

Article

Singularly Perturbed Cauchy Problem for a Parabolic Equation with a Rational “Simple” Turning Point

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Abstract: The aim of the research is to develop the regularization method. By Lomov’s regularization method, we constructed a uniform asymptotic solution of the singularly perturbed Cauchy problem for a parabolic equation in the case of violation of stability conditions of the limit-operator spectrum. The problem with a “simple” turning point is considered in the case, when the eigenvalue vanishes at $t = 0$ and has the form $t^{m/n}a(t)$. The asymptotic convergence of the regularized series is proved.

Keywords: singularly perturbed Cauchy problem; parabolic equation; asymptotic solution; rational “simple” turning point

1. Introduction

Singularly perturbed problems with an unstable spectrum of the limit operator for ordinary partial differential equations and partial differential equations have been studied by many authors [1–8]. The most difficult of them are problems with point instability, namely turning points.

For the first time, these problems arose, in particular, in quantum mechanics. One of the first methods of solution was the WKB method, a method of semiclassical calculation, which was developed in 1926 by G. Wentzel, H.A. Kramers, and L. Brillouin. At the same time, H. Jeffreys generalized the method of approximate solution of linear differential equations of the second order, including the solution of the Schrodinger equation. Methods for solving problems with spectral features were and are being developed by: the school of V.P. Maslov, the school of A.B. Vasilieva – V.F. Butuzov – N.N. Nefedov, and the school of S.A. Lomov, among others.

The regularization method defines three groups of turning points.

1. “Simple” turning point: The eigenvalues of the limit operator are isolated from each other, and one eigenvalue at separate points of t vanishes.

2. “Weak” turning point: At least one pair of eigenvalues intersect at separate points of t , but the limit operator preserves the diagonal structure up to the intersection points. The eigenvector basis remains smooth in t .

3. “Strong” turning point: At least one pair of eigenvalues intersect at separate points of t , but the limit operator changes the diagonal structure to Jordan at the intersection points. The basis at the intersection points loses its smoothness in t .

Classic turning points are of the third type. A feature of the problem with a “simple” turning point presented in the article is the pointwise irreversibility of the limit operator $t^{m/n}$.

In the present paper, using the Lomov’s regularization method [1], a regularized asymptotic solution of the singularly perturbed Cauchy problem on the entire segment $[0, T]$ for a parabolic equation is constructed in the presence of a “simple” rational turning point of the limit operator.

The point $\varepsilon = 0$ for the singularly perturbed Cauchy problem is singular in the sense that the classical existence theorems for the solution of the Cauchy problem do not hold at this point. Therefore, in the solution of singularly perturbed problems, essentially special singularities arise that describe the irregular dependence of the solution on ε [9].

One of the advantages of the regularization method is that it makes it possible to construct a global asymptotic solution over the entire domain of integration, and, under certain conditions on the coefficients of the equation, it gives an exact solution.

The fractional turning point of order $1/2$ for an ordinary differential equation of the first order was considered by the method of boundary functions by [6].

In this article, we continue on the “simple” rational turning point work carried out in [10,11].

2. Formulation of the Problem and Construction of an Asymptotic Solution

Consider the Cauchy problem

$$\begin{cases} \varepsilon \frac{\partial u}{\partial t} = \varepsilon^2 \left(\frac{\partial}{\partial x} \left(k(x) \frac{\partial u}{\partial x} \right) \right) - t^{m/n} a(t) u + h(x, t), \\ u(x, 0) = f(x), \quad -\infty < x < +\infty, \end{cases} \quad (1)$$

where $u = u(x, t, \varepsilon)$ is a function depending on variables x and t and real parameters ε , $\varepsilon > 0$ and $\varepsilon < 1$.

Let the following conditions be satisfied:

- (1) $h(x, t) \in C^\infty(R \times [0, T])$, the function $h(x, t)$ and all its derivatives are bounded on $\{R \times [0, T]\}$;
- (2) $k(x) \in C^\infty(R)$, $k(x) \geq k_0 > 0$;
- (3) $f(x) \in C^\infty(R)$, the function $f(x)$ and all its derivatives are bounded to R ;
- (4) $a(t) \in C^\infty([0, T])$, $a(t) > 0$; and
- (5) $m, n \in N$, $p = m + n - 1$, $m/n = r$ is a fractional number.

