources that have not yet reached spin equilibrium.

Here, we assumed the same peak accretion rate ($\dot{M}_{\rm peak}$) for all outbursts of a given MSP. In reality, $\dot{M}_{\rm peak}$ could have a wide range, and some outbursts may not even enter the accretion phase. However, even then an effective spin equilibrium could be established and an equilibrium spin frequency could be estimated, as shown in a recent paper [34].

Another important aspect not considered here is a slow evolution of the long-term average accretion rate $\dot{M}_{\rm av}$, which depends on the binary evolution. Depending on the $\dot{M}_{\rm av}$ value relative to a critical average accretion rate $\dot{M}_{\rm av,crit}$, the same MSP could accrete persistently in one phase and transiently in another phase [36]. Thus, the star should approach a lower equilibrium frequency $\nu_{\rm eq}$ (Equation (3)) in the former phase, and a much higher equilibrium frequency $\nu_{\rm eq,eff}$ (Equation (13)) in the latter phase. This shows that the spin evolution of MSPs could be complex and the traditional computation of ν-evolution without transient accretion may not be adequate in most cases [36].

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Note

[1] http://www.atnf.csiro.au/research/pulsar/psrcat/ (accessed on 21 September 2023)

References

1. Bhattacharyya, B.; Roy, J. Radio Millisecond Pulsars. In *Astrophysics and Space Science Library Volume 465, Millisecond Pulsars*; Bhattacharyya, S., Papitto, A., Bhattacharya, D., Eds.; Springer: Cham, Switzerland, 2022; pp. 1–32.
2. Di Salvo, T.; Sanna, A. Accretion-powered X-ray Millisecond Pulsars. In *Astrophysics and Space Science Library, Volume 465, Millisecond Pulsars*; Bhattacharyya, S., Papitto, A., Bhattacharya, D., Eds.; Springer: Cham, Switzerland, 2022; pp. 87–124.
3. Torres, D.F.; Li, J. The High-Energy Emission of Millisecond Pulsars. In *Astrophysics and Space Science Library, Volume 465, Millisecond Pulsars*; Bhattacharyya, S., Papitto, A., Bhattacharya, D., Eds.; Springer: Cham, Switzerland, 2022; pp. 33–55.
4. Papitto, A.; de Martino, D. Transitional Millisecond Pulsars. In *Astrophysics and Space Science Library, Volume 465, Millisecond Pulsars*; Bhattacharyya, S., Papitto, A., Bhattacharya, D., Eds.; Springer: Cham, Switzerland, 2022; pp. 157–200.
5. Bhattacharyya, S. Nuclear-Powered X-ray Millisecond Pulsars. In *Astrophysics and Space Science Library, Volume 465, Millisecond Pulsars*; Bhattacharyya, S., Papitto, A., Bhattacharya, D., Eds.; Springer: Cham, Switzerland, 2022; pp. 125–155.
6. Bhattacharya, D.; van den Heuvel, E.P.J. Formation and evolution of binary and millisecond radio pulsars. *Phys. Rep.* **1991**, *203*, 1–124. [CrossRef]
7. Strohmayer, T.E.; Bildsten, L. New Views of Thermonuclear Bursts. In *Compact Stellar X-ray Sources*; Lewin, W.H.G., van der Klis, M., Eds.; Cambridge University Press: Cambridge, UK, 2006; Volume 39, pp. 113–156.

