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Inertial Iterative Self-Adaptive Step Size Extragradient-Like Method for Solving Equilibrium Problems in Real Hilbert Space with Applications

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Abstract: A number of applications from mathematical programmings, such as minimization problems, variational inequality problems and fixed point problems, can be written as equilibrium problems. Most of the schemes being used to solve this problem involve iterative methods, and for that reason, in this paper, we introduce a modified iterative method to solve equilibrium problems in real Hilbert space. This method can be seen as a modification of the paper titled "A new two-step proximal algorithm of solving the problem of equilibrium programming" by Lyashko et al. (Optimization and its applications in control and data sciences, Springer book pp. 315–325, 2016). A weak convergence result has been proven by considering the mild conditions on the cost bifunction. We have given the application of our results to solve variational inequality problems. A detailed numerical study on the Nash–Cournot electricity equilibrium model and other test problems is considered to verify the convergence result and its performance.

Keywords: pseudomonotone bifunction; Lipschitz-type conditions; equilibrium problem; variational inequalities

1. Introduction

An equilibrium problem (EP) is a generalized concept that unifies several mathematical problems, such as the variational inequality problems, minimization problems, complementarity problems, the fixed point problems, non-cooperative games of Nash equilibrium, the saddle point problems and scalar and vector minimization problems (see e.g., [1–3]). The particular form of an equilibrium problem was firstly established in 1992 by Muu and Oettli [4] and then further elaborated by Blum and Oettli [1]. Next, we consider the concept of an equilibrium problem introduced by Blum and Oettli in [1]. Let C be a non-empty, closed and convex subset $\mathbb{H}$ of a real Hilbert space and $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is bifunction with $f(v, v) = 0$, for each $v \in C$. A equilibrium problem regarding f on the set C is defined in the following way:

$$\text{Find } p \in C \text{ such that } f(p, v) \geq 0, \text{ for all } v \in C. \tag{1}$$

Many methods have been already established over the past couple of years to figure out the equilibrium problem in Hilbert spaces [5–15], the inertial methods [11,16–18] and others in [18–24]. In particular, Tran et al. introduced an iterative scheme in [8], in that a sequence $\{u_n\}$ was generated in the following way:

$$\begin{cases} u_0 \in C, \\ v_n = \arg\min\{\lambda f(u_n, y) + \frac{1}{2}\|u_n - y\|^2 : y \in C\}, \\ u_{n+1} = \arg\min\{\lambda f(v_n, y) + \frac{1}{2}\|u_n - y\|^2 : y \in C\}, \end{cases} \tag{2}$$

where $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$ and c_1, c_2 are Lipschitz constants. Lyashko et al. [25] in 2016 introduced an improvement of the method (2) to solve equilibrium problem and sequence $\{u_n\}$ was generated in the following way:

$$\begin{cases} u_0, v_0 \in C, \\ u_{n+1} = \arg\min\{\lambda f(v_n, y) + \frac{1}{2}\|u_n - y\|^2 : y \in C\}, \\ v_{n+1} = \arg\min\{\lambda f(v_n, y) + \frac{1}{2}\|u_{n+1} - y\|^2 : y \in C\}, \end{cases} \tag{3}$$

where $0 < \lambda < \frac{1}{2c_2 + 4c_1}$ and c_1, c_2 are Lipschitz constants.

In this paper, we consider the extragradient method in (3) and to provide its improvement by using the inertial scheme [26] and continue to improve the step size rule for its second step. The step size is not fixed in our case, but it is dependent on a particular formula by using prior information of the bifunction values. A weak convergence theorem dealing with the suggested technique is presented by having the specific bi-functional condition. We have also considered how our results are presented to the problems of a variational inequality. A few other formulations of the problem of variational inequalities are discussed, and many computational examples in finite and infinite dimensions spaces are also presented to demonstrate the applicability of our proposed results.

In this study, we study the equilibrium problem through the following assumptions:

(f_1) A bifunction $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is said to be (see [1,27]) *pseudomonotone* on C if

$$f(v_1, v_2) \geq 0 \Longrightarrow f(v_2, v_1) \leq 0, \text{ for all } v_1, v_2 \in C.$$

(f_2) A bifunction $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is said to be Lipschitz-type continuous [28] on C if there exist $c_1, c_2 > 0$ such that

$$f(v_1, v_3) \leq f(v_1, v_2) + f(v_2, v_3) + c_1\|v_1 - v_2\|^2 + c_2\|v_2 - v_3\|^2, \text{ for all } v_1, v_2, v_3 \in C.$$

(f_3) $\limsup_{n \to +\infty} f(v_n, z) \leq f(v^*, z)$ for each $z \in C$ and $\{v_n\} \subset C$ satisfying $v_n \rightharpoonup v^*$;

(f_4) $f(u, \cdot)$ is convex and subdifferentiable on $\mathbb{H}$ for each $u \in \mathbb{H}$.

The rest of this paper will be organized as follows: In Section 2, we give a few definitions and important lemmas to be used in this paper. Section 3 includes the main algorithm involving pseudomonotone bifunction and provides a weak convergence theorem. Section 4 describes some applications in the variational inequality problems. Section 5 sets out the numerical studies to demonstrate the algorithmic efficiency.

2. Preliminaries

In this section, some important lemmas and basic definitions are provided. Moreover, $EP(f, C)$ denotes the solution set of an equilibrium problem on the set C and p is any arbitrary element of $EP(f, C)$.

A *metric projection* $P_C(u)$ of u onto a closed, convex subset C of $\mathbb{H}$ is defined by

$$P_C(u) = \arg\min_{v \in C}\{\|v - u\|\}.$$

Lemma 1. [29] *Let $P_C : \mathbb{H} \to C$ be a metric projection from $\mathbb{H}$ onto C. Then*

(i) *For all $u \in C$, $v \in \mathbb{H}$ and*

$$\|u - P_C(v)\|^2 + \|P_C(v) - v\|^2 \leq \|u - v\|^2.$$

(ii) $w = P_C(u)$ *if and only if*

$$\langle u - w, v - w \rangle \leq 0, \text{ for all } v \in C.$$

Lemma 2. [29] *For all $u, v \in \mathbb{H}$ with $\daleth \in \mathbb{R}$. Then, the following relationship is holds.*

$$\|\daleth u + (1 - \daleth)v\|^2 = \daleth\|u\|^2 + (1 - \daleth)\|v\|^2 - \daleth(1 - \daleth)\|u - v\|^2.$$

Assume that $g : C \to \mathbb{R}$ be a convex function and subdifferential of g at $u \in C$ is defined by

$$\partial g(u) = \{w \in C : g(v) - g(u) \geq \langle w, v - u \rangle, \text{ for all } v \in C\}.$$

Given that $f(u, .)$ is convex and subdifferentiable on $\mathbb{H}$ for each fixed $u \in \mathbb{H}$ and *subdifferential of* $f(u, .)$ at $x \in \mathbb{H}$ defined by

$$\partial_2 f(u, .)(x) = \partial_2 f(u, x) = \{z \in \mathbb{H} : f(u, v) - f(u, x) \geq \langle z, v - x \rangle, \text{ for all } v \in \mathbb{H}\}.$$

A normal cone of C at $u \in C$ is defined by

$$N_C(u) = \{w \in \mathbb{H} : \langle w, v - u \rangle \leq 0, \text{ for all } v \in C\}.$$

Lemma 3. [30] *Assume that C is a nonempty, closed and convex subset of a real Hilbert space $\mathbb{H}$ and $h : C \to \mathbb{R}$ be a convex, lower semi-continuous and subdifferentiable function on C. Then, $u \in C$ is a minimizer of a function h if and only if $0 \in \partial h(u) + N_C(u)$ where $\partial h(u)$ and $N_C(u)$ denotes the subdifferential of h at u and the normal cone of C at u, respectively.*

Lemma 4. [31] *Let a_n, b_n and c_n are non-negative real sequences such that*

$$a_{n+1} \leq a_n + b_n(a_n - a_{n-1}) + c_n, \text{ for all } n \geq 1, \text{ with } \sum_{n=1}^{+\infty} c_n < +\infty,$$

where $b > 0$ such that $0 \leq b_n \leq b < 1$ for all $n \in \mathbb{N}$. Then, the following relations are true.

(i) $\sum_{n=1}^{+\infty} [a_n - a_{n-1}]_+ < +\infty$, *with* $[s]_+ := \max\{s, 0\}$;
(ii) $\lim_{n \to +\infty} a_n = a^* \in [0, +\infty)$.

Lemma 5. [32] *Let a sequence $\{a_n\}$ in $\mathbb{H}$ and $C \subset \mathbb{H}$ and the following conditions have been met:*

(i) *for each $a \in C$, $\lim_{n \to +\infty} \|a_n - a\|$ exists;*
(ii) *each weak sequentially limit point of $\{a_n\}$ belongs to set C.*

Then, $\{a_n\}$ weakly converges to an element in C.

3. Main Results

In this section, we present our main algorithm and provide a weak convergence theorem for our proposed method. The detailed method is given below.