These conditions ensure the existence and uniqueness of the bounded solution and the possibility of constructing the asymptotic series of the problem (1) [12].

Singularly perturbed problems arise when the domain of definition of the initial operator depending on ε at $\varepsilon \neq 0$ does not coincide with the domain of definition of the limit operator at $\varepsilon = 0$.

Under the stability condition for the spectrum of the limiting operator, essentially singular singularities are described using exponentials of the form $e^{\varphi_i(t)/\varepsilon}$, $\varphi_i(t) = \int_0^t \lambda_i(s) ds$, $i = \overline{1, n}$, where $\lambda_i(t)$ are the eigenvalues of the limit operator and $\varphi_i(t)$ are smooth (in general, complex) functions of a real variable t .

If the stability conditions for the limit operator are violated for at least one point of the spectrum of the limit operator, then new singularities arise in the solution of the inhomogeneous equation. When studying problems with a “simple” turning point, we are faced with a problem when the range of values of the original operator does not coincide with the range of values of the limit operator.

Special singularities of the problem (1) have the form:

$$e^{\varphi_i(t)/\varepsilon}, \quad \varphi(t) = \int_0^t s^{m/n} a(s) ds, \quad \sigma_i = e^{-\varphi(t)/\varepsilon} \int_0^t e^{\varphi(s)/\varepsilon} s^{(i+1-n)/n} ds, \quad i = \overline{0, (p-1)}.$$

We look for a solution $u(x, t, \varepsilon)$ in the form [1,9]:

$$u(x, t, \varepsilon) = e^{-\varphi(t)/\varepsilon} X(x, t, \varepsilon) + \sum_{i=0}^{p-1} Z^i(x, t, \varepsilon) \sigma_i + W(x, t, \varepsilon). \quad (3)$$

We denote differentiation with respect to t by “ \cdot ” and differentiation with respect to x by “ $'$ ” or $\frac{\partial}{\partial x}$. Substituting (3) into (1) we get the equation

$$\begin{aligned} & \varepsilon \left[e^{-\varphi(t)/\varepsilon} \dot{X} + \sum_{i=0}^{p-1} \left(t^{(i+1-n)/n} Z^i + \sigma_i \dot{Z}^i \right) + \dot{W} \right] = \\ & = \varepsilon^2 \left[e^{-\varphi(t)/\varepsilon} \frac{\partial}{\partial x} \left(k(x) \frac{\partial X}{\partial x} \right) + \sum_{i=0}^{p-1} \frac{\partial}{\partial x} \left(k(x) \frac{\partial Z^i}{\partial x} \right) \sigma_i + \frac{\partial}{\partial x} \left(k(x) \frac{\partial W}{\partial x} \right) \right] - t^{m/n} a(t) W + h(x, t). \end{aligned}$$

By identifying the coefficients in the linear combination of $e^{-\varphi(t)/\varepsilon}$, σ_i and 1, we obtain the system

$$\begin{cases} \dot{X} = \varepsilon \frac{\partial}{\partial x} \left(k(x) \frac{\partial X}{\partial x} \right), \\ \dot{Z}^i = \varepsilon \frac{\partial}{\partial x} \left(k(x) \frac{\partial Z^i}{\partial x} \right), \quad i = \overline{0, (p-1)}, \\ t^{m/n} a(t) W = -\varepsilon \dot{W} + \varepsilon^2 \frac{\partial}{\partial x} \left(k(x) \frac{\partial W}{\partial x} \right) - \varepsilon \sum_{i=0}^{p-1} t^{(i+1-n)/n} Z^i + h(x, t), \\ X(x, 0) + W(x, 0) = f(x). \end{cases} \quad (4)$$

