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On the Periodic Orbits of the Perturbed Two- and Three-Body Problems

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Abstract: In this work, a perturbed system of the restricted three-body problem is derived when the perturbation forces are conservative alongside the corresponding mean motion of two primaries bodies. Thus, we have proved that the first and second types of periodic orbits of the rotating Kepler problem can persist for all perturbed two-body and circular restricted three-body problems when the perturbation forces are conservative or the perturbed motion has its own extended Jacobian integral.

Keywords: restricted 3−body problem; first- (second)-type periodic orbits; conservative forces; first integral of motion

1. Introduction

In *celestial mechanics*, the periodic orbits have a considerable importance due to the existence of a direct relationship among the periodic orbits and the motion of planetary systems, as well as the motion of most stellar systems. Thus, in the literature of celestial mechanics, there are many studies on periodic orbits. In 1897, Henri Poincaré studied periodic orbits, and he considered that the exploration of such orbits is important for the understanding of the dynamics of differential systems. Through his study on the restricted three-body problem, he distinguished three classes of periodic orbits [1]:

- First, those emerging from the circular periodic orbits of the rotating Kepler problem;
- Second, those coming from the elliptical periodic orbits of the rotating Kepler problem;
- Third, those generated from the spatial rotating Kepler problem, where the plane of motion is inclined.

The analytical studies on these three types of periodic orbits are developed in [2,3].

In space science, the importance of periodic orbits is not only restricted to *celestial mechanics* but also to *astrodynamics*. There is a need to find periodic orbits to design different types of space missions. For example, in [4], the presence of periodic orbits of the first type is confirmed using the small-parameter technique when the period of a spacecraft's motion and the undisturbed circular orbit of the primaries are equal. Furthermore, the obtained results can be applied to the design and ballistic analysis of electrically powered thrusting spacecraft. Additionally, polar periodic orbits with $\pi/2$ inclination are employed to observe planet surfaces subjected to space missions [5,6].

Periodic solutions or periodic orbits play outstanding roles not only in physical, mathematical, engineering systems but also in biological biosphere systems, where a paramount problem is to estimate or explore if the automatic oscillatory activity can be continued when it is subjected to a small external effect. Many researchers have studied the perseverance and continuation problems of the periodic orbits. Although these studies are

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carried out from several standpoints all of them obtained the same basic outcome under an assumption regarding the period manifold, known as normal non-degeneracy, where this assumption is expressible as a condition on the unperturbed flow over the manifold [7–13].

Using Delaunay variables, the periodic orbits of the second type were investigated in [14]. The presence of second-type period orbits was developed under the perturbation secular potential for the perturbed three-body problem [15]. The existence of first-type orbits in the framework of the restricted photo-gravitational four-body problem was studied in [16]. Recently, in [17], the authors studied the resonance transition structures using the continuation method. Then, they described how a $1:1$ resonance is progressively overlapped in the proximity of the first-order resonance through the increases in the perturbation. They also proposed a technique to find these orbits as well as the transition resonance with respect to the system of the Sun and Jupiter. Further considerable studies on periodic orbits that would serve as an excellent guide to readers were performed in [18–27]

The innovative part of this study is the construction of the general perturbed equations of an infinitesimal body under many different perturbations, which come from conservative forces, such as a lack of sphericity and the radiation pressure or asteroid belt effect. Thus, we will prove that the circular and elliptic periodic orbits of the planar rotating Kepler problem can be prolonged to the periodic orbits of first and second type for the perturbed two-body problem and the perturbed planar circular restricted three-body problem when the sources of perturbations are conservative forces.

2. Perturbation and Mean Motion

2.1. Perturbed Forces

In dynamical astronomy or in stellar and solar system motion, perturbation is the complex motion of a celestial body that is moving not only under the effect of the main force of gravitation but is also subjected to other forces. These forces can involve the gravitational attraction of another body; two, three, or more bodies instead of the resistance of drag forces; the off-center oblate (or triaxial and misshapen body) attraction; or the effect from asteroid belts and Kirkwood gaps.