Remark 1. *By Expression* (5), *we obtain*

$$\lambda_{n+1}\Big[f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) - c_1\|v_{n-1} - v_n\|^2 - c_2\|v_n - u_{n+1}\|^2\Big] \leq \mu f(v_n, u_{n+1}). \quad (4)$$

Lemma 6. *Let $\{u_n\}$ be a sequence generated by Algorithm 1. Then, the following inequality holds.*

$$\mu\lambda_n f(v_n, y) - \mu\lambda_n f(v_n, u_{n+1}) \geq \langle \rho_n - u_{n+1}, y - u_{n+1}\rangle, \textit{ for all } y \in C.$$

Algorithm 1 Modified Popov's subgradient extragradient-like iterative scheme.

Step 1: Choose $u_{-1}, v_{-1}, u_0, v_0 \in \mathbb{H}$ and a sequence $\wp_n$ is non-decreasing such that $0 \leq \wp_n \leq \wp < \frac{1}{3}$, $\lambda_0 > 0$ and $0 < \sigma < \min\left\{\frac{1-3\wp}{(1-\wp)^2+4c_1(\wp+\wp^2)}, \frac{1}{2c_2+4c_1(1+\wp)}\right\}$ and $\mu \in (0, \sigma)$.

Step 2: Evaluate

$$u_{n+1} = \arg\min\{\mu\lambda_n f(v_n, y) + \frac{1}{2}\|\rho_n - y\|^2 : y \in C\},$$

where $\rho_n = u_n + \wp_n(u_n - u_{n-1})$.

Step 3: Updated the step size in the following order:

$$\lambda_{n+1} = \begin{cases} \min\left\{\sigma, \frac{\mu f(v_n,u_{n+1})}{f(v_{n-1},u_{n+1})-f(v_{n-1},v_n)-c_1\|v_{n-1}-v_n\|^2-c_2\|u_{n+1}-v_n\|^2+1}\right\}, \\ \quad \text{if } \frac{\mu f(v_n,u_{n+1})}{f(v_{n-1},u_{n+1})-f(v_{n-1},v_n)-c_1\|v_{n-1}-v_n\|^2-c_2\|u_{n+1}-v_n\|^2+1} > 0, \\ \lambda_0 \qquad \qquad \qquad \qquad \qquad \qquad \textit{else}. \end{cases} \tag{5}$$

Step 4: Evaluate

$$v_{n+1} = \arg\min\{\lambda_{n+1} f(v_n, y) + \frac{1}{2}\|\rho_{n+1} - y\|^2 : y \in C\},$$

where $\rho_{n+1} = u_{n+1} + \wp_{n+1}(u_{n+1} - u_n)$. If $u_{n+1} = v_n = \rho_n$ or $\rho_{n+1} = v_{n+1} = v_n$ then Stop. Else, take $n := n + 1$ and go back to **Step 2**.

Proof. By the use of Lemma 3, we get

$$0 \in \partial_2\Big\{\mu\lambda_n f(v_n, y) + \frac{1}{2}\|\rho_n - y\|^2\Big\}(u_{n+1}) + N_C(u_{n+1}).$$

From above there is a $\omega \in \partial_2 f(v_n, u_{n+1})$ and $\overline{\omega} \in N_C(u_{n+1})$ such that

$$\mu\lambda_n\omega + u_{n+1} - \rho_n + \overline{\omega} = 0.$$

Therefore, we obtain

$$\langle \rho_n - u_{n+1}, y - u_{n+1}\rangle = \mu\lambda_n\langle \omega, y - u_{n+1}\rangle + \langle \overline{\omega}, y - u_{n+1}\rangle, \text{ for all } y \in C.$$

Due to $\overline{\omega} \in N_C(u_{n+1})$ then $\langle \overline{\omega}, y - u_{n+1}\rangle \leq 0$, for each $y \in C$. It implies that

$$\mu\lambda_n\langle \omega, y - u_{n+1}\rangle \geq \langle \rho_n - u_{n+1}, y - u_{n+1}\rangle, \text{ for all } y \in C. \tag{6}$$

Given that $\omega \in \partial_2 f(v_n, u_{n+1})$, we have

$$f(v_n, y) - f(v_n, u_{n+1}) \geq \langle \omega, y - u_{n+1}\rangle, \text{ for all } y \in \mathbb{H}. \tag{7}$$

By combining Expressions (6) and (7), we obtain

$$\mu\lambda_n f(v_n, y) - \mu\lambda_n f(v_n, u_{n+1}) \geq \langle \rho_n - u_{n+1}, y - u_{n+1}\rangle, \text{ for all } y \in C.$$

□

Lemma 7. *Let $\{v_n\}$ be a sequence generated by Algorithm 1. Then, the following inequality holds.*

$$\lambda_{n+1} f(v_n, y) - \lambda_{n+1} f(v_n, v_{n+1}) \geq \langle \rho_{n+1} - v_{n+1}, y - v_{n+1} \rangle, \text{ for all } y \in C.$$

Proof. The proof is same as the proof of Lemma 6. □

Lemma 8. *If $u_{n+1} = v_n = \rho_n$ and $\rho_{n+1} = v_{n+1} = v_n$ in Algorithm 1, then $v_n \in EP(f, C)$.*

Proof. The proof of this can easily be seen from Lemmas 6 and 7. □

Lemma 9. *Let $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a bifunction and satisfies the conditions (f_1)–(f_4). Then, for each $p \in EP(f, C) \neq \emptyset$, we have*

$$\begin{aligned}
&\|u_{n+1} - p\|^2 \\
&\leq \|\rho_n - p\|^2 - (1 - \lambda_{n+1})\|u_{n+1} - \rho_n\|^2 + 4c_1\lambda_{n+1}\lambda_n\|\rho_n - v_{n-1}\|^2 \\
&\quad - \lambda_{n+1}(1 - 4c_1\lambda_n)\|\rho_n - v_n\|^2 - \lambda_{n+1}(1 - 2c_2\lambda_n)\|u_{n+1} - v_n\|^2.
\end{aligned}$$

Proof. By Lemma 6, we obtain

$$\mu\lambda_n f(v_n, p) - \mu\lambda_n f(v_n, u_{n+1}) \geq \langle \rho_n - u_{n+1}, p - u_{n+1} \rangle. \tag{8}$$

Thus, $p \in EP(f, C)$ and the condition (f_1) implies that $f(v_n, p) \leq 0$. From (8), we have

$$\langle \rho_n - u_{n+1}, u_{n+1} - p \rangle \geq \mu\lambda_n f(v_n, u_{n+1}). \tag{9}$$

From Expression (4), we obtain

$$\mu f(v_n, u_{n+1}) \geq \lambda_{n+1}\big(f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) - c_1\|v_{n-1} - v_n\|^2 - c_2\|v_n - u_{n+1}\|^2\big). \tag{10}$$

Combining expression (9) and (10), implies that

$$\begin{aligned}
\langle \rho_n - u_{n+1}, u_{n+1} - p \rangle \geq \lambda_{n+1}\Big[&\lambda_n\{f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n)\} \\
&- c_1\lambda_n\|v_{n-1} - v_n\|^2 - c_2\lambda_n\|u_{n+1} - v_n\|^2\Big].
\end{aligned} \tag{11}$$

By Lemma 7, we have

$$\lambda_n\{f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n)\} \geq \langle \rho_n - v_n, u_{n+1} - v_n \rangle. \tag{12}$$

Thus, combining (11) and (12) we get

$$\begin{aligned}
\langle \rho_n - u_{n+1}, u_{n+1} - p \rangle \geq \lambda_{n+1}\Big[&\langle \rho_n - v_n, u_{n+1} - v_n \rangle \\
&- c_1\lambda_n\|v_{n-1} - v_n\|^2 - c_2\lambda_n\|u_{n+1} - v_n\|^2\Big].
\end{aligned} \tag{13}$$

We have the following mathematical expressions:

$$2\langle \rho_n - u_{n+1}, u_{n+1} - p \rangle = \|\rho_n - p\|^2 - \|u_{n+1} - \rho_n\|^2 - \|u_{n+1} - p\|^2.$$

$$2\langle \rho_n - v_n, u_{n+1} - v_n \rangle = \|\rho_n - v_n\|^2 + \|u_{n+1} - v_n\|^2 - \|\rho_n - u_{n+1}\|^2.$$

From the above equation and (13), we have

$$
\begin{aligned}
&\|u_{n+1} - p\|^2 \\
&\le \|\rho_n - p\|^2 - (1-\lambda_{n+1})\|u_{n+1} - \rho_n\|^2 - \lambda_{n+1}(1 - 2c_2\lambda_n)\|u_{n+1} - v_n\|^2 \\
&\quad - \lambda_{n+1}\|\rho_n - v_n\|^2 + \lambda_{n+1}(2c_1\lambda_n)\|v_{n-1} - v_n\|^2
\end{aligned}
$$

We also have

$$
\|v_{n-1} - v_n\|^2 \le \big(\|v_{n-1} - \rho_n\| + \|\rho_n - v_n\|\big)^2 \le 2\|v_{n-1} - \rho_n\|^2 + 2\|\rho_n - v_n\|^2.
$$