We look for a solution (4) in the form of power series in ε :

$$\begin{cases} X(x, t, \varepsilon) = \sum_{k=-1}^{\infty} \varepsilon^k X_k(x, t), \\ Z^i(x, t, \varepsilon) = \sum_{k=-1}^{\infty} \varepsilon^k Z_k^i(x, t), \quad i = \overline{0, (p-1)}, \\ W(x, t, \varepsilon) = \sum_{k=-1}^{\infty} \varepsilon^k W_k(x, t). \end{cases} \quad (5)$$

Substituting (5) into (4) we get a series of iterative tasks:

$$\begin{cases} \dot{X}_k = \frac{\partial}{\partial x} \left(k(x) \frac{\partial X_{k-1}}{\partial x} \right), \\ \dot{Z}_k^i = \frac{\partial}{\partial x} \left(k(x) \frac{\partial Z_{k-1}^i}{\partial x} \right), \quad i = \overline{0, (p-1)}, \\ t^{m/n} a(t) W_k = -\dot{W}_{k-1} + \frac{\partial}{\partial x} \left(k(x) \frac{\partial W_{k-2}}{\partial x} \right) - \sum_{i=0}^{p-1} t^{(i+1-n)/n} Z_{k-1}^i + \delta_0^k h(x, t), \\ X_k(x, 0) + W_k(x, 0) = \delta_0^k f(x), \quad k = \overline{-1, \infty}; \text{ if the index } (k-1) \leq -2, \\ \quad \quad \quad (k-2) \leq -2, \text{ then the term is by definition 0.} \end{cases} \quad (6)$$

To solve iterative problems, the solvability theorem is used (see Section 4).

Consider the system (6) with $k = -1$:

$$\begin{cases} \dot{X}_{-1} = 0, \\ \dot{Z}_{-1}^i = 0, \quad i = \overline{0, (p-1)}, \\ t^{m/n} a(t) W_{-1} \equiv 0, \\ X_{-1}(x, 0) + W_{-1}(x, 0) = 0. \end{cases} \quad (7)$$

From (7) it follows that

$$\begin{cases} X_{-1}(x, t) = X_{-1}(x, 0) = 0, \\ Z_{-1}^i(x, t) = Z_{-1}^i(x, 0), \quad i = \overline{0, (p-1)}, \\ W_{-1}(x, t) \equiv 0. \end{cases} \quad (8)$$

The functions $Z_{-1}^i(x, 0)$ are found from the solvability condition for the system (6) for $k = 0$:

$$\begin{cases} \dot{X}_0 = 0, \\ \dot{Z}_0^i = \frac{\partial}{\partial x} \left(k(x) \frac{\partial Z_{-1}^i(x, 0)}{\partial x} \right), \quad i = \overline{0, (p-1)}, \\ t^{m/n} a(t) W_0 = - \sum_{i=0}^{p-1} t^{(i+1-n)/n} Z_{-1}^i(x, 0) + h(x, t), \\ X_0(x, 0) + W_0(x, 0) = f(x). \end{cases} \quad (9)$$

Let us expand $h(x, t)$ by Maclaurin's formula in t :

$$h(x, t) = h(x, 0) + t \dot{h}(x, 0) + \dots + t^{[m/n]} h^{[m/n]}(x, 0) + t^{[m/n]+1} h_0(x, t).$$

From the condition of solvability of the equation for W_0 at $t = 0$, it follows that, if $\frac{i+1-n}{n} = j$, $i = \overline{0, (p-1)}$, $0 \leq j \leq [\frac{m}{n}]$, then

$$Z_{-1}^{(j+1)n-1}(x, 0) = \frac{h^{(j)}(x, 0)}{j!}, \quad (10)$$

where $[\frac{m}{n}]$ is the whole part of m/n ; if $\frac{i+1-n}{n} \neq j$, $i = \overline{0, (p-1)}$, $0 \leq j \leq [\frac{m}{n}]$, then

$$Z_{-1}^i(x, 0) = 0.$$

As a result, we get the solution at the (-1) iteration step:

$$u_{-1}(x, t, \varepsilon) = \frac{1}{\varepsilon} \sum_{j=0}^{[m/n]} Z_{-1}^{(j+1)n-1}(x, 0) \sigma_{(j+1)n-1}(t, \varepsilon) = \frac{1}{\varepsilon} \sum_{j=0}^{[m/n]} \frac{h^{(j)}(x, 0)}{j!} \sigma_{(j+1)n-1}(t, \varepsilon). \quad (11)$$