The simple motion of a body under a mutual gravitation effect of another body can be described in geometrical terms as conic sections, which are two unperturbed bodies in Kepler motion, but this description originates only from a hypothetical or theoretical point of view. The difference between the theoretical and real motion of celestial bodies considers the effect of perturbation forces or the additional gravitational forces of the surrounding and enclosure bodies.

If there is a third body (or multiple bodies) affecting the two-body motion, then the motion becomes perturbed, and it can be considered a three-body (or N-body) motion. However, the analytical solution with a closed form or a specified mathematical expression to identify the locations and velocities of the body at any time only exists for Kepler motion. Thus, the two-body problem becomes unsolvable if at least one of the two bodies has an irregular shape.

In the motion of most dynamical systems, which include many gravitational attractions (or other perturbed forces), there is one main body, which is predominant in its effect. The other gravitational effects of the remaining bodies (or other perturbed forces) can be treated as a perturbation for the hypothetical unperturbed motion of the planet (or satellite) around its star or planet. In this context, we will construct the perturbed mean motion of a two-body system in the next subsection.

2.2. Perturbed Mean Motion

We assume that the perturbed potential between two masses m_i and m_j can be written in the following form:

$$\Gamma_{ij} = G m_i m_j \left[\frac{1}{\rho_{ij}} + \digamma_{ij}(\rho_{ij}, \varepsilon_{ij}) \right], \tag{1}$$

where ρ_{ij} is the distance between the two centers of the masses m_i and m_j. $\digamma_{ij}(\rho_{ij}, \varepsilon_{ij})$ is the perturbation force such that it tends to zero (infinity) when ρ_{ij} tends to infinity (zero); it can be also written as $\digamma_{ij}(\rho_{ij}, \varepsilon_{ij}) = \varepsilon_{ij} \Upsilon_{ij}(\rho_{ij})$, where $\Upsilon_{ij}(\rho_{ij})$ is differentiable $n-$times with respect to ρ_{ij}. In addition, ε_{ij} is a small parameter, and its value depends on the kind of perturbation force.

Let $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ be the position vectors of masses m_1 and m_2, respectively, in the inertial frame, while $\boldsymbol{\rho}_{12} = \boldsymbol{\rho}_2 - \boldsymbol{\rho}_1$ is the relative position vector of m_2 with respect to m_1. In addition, let $\mathbf{F}_r$ $(r = 1, 2)$ be the force between the masses m_2 and m_1.

$$\mathbf{F}_1 = -\mathbf{F}_2 = \frac{\partial \Gamma_{12}}{\partial \rho_{12}} \hat{\boldsymbol{\rho}}_{12}, \tag{2}$$

where $\hat{\boldsymbol{\rho}}_{12}$ is the unit vector in the direction of $\boldsymbol{\rho}_{12}$ and Γ_{12} is the perturbed potential experienced by the second body given by Equation (1). Thus, the equations of motion of the two bodies are

$$\begin{aligned} m_1 \ddot{\boldsymbol{\rho}}_1 &= \mathbf{F}_1, \\ m_2 \ddot{\boldsymbol{\rho}}_2 &= \mathbf{F}_2. \end{aligned} \tag{3}$$

Utilizing Equations (1)–(3), the vectorial equation of motion of the mass m_2 with respect to the first mass m_1 is

$$\ddot{\boldsymbol{\rho}}_{12} = -\left(\frac{m_1 + m_2}{m_1\, m_2}\right) \nabla \Gamma_{12}. \tag{4}$$

