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Linear Matrix Inequalities in Fault Detection Filter Design for Linear Ostensible Metzler Systems [†]

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Abstract: The article deals with the properties of fault detection filters when applying their structure to a class of linear, continuous-time systems, with dynamics being specified by the system matrix of the ostensible Metzler structure. The proposed solution is reduced to the use of diagonal stabilization in the synthesis of the state observer and uses the decomposition of the ostensible Metzler matrix. The approach creates a unified framework that covers the compactness of parametric constraints on Metzler matrices and their quadratic stability. Due to the complexity of such constraints, the design conditions are formulated using sharp linear matrix inequalities. For potential application in network control structures, the problem is formulated and solved for linear discrete-time ostensible positive systems. Finally, a linearized model of the B747-100/200 aircraft is used to validate the proposed method. The numerical solution and simulation results show that the proposed approach provides superior sensitivity of the fault detection filter in detecting faults, compared to synthesis methods that do not guarantee the positivity of the filter gain.

Keywords: ostensible Metzler systems; positive systems; parametric constraints; linear matrix inequalities; state estimation; fault residual filters



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1. Introduction

Given the complexity of control design tasks, it is necessary to respect not only the boundaries of system quantities, but also the limitations of system parameters and the balance of conflicting goals. Such a class of systems includes positive systems [1,2], with hard parametric constraints on the elements of the dynamics matrix of Metzler systems [3], where most of the procedures that are valid for linear systems [4–6] cannot be used due to the requirement of their diagonal stabilization [7]. In these applications, significant simplifications can be achieved by exploiting the principle of diagonal parameterization of the strictly Metzler matrix [8], combining the quadratic Lyapunov function with the definition of constraints in the form of linear matrix inequalities (LMI). If the description of the system dynamics does not have a strictly Metzler matrix structure (ostensible Metzler systems), or if the positivity of the input matrix parameters varies depending on the operating environment [9], there is a risk of reduction or loss of the positive properties of the system.

System faults are dangerous for practical systems; because they can drastically change the behavior of the system, it is necessary to design control structures to correct the behavior of the system at least within the range of acceptable degradation. Fault-tolerant control (FTC) schemes provide procedures that improve system tolerance to certain classes of

faults [10,11]. The motivation comes from the fact that the most common model-based fault detection approaches use state observers to construct residual fault signals [12–14]. In standard control structures of multi-agent systems, controller faults can be transmitted to neighboring agents; thus, a single fault in one agent can cause a cascade of subsequent faults. The principle of early detection of small loss of actuator gain characterizes new trends in FTC. It should be noted that the principles presented in [15–17] cannot be used to directly solve the problem of detecting faults in ostensible Metzler systems because they do not support the constraints defined by the ostensible structure of Metzler matrices.

Another interesting class is linear positive switched systems, consisting of a set of linear strictly positive subsystems, where the active subsystem is defined by a switching signal [18]. Their mathematical models for diagnostic purposes are usually given via discrete-time representation and the synthesis is linked to the linear programming technique [19,20], which implies stability conditions using a linear co-positive Lyapunov functional [21,22]. These results can only be marginally used for non-switching positive systems as a special case because they are limited in estimating the constrained and distributed states [23,24]. The same difficulties in FDF design and fault detection in positive complex network structures can be noted in [25–28] and in stochastic linear dynamical systems, such as positive Markov jump systems, whose structures or parameters may change probabilistically at a certain transition [29–31].

This paper deals with the design of a fault detection filter (FDF) for ostensible linear Metzler systems. New conditions for the stability of the FDF based on the system state observer are formulated using a set of LMIs, taking into account the H_∞ norm of the FDF transfer function matrix. Given the necessity of diagonal stabilization of the state observer, the proposed approach uses a suitable form of decomposition of the ostensible Metzler matrix and defines parametric constraints for the one of the separated matrices that is strictly Metzler. The FDF scheme is based on the Luenberger observer, which generates residuals with high sensitivity to small losses in the actuator gain, which is guaranteed by the dynamic properties of the strictly Metzler part of the observer state dynamics. Considering applications in network control structures, the problems are also reformulated and solved in the context of linear discrete-time ostensible positive systems. The parameterized matrices of this form of FDFs are constructed by solving the corresponding set of LMIs; the Lyapunov method forms the basis of stability analyses under the required positivity and synthesis conditions with given parameter bounds. A linearized model of the B747-100/200 aircraft is used to illustrate the proposed procedure and present its properties. Considering stability in the presence of actuator gain loss, the FDF is designed as a quadratic stable observer with ostensible Metzler dynamics. The most likely extension of the presented approach appears to be the area of fault detection of systems, whose mathematical models have the characteristics of structures that temporarily overlap with those of the structures of linear ostensible Metzler systems.

The paper is organized in the following order. Following the properties of linear strictly positive systems presented in Section 2, the parametric bounds and representation approaches for linear, continuous-time ostensible Metzler systems and linear, discrete-time ostensible positive systems are given in Section 3. The application of the H_∞ FDF design principle to these classes of systems is given in Section 4. To illustrate the proposed digital implementation of FDFs, essential results are demonstrated on an example in Section 4 and concluding remarks are given in Section 6.

For convenience, the notations used in this document reflect common convention: x^T , X^T denote the transpose of the vector x , and the matrix X , respectively; $\text{diag}[\cdot]$ denotes the (block) diagonal matrix; for a square symmetric matrix, $X \prec 0$ denotes its negative definiteness; I_n denotes the n -th order identity matrix; X^{-1} and $\rho(X)$ denote the inverse

matrix and the eigenvalue spectrum of a square matrix X ; the maximal positive real part of the eigenvalue spectrum of X is denoted by $\lambda_0 := \max_{\lambda \in \rho(X)} \Re(\lambda)$; $\mathbf{1} \in \mathbb{R}^n$ denotes a vector whose elements are all 1; for a real matrix, $X \geq 0$ means that its components are non-negative; $\mathbb{R}(\mathbb{R}_+)$ denotes the set of (non-negative) real numbers; $\mathbb{R}^{n \times n}(\mathbb{R}_+^{n \times n})$ refers to the set of (non-negative) real matrices; $\mathbb{M}_{-+}^{n \times n}(\mathbb{M}_{-\oplus}^{n \times n})$ denotes the set of strictly (ostensible) Metzler matrices.

