axioms

MDPI

Article

An Accelerated Extragradient Method for Solving Pseudomonotone Equilibrium Problems with Applications

Nopparat Wairojjana [1], Habib ur Rehman [2], Ioannis K. Argyros [3,*] and Nuttapol Pakkaranang [2]

1 Applied Mathematics Program, Faculty of Science and Technology, Valaya Alongkorn Rajabhat University under the Royal Patronage (VRU), 1 Moo 20 Phaholyothin Road, Klong Neung, Klong Luang, Pathumthani 13180, Thailand; nopparat@vru.ac.th

2 KMUTTFixed Point Research Laboratory, KMUTT-Fixed Point Theory and Applications Research Group, SCL 802 Fixed Point Laboratory, Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand; habib.rehman@mail.kmutt.ac.th (H.u.R.); nuttapol.pak@mail.kmutt.ac.th (N.P.)

3 Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA

* Correspondence: iargyros@cameron.edu

Received: 7 July 2020; Accepted: 13 August 2020; Published: 17 August 2020

Abstract: Several methods have been put forward to solve equilibrium problems, in which the two-step extragradient method is very useful and significant. In this article, we propose a new extragradient-like method to evaluate the numerical solution of the pseudomonotone equilibrium in real Hilbert space. This method uses a non-monotonically stepsize technique based on local bifunction values and Lipschitz-type constants. Furthermore, we establish the weak convergence theorem for the suggested method and provide the applications of our results. Finally, several experimental results are reported to see the performance of the proposed method.

Keywords: Lipschitz-type conditions; pseudomonotone bifunction; equilibrium problem; variational inequality problems; weak convergence; fixed point problems

1. Introduction

Assume $\mathcal{K}$ to be a subset of a Hilbert space $\mathcal{E}$ with $f : \mathcal{E} \times \mathcal{E} \to \mathcal{R}$ a bifunction with $f(z_1, z_1) = 0$, for every $z_1 \in \mathcal{K}$. An equilibrium problem [1,2] for f on $\mathcal{K}$ is formulated in the following way:

$$\text{Find } q^* \in \mathcal{K} \text{ such that } f(q^*, z_1) \geq 0, \ \forall z_1 \in \mathcal{K}. \tag{1}$$

The equilibrium problem (1) has many mathematical problems as particular instances such as the variational inequality problems (VIP), the minimization problems, the fixed point problems, the complementarity problems, the Nash equilibrium of non-cooperative games, the saddle point problems, and the vector optimization problem (see [1,3,4] for more details). The term "equilibrium problem" in a particular format was established in 1992 by Muu and Oettli [2], and it was further promoted by Blum and Oettli in the article [1]. Some of the most popular ones, interesting and worthwhile research fields in equilibrium problem theory, are to develop new iterative schemes, improve the convergence rate and efficiency of the already existing methods, and study their converging analysis with optimal conditions. Several methods have been established in the past few years to solve the equilibrium problems in real Hilbert spaces, i.e., the extragradient methods [5–14], the inertial methods [15–20], for particular classes of equilibrium problems [21–28], and others in [29–37].

The proximal-like method [38] is one of the famous and efficient techniques to solve equilibrium problems. This technique is equivalent to solving minimization problems on each iteration. This approach was also considered as the two-step extragradient method in [5] due to the previous contribution of the Korpelevich method [39] to numerically solve the saddle point problems. Tran et al. in [5] generated the sequence $\{u_n\}$ in the following way:

$$\begin{cases} u_0 \in \mathcal{K}, \\ v_n = \arg\min_{v \in \mathcal{K}} \{\lambda f(u_n, v) + \frac{1}{2}\|u_n - v\|^2\}, \\ u_{n+1} = \arg\min_{v \in \mathcal{K}} \{\lambda f(v_n, v) + \frac{1}{2}\|u_n - v\|^2\}, \end{cases}$$

where $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$. The iterative sequence generated by the above-written method provides the weak convergence of the iterative sequence, and in order to operate it, previous knowledge of Lipschitz-type constants is required that help to choose the value of the stepsize.