8. Lyne, A.; Graham-Smith, F. *Pulsar Astronomy*, 4th ed.; Cambridge University Press: Cambridge, UK, 2012.
9. Radhakrishnan, V.; Srinivasan, G. On the origin of the recently discovered ultra-rapid pulsar. *Curr. Sci.* **1982**, *51*, 1096–1099.
10. Alpar, M.A.; Cheng, A.F.; Ruderman, M.A.; Shaham, J. A new class of radio pulsars. *Nature* **1982**, *300*, 728–730. [CrossRef]
11. Wijnands, R.; van der Klis, M. A millisecond pulsar in an X-ray binary system. *Nature* **1998**, *394*, 344–346. [CrossRef]
12. Chakrabarty, D.; Morgan, E.H. The two-hour orbit of a binary millisecond X-ray pulsar. *Nature* **1998**, *394*, 346–348. [CrossRef]
13. Archibald, A.M.; Stairs, I.H.; Ransom, S.M.; Kaspi, V.M.; Kondratiev, V.I.; Lorimer, D.R.; McLaughlin, M.A.; Boyles, J.; Hessels, J.W.T.; Lynch, R.; et al. A Radio Pulsar/X-ray Binary Link. *Science* **2009**, *324*, 1411. [CrossRef]
14. Papitto, A.; Ferrigno, C.; Bozzo, E.; Rea, N.; Pavan, L.; Burderi, L.; Burgay, M.; Campana, S.; Di Salvo, T.; Falanga, M.; et al. Swings between rotation and accretion power in a binary millisecond pulsar. *Nature* **2013**, *501*, 517. [CrossRef]
15. de Martino, D.; Belloni, T.; Falanga, M.; Papitto, A.; Motta, S.; Pellizzoni, A.; Evangelista, Y.; Piano, G.; Masetti, N.; Bonnet-Bidaud, J.-M.; et al. X-ray follow-ups of XSS J12270-4859: A low-mass X-ray binary with gamma-ray Fermi-LAT association. *Astron. Astrophys.* **2013**, *550*, A89. [CrossRef]
16. Bassa, C.G.; Patruno, A.; Hessels, J.W.T.; Keane, E.F.; Monard, B.; Mahony, E.K.; Bogdanov, S.; Corbel, S.; Edwards, P.G.; Archibald, A.M.; et al. A state change in the low-mass X-ray binary XSS J12270-4859. *Mon. Not. R. Astron. Soc.* **2014**, *441*, 1825–1830. [CrossRef]
17. Mukherjee, D.; Bult, P.; van der Klis, M.; Bhattacharya, D. The magnetic-field strengths of accreting millisecond pulsars. *Mon. Not. R. Astron. Soc.* **2015**, *452*, 3994–4012. [CrossRef]
18. Konar, S. Magnetic Fields of Neutron Stars. *J. Astrophys. Astr.* **2017**, *38*, 47. [CrossRef]
19. Konar, S.; Choudhuri, A.R. Diamagnetic screening of the magnetic field in accreting neutron stars—II. The effect of polar cap widening. *Mon. Not. R. Astron. Soc.* **2004**, *348*, 661–668. [CrossRef]
20. Bhattacharyya, S.; Chakrabarty, D. The effect of transient accretion on the spin-up of millisecond pulsars. *Astrophys. J.* **2017**, *835*, 4. [CrossRef]
21. Hessels, J.W.; Ransom, S.M.; Stairs, I.H.; Freire, P.C.; Kaspi, V.M.; Camilo, F. Radio Pulsar Spinning at 716 Hz. *Science* **2008**, *311*, 1901–1904. [CrossRef] [PubMed]
22. Chakrabarty, D.; Morgan, E.H.; Muno, M.P.; Galloway, D.K.; Wijn, S.R.; Van Der Klis, M.; Markwardt, C.B. Nuclear-powered millisecond pulsars and the maximum spin frequency of neutron stars. *Nature* **2003**, *424*, 6944. [CrossRef] [PubMed]
23. Patruno, A. The Accreting Millisecond X-ray Pulsar IGR J00291 + 5934: Evidence for a Long Timescale Spin Evolution. *Astrophys. J.* **2010**, *722*, 909–918. [CrossRef]
24. Cook, G.B.; Shapiro, S.L.; Teukolsky, S.A. Recycling Pulsars to Millisecond Periods in General Relativity. *Astrophys. J.* **1994**, *423*, L117. [CrossRef]
25. Bhattacharyya, S.; Bombaci, I.; Logoteta, D.; Thampan, A.V. Fast spinning strange stars: Possible ways to constrain interacting quark matter parameters. *Mon. Not. R. Astron. Soc.* **2016**, *457*, 3101–3114. [CrossRef]
26. Ghosh, P.; Lamb, F.K. Accretion by rotating magnetic neutron stars. III. Accretion torques and period changes in pulsating X-ray sources. *Astrophys. J.* **1979**, *234*, 296–316. [CrossRef]
27. Wang, Y.-M. Location of the Inner Radius of a Magnetically Threaded Accretion Disk. *Astrophys. J. Lett.* **1996**, *465*, L111. [CrossRef]
28. Pringle, J.E.; Rees, M.J. Accretion Disc Models for Compact X-ray Sources. *Astron. Astrophys.* **1972**, *21*, 1 .
29. Lamb, F.K.; Pethick, C.J.; Pines, D. A Model for Compact X-ray Sources: Accretion by Rotating Magnetic Stars. *Astrophys. J.* **1973**, *184*, 271–290. [CrossRef]
30. Illarionov, A.F.; Sunyaev, R.A. Why the Number of Galactic X-ray Stars Is so Small? *Astron. Astrophys.* **1975**, *39*, 185.
31. Ustyugova, G.V.; Koldoba, A.V.; Romanova, M.M.; Lovelace, R.V.E. "Propeller" Regime of Disk Accretion to Rapidly Rotating Stars. *Astrophys. J.* **2006**, *646*, 304–318. [CrossRef]
32. D'Angelo, C.R.; Spruit, H.C. Episodic accretion on to strongly magnetic stars. *Mon. Not. R. Astron. Soc.* **2010**, *406*, 1208–1219. [CrossRef]
33. Stella, L.; Campana, S.; Colpi, M.; Mereghetti, S.; Tavani, M. Do Quiescent Soft X-ray Transients Contain Millisecond Radio Pulsars? *Astrophys. J. Lett.* **1994**, *423*, L47. [CrossRef]
34. Bhattacharyya, S. Application of a New Method to Study the Spin Equilibrium of Aql X-1: The Possibility of Gravitational Radiation. *Astrophys. J.* **2017**, *847*, 2. [CrossRef]
35. Rappaport, S.A.; Fregeau, J.M.; Spruit, H. Accretion onto Fast X-ray Pulsars. *Astrophys. J.* **2004**, *606*, 436–443. [CrossRef]
36. Bhattacharyya, S. Spin evolution of neutron stars in two modes: Implication for millisecond pulsars. *Mon. Not. R. Astron. Soc. Lett.* **2021**, *502*, L45–L49. [CrossRef]
37. Bildsten, L. Gravitational Radiation and Rotation of Accreting Neutron Stars. *Astrophys. J. Lett.* **1998**, *501*, L89–L93. [CrossRef]
38. Yan, Z.; Yu, W. X-ray Outbursts of Low-mass X-ray Binary Transients Observed in the RXTE Era. *Astrophys. J.* **2015**, *805*, 87. [CrossRef]
39. Lasota, J.-P. The disc instability model of dwarf novae and low-mass X-ray binary transients. *New Astron. Rev.* **2001**, *45*, 449–508. [CrossRef]
40. Done, C.; Gierliński, M.; Kubota, A. Modelling the behaviour of accretion flows in X-ray binaries. Everything you always wanted to know about accretion but were afraid to ask. *Astron. Astrophys. Rev.* **2007**, *15*, 1–66. [CrossRef]
41. Geng, J.; Li, B.; Huang, Y. Repeating fast radio bursts from collapses of the crust of a strange star. *Innovation* **2021**, *2*, 100152. [CrossRef] [PubMed]