In 2dimensions, the operator ∇ can be written in the polar coordinates (ρ_{12}, φ) as

$$\nabla = \hat{\boldsymbol{\rho}}_{12} \frac{\partial}{\partial \rho_{12}} + \hat{\boldsymbol{\varphi}} \frac{1}{\rho_{12}} \frac{\partial}{\partial \varphi}, \tag{5}$$

where ρ_{12} is the radial distance and $\varphi \in [0, 2\pi]$ is the polar angle. Of course, $\hat{\boldsymbol{\rho}}_{12}$ and $\hat{\boldsymbol{\varphi}}$ are two orthogonal unit vectors such that the first refers to the direction of the second body related to the first one, and $\hat{\boldsymbol{\varphi}}$ indicates the direction of the increasing polar angle φ. Hence, the vector acceleration in polar coordinates is identified by

$$\ddot{\boldsymbol{\rho}}_{12} = \left(\ddot{\rho}_{12} - \rho_{12}\dot{\varphi}^2\right)\hat{\boldsymbol{\rho}}_{12} + \frac{1}{\rho_{12}} \frac{d}{dt}\left(\rho_{12}^2 \dot{\varphi}\right)\hat{\boldsymbol{\varphi}}. \tag{6}$$

Substituting Equations (5) and (6) into (4), we obtain

$$\left(\ddot{\rho}_{12} - \rho_{12}\dot{\varphi}^2\right)\hat{\boldsymbol{\rho}}_{12} + \frac{1}{\rho_{12}} \frac{d}{dt}\left(\rho_{12}^2 \dot{\varphi}\right)\hat{\boldsymbol{\varphi}} = -\frac{(m_1 + m_2)}{m_1\, m_2} \frac{\partial \Gamma_{12}}{\partial \rho_{12}} \hat{\boldsymbol{\rho}}_{12}. \tag{7}$$

From Equation (1), we obtain

$$\frac{\partial \Gamma_{12}}{\partial \rho_{12}} = -G m_1 m_1 \left[\frac{1}{\rho_{12}^2} - \varepsilon_{12} \frac{d\Upsilon_{12}}{d\rho_{ij}}\right]. \tag{8}$$

Now, utilizing Equations (7) and (8), we obtain

$$\left(\ddot{\rho}_{12} - \rho_{12}\dot{\varphi}^2\right)\hat{\boldsymbol{\rho}}_{12} + \frac{1}{\rho_{12}} \frac{d}{dt}\left(\rho_{12}^2 \dot{\varphi}\right)\hat{\boldsymbol{\varphi}} = -G(m_1 + m_1)\left[\frac{1}{\rho_{12}^2} - \varepsilon_{12} \frac{d\Upsilon_{12}}{d\rho_{ij}}\right]\hat{\boldsymbol{\rho}}_{12}. \tag{9}$$

Equation (9) has particular solutions as $\rho_{12} = constant$ and $\dot{\varphi} = \varpi$, where ϖ is the mean motion, which can be written as

$$\varpi^2 = \frac{G(m_1 + m_1)}{\rho_{12}} \left[\frac{1}{\rho_{12}^2} - \varepsilon_{12} \frac{d\Upsilon_{12}}{d\rho_{ij}} \right]. \tag{10}$$

Expression (10) represents the general formula for the perturbed mean motion when the two bodies are moving under the effect of some conservative forces, such as a lack of sphericity, oblateness, and triaxial and radiation pressure effects [28–31].

Kepler's laws can be employed with Newton's laws in the case where the force is oriented toward a central fixed point and proportional to the inverse of distance square. Although the sun is considered the source of the main force in solar system, it is not fixed; rather, it experiences small accelerations due to the motions of the planets. In addition, the planets attract each other, and their motions are affected in accordance with Newton's second and third laws. Thus, the Sun is not only the source of effective force on the motions of the planets. However, the other planets disturb the elliptical motion of the planet that would have happened if the planet was the only body moving around the Sun. Thereby, Kepler's laws are considered as the only approximation for real motion, which motivated us to find the mean motion in the perturbed sense.