Considering (10) we can write the system (9) as:

$$\begin{cases} \dot{X}_0 = 0, \\ \dot{Z}_0^i = 0, \quad \text{when } i \neq (j+1)n-1, \quad 0 \leq j \leq [m/n], \\ \dot{Z}_0^i = \frac{\partial}{\partial x} \left(k(x) \frac{\partial Z_{-1}^i(x, 0)}{\partial x} \right), \quad i = (j+1)n-1, \quad 0 \leq j \leq [m/n], \\ t^{m/n} a(t) W_0 = - \sum_{j=0}^{[m/n]} t^j Z_{-1}^{(j+1)n-1}(x, 0) + h(x, t), \\ X_0(x, 0) + W_0(x, 0) = f(x). \end{cases} \quad (12)$$

The right-hand side of the equation with respect to W_0 of the system (12), due to the choice of $Z_{-1}^i(x, 0)$, satisfies the solvability theorem (see Section 4):

$$W_0(x, t) = \frac{h(x, t) - \sum_{j=0}^{[m/n]} t^j Z_{-1}^{(j+1)n-1}(x, 0)}{t^{m/n} a(t)} = t^{s/n} h_0(x, t), \quad 1 \leq s \leq n-1 \quad (13)$$

(s is an integer). Notice, that $W_0(x, 0) = 0$.

Solving (12) we get

$$\begin{cases} X_0(x, t) = X_0(x, 0) = f(x), \\ Z_0^i(x, t) = Z_0^i(x, 0), \quad i \neq (j+1)n-1, \quad 0 \leq j \leq [m/n], \\ Z_0^i(x, t) = t \frac{\partial}{\partial x} \left(k(x) \frac{\partial Z_{-1}^i(x, 0)}{\partial x} \right) + Z_0^i(x, 0), \quad i = (j+1)n-1, \quad i = \overline{0, (p-1)}, \\ W_0(x, t) = t^{s/n} h_0(x, t), \end{cases} \quad (14)$$

where $Z_{-1}^i(x, 0)$ are determined from (10).

$Z_0^i(x, 0)$ at the iteration step $k = 0$ are unknown. The functions $Z_0^i(x, 0)$ are found from the condition of solvability for $t = 0$ of the iterative problem for $k = 1$:

$$\begin{cases} \dot{X}_1(x, t) = f(x), \\ \dot{Z}_1^i(x, t) = \frac{\partial}{\partial x} \left(k(x) \frac{\partial Z_0^i(x, t)}{\partial x} \right), \quad i = \overline{0, (p-1)}, \\ t^{m/n} a(t) W_1(x, t) = -\dot{W}_0 - \sum_{i=0}^{p-1} t^{(i+1-n)/n} Z_0^i(x, t), \\ X_1(x, 0) + W_1(x, 0) = 0; \end{cases} \quad (15)$$

$$\begin{aligned} \dot{W}_0(x, t) &= \frac{s}{n} t^{\frac{s}{n}-1} h_0(x, t) + t^{\frac{s}{n}} \dot{h}_0(x, t) = t^{\frac{s}{n}-1} \left[\frac{s}{n} h_0(x, t) + \dot{h}(x, t) t \right] = \\ &= t^{\frac{s}{n}-1} h_1(x, t) = t^{-\{\frac{m}{n}\}} h_1(x, t), \end{aligned}$$

where $\{\frac{m}{n}\}$ is the fractional part of m/n .