Finally, we get

$$
\begin{aligned}
&\|u_{n+1} - p\|^2 \\
&\le \|\rho_n - p\|^2 - (1-\lambda_{n+1})\|u_{n+1} - \rho_n\|^2 + 4c_1\lambda_n\lambda_{n+1}\|\rho_n - v_{n-1}\|^2 \\
&\quad - \lambda_{n+1}(1 - 4c_1\lambda_n)\|\rho_n - v_n\|^2 - \lambda_{n+1}(1 - 2c_2\lambda_n)\|u_{n+1} - v_n\|^2.
\end{aligned}
$$

□

Theorem 1. *Assume that $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ satisfies the conditions (f_1)–(f_4). Then, for some $p \in EP(f, C) \neq \emptyset$, the sequence $\{\rho_n\}$, $\{u_n\}$ and $\{v_n\}$ generated by Algorithm 1, weakly converge to $p \in EP(f, C)$.*

Proof. By Lemma 9, we obtain

$$
\begin{aligned}
&\|u_{n+1} - p\|^2 \\
&\le \|\rho_n - p\|^2 - (1-\lambda_{n+1})\|u_{n+1} - \rho_n\|^2 + 4c_1\lambda_n\lambda_{n+1}\|\rho_n - v_{n-1}\|^2 \\
&\quad - \lambda_{n+1}(1 - 4c_1\lambda_n)\|\rho_n - v_n\|^2 - \lambda_{n+1}(1 - 2c_2\lambda_n)\|u_{n+1} - v_n\|^2.
\end{aligned} \tag{14}
$$

By definition of ρ_n in the Algorithm 1, we have

$$
\begin{aligned}
\|\rho_n - v_{n-1}\|^2 &= \|u_n + \wp_n(u_n - u_{n-1}) - v_{n-1}\|^2 \\
&= \|(1+\wp_n)(u_n - v_{n-1}) - \wp_n(u_{n-1} - v_{n-1})\|^2 \\
&= (1+\wp_n)\|u_n - v_{n-1}\|^2 - \wp_n\|u_{n-1} - v_{n-1}\|^2 + \wp_n(1+\wp_n)\|u_n - u_{n-1}\|^2 \\
&\le (1+\wp)\|u_n - v_{n-1}\|^2 + \wp(1+\wp)\|u_n - u_{n-1}\|^2.
\end{aligned} \tag{15}
$$

Adding the term $4c_1\sigma\lambda_{n+1}(1+\wp)\|u_{n+1}-v_n\|^2$ on both sides in (14) with (15) for $n \geq 1$, we have

$$\begin{aligned}
&\|u_{n+1}-p\|^2 + 4c_1\sigma\lambda_{n+1}(1+\wp)\|u_{n+1}-v_n\|^2 \\
&\leq \|\rho_n - p\|^2 - (1-\sigma)\|u_{n+1}-\rho_n\|^2 + 4c_1\sigma\lambda_{n+1}(1+\wp)\|u_{n+1}-v_n\|^2 \\
&\quad + 4c_1\sigma\lambda_n\big[(1+\wp)\|u_n - v_{n-1}\|^2 + \wp(1+\wp)\|u_n-u_{n-1}\|^2\big] \\
&\quad - \lambda_{n+1}(1-4c_1\sigma)\|\rho_n - v_n\|^2 - \lambda_{n+1}(1-2c_2\sigma)\|u_{n+1}-v_n\|^2 && (16)\\
&\leq \|\rho_n - p\|^2 - (1-\sigma)\|u_{n+1}-\rho_n\|^2 + 4c_1\sigma\lambda_n(1+\wp)\|u_n-v_{n-1}\|^2 \\
&\quad + 4c_1\sigma(\wp+\wp^2)\|u_n-u_{n-1}\|^2 - \lambda_{n+1}(1-4c_1\sigma)\|\rho_n-v_n\|^2 \\
&\quad - \lambda_{n+1}(1-2c_2\sigma - 4c_1\sigma(1+\wp))\|u_{n+1}-v_n\|^2 && (17)\\
&\leq \|\rho_n - p\|^2 - (1-\sigma)\|u_{n+1}-\rho_n\|^2 + 4c_1\sigma\lambda_n(1+\wp)\|u_n-v_{n-1}\|^2 \\
&\quad + 4c_1\sigma(\wp+\wp^2)\|u_n-u_{n-1}\|^2 \\
&\quad - \frac{\lambda_{n+1}}{2}(1-2c_2\sigma-4c_1\sigma(1+\wp))\big[2\|u_{n+1}-v_n\|^2 + 2\|\rho_n - v_n\|^2\big] && (18)\\
&\leq \|\rho_n - p\|^2 - (1-\sigma)\|u_{n+1}-\rho_n\|^2 + 4c_1\sigma\lambda_n(1+\wp)\|u_n-v_{n-1}\|^2 \\
&\quad + 4c_1\sigma(\wp+\wp^2)\|u_n-u_{n-1}\|^2 \\
&\quad - \frac{\lambda_{n+1}}{2}(1-2c_2\sigma-4c_1\sigma(1+\wp))\|u_{n+1}-\rho_n\|^2. && (19)
\end{aligned}$$

Given that $0 < \lambda_n \leq \sigma < \frac{1}{2c_2+4c_1(1+\wp)}$, then the last inequality turns into

$$\begin{aligned}
&\|u_{n+1}-p\|^2 + 4c_1\sigma\lambda_{n+1}(1+\wp)\|u_{n+1}-v_n\|^2 \\
&\leq \|\rho_n - p\|^2 - (1-\sigma)\|u_{n+1}-\rho_n\|^2 + 4c_1\sigma\lambda_n(1+\wp)\|u_n-v_{n-1}\|^2 \\
&\quad + 4c_1\sigma(\wp+\wp^2)\|u_n-u_{n-1}\|^2. && (20)
\end{aligned}$$

From the definition of ρ_n, we have

$$\begin{aligned}
\|\rho_n - p\|^2 &= \|u_n + \wp_n(u_n-u_{n-1}) - p\|^2 \\
&= \|(1+\wp_n)(u_n-p) - \wp_n(u_{n-1}-p)\|^2 \\
&= (1+\wp_n)\|u_n-p\|^2 - \wp_n\|u_{n-1}-p\|^2 + \wp_n(1+\wp_n)\|u_n-u_{n-1}\|^2. && (21)
\end{aligned}$$

From ρ_{n+1}, we obtain

$$\begin{aligned}
\|u_{n+1}-\rho_n\|^2 &= \|u_{n+1}-u_n-\wp_n(u_n-u_{n-1})\|^2 \\
&= \|u_{n+1}-u_n\|^2 + \wp_n^2\|u_n-u_{n-1}\|^2 - 2\wp_n\langle u_{n+1}-u_n, u_n-u_{n-1}\rangle && (22)\\
&\geq \|u_{n+1}-u_n\|^2 + \wp_n^2\|u_n-u_{n-1}\|^2 - 2\wp_n\|u_{n+1}-u_n\|\|u_n-u_{n-1}\| \\
&\geq \|u_{n+1}-u_n\|^2 + \wp_n^2\|u_n-u_{n-1}\|^2 - \wp_n\|u_{n+1}-u_n\|^2 - \wp_n\|u_n-u_{n-1}\|^2 \\
&= (1-\wp_n)\|u_{n+1}-u_n\|^2 + (\wp_n^2-\wp_n)\|u_n-u_{n-1}\|^2. && (23)
\end{aligned}$$

Combining the Expressions (20), (21) and (23) we have

$$\begin{aligned}
&\|u_{n+1}-p\|^2+4c_1\sigma\lambda_{n+1}(1+\wp)\|u_{n+1}-v_n\|^2\\
&\leq(1+\wp_n)\|u_n-p\|^2-\wp_n\|u_{n-1}-p\|^2+\wp_n(1+\wp_n)\|u_n-u_{n-1}\|^2\\
&\quad-(1-\sigma)\big[(1-\wp_n)\|u_{n+1}-u_n\|^2+(\wp_n^2-\wp_n)\|u_n-u_{n-1}\|^2\big]\\
&\quad+4c_1\sigma\lambda_n(1+\wp)\|u_n-v_{n-1}\|^2+4c_1\sigma(\wp+\wp^2)\|u_n-u_{n-1}\|^2 \qquad (24)\\
&\leq(1+\wp_n)\|u_n-p\|^2-\wp_n\|u_{n-1}-p\|^2+4c_1\sigma\lambda_n(1+\wp)\|u_n-v_{n-1}\|^2\\
&\quad+\Big[\wp(1+\wp)-(1-\sigma)(\wp_n^2-\wp_n)+4c_1\sigma(\wp+\wp^2)\Big]\|u_n-u_{n-1}\|^2\\
&\quad-(1-\sigma)(1-\wp_n)\|u_{n+1}-u_n\|^2 \qquad (25)\\
&\leq(1+\wp_n)\|u_n-p\|^2-\wp_n\|u_{n-1}-p\|^2+4c_1\sigma\lambda_n(1+\wp)\|u_n-v_{n-1}\|^2\\
&\quad+r_n\|u_n-u_{n-1}\|^2-q_n\|u_{n+1}-u_n\|^2, \qquad (26)
\end{aligned}$$

where

$$r_n=\Big[\wp(1+\wp)-(1-\sigma)(\wp_n^2-\wp_n)+4c_1\sigma(\wp+\wp^2)\Big];$$
$$q_n=(1-\sigma)(1-\wp_n).$$