3. Restricted Three-Body Motion

3.1. Unperturbed Restricted Three–Body Motion

We assume that m_1 and m_2 are the two primary masses and m is the mass of the test particle, where its positions vectors with respect to the masses m_1 and m_2 are $\boldsymbol{\rho}_1 = (x_1, y_1, z_1)$, $\boldsymbol{\rho}_2 = (x_2, y_2, z_2)$ while its position with respect to the origin of the inertial reference frame $OXYZ$ is $\boldsymbol{\rho} = (x, y, z)$. We also suppose that the test particle does not affect the motion of the primaries and moves under their mutual gravitational.

Now, the motion of the test particle can be written in the following vectorial form:

$$m\ddot{\boldsymbol{\rho}} = \nabla V_I \tag{11}$$

where V_I is the potential function with the following form:

$$V_I(\rho_1, \rho_2) = m\left(\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}\right) \tag{12}$$

The relative distances of the test particle are given by

$$\begin{aligned} \rho_1^2 &= (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \\ \rho_2^2 &= (x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2 \\ \rho^2 &= \xi^2 + y^2 + z^2 \end{aligned} \tag{13}$$

We intend to study the periodic orbits of the perturbed motion for the test particle, which has negligible mass with respect to the primary masses. Thus, we consider a synodic frame $O\xi\eta\zeta$, which is moving around the $\zeta-$axis with an angular velocity ϖ where ζ and Zaxes are congruent.

To normalise the variable and use dimensionless coordinates, it is assumed that the universal constant of gravitation, the distance between the two primaries, the sum of their masses, and the angular velocity on the circular orbit are equal to one; see for details [32]. Thus, the mass of the massive big primary is $m_1 = 1 - \mu$, and the mass of the small primary is $m_2 = \mu$, with $\mu = m_2/(m_1 + m_2) \in (0, 1/2)$. In synodical coordinates, the position of the massive primary is $(\xi_1, \eta_1, \zeta_1) = (-\mu, 0, 0)$ and the position of the small one is $(\xi_2, \eta_2, \zeta_2) = (1 - \mu, 0)$, while the position of the test particle is (ξ, η, ζ). Furthermore, we denote the magnitudes of the vectors $\boldsymbol{\rho}$, $\boldsymbol{\rho}_1$, and $\boldsymbol{\rho}$ by ρ, ρ_2, and ρ_2, respectively. Then, the following vectorial equation of motion of the test particle in the framework of the classical restricted three-body problem in normalized synodic coordinates is given in [32]:

$$\ddot{\boldsymbol{\rho}} + 2\boldsymbol{\varpi} \wedge \dot{\boldsymbol{\rho}} = \nabla V, \tag{14}$$

where

$$V = \frac{1}{2}|\boldsymbol{\varpi} \wedge \boldsymbol{\rho}|^2 + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}. \tag{15}$$

The angular velocity and the position vectors in synodic coordinates are defined as follows:

$$\begin{aligned} \boldsymbol{\varpi} &= [0 \quad 0 \quad \varpi]^T, \\ \boldsymbol{\rho}_1 &= [(\xi+\mu) \quad \eta \quad \zeta]^T, \\ \boldsymbol{\rho}_2 &= [(\xi+\mu-1) \quad \eta \quad \zeta]^T, \\ \boldsymbol{\rho} &= [\xi \quad \eta \quad \zeta]^T, \\ \dot{\boldsymbol{\rho}} &= [\dot{\xi} \quad \dot{\eta} \quad \dot{\zeta}]^T, \end{aligned} \tag{16}$$

where ϖ equals one. In Equation (16), the distances among the infinitesimal body and the primaries are

$$\begin{aligned} \rho_1^2 &= (\xi+\mu)^2 + \quad \eta^2 + \zeta^2, \\ \rho_2^2 &= [(\xi+\mu-1)^2 + \eta^2 + \zeta^2, \\ \rho^2 &= \xi^2 + \eta^2 + \zeta^2. \end{aligned} \tag{17}$$