To determine $W_1(x, t)$, we subordinate the right side of the equation to the conditions of point solvability. To do this, we expand $h_1(x, t)$ by Maclaurin's formula in t :

$$\dot{W}_0(x, t) = t^{-\{\frac{m}{n}\}} h_1(x, 0) + t^{-\{\frac{m}{n}\}+1} h_1(x, 0) + \dots + t^{-\{\frac{m}{n}\}+k} \frac{h^k(x, 0)}{k!} + t^{-\{\frac{m}{n}\}+k+1} h_2(x, t), \quad (16)$$

where $k = [\frac{m}{n} + \{\frac{m}{n}\}]$, $\{\frac{m}{n}\} = \frac{s}{n}$, $1 \leq s \leq n-1$. Then,

$$\begin{aligned} \text{(a) if } j - \{\frac{m}{n}\} &= \frac{i+1-n}{n}, \text{ i.e } i = n(j+1) - s - 1, \quad j = \overline{0, k}, \text{ then } Z_0^i(x, 0) = -\frac{h_1^{(j)}(x, 0)}{j!}; \\ \text{(b) if } i &\neq (j+1)n - s - 1, \quad j = \overline{0, k}, \text{ then } Z_0^i(x, 0) = 0. \end{aligned} \quad (17)$$

Using this scheme, you can find a solution at any iteration step.

For $u_0(x, t, \varepsilon)$, we get

$$u_0(x, t, \varepsilon) = e^{-\varphi(t)/\varepsilon} f(x) + \sum_{i=0}^{p-1} \sigma_i Z_0^i(x, t) + \frac{h(x, t) - \sum_{j=0}^{[m/n]} \frac{t^j h^{(j)}(x, 0)}{j!}}{t^{m/n} a(t)}, \quad (18)$$

where $Z_0^i(x, t)$ are determined from (14) and (17).

Thus, the main term of the asymptotics of the solution is written in the form

$$u_{\text{main}}(x, t, \varepsilon) = \frac{1}{\varepsilon} \sum_{j=0}^{[m/n]} \frac{h^{(j)}(x, 0)}{j!} \sigma_{(j+1)n-1}(t, \varepsilon) + e^{-\varphi(t)/\varepsilon} f(x) + \sum_{i=0}^{p-1} \sigma_i(t, \varepsilon) Z_0^i(x, t) +$$

$$h(x, t) - \sum_{j=0}^{[m/n]} \frac{h^{(j)}(x, 0)}{j!} t^j + \frac{t^{m/n} a(t)}{t^{m/n} a(t)}.$$

3. Remainder Estimate

Let $(N + 1)$ iterative problems be solved. Then, the solution to the Cauchy problem can be represented in the form

$$u(x, t, \varepsilon) = \sum_{k=-1}^N u_k(x, t, \varepsilon) \varepsilon^k + \varepsilon^{N+1} R_N(x, t, \varepsilon). \quad (19)$$

Substituting (19) into (1) and taking into account that $u_k(x, t, \varepsilon)$ are solutions to iterative problems, we obtain the Cauchy problem for determining the remainder $R_N(x, t, \varepsilon)$:

$$\begin{cases} L(R_N) = \varepsilon \dot{R}_N - \varepsilon^2 \frac{\partial}{\partial x} \left(k(x) \frac{\partial R_N}{\partial x} \right) + t^{m/n} a(t) R_N = H(x, t, \varepsilon), \\ R_N(x, 0, \varepsilon) = 0, \quad -\infty < x < +\infty, \end{cases} \quad (20)$$

where

$$\begin{aligned} H(x, t, \varepsilon) &= H_1(x, t) + \varepsilon H_2(x, t, \varepsilon), \\ H_1(x, t) &= -\dot{W}_N + \frac{\partial}{\partial x} \left(k(x) \frac{\partial W_{N-1}}{\partial x} \right) - \sum_{i=0}^{p-1} t^{\frac{i+1}{n}-1} Z_N^i(x, t), \\ H_2(x, t, \varepsilon) &= -e^{-\varphi(t)/\varepsilon} \frac{\partial}{\partial x} \left(k(x) \frac{\partial X_N}{\partial x} \right) - \frac{\partial}{\partial x} \left(k(x) \frac{\partial W_N}{\partial x} \right) - \sum_{i=0}^{p-1} \frac{\partial}{\partial x} \left(k(x) \frac{\partial Z_N^i}{\partial x} \right) \sigma_i(t, \varepsilon). \end{aligned}$$

The remainder estimate is based on the maximum principle for parabolic problems [12]. This principle is used in the form of generality that we need to estimate the remainder. The classical solution to the problem (1) is a function $R(x, t, \varepsilon)$, continuous in $Q_T = (-\infty, +\infty) \times [0, T] \times (0, \varepsilon]$, having continuous $\frac{\partial R}{\partial t}$, $\frac{\partial R}{\partial x}$, $\frac{\partial^2 R}{\partial x^2}$ in Q_T , and satisfies Equation (1) and the initial conditions at $t = 0$ at all points of Q_T .