2. Consequences of Internal System Positivity

This section deals with internally positive linear, time-invariant systems described by state-space realizations for (A, B, C) and (F, G, C)

$$(\Xi_c): \quad \begin{aligned} \dot{q}(t) &= Aq(t) + Bu(t), \\ y(t) &= Cq(t), \end{aligned} \quad (\Xi_d): \quad \begin{aligned} q(i+1) &= Fq(i) + Gu(i), \\ y(i) &= Cq(i), \end{aligned} \quad (1)$$

where $q(t), q(i) \in \mathbb{R}_+^n$, $y(t), y(i) \in \mathbb{R}_+^m$ are non-negative, $B, G \in \mathbb{R}_+^{n \times r}$, $C \in \mathbb{R}_+^{m \times n}$ are non-negative and with a suitably chosen sampling period Δt

$$F = I_n + A\Delta t, \quad G = B\Delta t, \quad (2)$$

If $A \in \mathbb{M}_{-+}^{n \times n}$ is strictly Metzler, all its diagonal elements are negative and its off-diagonal elements are strictly positive, then $F \in \mathbb{R}_+^{n \times n}$ is strictly positive (all its elements are strictly positive) [32], i.e., their elements have parametric bounds

$$\begin{aligned} a_{ii} &< 0, \quad a_{ij} > 0, \quad i \neq j, \\ f_{ij} &> 0, \end{aligned} \quad \forall i, j \in \langle 1, n \rangle \quad (3)$$

and according to (3) these matrix parameterizations imply:

Lemma 1 (Parameterization of $A \in \mathbb{M}_{-+}^{n \times n}$, $F \in \mathbb{R}_+^{n \times n}$ [33]). For $A = \{a_{ij}\}_{i,j=1}^n \in \mathbb{M}_{-+}^{n \times n}$, $F = \{f_{ij}\}_{i,j=1}^n \in \mathbb{R}_+^{n \times n}$, $h = 0, 1, \dots, n-1$ the following statements hold equivalently

$$\begin{aligned} (i)_A \quad A &= \sum_{h=0}^{n-1} A_\Theta(v, v+h) L^h, & (i)_F \quad F &= \sum_{h=0}^{n-1} F_\Theta(v, v+h) L^h, \\ (ii)_A \quad A &= \sum_{h=0}^{n-1} L^h A_\Theta(v+h, v), & (ii)_F \quad F &= \sum_{h=0}^{n-1} L^h F_\Theta(v+h, v), \end{aligned} \quad (4)$$

for the notations

$$\begin{aligned} (i)_A \quad A_\Theta(v, v+h) &= \text{diag} \left[a_{1,1+h} \cdots a_{n-h,n} \ a_{n-h+1,1} \cdots a_{n,h} \right], \\ (ii)_A \quad A_\Theta(v+h, v) &= \text{diag} \left[a_{1+h,1} \cdots a_{n,n-h} \ a_{1,n-h+1} \cdots a_{h,n} \right], \\ (i)_F \quad F_\Theta(v, v+h) &= \text{diag} \left[f_{1,1+h} \cdots f_{n-h,n} \ f_{n-h+1,1} \cdots f_{n,h} \right], \\ (ii)_F \quad F_\Theta(v+h, v) &= \text{diag} \left[f_{1+h,1} \cdots f_{n,n-h} \ f_{1,n-h+1} \cdots f_{h,n} \right], \end{aligned} \quad L = \begin{bmatrix} 0^T & 1 \\ I_{n-1} & 0 \end{bmatrix}, \quad (5)$$

where $L \in \mathbb{R}^{n \times n}$ is the circulant form of a permutation matrix [34].

The following remark is a slightly generalized characterization of the parametric representation of the boundaries of positive systems.

Remark 1. If the matrix $\Gamma = \{\gamma_{ij}\}_{i,j=1}^n \in \mathbb{R}^{n \times n}$ is represented in the equivalent rhombic forms

$$(i) \Gamma_{\Theta} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & \cdots & \gamma_{1n} \\ & \gamma_{22} & \gamma_{23} & \cdots & \gamma_{2n} & \gamma_{21} \\ & & \gamma_{33} & \cdots & \gamma_{3n} & \gamma_{31} & \gamma_{32} \\ & & & \ddots & \vdots & \vdots & \vdots & \ddots \\ & & & & \gamma_{nn} & \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{n,n-1} \end{bmatrix}, (ii) \Gamma_{\Theta} = \begin{bmatrix} \gamma_{11} \\ \gamma_{21} & \gamma_{22} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \\ \vdots & \vdots & \vdots & \ddots \\ \gamma_{n1} & \gamma_{n2} & \gamma_{n3} & \cdots & \gamma_{nn} \\ & \gamma_{12} & \gamma_{13} & \cdots & \gamma_{1n} \\ & & \gamma_{23} & \cdots & \gamma_{2n} \\ & & & \ddots & \vdots \\ & & & & \gamma_{n-1,n} \end{bmatrix}, \quad (6)$$

then, in the sense of (5), the diagonal matrices are defined by the diagonals (6 (i)), (6 (ii)) such that for $h = 0, 1, \dots, n-1$,

$$\begin{aligned} (i)_{\Gamma} \quad \Gamma_{\Theta}(v, v+h) &= \text{diag}[\gamma_{1,1+h} \cdots \gamma_{n-h,n} \gamma_{n-h+1,1} \cdots \gamma_{n,h}], \\ (ii)_{\Gamma} \quad \Gamma_{\Theta}(v+h, v) &= \text{diag}[\gamma_{1+h,1} \cdots \gamma_{n,n-h} \gamma_{1,n-h+1} \cdots \gamma_{h,n}]. \end{aligned} \quad (7)$$

As a result, the structural constraints (3) of the Metzler matrix $A \in \mathbb{M}_{-+}^{n \times n}$ can be equivalently formalized as

$$(i) \begin{cases} A_{\Theta}(v, v+h) \prec 0, h=0, \\ A_{\Theta}(v, v+h) \succ 0, h=1, \dots, n-1, \end{cases} \quad (ii) \begin{cases} A_{\Theta}(v+h, v) \prec 0, h=0, \\ A_{\Theta}(v+h, v) \succ 0, h=1, \dots, n-1, \end{cases} \quad (8)$$

ad the structural constraints (3) of the strictly positive matrix $F \in \mathbb{R}_{+}^{n \times n}$ can be formalized as

$$(i) F_{\Theta}(v, v+h) \succ 0, h=0, 1, \dots, n-1, \quad (ii) F_{\Theta}(v+h, v) \succ 0, h=0, 1, \dots, n-1. \quad (9)$$

Moreover, for example with $h=0$, the following holds

$$\begin{aligned} L\Gamma_{\Theta}(v, v)L^T &= L\text{diag}[\gamma_{1,1} \cdots \gamma_{n,n}]L^T = \text{diag}[\gamma_{n,n} \gamma_{1,1} \cdots \gamma_{n-1,n-1}], \\ L^2\Gamma_{\Theta}(v, v)L^{2T} &= L^2\text{diag}[\gamma_{1,1} \cdots \gamma_{n,n}]L^{2T} = \text{diag}[\gamma_{n-1,n-1} \gamma_{n,n} \gamma_{1,1} \cdots \gamma_{n-2,n-2}], \end{aligned} \quad (10)$$

and these circular shift operations do not change the existing definiteness of diagonal matrices. Analogously, these procedures can be applied to (8) and (9) using L^h, L^{hT} in constructions with multiple circular shifts.

The defined constructions of positivity constraints reflect the dual formulation of the state control design conditions and the system state estimation conditions, necessary with respect to the parameterization of system dynamics matrices in positive systems. Diagonal formulations follow from the restriction that positive systems are only diagonally stabilized [35].