42. Burderi, L.; Possenti, A.; Colpi, M.; Di Salvo, T.; D'Amico, N. Neutron Stars with Submillisecond Periods: A Population of High-Mass Objects? *Astrophys. J.* **1999**, *519*, 285–290. [CrossRef]
43. Asai, K.; Matsuoka, M.; Mihara, T.; Sugizaki, M.; Serino, M.; Nakahira, S.; Negoro, H.; Ueda, Y.; Yamaoka, K. A Propeller-effect Interpretation of MAXI/GSC Light Curves of 4U 1608-52 and Aql X-1 and Application to XTE J1701-462. *Astrophys. J.* **2013**, *773*, 117. [CrossRef]
44. Campana, S.; Brivio, F.; Degenaar, N.; Mereghetti, S.; Wijn, S.R.; D'Avanzo, P.; Israel, G.L.; Stella, L. The return to quiescence of Aql X-1 following the 2010 outburst. *Mon. Not. R. Astron. Soc.* **2014**, *441*, 1984–1991. [CrossRef]
45. Li, J.; Wickramasinghe, D.T. On spin-up/spin-down torque reversals in disc accreting pulsars. *Mon. Not. R. Astron. Soc.* **1998**, *300*, 1015–1022. [CrossRef]
46. Romanova, M.M.; Ustyugova, G.V.; Koldoba, A.V.; Lovelace, R.V.E. Magnetohydrodynamic Simulations of Disk-Magnetized Star Interactions in the Quiescent Regime: Funnel Flows and Angular Momentum Transport. *Astrophys. J.* **2002**, *578*, 420–438. [CrossRef]
47. Campbell, C.G. Angular momentum transport through magnetic accretion curtains inside disrupted discs. *Mon. Not. R. Astron. Soc.* **2012**, *420*, 1034–1044. [CrossRef]