Assume that

$$\Gamma_n=\Psi_n+r_n\|u_n-u_{n-1}\|^2,$$

where $\Psi_n=\|u_n-p\|^2-\wp_n\|u_{n-1}-p\|^2+4c_1\sigma\lambda_n(1+\wp)\|u_n-v_{n-1}\|^2$. Next, (26) implies that

$$\begin{aligned}
&\Gamma_{n+1}-\Gamma_n\\
&=\|u_{n+1}-p\|^2-\wp_{n+1}\|u_n-p\|^2+4c_1\sigma\lambda_{n+1}(1+\wp)\|u_{n+1}-v_n\|^2+r_{n+1}\|u_{n+1}-u_n\|^2\\
&\quad-\|u_n-p\|^2+\wp_n\|u_{n-1}-p\|^2-4c_1\sigma\lambda_n(1+\wp)\|u_n-v_{n-1}\|^2-r_n\|u_n-u_{n-1}\|^2\\
&\leq\|u_{n+1}-p\|^2-(1+\wp_n)\|u_n-p\|^2+\wp_n\|u_{n-1}-p\|^2+4c_1\sigma\lambda_{n+1}(1+\wp)\|u_{n+1}-v_n\|^2\\
&\quad+r_{n+1}\|u_{n+1}-u_n\|^2-4c_1\sigma\lambda_n(1+\wp)\|u_n-v_{n-1}\|^2-r_n\|u_n-u_{n-1}\|^2\\
&\leq-(q_n-r_{n+1})\|u_{n+1}-u_n\|^2. \qquad (27)
\end{aligned}$$

Next, we have to compute

$$\begin{aligned}
(q_n-r_{n+1})&=(1-\sigma)(1-\wp_n)-\wp(1+\wp)+(1-\sigma)(\wp_n^2-\wp_n)-4c_1\sigma(\wp+\wp^2)\\
&\geq(1-\sigma)(1-\wp)^2-\wp(1+\wp)-4c_1\sigma(\wp+\wp^2)\\
&=(1-\wp)^2-\wp(1+\wp)-\sigma(1-\wp)^2-4c_1\sigma(\wp+\wp^2)\\
&=1-3\wp-\sigma\big((1-\wp)^2+4c_1(\wp+\wp^2)\big)\\
&\geq 0. \qquad (28)
\end{aligned}$$

By the use of (27) and (28) for some $\delta\geq 0$ implies that

$$\Gamma_{n+1}-\Gamma_n\leq-(q_n-r_{n+1})\|u_{n+1}-u_n\|^2\leq-\delta\|u_{n+1}-u_n\|^2\leq 0. \qquad (29)$$

The relation (29) implies that $\{\Gamma_n\}$ is non-increasing. From Γ_{n+1} we have

$$\begin{aligned}
\Gamma_{n+1}&=\|u_{n+1}-p\|^2-\wp_{n+1}\|u_n-p\|^2+r_{n+1}\|u_{n+1}-u_n\|^2+4c_1\sigma\lambda_{n+1}(1+\wp)\|u_{n+1}-v_n\|^2\\
&\geq-\wp_{n+1}\|u_n-p\|^2. \qquad (30)
\end{aligned}$$

By definition Γ_n, we have

$$\begin{aligned}\|u_n - p\|^2 &\le \Gamma_n + \wp_n \|u_{n-1} - p\|^2 \\ &\le \Gamma_1 + \wp \|u_{n-1} - p\|^2 \\ &\le \cdots \le \Gamma_1(\wp^{n-1} + \cdots + 1) + \wp^n \|u_0 - p\|^2 \\ &\le \frac{\Gamma_1}{1-\wp} + \wp^n \|u_0 - p\|^2. \end{aligned} \tag{31}$$

From Equations (30) and (31), we obtain

$$\begin{aligned}-\Gamma_{n+1} &\le \wp_{n+1} \|u_n - p\|^2 \\ &\le \wp \|u_n - p\|^2 \\ &\le \wp \frac{\Gamma_1}{1-\wp} + \wp^{n+1} \|u_0 - p\|^2. \end{aligned} \tag{32}$$

It follows (29) and (32) that

$$\begin{aligned}\delta \sum_{n=1}^{k} \|u_{n+1} - u_n\|^2 &\le \Gamma_1 - \Gamma_{k+1} \\ &\le \Gamma_1 + \wp \frac{\Gamma_1}{1-\wp} + \wp^{k+1} \|u_0 - p\|^2 \\ &\le \frac{\Gamma_1}{1-\wp} + \|u_0 - p\|^2. \end{aligned} \tag{33}$$

By letting $k \to +\infty$ in (33), we obtain

$$\sum_{n=1}^{+\infty} \|u_{n+1} - u_n\|^2 < +\infty \quad \text{implies that} \quad \lim_{n \to +\infty} \|u_{n+1} - u_n\| = 0. \tag{34}$$

From Expressions (22) with (34), we obtain

$$\|u_{n+1} - \rho_n\| \to 0 \quad \text{as} \quad n \to +\infty. \tag{35}$$

From (32), we have

$$-\Psi_{n+1} \le \wp \frac{\Gamma_1}{1-\wp} + \wp^{n+1} \|u_0 - p\|^2 + r_{n+1} \|u_{n+1} - u_n\|^2. \tag{36}$$

From Expression (18) and using (21), we have

$$\begin{aligned}&\lambda_{n+1}(1 - 2c_2\sigma - 4c_1\sigma(1+\wp)) \left[\|u_{n+1} - v_n\|^2 + \|\rho_n - v_n\|^2 \right] \\ &\le \Psi_n - \Psi_{n+1} + \wp(1+\wp)\|u_n - u_{n-1}\|^2 + 4c_1\sigma\wp(1+\wp)\|u_n - u_{n-1}\|^2. \end{aligned} \tag{37}$$

Fix $k \in \mathbb{N}$ and using above expression for $n = 1, 2, \cdots, k$. Summing them up, we obtain

$$\begin{aligned}
&\lambda_{n+1}(1 - 2c_2\sigma - 4c_1\sigma(1+\wp)) \sum_{n=1}^{k} \left[\|u_{n+1} - v_n\|^2 + \|\rho_n - v_n\|^2 \right] \\
&\leq \Psi_0 - \Psi_{k+1} + \wp(1+\wp) \sum_{n=1}^{k} \|u_n - u_{n-1}\|^2 + 4c_1\sigma\wp(1+\wp) \sum_{n=1}^{k} \|u_n - u_{n-1}\|^2 \\
&\leq \Psi_0 + \wp\frac{\Gamma_1}{1-\wp} + \wp^{k+1}\|u_0 - p\|^2 + r_{k+1}\|u_{k+1} - u_k\|^2 \\
&\quad + \wp(1+\wp) \sum_{n=1}^{k} \|u_n - u_{n-1}\|^2 + 4c_1\sigma\wp(1+\wp) \sum_{n=1}^{k} \|u_n - u_{n-1}\|^2,
\end{aligned} \tag{38}$$

letting $k \to +\infty$, and due to sum of the positive terms series, we obtain

$$\sum_{n=1}^{+\infty} \|u_{n+1} - v_n\|^2 < +\infty \quad \text{and} \quad \sum_{n=1}^{+\infty} \|\rho_n - v_n\|^2 < +\infty. \tag{39}$$

Moreover, we obtain

$$\lim_{n\to+\infty} \|u_{n+1} - v_n\| = \lim_{n\to+\infty} \|\rho_n - v_n\| = 0. \tag{40}$$

By using the triangular inequality, we get

$$\lim_{n\to+\infty} \|u_n - v_n\| = \lim_{n\to+\infty} \|u_n - \rho_n\| = \lim_{n\to+\infty} \|v_{n-1} - v_n\| = 0. \tag{41}$$