3.2. Perturbed Restricted Three–Body Motion

In the case that the motion of the infinitesimal body is perturbed by very small forces with respect to the main gravitational forces of the primary bodies, the infinitesimal body will be affected by an additional acceleration in a synodic frame, which is called the perturbing acceleration $\mathcal{A}$. Using Equations (8), (14) and (15), the perturbed motion of the infinitesimal body in synodic frame is given by

$$\ddot{\boldsymbol{\rho}} + 2\boldsymbol{\varpi} \wedge \dot{\boldsymbol{\rho}} = \nabla \Psi, \tag{18}$$

where $\nabla\Psi = \nabla(V + \mathcal{A})$, and $\mathcal{A}$ may be written as $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3$. Here, the magnitudes of the perturbing accelerations are $\mathcal{A}_1 = \varepsilon_1 \Upsilon(\rho_1)$ and $\mathcal{A}_2 = \varepsilon_2 \Upsilon(\rho_2)$ due to the massive and smaller primaries, respectively, while $\mathcal{A}_3 = \varepsilon_3 \Upsilon(\rho)$, may represent some additional perturbed forces, for example coming from asteroids belt. We mean that the first two perturbed forces are regarding to the two primaries, such that they may be are radiating bodies or have non–sphericity in their shapes (i.e. they are irregular bodies, these mean that they are oblate or triaxial bodies). In small word, the sources of all perturbations are conservative forces, which lead to total energy of the perturbed system is always has a constant amount. Hence, Ψ will be given by

$$\Psi = \frac{1}{2}\varpi^2(\xi^2 + \eta^2) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} + \mathcal{A}, \tag{19}$$

where ρ_1 and ρ_2 are given in (17). Using Equation (10), the perturbed mean motion ϖ can be calculated by

$$\varpi^2 = 1 - \varepsilon_{12}\Upsilon_{12}', \tag{20}$$

where $\Upsilon_{12}' = d\Upsilon_{12}/d\rho_{12}$ when $\rho_{12} = 1$.

Utilizing Equations (18) and (19), the three dimensional equations of motion in the Cartesian synodic frame are controlled by

$$\begin{aligned} \ddot{\xi} - 2\varpi\dot{\eta} &= \Psi_{\xi}, \\ \ddot{\eta} + 2\varpi\dot{\xi} &= \Psi_{\eta}, \\ \ddot{\zeta} &= \Psi_{\zeta}, \end{aligned} \tag{21}$$

where $\Psi_{\xi} = \partial\Psi/\partial\xi$, $\Psi_{\eta} = \partial\Psi/\partial\eta$, $\Psi_{\zeta} = \partial\Psi/\partial\zeta$.

Equations (21) represents the dynamical system of the perturbed motion of the infinitesimal body in the framework of the spatial-restricted three-body problem when the perturbing forces are conservative. Thus, Equations (19) and (20) admit a Jacobian integral of the following form:

$$\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 = 2\Psi - \mathcal{C}, \tag{22}$$

where $\mathcal{C}$ is the Jacobian constant, which can be used to identify the real motion as well as the possible regions of motion.

4. Hamiltonian of Perturbed Motion

In *classical mechanics*, the *Hamiltonian function* is a mathematical expression constructed in 1835 by Sir William Rowan Hamilton to characterize the evolution of a physical system that is composed of a set of differential equations. The features of this expression are that it provides significant insight into the properties of dynamical motion even if the physical system has neither an analytical solution nor is solvable. One of the famous examples is three-body systems in planetary and solar systems or stellar systems of motions. These systems have neither analytical nor closed-form solutions to their own systems, where Poincaré investigated that these systems revelation deterministic chaos in motion.

The total energy of a physical system (i.e., the sum of its kinetic energy and its potential energy) is identified by the Hamiltonian function. If this function does not depend on time, then the total energy of the system is conserved. One of the important aspects of this feature is that an infinitesimal phase–space volume is preserved [33]. A direct result to this property is Liouville's theorem, which states that in the Hamiltonian system, the phase–space volume of a closed surface is preserved under time evolution.