Theorem 1 ([36]). In limiting the positivity of Luenberger state observers on

$$\begin{aligned} (c) \quad \dot{e}(t) &= (A - JC)e(t) = A_e e(t), \\ (d) \quad e(i+1) &= (F - HC)e(i) = F_e e(i), \end{aligned} \quad (11)$$

with (c) $A \in \mathbb{M}_{-+}^{n \times n}$, $J \in \mathbb{R}_{+}^{n \times m}$, $C \in \mathbb{R}_{+}^{m \times n}$ or (d) $F \in \mathbb{R}_{+}^{n \times n}$, $H \in \mathbb{R}_{+}^{n \times m}$, $C \in \mathbb{R}_{+}^{m \times n}$, diagonal parameterization $A_e \in \mathbb{M}_{-+}^{n \times n}$ is prescribed using the diagonal matrices $A_{\Theta}(v+h, v) \in \mathbb{R}_{+}^{n \times n}$, $C_k \in \mathbb{R}_{+}^{n \times \text{times} n}$, $J_k \in \mathbb{R}_{+}^{n \times n}$, $J_{kh} = L^{hT} J_k L^h \in \mathbb{R}_{+}^{n \times n}$ and diagonal parameterization $F_e \in \mathbb{R}_{+}^{n \times n}$

is prescribed using the diagonal matrices $F_{\Theta}(v+h, v) \in \mathbb{R}_+^{n \times n}$, $C_k \in \mathbb{R}_+^{n \times n}$, $H_k \in \mathbb{R}_+^{n \times n}$, $H_{kh} = L^{hT} H_k L^h \in \mathbb{R}_+^{n \times n}$ constructed so that

$$C = \begin{bmatrix} c_1^T \\ \vdots \\ c_m^T \end{bmatrix}, \quad C_k = \text{diag}[c_k^T] = \text{diag}[c_{k1} \cdots c_{kn}], \quad (12)$$

$$\begin{aligned} (c) \quad J &= [j_1 \cdots j_m], \quad J_k = \text{diag}[j_k] = \text{diag}[j_{k1} \cdots j_{kn}], \\ (d) \quad H &= [h_1 \cdots h_m], \quad H_k = \text{diag}[h_k] = \text{diag}[h_{k1} \cdots h_{kn}], \end{aligned} \quad (13)$$

and the observer system matrices $A_e = A - JC \in \mathbb{M}_{-+}^{n \times n}$, $F_e = F - HC \in \mathbb{R}_+^{n \times n}$ are

$$\begin{aligned} (c) \quad A_e &= \sum_{h=0}^{n-1} L^h \left(A_{\Theta}(v+h, v) - \sum_{k=0}^m J_{kh} C_k \right), \\ (d) \quad F_e &= \sum_{h=0}^{n-1} L^h \left(F_{\Theta}(v+h, v) - \sum_{k=0}^m H_{kh} C_k \right). \end{aligned} \quad (14)$$

The parameterization defined in this way does not guarantee that the strictly Metzler matrix A_e is Hurwitz, or that the strictly positive matrix F_e is Schur. In the Lyapunov sense, these matrices must still have a solution for a positive definite diagonal matrix $Q \in \mathbb{R}_+^{n \times n}$.

Regarding the control design task, the dual parameterization process is obvious, where the duality of positive parameterization yields.

3. Ostensible System Characterization

Let the matrix $A \in \mathbb{M}_{-\ominus}^{n \times n}$ in the system (1) be an ostensible Metzler matrix. Then, since $A \in \mathbb{M}_{-\ominus}^{n \times n}$ is not strictly Metzler, the autonomous part of the system Ξ is not internally positive.

Definition 1. A matrix $A \in \mathbb{M}_{-\ominus}^{n \times n}$ is ostensible Metzler if there is at least one negative off-diagonal element in the structure A while the number of positive off-diagonal elements of A is predominant and all diagonal elements of the matrix A are negative.

Theorem 2 ([37]). If for U is prescribed $V \in \mathbb{R}^{n \times n}$ in the form

$$V = cU + dI_n \quad (15)$$

with $c, d \in \mathbb{R}$, $c \neq 0$, $I_n \in \mathbb{R}^{n \times n}$, then the eigenvectors U and V are identical and

$$\eta_k = c\lambda_k + d \quad (16)$$

where η_k , $k = 1, \dots, n$ are the eigenvalues of V and λ_k , $k = 1, \dots, n$ are the eigenvalues of U .

By exploiting the above property, the compatibility of the parametric bounds of the ostensible Metzler matrix $A \in \mathbb{M}_{-\ominus}^{n \times n}$ can be achieved by using the constructive separation

$$A = A_p + A_m \quad (17)$$

where $A_p \in \mathbb{M}_{-+}^{n \times n}$ is strictly Metzler and $A_m \in \mathbb{R}_-^{n \times n}$ is element-wise negative and Hurwitz. This concept transforms the synthesis task for ostensible Metzler systems into the synthesis task for strictly Metzler systems, parameterizing $A_p \in \mathbb{M}_{-+}^{n \times n}$ by its rhombic diagonals.

Lemma 2 ([38]). For ostensible Metzler $A \in \mathbb{M}_{-\ominus}^{n \times n}$ there exists a strictly Metzler $A_p \in \mathbb{M}_{-+}^{n \times n}$ and an element-wise negative and Hurwitz $A_m \in \mathbb{R}_{-}^{n \times n}$ satisfying (17) with relation to positive scalars $\eta, \delta \in \mathbb{R}_+$ only if

$$A_p = A_d + A^+ + \eta \Sigma + p I_n = A_p^\circ + p I_n, \quad (18)$$

$$A_m = A^- - \eta \Sigma - p I_n = A_m^\circ - p I_n, \quad (19)$$

$$A_d = \text{diag}[-a_{11} - a_{22} \cdots - a_{nn}], \quad A_d + p I_n \prec 0, \quad (20)$$

$$A^+ = \begin{bmatrix} 0 & a_{12}^+ & \cdots & a_{1,n-1}^+ & a_{1n}^+ \\ a_{21}^+ & 0 & \cdots & a_{2,n-1}^+ & a_{2n}^+ \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1}^+ & a_{n-1,2}^+ & \cdots & 0 & a_{n-1,n}^+ \\ a_{n1}^+ & a_{n2}^+ & \cdots & a_{n-1,n-1}^+ & 0 \end{bmatrix}, \quad A^- = \begin{bmatrix} 0 & a_{12}^- & \cdots & a_{1,n-1}^- & a_{1n}^- \\ a_{21}^- & 0 & \cdots & a_{2,n-1}^- & a_{2n}^- \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1}^- & a_{n-1,2}^- & \cdots & 0 & a_{n-1,n}^- \\ a_{n1}^- & a_{n2}^- & \cdots & a_{n-1,n-1}^- & 0 \end{bmatrix}, \quad (21)$$

$$a_{ij}^+ = \begin{cases} a_{ij} & \text{if } a_{ij} > 0, \\ 0 & \text{if } a_{ij} < 0, \end{cases} \quad a_{ij}^- = \begin{cases} a_{ij} & \text{if } a_{ij} < 0, \\ 0 & \text{if } a_{ij} > 0, \end{cases} \quad i \neq j, \quad i, j \in \langle 1, n \rangle, \quad (22)$$

$$\lambda_o = \max_k (\lambda_k^+ | \lambda_k^+ = \text{real}(\lambda_k) > 0, \lambda_k \in \rho(A_m^\circ)), \quad p = \lambda_o + \delta, \quad (23)$$

$$\Sigma = \begin{bmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \quad (24)$$

The parameterization defined in this way does not guarantee that the desired ostensible Metzler matrix $A_e = A - JC \in \mathbb{M}_{-\oplus}^{n \times n}$ is Hurwitz. In the Lyapunov sense, this matrix must still have a solution for a positive definite diagonal matrix $Q \in \mathbb{R}_+^{n \times n}$.