Recently, the authors introduced an inertial iterative scheme in [19] to determine a numerical solution of pseudomonotone equilibrium problems. The key contribution is an inertial factor that has helped to enhance the rate of convergence of the iterative sequence $\{u_n\}$. The detailed method is provided as follows:

Step 1: Choose $u_{-1}, u_0 \in \mathcal{K}$, $\theta \in [0, 1)$, $0 < \lambda < \{\frac{1}{2c_1}, \frac{1}{2c_2}\}$ and a sequence $\{\epsilon_n\} \subset [0, +\infty)$ such that:

$$\sum_{n=0}^{+\infty} \epsilon_n < +\infty, \tag{2}$$

holds. Let θ_n satisfy $0 \leq \theta_n \leq \bar{\theta_n}$ such that:

$$\bar{\theta_n} = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|u_n - u_{n-1}\|}\right\} & \text{if} \quad u_n \neq u_{n-1}, \\ \theta & \text{otherwise.} \end{cases} \tag{3}$$

Step 2: Compute:

$$\begin{cases} \rho_n = u_n + \theta_n(u_n - u_{n-1}), \\ v_n = \arg\min_{v \in \mathcal{K}} \{\lambda f(\rho_n, v) + \frac{1}{2}\|\rho_n - v\|^2\}, \\ u_{n+1} = \arg\min_{v \in \mathcal{K}} \{\lambda f(v_n, v) + \frac{1}{2}\|\rho_n - v\|^2\}. \end{cases}$$

In this study, we concentrate on projection methods that are well known and practically straightforward to operate due to their simple numerical computation. Motivated by the works of [19,40], we propose an inertial explicit extragradient method to solve pseudomonotone equilibrium problems and other particular classes of equilibrium problems such as the fixed point problems and the variational inequality problems. The proposed method can be considered as the modification of the methods that appeared in [5,19,39,40]. Under certain mild conditions, the weak convergence results are established corresponding to the proposed method. Numerical studies have been demonstrated that show that the suggested method is more efficient than the existing method in [19].

The remainder of the paper is arranged in the following way: Section 2 includes some preliminary and necessary results that will be used throughout the paper. Section 3 contains our main method, as well as the weak convergence theorem. Section 4 covers the applications of the proposed method. Section 5 demonstrates the numerical results that provide the computational performance of our proposed method.

2. Preliminaries

Assume $\mathcal{K} \subset \mathcal{E}$ to be a convex and closed subset of a real Hilbert space $\mathcal{E}$, and $\mathcal{R}$ and $\mathcal{N}$ denote the set of real numbers and the set of a natural numbers, respectively. Let $f : \mathcal{E} \times \mathcal{E} \to \mathcal{R}$ be a bifunction and $EP(f, \mathcal{K})$ be the solution set of an equilibrium problem on the set $\mathcal{K}$, q^* being an arbitrary element of $EP(f, \mathcal{K})$. Next, we consider the definitions of a bifunction monotonicity (see [1,41] for more details). A bifunction $f : \mathcal{E} \times \mathcal{E} \to \mathcal{R}$ on $\mathcal{K}$ for $\gamma > 0$ is said to be:

(1) strongly monotone if:

$$f(z_1, z_2) + f(z_2, z_1) \leq -\gamma \|z_1 - z_2\|^2, \ \forall z_1, z_2 \in \mathcal{K};$$

(2) monotone if:

$$f(z_1, z_2) + f(z_2, z_1) \leq 0, \ \forall z_1, z_2 \in \mathcal{K};$$

(3) strongly pseudomonotone if:

$$f(z_1, z_2) \geq 0 \Longrightarrow f(z_2, z_1) \leq -\gamma \|z_1 - z_2\|^2, \ \forall z_1, z_2 \in \mathcal{K};$$