It is follow from the relation (24), we obtain

$$\begin{aligned}
&\|u_{n+1} - p\|^2 \\
&\leq (1+\wp_n)\|u_n - p\|^2 - \wp_n\|u_{n-1} - p\|^2 + \wp(1+\wp)\|u_n - u_{n-1}\|^2 \\
&\quad + 4c_1\sigma(1+\wp)\|u_n - v_{n-1}\|^2 + 4c_1\sigma(\wp + \wp^2)\|u_n - u_{n-1}\|^2,
\end{aligned} \tag{42}$$

with (34), (39) and Lemma 4 imply that the sequences $\|u_n - p\|$, $\|\rho_n - p\|$ and $\|v_n - p\|$ limits exist for every $p \in EP(f, C)$. It means that $\{u_n\}$, $\{\rho_n\}$ and $\{v_n\}$ are bounded sequences. Take z an arbitrary sequential cluster point of the sequence $\{u_n\}$. Now our aim to prove that $z \in EP(f, C)$. By Lemma 6 with Expressions (10) and (12), we write

$$\begin{aligned}
\mu\lambda_{n_k} f(v_{n_k}, y) &\geq \mu\lambda_{n_k} f(v_{n_k}, u_{n_k+1}) + \langle \rho_{n_k} - u_{n_k+1}, y - u_{n_k+1} \rangle \\
&\geq \lambda_{n_k}\lambda_{n_{k+1}} f(v_{n_k-1}, u_{n_k+1}) - \lambda_{n_k}\lambda_{n_{k+1}} f(v_{n_k-1}, v_{n_k}) - c_1\lambda_{n_k}\lambda_{n_{k+1}}\|v_{n_k-1} - v_{n_k}\|^2 \\
&\quad - c_2\lambda_{n_k}\lambda_{n_{k+1}}\|v_{n_k} - u_{n_k+1}\|^2 + \langle \rho_{n_k} - u_{n_k+1}, y - u_{n_k+1} \rangle \\
&\geq \lambda_{n_{k+1}} \langle \rho_{n_k} - v_{n_k}, u_{n_k+1} - v_{n_k} \rangle - c_1\lambda_{n_k}\lambda_{n_{k+1}}\|v_{n_k-1} - v_{n_k}\|^2 \\
&\quad - c_2\lambda_{n_k}\lambda_{n_{k+1}}\|v_{n_k} - u_{n_k+1}\|^2 + \langle \rho_{n_k} - u_{n_k+1}, y - u_{n_k+1} \rangle
\end{aligned} \tag{43}$$

where y in C. Next, from (35), (40), (41) and due to boundedness of $\{u_n\}$ gives that the right hand side reaches to zero. Due to μ, $\lambda_{n_k} > 0$ and $v_{n_k} \rightharpoonup z$, we have

$$0 \leq \limsup_{k\to+\infty} f(v_{n_k}, y) \leq f(z, y), \ \text{ for all } y \in C. \tag{44}$$

Thus, $z \in C$ implies that $f(z, y) \geq 0$, for all $y \in C$. It proves that $z \in EP(f, C)$. By Lemma 5, the sequence $\{u_n\}$ converges weakly to $p \in EP(f, C)$. □

If $\wp_n = 0$ in Algorithm 1, we have a better version of Lyashko et al. [25] extragradient method in terms of step size improvement.

Corollary 1. *Let $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ satisfy the conditions (f_1)-(f_4). For some $p \in EP(f, C) \neq \emptyset$, the sequence $\{u_n\}$ and $\{v_n\}$ generated in the following way:*

(i) *Given $u_0, v_{-1}, v_0 \in \mathbb{H}$, $0 < \sigma < \min\left\{1, \frac{1}{2c_2+4c_1}\right\}$, $\mu \in (0, \sigma)$ and $\lambda_0 > 0$.*

(ii) *Compute*

$$\begin{cases} u_{n+1} = \arg\min\limits_{y \in C}\{\mu\lambda_n f(v_n, y) + \frac{1}{2}\|u_n - y\|^2\}, \\ v_{n+1} = \arg\min\limits_{y \in C}\{\lambda_{n+1} f(v_n, y) + \frac{1}{2}\|u_{n+1} - y\|^2\}, \end{cases}$$

where

$$\lambda_{n+1} = \min\left\{\sigma, \frac{\mu f(v_n, u_{n+1})}{f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) - c_1\|v_{n-1} - v_n\|^2 - c_2\|u_{n+1} - v_n\|^2 + 1}\right\}.$$

Then, the sequences $\{u_n\}$ and $\{v_n\}$ converge weakly to $p \in EP(f, C)$.

4. Applications

Now, we consider the applications of Theorem 1 to solve the variational inequality problems involving pseudomonotone and Lipschitz continuous operator. A variational inequality problem is defined in the following way:

$$\text{Find } p^* \in C \text{ such that } \langle F(p^*), v - p^* \rangle \geq 0, \text{ for all } v \in C.$$

We consider that F meets the following conditions.

(F_1) Solution set $VI(F, C)$ is non-empty and F is pseudomonotone on C, i.e.,

$$\langle F(u), v - u \rangle \geq 0 \quad \text{implies that} \quad \langle F(v), u - v \rangle \leq 0, \text{ for all } u, v \in C;$$

(F_2) F is L-Lipschitz continuous on C if there exists a positive constants $L > 0$ such that

$$\|F(u) - F(v)\| \leq L\|u - v\|, \text{ for all } u, v \in C.$$

(F_3) $\limsup\limits_{n \to +\infty} \langle F(u_n), v - u_n \rangle \leq \langle F(p^*), v - p^* \rangle$ for every $v \in C$ and $\{u_n\} \subset C$ satisfying $u_n \rightharpoonup p^*$.

Corollary 2. *Assume that $F : C \to \mathbb{H}$ meet the conditions (F_1)–(F_3). Let $\{\rho_n\}$, $\{u_n\}$ and $\{v_n\}$ be the sequences are generated in the following way:*

(i) *Choose $u_{-1}, v_{-1}, u_0, v_0 \in \mathbb{H}$ and a sequence $\wp_n$ is non-decreasing such that $0 \leq \wp_n \leq \wp < \frac{1}{3}$, $\lambda_0 > 0$, $0 < \sigma < \min\left\{\frac{1-3\wp}{(1-\wp)^2 + 2L(\wp+\wp^2)}, \frac{1}{3L+2\wp L)}\right\}$ and $\mu \in (0, \sigma)$.*

(ii) *Compute*

$$\begin{cases} u_{n+1} = P_C(\rho_n - \mu\lambda_n F(v_n)), \quad \text{where} \quad \rho_n = u_n + \wp_n(u_n - u_{n-1}), \\ v_{n+1} = P_C(\rho_{n+1} - \lambda_{n+1} F(v_n)), \quad \text{where} \quad \rho_{n+1} = u_{n+1} + \wp_{n+1}(u_{n+1} - u_n), \end{cases}$$

while

$$\lambda_{n+1} = \min\left\{\sigma, \frac{\mu\langle Fv_n, u_{n+1} - v_n\rangle}{\langle Fv_{n-1}, u_{n+1} - v_n\rangle - \frac{L}{2}\|v_{n-1} - v_n\|^2 - \frac{L}{2}\|u_{n+1} - v_n\|^2 + 1}\right\}.$$

Then, the sequence $\{\rho_n\}$, $\{u_n\}$ and $\{v_n\}$ converge weakly to p.

Corollary 3. *Assume that $F : C \to \mathbb{H}$ meets the conditions (F_1)-(F_3). Let $\{u_n\}$ and $\{v_n\}$ be the sequences are generated in the following way:*

(i) *Choose $v_{-1}, u_0, v_0 \in \mathbb{H}$, $0 < \sigma < \min\{1, \frac{1}{3L}\}$ and $\lambda_0 > 0$.*

(ii) *Compute*

$$\begin{cases} u_{n+1} = P_C(u_n - \mu\lambda_n F(v_n)), \\ v_{n+1} = P_C(u_{n+1} - \lambda_{n+1} F(v_n)), \end{cases}$$

while

$$\lambda_{n+1} = \min\left\{\sigma, \frac{\mu\langle Fv_n, u_{n+1} - v_n\rangle}{\langle Fv_{n-1}, u_{n+1} - v_n\rangle - \frac{L}{2}\|v_{n-1} - v_n\|^2 - \frac{L}{2}\|u_{n+1} - v_n\|^2 + 1}\right\}.$$

Then, the sequence $\{u_n\}$ and $\{v_n\}$ converge weakly to p.

5. Computational Illustration

Numerical findings are discussed in this section to show the efficiency of our suggested method. Moreover, for Lyashko et al.'s [25] method (`L.EgA`), our proposed algorithm (`Algo.1`) and we use error term $D_n = \|u_{n+1} - u_n\|$.

Example 1. *Consider the Nash–Cournot equilibrium of electricity markets as in [7]. In this problem, there are total three electricity producing firms: i $(i = 1, 2, 3)$. The firm's 1,2,3 have generating units named as $I_1 = \{1\}$, $I_2 = \{2, 3\}$ and $I_3 = \{4, 5, 6\}$, respectively. Assume that u_j denote the producing power of the unit for $i = \{1, 2, 3, 4, 5, 6\}$. Suppose that the value p of electricity can be taken as $p = 378.4 - 2\sum_{j=1}^{6} u_j$. The cost of the manufacture j unit follows:*

$$c_j(u_j) := \max\{\mathring{c}_j(u_j), \overset{\bullet}{c}_j(u_j)\},$$

where $\mathring{c}_j(u_j) := \frac{\mathring{\alpha}_j}{2}u_j^2 + \mathring{\beta}_j u_j + \mathring{\gamma}_j$ and $\overset{\bullet}{c}_j(u_j) := \overset{\bullet}{\alpha}_j u_j + \frac{\overset{\bullet}{\beta}_j}{\overset{\bullet}{\beta}_j + 1}\overset{\bullet}{\gamma}_j^{\frac{-1}{\overset{\bullet}{\beta}_j}}(u_j)^{\frac{(\overset{\bullet}{\beta}_j+1)}{\overset{\bullet}{\beta}_j}}$. The values are provided in $\mathring{\alpha}_j$, $\mathring{\beta}_j$, $\mathring{\gamma}_j$, $\overset{\bullet}{\alpha}_j$, $\overset{\bullet}{\beta}_j$ and $\overset{\bullet}{\gamma}_j$ in Table 1. Profit of the firm i is

$$f_i(u) := p\sum_{j \in I_i} u_j - \sum_{j \in I_i} c_j(u_j) = \left(378.4 - 2\sum_{l=1}^{6} u_l\right)\sum_{j \in I_i} u_j - \sum_{j \in I_i} c_j(u_j),$$

where $u = (u_1, \cdots, u_6)^T$ with reference to set $u \in C := \{u \in \mathbb{R}^6 : u_j^{\min} \le u_j \le u_j^{\max}\}$, with $u_j^{\min}$ and $u_j^{\max}$ give in Table 2. Define the equilibrium bifunction f in the following way:

$$f(u, v) := \sum_{i=1}^{3}\left(\phi_i(u, u) - \phi_i(u, v)\right),$$

where

$$\phi_i(u, v) := \left[378.4 - 2\Big(\sum_{j \notin I_i} u_j + \sum_{j \in I_i} v_j\Big)\right]\sum_{j \in I_i} v_j - \sum_{j \in I_i} c_j(v_j).$$