Theorem 1 (remainder estimate). *Let the conditions be satisfied:*

- (1) Conditions (1)–(5) of the Cauchy problem (1);
- (2) $|H(x, t, \varepsilon)| \leq M_1 \forall (x, t) \in (-\infty, +\infty) \times [0, T] \forall \varepsilon \in (0, \varepsilon_0], M_1 > 0$;
- (3) $k(x) < M(x^2 + 1)$, $|k'(x)| < M\sqrt{x^2 + 1}$, $M > 0$; and
- (4) $R_N > -m$, $m > 0$.

Then, $|R_N(x, t, \varepsilon)| \leq M_2 \forall (x, t) \in (-\infty, +\infty) \times [0, T] \forall \varepsilon \in (0, \varepsilon_0], M_2 > 0$.

Proof of Theorem 1. We denote $t^{m/n} a(t) = q(t)$. We prove the theorem in two stages.

Stage 1. Consider the homogeneous Cauchy problem in the domain $D_L = \{[-L, L] \times [0, T]\}$:

$$\begin{cases} L(u) = \varepsilon \frac{\partial u}{\partial t} - \varepsilon^2 \frac{\partial}{\partial x} \left(k(x) \frac{\partial u}{\partial x} \right) + q(t) u = 0, \\ u(x, 0, \varepsilon) = 0. \end{cases} \quad (21)$$

In the proof of this theorem, ideas from [12] are used to estimate the solution. We introduce the function $w = u + e^{\alpha t/\varepsilon} \frac{m}{L^2} (x^2 + pt)$. Then,

$$\begin{aligned} L(w) &= L(u) + L\left(e^{\alpha t/\varepsilon} \frac{m}{L^2} (x^2 + pt)\right) = \\ &= [(\alpha + q)(x^2 + pt) + \varepsilon p - \varepsilon^2 2xk'(x) - \varepsilon^2 2k(x)] e^{\alpha t/\varepsilon} \frac{m}{L^2} \geq \\ &\geq e^{\alpha t/\varepsilon} \frac{m}{L^2} [(\alpha + q)(x^2 + pt) + \varepsilon p - \varepsilon^2 2Mx\sqrt{x^2 + 1} - \varepsilon^2 2M(x^2 + 1)]. \end{aligned}$$

(1) For $|x| \geq 1$,

$$L(w) \geq e^{\alpha t/\varepsilon} \frac{m}{L^2} [(\alpha + q)(x^2 + pt) + \varepsilon p - \varepsilon^2 8Mx^2] \geq e^{\alpha t/\varepsilon} \frac{m}{L^2} [\alpha - \varepsilon^2 8M] x^2.$$

Take $\alpha > \varepsilon_0^2 8M$, then $L(w) \geq 0$.

(2) For $|x| < 1$,

$$L(w) \geq e^{\alpha t/\varepsilon} \frac{m}{L^2} [(\alpha + q)(x^2 + pt) + \varepsilon p - \varepsilon^2 8M] \geq e^{\alpha t/\varepsilon} \frac{m}{L^2} [p - \varepsilon 8M] \varepsilon.$$

Take $p > \varepsilon_0 8M$, then $L(w) \geq 0$.

Besides,

$$\begin{aligned} w|_{t=0} &= u|_{t=0} + \frac{m}{L^2} x^2 \geq 0, \\ w|_{\pm L} &= u|_{\pm L} + e^{\alpha t/\varepsilon} \frac{m}{L^2} (L^2 + pt) \geq -m + m = 0. \end{aligned}$$

Hence, by the maximum theorem in a bounded domain, we have $w \geq 0$ in D_L , i.e.

$$u + e^{\alpha t/\varepsilon} \frac{m}{L^2} (x^2 + pt) \geq 0.$$

Letting $L \rightarrow +\infty$, we get $u \geq 0$.