In general, Hamiltonian systems are also called canonical systems, and if it has the properties of the autonomous system (we mean that the Hamilton function is not an explicit function of time), it may be indicated as a conservative system because the total energy of system will be a constant along time of motion and the Hamiltonian functions will also be called the first integral of motion. Thus, we will use the Hamiltonian function to describe the perturbed motion of the restricted three-body problem to prove the main result of this study, which is included in Theorem 1.

From [34] and using Equations (14) and (15), the Hamiltonian ($\mathcal{H}_0$) of the spatial unperturbed circular restricted 3−body problem in synodical coordinates is

$$\mathcal{H}_0 = \frac{1}{2}(\mathcal{P}_{\xi}^2 + \mathcal{P}_{\eta}^2 + \mathcal{P}_{\zeta}^2) + \eta\mathcal{P}_{\xi} - \xi\mathcal{P}_{\eta} - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2}, \tag{23}$$

where $\mathcal{H}_0 \equiv \mathcal{H}_0(\xi, \eta, \zeta, \mathcal{P}_{\xi}, \mathcal{P}_{\eta}, \mathcal{P}_{\zeta})$, and $(\mathcal{P}_{\xi}, \mathcal{P}_{\eta}, \mathcal{P}_{\zeta})$ is the conjugate momentum of the infinitesimal body at the position (ξ, η, ζ).

In the case that the infinitesimal body is moving under some conservative perturbing force or its motion is governed by System (21) and admits the Jacobian integral (22), the Hamiltonian of the perturbed motion ($\mathcal{H} \equiv \mathcal{H}(\xi, \eta, \zeta, \mathcal{P}_{\xi}, \mathcal{P}_{\eta}, \mathcal{P}_{\zeta})$) is given by

$$\mathcal{H} = \mathcal{H}_0 + \varepsilon_{12}\Upsilon'_{12}(\eta\mathcal{P}_{\xi} - \xi\mathcal{P}_{\eta}) + \varepsilon_1\Upsilon(\rho_1) + \varepsilon_2\Upsilon(\rho_2). \tag{24}$$

It is clear that the Hamiltonians (23) and (24) are equivalent in the absence of the perturbing forces.

Since the quantities ε_{12}, ε_1, ε_2, and ε_3 are small, they can be written with a factor $\lambda \ll 1$, i.e., $\varepsilon_{12} = \lambda\,\bar{\varepsilon}_{12}$, $\varepsilon_1 = \lambda\,\bar{\varepsilon}_1$, $\varepsilon_2 = \lambda\,\bar{\varepsilon}_2$ and $\varepsilon_3 = \lambda\,\bar{\varepsilon}_3$. In this context, the Hamiltonian (24) becomes

$$\mathcal{H} = \frac{1}{2}(\mathcal{P}_\xi^2 + \mathcal{P}_\eta^2 + \mathcal{P}_\zeta^2) + \eta\mathcal{P}_\xi - \xi\mathcal{P}_\eta - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2} + \mathcal{O}(\lambda). \tag{25}$$

We remark that the unperturbed motion of the infinitesimal body in the framework of the restricted three–body problem can be represented by the Hamiltonian (25) in the case of the whole perturbed forces are ignored. Moreover, if the mass ratio μ is very small, such that $\mu = \lambda\,\bar{\mu}$, we have that the Hamiltonian (25) becomes

$$\mathcal{H} = \frac{1}{2}(\mathcal{P}_\xi^2 + \mathcal{P}_\eta^2 + \mathcal{P}_\zeta^2) + \eta\mathcal{P}_\xi - \xi\mathcal{P}_\eta - \frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}} + \mathcal{O}(\lambda). \tag{26}$$

We must take care with $O(\lambda)$ because it has terms that go to infinity near the primaries; therefore, we must exclude a neighborhood of the primaries. When the infinitesimal body is moving in the primaries' plane, we have that $\zeta = 0$ and $\mathcal{P}_\zeta = 0$; hence, the Hamiltonian (26) becomes

$$\mathcal{H}(\xi,\eta,\mathcal{P}_\xi,\mathcal{P}_\eta) = \frac{1}{2}(\mathcal{P}_\xi^2 + \mathcal{P}_\eta^2) + \eta\mathcal{P}_\xi - \xi\mathcal{P}_\eta - \frac{1}{\sqrt{\xi^2+\eta^2}} + \mathcal{O}(\lambda). \tag{27}$$