This model of electricity markets can be viewed as an equilibrium problem

$$\text{Find } u^* \in C \text{ such that } f(u^*, v) \ge 0, \text{ for all } v \in C.$$

Numerical conclusions have shown in Figures 1–4 and Table 3. For these numerical experiments we take $u_{-1} = v_{-1} = u_0 = v_0 = (48, 48, 30, 27, 18, 24)^T$ and $\lambda = 0.01$, $\sigma = 0.026$, $\mu = 0.024$, $\wp_n = 0.20$, $\lambda_0 = 0.1$.

Table 1. Parameters for cost bi-function.

Unit j	$\overset{\circ}{\alpha}_j$	$\overset{\circ}{\beta}_j$	$\overset{\circ}{\gamma}_j$	$\overset{\bullet}{\alpha}_j$	$\overset{\bullet}{\beta}_j$	$\overset{\bullet}{\gamma}_j$
1	0.04	2	0	2	1	25
2	0.035	1.75	0	1.75	1	28.5714
3	0.125	1	0	1	1	8
4	0.0116	3.25	0	3.25	1	86.2069
5	0.05	3	0	3	1	20
6	0.05	3	0	3	1	20

Table 2. Values used for constraint set.

j	u_j^{min}	u_j^{max}
1	0	80
2	0	80
3	0	50
4	0	55
5	0	30
6	0	40

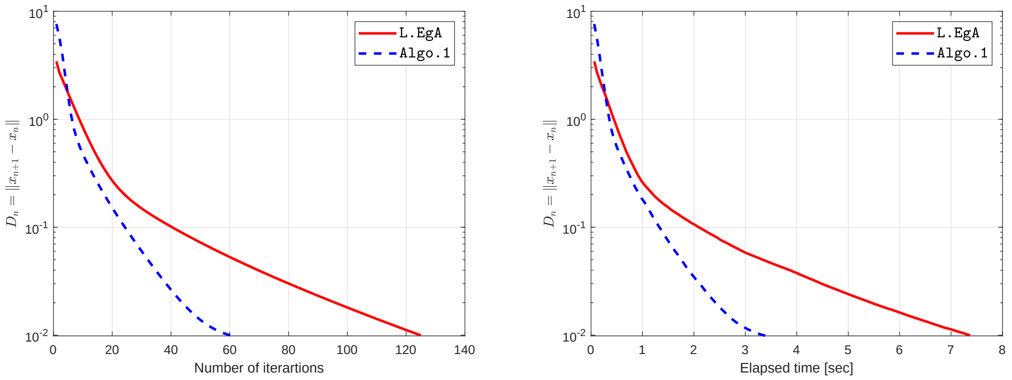


Figure 1. Example 1 while tolerance is 0.01.

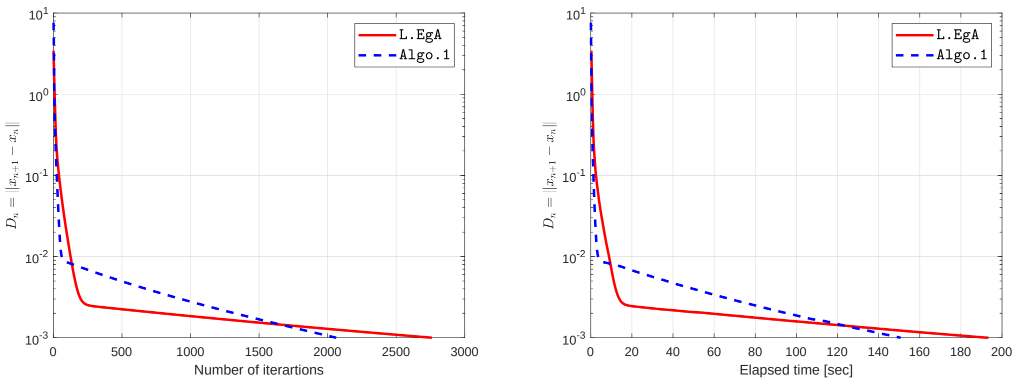


Figure 2. Example 1 while tolerance is 0.001.

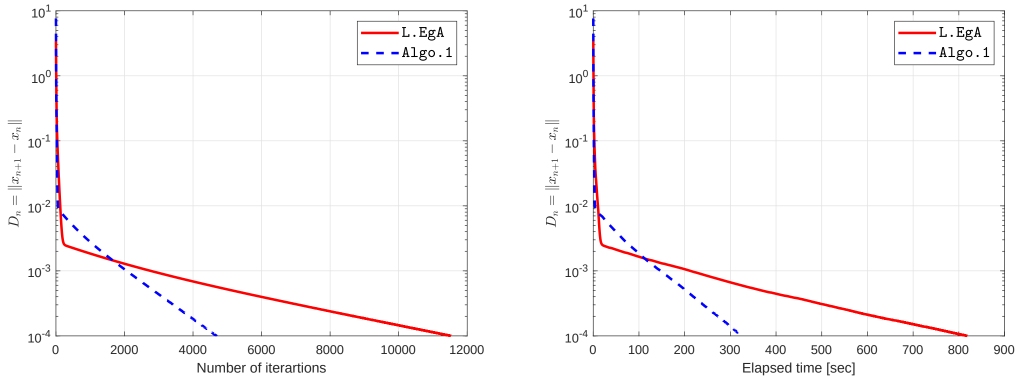


Figure 3. Example 1 while tolerance is 0.0001.

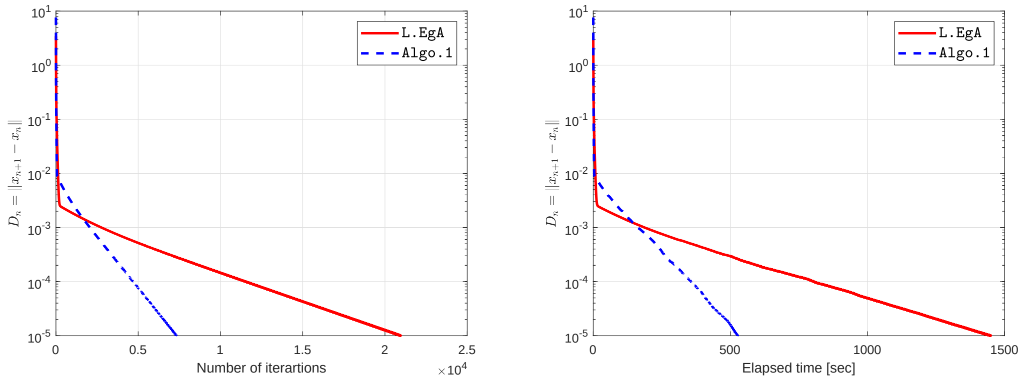


Figure 4. Example 1 while tolerance is 0.00001.

Table 3. Figures 1–4 numerical values.