Corollary 1. When working with Ξ_d from (1), it is necessary to define ostensible positivity constraints for $F \in \mathbb{R}^{n \times n}$ and obtain a strictly positive matrix $F_p \in \mathbb{R}_+^{n \times n}$ and a strictly negative and Schur matrix $F_m \in \mathbb{R}_{-}^{n \times n}$. If a solution satisfying Lemma 2 is known for the parameters of a continuous-time system Ξ_c from (1), the simplest way is to use the definite relations for the conversion of the system description and calculate the parameters according to the relations (2) as follows:

$$F_p = I_n + A_p \Delta t, \quad F_m = A_m \Delta t, \quad G = B \Delta t, \quad (25)$$

where Δt is a suitably chosen sampling period.

Subsequently, the structural constraints $F \in \mathbb{R}^{n \times n}$ will be automatically satisfied.

The parameterization defined in this way does not guarantee that the ostensible positive matrix $F_e = F - HC \in \mathbb{R}_+^{n \times n}$ is Schur.

In the Lyapunov sense, this matrix must still have a solution for a positive definite diagonal matrix $P \in \mathbb{R}_+^{n \times n}$.

To complete the above procedures, it is necessary to find J, H such that A_e is ostensible Metzler and Hurwitz and F_e is ostensible positive and Schur.

4. Design of Fault Detection Filters

Let the description of the faulty state of the system be extended

$$(\Xi_{cf}) : \begin{cases} \dot{q}(t) = Aq(t) + Bu(t) + Mf(t), \\ y(t) = Cq(t), \end{cases} \quad (\Xi_{df}) : \begin{cases} q(i+1) = Fq(i) + Gu(i) + Nf(i), \\ y(i) = Cq(i), \end{cases} \quad (26)$$

where $M, N \in \mathbb{R}_+^{n \times p}$, $f(t), f(i) \in \mathbb{R}_+^p$ the cores of FDFs are state observers

$$\begin{aligned} (\Xi_{cef}) : \begin{cases} \dot{q}_e(t) = Aq_e(t) + Bu(t) + J(y(t) - y_e(t)), \\ y_e(t) = Cq_e(t), \end{cases} \\ (\Xi_{def}) : \begin{cases} q_e(i+1) = Fq_e(i) + Gu(i) + H(y(i) - y_e(i)), \\ y_e(i) = Cq_e(i). \end{cases} \end{aligned} \quad (27)$$

Using the estimation errors $e(t) = q(t) - q_e(t)$, $e(i) = q(i) - q_e(i)$, then from (26), (27) follows

$$\begin{aligned} (c) \quad \dot{e}(t) &= (A - JC)e(t) + Mf(t) = A_e e(t) + Mf(t), \\ (d) \quad e(i+1) &= (F - HC)e(i) + Nf(i) = F_e e(i) + Nf(i), \end{aligned} \quad (28)$$

with $A_e \in \mathbb{M}_{+\ominus}^{n \times n}$, $F_e \in \mathbb{R}^{n \times n}$ and respecting principle of constructive separation (17), then

$$(c) \quad \dot{e}(t) = A_{ep} e(t) + A_m e(t) + Mf(t), \quad (d) \quad e(i+1) = F_{ep} e(i) + F_m e(i) + Nf(i), \quad (29)$$

where $A_{ep} \in \mathbb{M}_{+\ominus}^{n \times n}$ must be strictly Metzler and Hurwitz, $A_m \in \mathbb{R}_{-}^{n \times n}$ is element-wise negative and Hurwitz, $F_{ep} \in \mathbb{R}^{n \times n}$ must be strictly positive and Schur and $F_m \in \mathbb{R}_{-}^{n \times n}$ is element-wise negative and Schur.

By defining the form of the residual weighting depending on the observer error innovation, the additive fault detection filters are defined as

$$\begin{aligned} (\Xi_{cr}) : \begin{cases} \dot{e}(t) = (A_p - JC + A_m)e(t) + Mf(t), \\ r(t) = RCe(t), \end{cases} \\ (\Xi_{dr}) : \begin{cases} e(i+1) = (F_p - HC + F_m)e(i) + Nf(i), \\ r(i) = RCe(i), \end{cases} \end{aligned} \quad (30)$$

where the weight matrix $R \in \mathbb{R}_+^{m \times m}$ is a positive definite diagonal matrix.

Using (30), it is possible to derive design conditions for FDF parameters considering only a pair of parameters (A_p, C) or (F_p, C) .

Theorem 3. Ξ_{cr} (30) is stable with ostensible Metzler and Hurwitz matrix $A_{ep} \in \mathbb{M}_{-\ominus}^{n \times n}$ if for a strictly Metzler matrix $A_p \in \mathbb{M}_{+\ominus}^{n \times n}$ and non-negative $C \in \mathbb{R}_+^{m \times n}$, $M \in \mathbb{R}_+^{n \times p}$ there exist positive definite diagonal matrices $Q, S_k \in \mathbb{R}_+^{n \times n}$, $R \in \mathbb{R}_+^{m \times m}$ and a positive scalar $\gamma_\infty \in \mathbb{R}_+$ such that for $h = 1, 2, \dots, n-1$ and $l^T = [1 \dots 1]$ the following set of LMIs is feasible:

$$Q \succ 0, S_k \succ 0, R \succ 0, \gamma_\infty > 0, \quad (31)$$

$$QA_{\ominus p}(v, v) - \sum_{k=1}^m S_k C_{dk} \prec 0, \quad QT^h A_{\ominus p}(v+h, v) T^{hT} - \sum_{k=1}^m S_k T^h C_k T^{hT} \succ 0, \quad (32)$$

$$\begin{bmatrix} QA_p + A_p^T Q - \sum_{k=1}^m S_k l l^T C_k - \sum_{k=1}^m C_k l l^T S_k & * & * \\ M^T Q & -\gamma_\infty I_p & * \\ RC & 0 & -\gamma_\infty I_m \end{bmatrix} \prec 0. \quad (33)$$

When the above conditions hold, the strictly positive observer gain matrix $J \in \mathbb{R}_+^{n \times m}$ is computed as

$$J_k = Q^{-1}S_k, \quad j_k = J_k I, \quad J = [j_1 \cdots j_m]. \quad (34)$$

Hereafter, $*$ is a symmetric entry in a symmetric matrix.