Then, equations of motion related to the Hamiltonian (27) will take the following form:

$$\begin{aligned}\ddot{\xi} &= 2n\dot{\eta} + n^2\xi - \frac{\xi}{(\xi^2+\eta^2)^{3/2}} + O(\lambda),\\ \ddot{\eta} &= -2n\dot{\xi} + n^2\eta - \frac{\eta}{(\xi^2+\eta^2)^{3/2}} + O(\lambda).\end{aligned} \tag{28}$$

Note that the Hamiltonian (27) and System (28) for $\lambda = 0$ coincide with the planar Kepler problem in a rotating reference frame. Hence, we will conclude the main result of this study in the following theorem:

Theorem 1. *The circular and elliptic periodic orbits of the planar rotating Kepler problem can be prolonged to the periodic orbits of first and second type for the perturbed two-body problem and the perturbed planar circular restricted three-body problem when the perturbation is made with conservative forces.*

5. Periodic Orbits of Perturbed Motion

In *celestial mechanics*, changes in two-body orbits occur rather slowly with respect to the perturbations, and it is considered a good first approximation for the perturbed motion. The perturbations can be applied if the perturbed forces are very small with respect to the main or fundamental gravitational forces. General perturbation techniques are preferred for some types of perturbed motions as the source of a certain predestined motion is easily found. This is not necessary with respect to particular perturbations where the motion would be predicted with similar reliability. However, there are no data on the configuration of the perturbing bodies.

In the *solar system*, planets are subjected to periodic perturbations from other planets, which pass in their orbit; these perturbations are composed of small impulses each time a planet crosses or passes the orbit of the other planet. This leads to the motion of bodies being periodic or quasi–periodic, such as the Moon in its strongly perturbed orbit. Thus, in the next subsection, we study the existence of first- and second-type periodic orbits for the perturbed problem.

Proof of Theorem 1. It is clear that System (28) represents Kepler motion in the rotating reference frame, also known as the so-called rotation Kepler problem when the parameter-to-mass ratio (μ equals zero. This means that the third body or the infinitesimal body will be moving under the effect of the gravitational field of the massive body only. In this case, the massive body will take its place at the origin of the reference frame.

We also remark that if $\lambda = 0$, the periodic solutions of System (28) are circular or elliptical orbits, namely $(\xi(t), (\eta(t))$ when the motion of the two-body system is bounded. In general, these orbits are called the first (second) type when the unperturbed motion is represented by circular (elliptical) orbits.

We emphasize that System (28) is the same or similar to the system in page 4 Equation (6) of [2]. Therefore in the case of $\lambda = 0$ (or $\lambda \neq 0$, but small, which means that the mass ratio μ is also very small, and $\mu \neq 0$), the obtained results in [2] can be applied to our system of perturbed motion (28). Hence, the results of Theorems 3.1, 3.2 and 5.1 in [2] can be applied to perturbed circular restricted three-body motion. These theorems will be stated and indexed in the present work as Theorems 2–4: □

Theorem 2. *Assume that the restricted three-body problem is perturbed through a very small perturbed force and that the perturbed system has a first integral. Additionally, we assume that $1/(1 - n\jmath^3) \notin \mathbb{Z}$, where $\jmath^3 \neq 1/n$ and $\mathbb{Z}$ is the set of integer numbers. Then, the periodic orbits of first type with angular momentum $\jmath$ can be continued to the perturbed circular restricted three-body problem.*