	L.EgA		Algo.1	
TOL	Iter.	time (s)	Iter.	time (s)
0.01 $u^*_{\texttt{L.EgA}} = (47.3245, 47.3245, 47.3245, 47.3245, 47.3245, 47.3245)$ $u^*_{\texttt{Algo.1}} = (47.3245, 47.3245, 47.3245, 47.3245, 47.3245, 47.3245)$	125	7.3692	61	3.4055
0.001 $u^*_{\texttt{L.EgA}} = (47.3245, 47.3245, 47.3245, 47.3245, 47.3245, 47.3245)$ $u^*_{\texttt{Algo.1}} = (47.3245, 47.3245, 47.3245, 47.3245, 47.3245, 47.3245)$	2761	193.3939	2063	150.6757
0.0001 $u^*_{\texttt{L.EgA}} = (47.3245, 47.3245, 47.3245, 47.3245, 47.3245, 47.3245)$ $u^*_{\texttt{Algo.1}} = (47.3245, 47.3245, 47.3245, 47.3245, 47.3245, 47.3245)$	11,526	818.7184	4687	324.3571
0.00001 $u^*_{\texttt{L.EgA}} = (47.3245, 47.3245, 47.3245, 47.3245, 47.3245, 47.3245)$ $u^*_{\texttt{Algo.1}} = (47.3245, 47.3245, 47.3245, 47.3245, 47.3245, 47.3245)$	20,946	1449.3959	7307	526.9766

Example 2. *Assume that the following cost bifunction f defined by*

$$f(u, v) = \langle (AA^T + B + C)u, v - u \rangle,$$

on the convex set $C = \{u \in \mathbb{R}^n : Du \leq d\}$ while D is an $100 \times n$ matrix and d is a non-negative vector. In the above bifunction definition we take A is an $n \times n$ matrix, B is an $n \times n$ skew-symmetric matrix, C is an $n \times n$ diagonal matrix having diagonal entries are non-negative. The matrices are generated as; A = rand(n), K = rand(n), $B = 0.5K - 0.5K^T$ and C = diag(rand(n,1)). The bifunction f is monotone and *having Lipschitz-type constants are* $c_1 = c_2 = \frac{1}{2}\|AA^T + B + C\|$. *Numerical results are presented in the Figures 5–8 and Table 4. For these numerical experiments we take* $u_{-1} = v_{-1} = u_0 = v_0 = (1, 1, \cdots, 1)^T$ *and* $\lambda = \frac{1}{10c_1}$, $\sigma = \frac{1}{8c_1}$, $\mu = \frac{1}{8.2c_1}$, $\wp_n = \frac{1}{5}$, $\lambda_0 = 1/4c_1$.

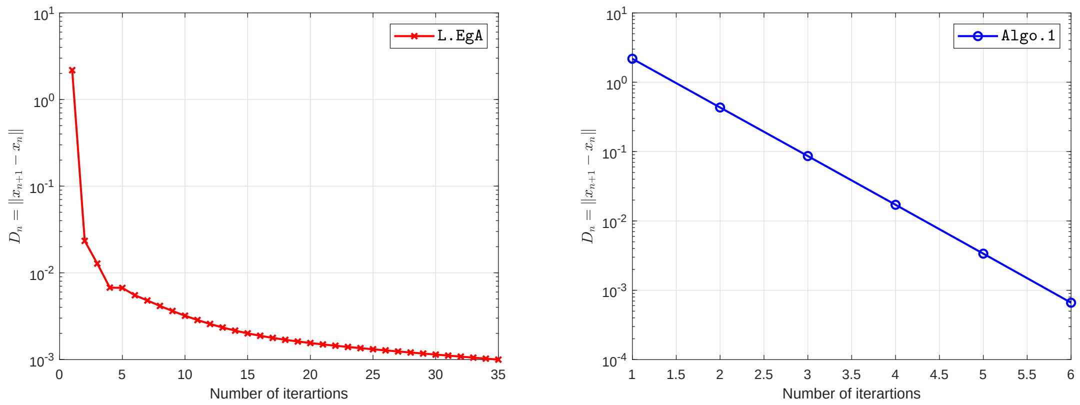


Figure 5. Example 2 for average number of iterations while $n = 5$.

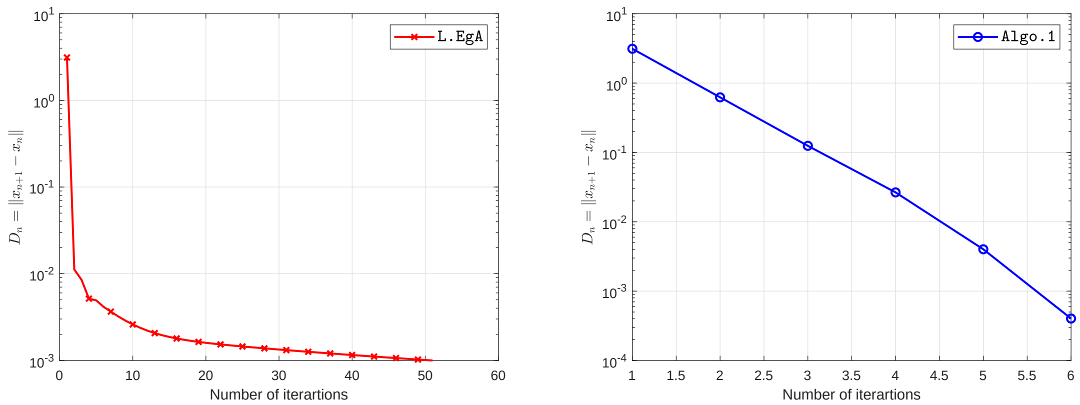


Figure 6. Example 2 for average number of iterations while $n = 10$.

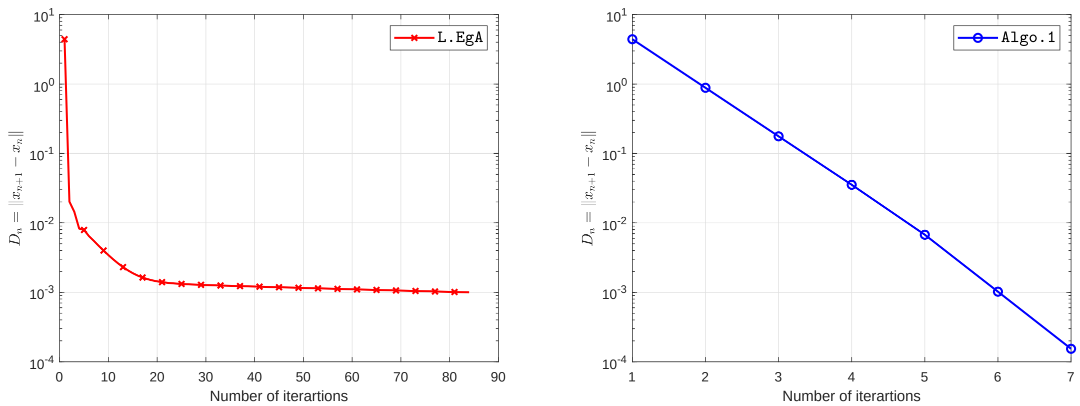
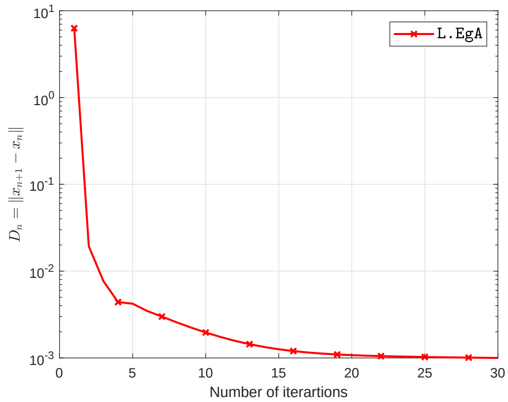


Figure 7. Example 2 for average number of iterations while $n = 20$.



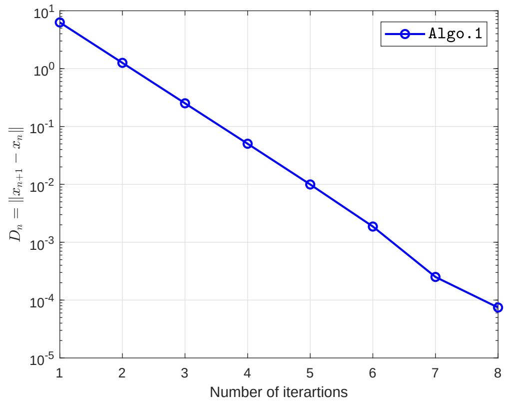


Figure 8. Example 2 for average number of iterations while $n = 40$.

Table 4. Numerical results for Figures 5–8.

		L.EgA		Algo.1	
n	**T. Samples**	Avg Iter.	Avg time(s)	Avg Iter.	Avg time(s)
5	10	35	0.8066	6	0.1438
10	10	51	1.1779	6	0.1302
20	10	84	1.7441	7	0.1801
40	10	30	0.6859	8	0.1999

Example 3. *Assume that $F : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by*

$$F(u) = \begin{pmatrix} 0.5u_1u_2 - 2u_2 - 10^7 \\ -4u_1 - 0.1u_2^2 - 10^7 \end{pmatrix}$$

with $C = \{u \in \mathbb{R}^2 : (u_1 - 2)^2 + (u_2 - 2)^2 \le 1\}$. It is not hard to check that F is Lipschitz continuous with $L = 5$ and pseudomonotone. The step size $\lambda = 10^{-6}$ for Lyashko et al. [25] and $\lambda_0 = 0.1$, $\sigma = 0.129$, $\wp_n = 0.20$ and $\mu = 0.119$. Computational results are shown in the Table 5 and in Figures 9–12.