Proof. For a positive definite diagonal matrix $Q \succ 0$, $Q \in \mathbb{R}_+^{n \times n}$ and an upper bound on the H_∞ norm $\gamma_\infty > 0$, $\gamma_\infty \in \mathbb{R}_+$ of the disturbance transfer function matrix the Lyapunov function has the form

$$v(e(t)) = e^T(t)Qe(t) + \gamma_\infty^{-1} \int_0^t (r^T(s)r(s) - \gamma_\infty^2 f^T(s)f(s))ds > 0 \quad (35)$$

and under this assumption, it must hold

$$\dot{v}(e(t)) = \dot{e}^T(t)Qe(t) + e^T(t)Q\dot{e}(t) + \gamma_\infty^{-1}r^T(t)r(t) - \gamma_\infty f^T(t)f(t) < 0. \quad (36)$$

If A_m is not considered when substituting (30) into (36), then

$$\begin{aligned} \dot{v}(e(t)) &= (A_{ep}e(t) + Mf(t))^T Qe(t) + e^T(t)Q(A_{ep}e(t) + Mf(t)) + \\ &+ e^T(t)(\gamma_\infty^{-1}C^T RRC)e(t) - \gamma_\infty f^T(t)f(t) \\ &< 0 \end{aligned} \quad (37)$$

and the redefinition with respect to $e^\circ(t) = [e^T(t) f^T(t)]$ means that (37) prescribes a solution in bilinear form

$$\dot{v}(e^\circ(t)) = e^{\circ T}(t)Te^\circ(t) < 0, \quad (38)$$

$$T = \begin{bmatrix} QA_{ep} + A_{ep}^T Q + \gamma_\infty^{-1}C^T RRC & QM \\ M^T Q & -\gamma_\infty I_p \end{bmatrix} \prec 0, \quad (39)$$

which, using the Schur complement property, can be linearized as

$$\begin{bmatrix} Q(A_p - JC) + (A_p - JC)^T Q & QM & C^T R \\ M^T Q & -\gamma_\infty I_p & 0 \\ RC & 0 & -\gamma_\infty I_m \end{bmatrix} \prec 0. \quad (40)$$

Analogously to (13), the following notations can be set

$$QJ = S, \quad S = [s_1 \cdots s_m], \quad S_k = \text{diag}[s_k], \quad QJ_k = S_k, \quad SC = \sum_{k=1}^m S_k I I^T C_k \quad (41)$$

and using (41) then (40) defines (34).

Multiplying the left-hand side (14c) for $h = 0$ by the matrix Q and applying it to A_p , the following condition arises

$$QA_{\Theta p}(v, v) - \sum_{k=1}^m QJ_k C_k \prec 0 \quad (42)$$

and with (41) then from (42) it follows the first inequality in (32).

Analogously, pre-multiplying the left-hand side by Q and post-multiplying the right-hand side by T^{hT} , then (14c), for A_p and any $h \in \langle 1, n-1 \rangle$ gives

$$\begin{aligned} & QT^h A_{\Theta p}(v+h, v) T^{hT} - \sum_{k=1}^m QT^h J_{kh} T^{hT} T^h C_k T^{hT} \\ &= QT^h A_{\Theta p}(v+h, v) T^{hT} - \sum_{k=1}^m QJ_k T^h C_k T^{hT} \\ &\succ 0 \end{aligned} \quad (43)$$

and (41), (43) give the rest inequalities in (32). This concludes the proof. $\square$

The basic idea is to construct a residual signal that remains at zero (or below a certain threshold) before the occurrence of a fault and reaches a non-zero value higher than a given threshold at the onset of the fault, allowing the fault to be detected.

Theorem 4. Ξ_{dr} (30) is stable with ostensible positive and Schur matrix $F_{ed} \in \mathbb{R}^{n \times n}$ if for a strictly positive matrix $F_p \in \mathbb{R}_+^{n \times n}$ and non-negative $C \in \mathbb{R}_+^{m \times n}$, $N \in \mathbb{R}_+^{n \times p}$ there exist positive definite diagonal matrices $P, Z_k \in \mathbb{R}_+^{n \times n}$, $R \in \mathbb{R}_+^{m \times m}$ and a positive scalar $\xi_\infty \in \mathbb{R}_+$ such that for $h = 0, 1, \dots, n-1$ and $L^T = [1 \cdots 1]$ the following set of LMIs is feasible:

$$P \succ 0, Z_k \succ 0, R \succ 0, \xi_\infty > 0, \quad (44)$$

$$PT^h F_{\Theta p}(v+h, v) T^{hT} - \sum_{k=1}^m Z_k T^h C_k T^{hT} \succ 0, \quad (45)$$

$$\begin{bmatrix} -P & * & * & * \\ 0 & -\xi_\infty I_p & * & * \\ PF_p - \sum_{k=1}^m Z_k L^T C_k & PN & -P & * \\ RC & 0 & 0 & -\xi_\infty I_n \end{bmatrix} \prec 0. \quad (46)$$

When the above conditions hold, the strictly positive observer gain matrix $H \in \mathbb{R}_+^{n \times m}$ is computed as

$$H_k = P^{-1} Z_k, \quad h_k = H_k L, \quad H = [h_1 \cdots h_m]. \quad (47)$$

Proof. For a positive definite diagonal matrix $P \succ 0$, $P \in \mathbb{R}_+^{n \times n}$ and an upper bound on the H_∞ norm $\xi_\infty > 0$, $\xi_\infty \in \mathbb{R}_+$ of the disturbance transfer function matrix, the Lyapunov function has the form

$$v(e(i)) = e^T(i) P e(i) + \xi_\infty^{-1} \sum_{j=0}^i (r^T(j) r(j) - \xi_\infty^2 f^T(j) f(j)) > 0 \quad (48)$$

and under these assumptions the forward difference of Lyapunov function at the time instant i must satisfy

$$\Delta v(e(i)) = e^T(i+1) P e(i+1) - e^T(i) P e(i) + \gamma_\infty^{-1} r^T(i) r(i) - \gamma_\infty f^T(i) f(i) < 0. \quad (49)$$

When substituting (30) into (49) not considering F_m then

$$\begin{aligned}\Delta v(e(i)) &= (F_{ep}e(i) + Nf(i))^T P(F_{ep}e(i) + Nf(i)) - e^T(i)Pe(i) + \\ &+ e^T(i)(\xi_\infty^{-1}C^T RRC)e(i) - \xi_\infty f^T(i)f(i) \\ &= e^T(i)(F_{ep}^T P F_{ep} + \xi_\infty^{-1}C^T RRC - P)e(i) + \\ &+ e^T(i)F_{ep}^T P N f(i) + f^T(i)N^T P F_{ep}e(i) - \xi_\infty f^T(i)f(i) \\ &< 0\end{aligned}\quad (50)$$

and the redefinition with respect to $e^\circ(i) = [e^T(i) f^T(i)]^T$ means that (50) prescribes a solution

$$\Delta v(e^\circ(i)) = e^{\circ T}(i) T_d e^\circ(i) < 0, \quad (51)$$

$$T_d = \begin{bmatrix} -P + \xi_\infty^{-1}C^T RRC + F_{ep}^T P F_{ep} + F_{ep}^T P N & \\ N^T P F_{ep} & -\xi_\infty I_p \end{bmatrix} \prec 0. \quad (52)$$