Theorem 3. *Assume that the restricted three-body problem is perturbed through a small perturbing force and that the perturbed system has a first integral. If $\tau_0 \neq (l + 1/2)\pi/n$ with $l \in \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of positive integer numbers and $\epsilon > 0$ is very small, then the perturbed restricted problem has periodic orbits, which tend to the periodic orbits of the first type during the τ_0- period when ϵ tends to zero.*

Theorem 4. *Assume that I and J are prime integers, $K \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of integer numbers, and $\tau = 2\pi I/|K|$. Then, the elliptical solution with a τ period for the rotating Kepler problem satisfying*

$$g(0) = -\pi\,, \qquad l(0) = \pi\,, \qquad L^3(0) = I/J,$$

and such that it does not cross the location of the smaller primary can be continued to the perturbed circular restricted three-body problem when $\epsilon > 0$ is small and its motion Equations (9) (in Reference [2]) come from Equations (6) (in Reference [2]), satisfying the following invariant symmetry by $(x, y, t) \to (x, -y, -t)$.

Furthermore, the proof of Theorem 1 comes directly from the proofs of Theorems 3.1 and 5.1 in [2], here Theorems 2 and 4. Thereby, the circular (elliptical) periodic orbits of the planar rotating Kepler problem can be prolonged in the periodic orbits of first (second) types for the perturbed two-body problem and for the perturbed restricted three-body problem by conservative perturbing forces. This means that both theorems are satisfied when the perturbed model has a first integral or the perturbed motion has the so-called Jacobian constant.

Now, we show that Theorem 1 can be applied to many models, for example, to the perturbed two–body problem, by different perturbed conservative forces, some of these models are listed in what follows.

1. Perturbed two-body problem where both bodies are oblate [35].
2. An anisotropic perturbed Kepler problem [36].
3. Relativistic effect is a source of perturbation [37].
4. Perturbed two-body problem when the main body has a continuation fraction potential. [38].

We can also list some related works for the perturbed restricted three-body problem, as follows:

1. The oblateness of a massive primary body is a source of perturbed force [28].
2. The oblateness of a smaller primary and the radiation of a bigger primary are sources of perturbed forces [29].
3. A lack of sphericity of the three participants is a source of perturbed forces as well as the primaries' radiations [30].
4. The effect of zonal harmonic coefficients (J_2 and J_4) of the massive body are considered [39].
5. The first primary is oblate, while the second is a source of radiation [40].
6. Both primaries are oblate and radiating bodies, as well as subject to the asteroid belt effect [41].
7. The effects of zonal harmonic coefficients (J_2 and J_4) of both primary bodies are considered [42].
8. The triaxial of a massive primary body and the oblateness of a smaller primary [43].
9. Both primaries are taken as triaxial rigid bodies [31].
10. The asteroid belt effect is considered a source of a perturbation force [44].
11. Quantum corrections are considered a source of perturbation forces [45].

The perturbed models for two and three bodies with their own first integral are not limited to those stated in the previous two lists, which are mentioned as examples. Furthermore, we demonstrate that this study is an extension and generalization of the obtained results in [2,46]. Theorem 1 can be applied in the case that the system is subjected to conservative forces, by which we mean that all forces can be derivable from the total energy of system, which must be a constant. However, the principle of conservation of energy is an expression relating the velocity of the mass and its own position at any time it has a fixed value for all time.

6. Conclusions

In this work, an overview of periodic orbits with their importance is stated in terms of both *celestial mechanics* and *astrodynamics*. The general perturbed mean motion of the two-body problem is derived if the perturbation forces are conservative, such as radiation pressure oblateness and the asteroid belt effects. Thereby, the unperturbed and perturbed motion of the test particle in the framework of the restricted three-body problem is also derived under the conservative perturbed forces. Thus, we have proved that the first and second types of periodic orbits of the rotating Kepler problem can persist for all perturbed two-body and circular restricted three-body problems when the perturbation forces are conservative or the perturbed motion has its own extended Jacobian integral. We emphasize that the terminologies of Poincaré and canonical transformation using Delaunay's variables can be applied to prove the obtained results.

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