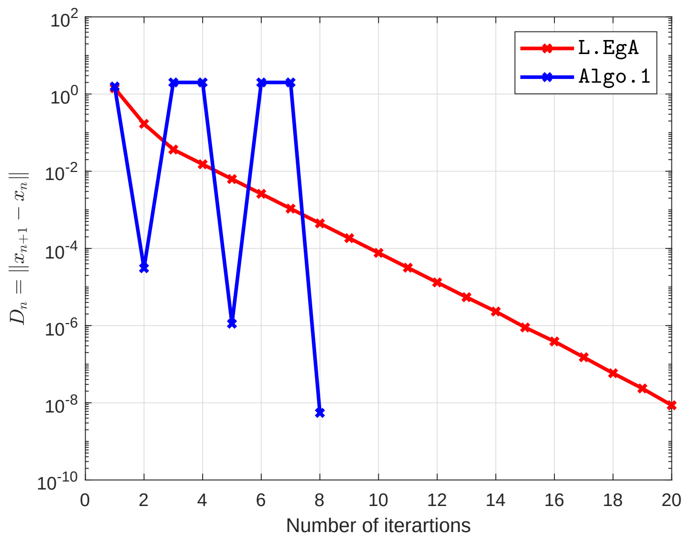


Figure 9. Example 3 while $u_0 = (1.5, 1.7)$.

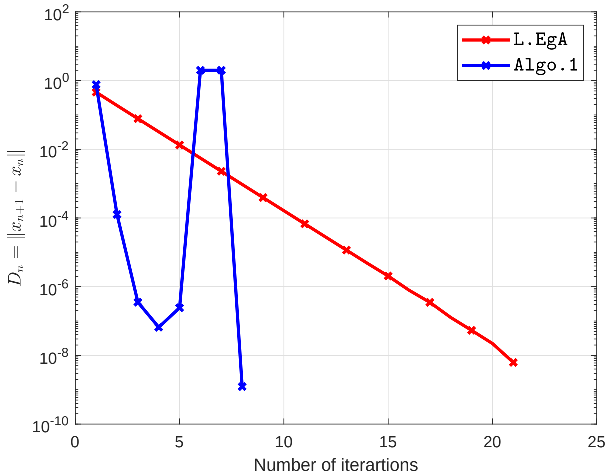


Figure 10. Example 3 while $u_0 = (2, 3)$.

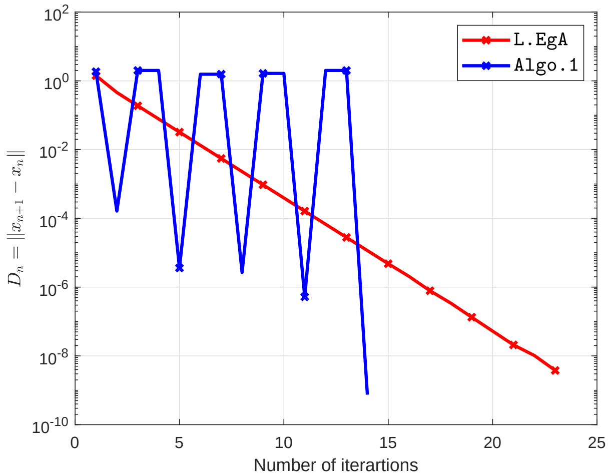


Figure 11. Example 3 while $u_0 = (1, 2)$.

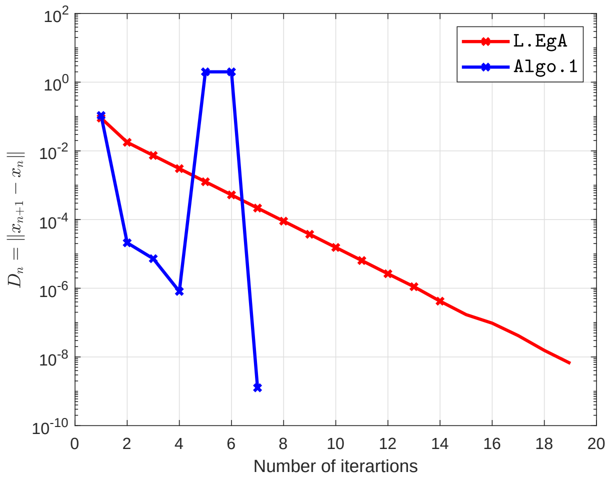


Figure 12. Example 3 while $u_0 = (2.7, 2.6)$.

Table 5. Numerical results for Figures 9–12.

u_0	L.EgA		Algo.1	
	Iter.	time(s)	Iter.	time(s)
$(1.5, 1.7)$	20	0.7506	8	0.5316
$(2.0, 3.0)$	21	0.7879	8	0.6484
$(1.0, 2.0)$	23	1.1450	14	0.9730
$(2.7, 2.6)$	19	0.7254	7	0.5835

Example 4. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by*

$$F(u) = \begin{pmatrix} (u_1^2 + (u_2 - 1)^2)(1 + u_2) \\ -u_1^3 - u_1(u_2 - 1)^2 \end{pmatrix}$$

and $C = \{u \in \mathbb{R}^2 : (u_1 - 2)^2 + (u_2 - 2)^2 \leq 1\}$. Here, F is not monotone but pseudomonotone on C and L-Lipschitz continuous through $L = 5$ (see, e.g., [33]). The stepsize $\lambda = 10^{-2}$ for Lyashko et al. [25] and $\lambda_0 = 0.01$, $\sigma = 0.129$, $\wp_n = 0.15$ and $\mu = 0.119$. The computational experimental findings are written in Table 6 and in Figures 13–15.

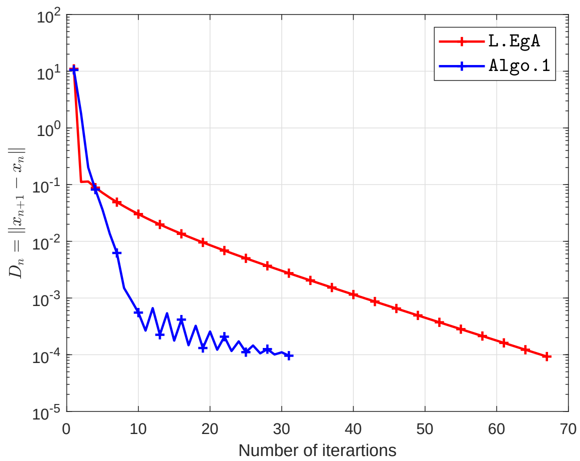


Figure 13. Example 4 while $u_0 = (10, 10)$.

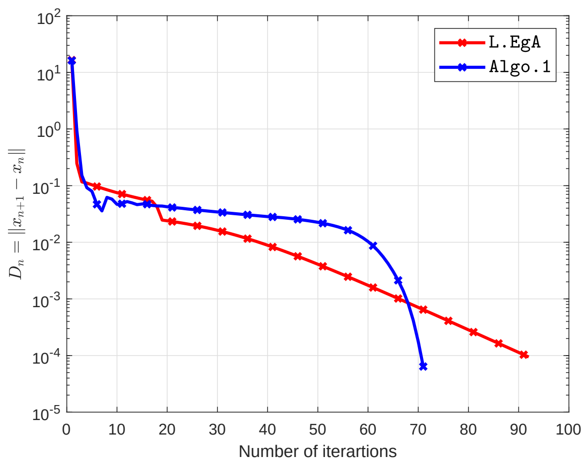


Figure 14. Example 4 while $u_0 = (-10, -10)$.

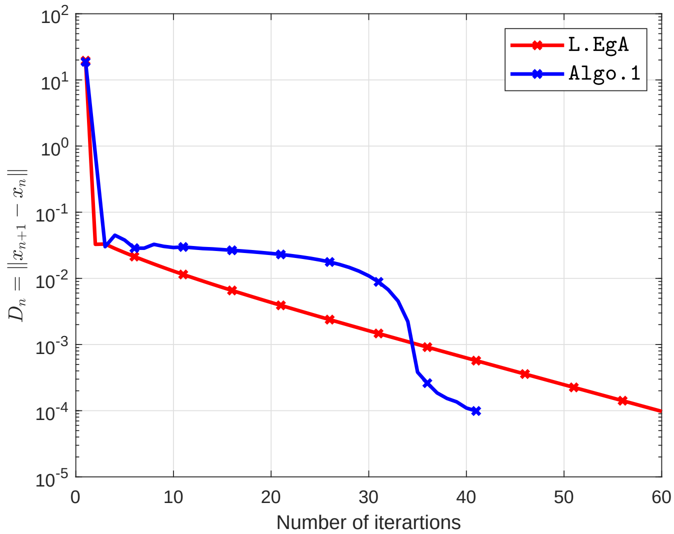


Figure 15. Example 4 while $u_0 = (10, 20)$.

Table 6. Figures 13–15 numerical values.

	L.EgA		Algo.1	
u_0	Iter.	time(s)	Iter.	time(s)
$(10, 10)$	67	1.9151	31	1.0752
$(-10, -10)$	92	2.5721	71	2.0469
$(10, 20)$	60	1.7689	41	1.1864

6. Conclusions

We have established an extragradient-like method to solve pseudomonotone equilibrium problems in real Hilbert space. The main advantage of the proposed method is that an iterative sequence has been incorporated with a certain step size evaluation formula. The step size formula is updated for each iteration based on the previous iterations. Numerical findings were presented to show our algorithm's numerical efficiency compared with other methods. Such numerical investigations indicate that inertial effects often generally improve the effectiveness of the iterative sequence in this context.

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