Consequently,

$$T_d = \begin{bmatrix} -P & 0 \\ 0 & -\xi_\infty I_p \end{bmatrix} + \begin{bmatrix} F_{ep}^T P \\ N^T P \end{bmatrix} P^{-1} [P F_{ep} P N] + \begin{bmatrix} C^T R \\ 0 \end{bmatrix} \xi_\infty^{-1} I_n [R C \ 0] \prec 0, \quad (53)$$

which, using the Schur complement property, can be written as

$$T_d = \begin{bmatrix} -P & 0 & (F_p - HC)^T P & C^T R \\ 0 & -\xi_\infty I_p & N^T P & 0 \\ P(F_p - HC) & P N & -P & 0 \\ R C & 0 & 0 & -\xi_\infty I_n \end{bmatrix} \prec 0. \quad (54)$$

Analogously to (41), the following notations can be set

$$P H = Z, \quad Z = [z_1 \cdots z_m], \quad Z_k = \text{diag}[z_k], \quad P H_k = Z_k, \quad Z C = \sum_{k=1}^m Z_k l^T C_k \quad (55)$$

and using (55), (54) defines (46).

Analogously to (43), pre-multiplying the left-hand side by P and post-multiplying the right-hand side by T^{hT} then (14d) for F_p and any $h \in \langle 0, n-1 \rangle$ gives

$$P T^h P_{\Theta p}(v + h, v) T^{hT} - \sum_{k=1}^m P Z_k T^h C_k T^{hT} \succ 0 \quad (56)$$

and (55), (56) give the inequalities in (45).

Based on the potential tuning optimization of the parameter ξ_∞ , this also allows for a relatively simple way of determining the detection threshold. This concludes the proof. $\square$

Corollary 2. For the design of a continuous-time Luenberger observer with ostensible Metzler and Hurwitz matrix $A_{ec} \in \mathbb{M}_{-\oplus}^{n \times n}$ in a disturbance-free environment ($M = 0, R = I_m$) or the design of a discrete-time Luenberger observer with ostensible positive and Schur matrix $F_{ed} \in \mathbb{R}^{n \times n}$ from Theorems 3 and 4, it follows:

$$\begin{aligned}
& Q \succ 0, S_k \succ 0, \\
& QA_{\Theta p}(v, v) - \sum_{k=1}^m S_k C_{dk} \prec 0, \\
(c): \quad & QT^h A_{\Theta p}(v + h, v) T^{hT} - \sum_{k=1}^m S_k T^h C_{dk} T^{hT} \succ 0, \quad h \in \langle 1, n-1 \rangle, \\
& QA_p + A_p^T Q - \sum_{k=1}^m (S_k l l^T C_{dk} + C_{dk} l l^T S_k) \prec 0,
\end{aligned} \tag{57}$$

$$\begin{aligned}
& P \succ 0, Z_k \succ 0, \\
& PT^h F_{\Theta p}(v + h, v) T^{hT} - \sum_{k=1}^m Z_k T^h C_{dk} T^{hT} \succ 0, \quad h \in \langle 0, n-1 \rangle, \\
(d): \quad & \begin{bmatrix} -P & * \\ PF_p - \sum_{k=1}^m Z_k l l^T C_k & -P \end{bmatrix} \prec 0.
\end{aligned} \tag{58}$$

where strictly positive gain matrices J, H are calculated as defined.

Remark 2. In the presented paradigm of continuous-time residual filter design, matrix inequalities (32) express parametric constraints imposed in the design on a strictly Metzler matrix $A_{pe} \in \mathbb{M}_{+}^{n \times n}$, (33) prescribes the asymptotic stability of the observer, and diagonal matrix variables in (31) reflect the principle of diagonal stabilization.

Analogously, in the design of a discrete-time residual filter, the matrix inequalities (45) express the parametric constraints given in the design for a strictly positive matrix $F_{pe} \in \mathbb{R}_{+}^{n \times n}$, (46) prescribes the asymptotic stability of the observer, and the diagonal matrix variables in (44) reflect the principle of diagonal stabilization.

In both cases, the diagonal weight matrix $R \in \mathbb{R}_{+}^{m \times m}$ is optimized with respect to the H_{∞} norm of the input disturbance transfer function matrix.

The threshold computation for additive fault detection and isolation can be solved as an optimization problem by [39].

5. Illustrative Example

To illustrate the effectiveness of the approach, an apparent Metzler linear system (1) is used for the stable linearized B747-100/200 aircraft model [40] with nominal and derived model matrices given by

$$\begin{aligned}
A &= \begin{bmatrix} -2.98 & 0.930 & 0 & -0.0340 \\ -0.99 & -0.210 & 0.035 & -0.0011 \\ 0 & 0 & -2 & 1 \\ 0.39 & -5.555 & 0 & -1.8900 \end{bmatrix}, \quad \rho(A) = \begin{bmatrix} -0.9181 \\ -1.2009 \\ -2.3219 \\ -2.6390 \end{bmatrix}, \quad A_d = \text{diag} \begin{bmatrix} -2.98 \\ -0.21 \\ -2.00 \\ -1.89 \end{bmatrix}^T, \\
C &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = M = \begin{bmatrix} -0.032 & 0.5 & 1.55 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1.600 & 1.8 & -2.00 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \\
A^- &= \begin{bmatrix} 0 & 0 & 0 & -0.0340 \\ -0.99 & 0 & 0 & -0.0011 \\ 0 & 0 & 0 & 0 \\ 0 & -5.555 & 0 & 0 \end{bmatrix}, \quad A^+ = \begin{bmatrix} 0 & 0.93 & 0 & 0 \\ 0 & 0 & 0.035 & 0 \\ 0 & 0 & 0 & 1 \\ 0.39 & 0 & 0 & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.
\end{aligned}$$

Choosing $\eta = 0.005$

$$A_m^\circ = \begin{bmatrix} 0 & -0.005 & -0.005 & -0.0390 \\ -0.995 & 0 & -0.005 & -0.0061 \\ -0.005 & -0.005 & 0 & -0.0050 \\ -0.005 & -5.560 & -0.005 & 0 \end{bmatrix}, \quad \rho(A_m^\circ) = \begin{Bmatrix} -0.6219 \\ 0.0007 \\ 0.3106 \pm 0.5007i \end{Bmatrix}$$

and for the resulting $\lambda_0 = 0.3106$ with the prescribed D -parameter of the D -stability region $\delta = 0.003$ the value of the tuning parameter will be $p = \lambda_0 + \delta = 0.3136$, which implies the strictly negative and Hurwitz matrix $A_m = A_m^\circ - pI_n$,

$$A_m = \begin{bmatrix} -0.3136 & -0.0050 & -0.0050 & -0.0390 \\ -0.9950 & -0.3136 & -0.0050 & -0.0061 \\ -0.0050 & -0.0050 & -0.3136 & -0.0050 \\ -0.0050 & -5.5600 & -0.0050 & -0.3136 \end{bmatrix}, \quad \rho(A_m) = \begin{Bmatrix} -0.3129 \\ -0.9355 \\ -0.0030 \pm 0.5007i \end{Bmatrix}$$

and the unstable diagonally corrupted Metzler matrix $A_p = A_d + A^+ + \eta\Sigma + pI_n$,

$$A_p = \begin{bmatrix} -2.6664 & 0.9350 & 0.0050 & 0.0050 \\ 0.0050 & 0.1036 & 0.0400 & 0.0050 \\ 0.0050 & 0.0050 & -1.6864 & 1.0050 \\ 0.3950 & 0.0050 & 0.0050 & -1.5764 \end{bmatrix}, \quad \rho(A_p) = \begin{Bmatrix} 0.1077 \\ -1.5898 \\ -1.6711 \\ -2.6724 \end{Bmatrix},$$

which solves the required separation of the matrix A .