Stage 2. Consider the inhomogeneous Cauchy problem

$$\begin{cases} L(R_N) = H(x, t, \varepsilon), \\ R_N(x, 0, \varepsilon) = 0. \end{cases} \quad (22)$$

We introduce the function

$$w = \pm R_N + \frac{M_1 t}{\varepsilon} + m.$$

Then,

$$L(w) = \pm H + M_1 + q(t) \left(\frac{M_1 t}{\varepsilon} + m \right) \geq 0, \quad w|_{t=0} = m \geq 0.$$

From the result of Stage 1, it follows that $w \geq 0$ in $D = (-\infty, +\infty) \times [0, T] \forall \varepsilon \in (0, \varepsilon_0]$, i.e. $\pm R_N + \frac{M_1 t}{\varepsilon} + m \geq 0$. Consequently, $|R_N| \leq \frac{M_1 t}{\varepsilon} + m \leq \frac{M_3}{\varepsilon}$, $M_3 > 0$. We write the remainder

$$R_N = u_N + \varepsilon R_{N+1}.$$

Then, $|R_N| \leq |u_N| + \varepsilon \frac{M_4}{\varepsilon} \leq M_2$, $M_4 > 0$. $\triangleright \square$

4. Appendix

Lemma 1 (on the solvability of iterative problems). *Let the equation be given*

$$t^{m/n} Z(x, t) = F(x, t) \quad (23)$$

and conditions are met

$$F(x, t) \in C^\infty(R \times [0, T]).$$

Then, Equation (23) is solvable in the class of smooth functions if and only if

$$\frac{\partial F^k(x, 0)}{\partial t} = 0, \quad k = 0, \overline{\left[\frac{m}{n}\right]}.$$

Proof of Lemma 1. We expand $F(x, t)$ by Maclaurin's formula in t :

$$F(x, t) = F(x, 0) + \dot{F}(x, 0)t + \dots + F^{[m/n]}(x, 0)t^{[m/n]} + t^{[m/n]+1}f(x, t).$$

Then, Equation (23) has the form

$$t^{m/n}Z(x, t) = F(x, 0) + \dot{F}(x, 0)t + \dots + F^{[m/n]}(x, 0)t^{[m/n]} + t^{[m/n]+1}f(x, t).$$

Necessity, Let Equation (23) have a solution. For $t = 0$, we have $0 = F(x, 0)$:

$$t^{m/n}Z(x, t) = F(x, t) - F(x, 0).$$

Dividing the equation by t if $\left[\frac{m}{n}\right] > 1$:

$$t^{\frac{m}{n}-1}Z(x, t) = \frac{F(x, t) - F(x, 0)}{t}.$$

For $t = 0$, we get $0 = \dot{F}(x, 0)$.

Continuing this process to step $\left[\frac{m}{n}\right]$, we get $F^k(x, 0) = 0 \quad \forall k = 0, \overline{\left[\frac{m}{n}\right]}$.

Adequacy: Let be $F(x, 0) = \dot{F}(x, 0) = \dots = F^{[m/n]}(x, 0) = 0$. Then, Equation (23) has the form:

$$t^{m/n}Z(x, t) = t^{[m/n]+1}f(x, t),$$

hence the decision $Z(x, t) = t^{1-\left\{\frac{m}{n}\right\}}f(x, t)$. \square

5. Conclusions

The novelty of the article lies in the construction by the regularization method of an asymptotic solution of a singularly perturbed Cauchy problem for a parabolic equation in the presence of a "simple" rational turning point of the limit operator. The nature of this "simple" turning point affects the structure of the functions describing the singular dependence of the solution on the parameter ε . The asymptotic expansion of the constructed solution is justified using the maximum principle. This approach can be applied both in the study of applied problems containing turning points and in the construction of numerical algorithms for solving problems with spectral features.

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