Since parameterization is necessary to address the consistency of design constraints, manipulation according to (5), (12) yields:

$$\begin{aligned} A_{\Theta p}(v, v) &= \text{diag}[-2.6664 \ 0.1036 \ -1.6864 \ -1.5764], \\ A_{\Theta p}(v+1, v) &= \text{diag}[0.005 \ 0.005 \ 0.005 \ 0.050], \\ A_{\Theta p}(v+2, v) &= \text{diag}[0.005 \ 0.005 \ 0.005 \ 0.050], \\ A_{\Theta p}(v+3, v) &= \text{diag}[0.395 \ 0.935 \ 0.040 \ 1.005], \end{aligned}$$

$$C_1 = \text{diag}[1 \ 0 \ 0 \ 0], \quad C_2 = \text{diag}[0 \ 1 \ 0 \ 0], \quad C_3 = \text{diag}[0 \ 0 \ 0 \ 1].$$

and solving the set of inequalities within (31)–(33) gives

$$Q = \text{diag}[0.5070 \ 3.0098 \ 1.1782 \ 0.2600], \quad R = \text{diag}[0.5603 \ 0.5499 \ 0.4886],$$

$$S_1 = \text{diag}[0.4865 \ 0.0075 \ 0.0029 \ 0.0495], \quad \gamma_\infty = 1.5789,$$

$$S_2 = \text{diag}[0.2447 \ 0.9688 \ 0.0029 \ 0.0006], \quad S_3 = \text{diag}[0.0013 \ 0.0075 \ 0.6174 \ 1.0700].$$

The corresponding strictly positive observer gain is determined as

$$J = \begin{bmatrix} 0.9595 & 0.4826 & 0.0025 \\ 0.0025 & 0.3219 & 0.0025 \\ 0.0025 & 0.0025 & 0.5240 \\ 0.1905 & 0.0025 & 4.1151 \end{bmatrix}$$

and the desired fault detection filter is obtained with the parameterized matrices as follows:

$$A_{pe} = \begin{bmatrix} -3.6259 & 0.4524 & 0.0050 & 0.0025 \\ 0.0025 & -0.2183 & 0.0400 & 0.0025 \\ 0.0025 & 0.0025 & -1.6864 & 0.4810 \\ 0.2045 & 0.0025 & 0.0050 & -5.6915 \end{bmatrix}, \rho(A_{pe}) = \begin{bmatrix} -0.2178 \\ -1.6860 \\ -3.6260 \\ -5.6923 \end{bmatrix},$$

$$A_e = \begin{bmatrix} -3.9395 & 0.4474 & 0 & -0.0365 \\ -0.9925 & -0.5319 & 0.0350 & -0.0036 \\ -0.0025 & -0.0025 & -2 & 0.4760 \\ 0.1995 & -5.5575 & 0 & -6.0051 \end{bmatrix}, \rho(A_e) = \begin{bmatrix} -0.6896 \\ -1.9809 \\ -3.7797 \\ -6.0262 \end{bmatrix},$$

prescribing a strictly Metzler and Hurwitz matrix A_{pe} and an ostensible Metzler and Hurwitz matrix A_{pe} , while the residual gain is optimized as

$$R = \begin{bmatrix} 0.5603 & 0.5499 & 0.4886 \end{bmatrix}.$$

Note that the positions of the negative off-diagonal elements in (A, A_e) are preserved and the elements of the third columns in (A, A_e) remain the same due to the specified structure of the matrix C .

All the results are obtained by SeDuMi package [41], implemented in Matlab R2018a.

As expected, a stable strictly Metzler part of the observer is designed which produces asymptotical observer state responses and aperiodically reacted actuator fault residuals.

For the simulation phase, the control law rule for $u(t)$ is as

$$u(t) = -Kq(t) + Ww$$

with gains K [15], W and desired output w of the form

$$K = \begin{bmatrix} -2.2853 & 18.1622 & 0.1677 & -0.6563 \\ 6.4656 & -51.6318 & -0.9831 & 1.3232 \\ 10.4894 & -84.2087 & -2.5128 & 1.1836 \end{bmatrix}, W = \text{diag}[-45.0 \ -9.5 \ -2.0], w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

and setting the initial conditions $q(0) = q_e(0) = 0$.

The simulations results are presented in Figures 1 and 2, where Figure 2 shows the evolution of the residual signals generated by the residual disturbance filter in response to a step loss of gain of the second actuator $f^T(t) = [0 \ 0.0005 \ 0]$, starting its operation at time $t = 6$ s. It is clear from Figure 1 that such a small change in gain is practically not reflected in the time expressions of the state variables and their estimated values. The cases of loss of gain of other actuators are manifested analogously.

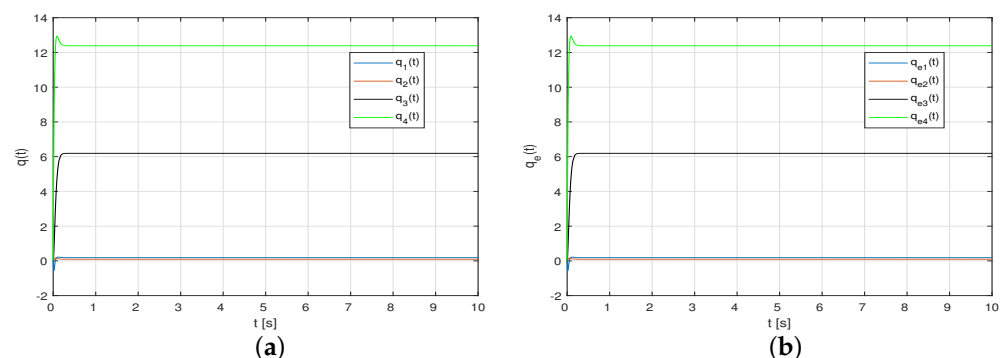


Figure 1. System state (a) and estimation of system state (b) in fault mode starting at time $t = 6$ s.

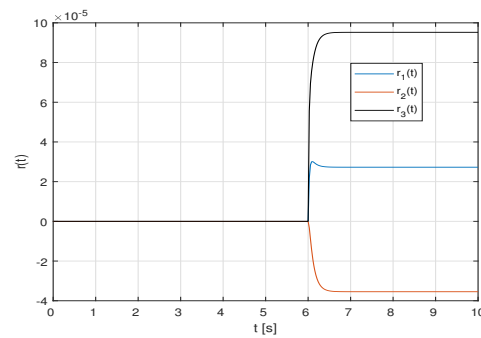


Figure 2. Fault residual filter response in fault mode starting at time $t = 6$ s.

In the discrete-time model, the matrix parameters are rewritten into expressions corresponding to (25), with the guarantee of preserving all parametric constraints, such that, with $\Delta t = 0.05$ s

$$F_m = \begin{bmatrix} -0.0157 & -0.0003 & -0.0003 & -0.0020 \\ -0.0498 & -0.0157 & -0.0003 & -0.0003 \\ -0.0003 & -0.0003 & -0.0157 & -0.0003 \\ -0.0003 & -0.2780 & -0.0003 & -0.0157 \end{bmatrix}, \quad \rho(F_m) = \begin{Bmatrix} -0.0156 \\ -0.0468 \\ -0.0001 \pm 0.0250i \end{Bmatrix},$$

$$F_p = \begin{bmatrix} 0.8667 & 0.0468 & 0.0003 & 0.0003 \\ 0.0003 & 1.0052 & 0.0020 & 0.0003 \\ 0.0003 & 0.0003 & 0.9157 & 0.0502 \\ 0.0198 & 0.0003 & 0.0003 & 0.9212 \end{bmatrix}, \quad \rho(F_p) = \begin{Bmatrix} 0.8664 \\ 0.9164 \\ 0.9205 \\ 1.0054 \end{Bmatrix},$$

$$N = \begin{bmatrix} -0.0016 & 0.0250 & 0.0775 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.0800 & 0.0900 & -0.100 \end{bmatrix}.$$

Analogously, by constructing the required diagonal parameterization F_p and solving LMIs according to Theorem 4

$$P = \text{diag}[12.3995 \ 2.2973 \ 1.3104 \ 2.3433], \quad R = \text{diag}[0.5577 \ 0.5400 \ 0.5606],$$

$$Z_1 = \text{diag}[1.2430 \ 0.0003 \ 0.0002 \ 0.0229], \quad \xi_\infty = 1.4856,$$

$$Z_2 = \text{diag}[0.0554 \ 1.4240 \ 0.0002 \ 0.0003], \quad Z_3 = \text{diag}[0.0003 \ 0.0003 \ 0.0326 \ 1.3259]$$

and with the strictly positive observer gain, it is

$$H = \begin{bmatrix} 0.5180 & 0.0231 & 0.0001 \\ 0.0001 & 0.6199 & 0.0001 \\ 0.0001 & 0.0001 & 0.0249 \\ 0.0098 & 0.0001 & 0.5658 \end{bmatrix}.$$

$$F_{pe} = \begin{bmatrix} 0.3487 & 0.0237 & 0.0003 & 0.0001 \\ 0.0001 & 0.3853 & 0.0020 & 0.0001 \\ 0.0001 & 0.0001 & 0.9157 & 0.0254 \\ 0.0100 & 0.0001 & 0.0003 & 0.3553 \end{bmatrix}, \quad \rho(F_{pe}) = \begin{Bmatrix} 0.3485 \\ 0.3555 \\ 0.3854 \\ 0.9157 \end{Bmatrix},$$

$$F_e = \begin{bmatrix} 0.3330 & 0.0234 & 0 & -0.0018 \\ -0.0496 & 0.3696 & 0.0018 & -0.0002 \\ -0.0001 & -0.0001 & 0.9000 & 0.0251 \\ 0.0097 & -0.2779 & 0 & 0.3397 \end{bmatrix}, \quad \rho(F_e) = \begin{Bmatrix} 0.3238 \\ 0.9000 \\ 0.3593 \pm 0.0338i \end{Bmatrix},$$

when constructing for

$$F = \begin{bmatrix} 0.8510 & 0.0465 & 0 & -0.0017 \\ -0.0495 & 0.9895 & 0.0018 & -0.0001 \\ 0 & 0 & 0.9000 & 0.0500 \\ 0.0195 & -0.2777 & 0 & 0.9055 \end{bmatrix}, \quad \rho(F) = \begin{Bmatrix} 0.9541 \\ 0.9400 \\ 0.8681 \\ 0.8839 \end{Bmatrix}.$$

Simulations with the discrete-time model structure show no significant change in terms of the residual signals presented in the figures of this section and confirm the effectiveness of this scheme as well.

The properties of the state estimator remain valid even in the case of the lossless actuator gain regime, which is characterized by zero values of the FDF outputs.

6. Conclusions

The article generalizes a design framework for constructing FDFs for linear, continuous-time ostensible Metzler systems and for linear, discrete-time ostensible positive systems. The LMIs-based design, which takes into account the parametric constraints and quadratic stability of the state-based observer FDF, exploits the ostensible matrix separation of the system dynamics to apply the diagonal stabilization principle for strictly Metzler systems. Such a philosophy means constructing a set of LMIs based on diagonal matrix variables only for a separated strictly Metzler dynamics matrix such that their fulfillment guarantees a separated state observer dynamics in the form of a strictly Metzler and Hurwitz matrix, while the remaining part of the observer dynamics is determined by a strictly negative and Hurwitz matrix. For potential use in the diagnosis of ostensible positive systems in network structures and distributed systems, the observer design problem for fault detection is defined and solved for linear ostensible positive systems with discrete-time state models. For the FDF design problem for these classes of models, conditions for the existence of such an FDF are established, where the required FDF parameter matrices can be easily determined from the solution of the matrix variables of the set of LMIs.

Using the example of a linearized model of the B747-100/200 aircraft, it is shown that the FDF with such a defined parameter design is able to detect relatively small gain losses of the actuators of the linear ostensible Metzler system, while all the aforementioned design constraints are mutually conditional.

Special cases of these problems may arise in various contexts related to actuator failure detection in the class of distributed multi-agent systems based on ostensible Metzler agents or in actuator failure detection in unmanned aerial vehicles (UAVs) because they are modeled by temporally overlapping forms with ostensible Metzler structures. All of these problems are currently being addressed by the authors.

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Abbreviations

The following abbreviations are used in this manuscript:

LMI	linear matrix inequality
MIMO	multiple-input multiple-output
FDF	fault detection filter
SeDuMi	self dual minimization

Notations

The following basic notations are used in this manuscript:

$q(t), q(i), u(t), u(i), y(t), y(i), f(t), f(i)$	state, input, output and fault variable vectors
$A, B, C, M, A_p, A_m, F, G, N, F_p, F_m$	system matrix parameters
$A_e, A_{ep}, F_e, F_{ep}, J, H$	observer's system matrices, observer gains
$Q, S_k, R, P, Z_k, \gamma_\infty, \xi_\infty$	LMI variables

All other notations are defined in the given context fluently.

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