(4) pseudomonotone if:

$$f(z_1, z_2) \geq 0 \Longrightarrow f(z_2, z_1) \leq 0, \ \forall z_1, z_2 \in \mathcal{K}.$$

The following implications can be seen from the definitions mentioned above:

$$(1) \Longrightarrow (2) \Longrightarrow (4) \text{ and } (1) \Longrightarrow (3) \Longrightarrow (4).$$

Generally speaking, the converse is not true. We say that a bifunction $f : \mathcal{E} \times \mathcal{E} \to \mathcal{R}$ satisfies the Lipschitz-type condition [42] on set $\mathcal{K}$ if there exist two constants $c_1, c_2 > 0$, such that:

$$f(z_1, z_3) \leq f(z_1, z_2) + f(z_2, z_3) + c_1 \|z_1 - z_2\|^2 + c_2 \|z_2 - z_3\|^2, \ \ \forall z_1, z_2, z_3 \in \mathcal{K}.$$

Let $h : \mathcal{K} \to \mathcal{R}$ be a convex function, and the subdifferential of h at $z_1 \in \mathcal{K}$ is defined by:

$$\partial h(z_1) \ = \ \{z_3 \in \mathcal{E} : h(z_2) - h(z_1) \geq \langle z_3, z_2 - z_1 \rangle, \ \forall z_2 \in \mathcal{K}\}.$$

A normal cone of $\mathcal{K}$ at $z_1 \in \mathcal{K}$ is defined by:

$$N_{\mathcal{K}}(z_1) \ = \ \{z_3 \in \mathcal{E} : \langle z_3, z_2 - z_1 \rangle \leq 0, \ \forall z_2 \in \mathcal{K}\}.$$

The metric projection $P_C(z_1)$ for $z_1 \in \mathcal{E}$ on $\mathcal{K}$ of $\mathcal{E}$ is defined by:

$$P_C(z_1) \ = \ \arg\min\{\|z_2 - z_1\| : z_2 \in \mathcal{K}\}.$$

Lemma 1. [43] *Let $h : \mathcal{K} \to \mathcal{R}$ be a subdifferentiable, convex, and lower semi-continuous function on $\mathcal{K}$, where $\mathcal{K}$ is a nonempty, convex, and closed subset of a real Hilbert space $\mathcal{E}$. An element $z_1 \in \mathcal{K}$ is a minimizer of a function h if and only if $0 \in \partial h(z_1) + N_{\mathcal{K}}(z_1)$, where $\partial h(z_1)$ and $N_{\mathcal{K}}(z_1)$ denote the subdifferential of h at $z_1 \in \mathcal{K}$ and the normal cone of $\mathcal{K}$ at z_1, respectively.*

Lemma 2. [44] *Suppose that a sequence $\{\eta_n\}$ in $\mathcal{E}$ and $\mathcal{K} \subset \mathcal{E}$ such that the following conditions are true.*

(i) *For every $\eta \in \mathcal{K}$, $\lim_{n\to\infty} \|\eta_n - \eta\|$ exists;*
(ii) *each sequentially weak cluster limit point of $\{\eta_n\}$ belongs to set $\mathcal{K}$.*

Then, $\{\eta_n\}$ weakly converges to a point in $\mathcal{K}$.

Lemma 3. [45] *For $u, v \in \mathcal{E}$ and $\tau \in \mathcal{R}$, then the following equality holds.*

$$\|\tau u + (1-\tau)v\|^2 = \tau\|u\|^2 + (1-\tau)\|v\|^2 - \tau(1-\tau)\|u-v\|^2.$$

Lemma 4. [46] *Assume that the sequence $\{q_n\}$ and $\{r_n\}$ of nonnegative real numbers satisfies $q_{n+1} \leq q_n + r_n, \forall n \in \mathcal{N}$. If $\sum r_n < \infty$, then $\lim_{n\to\infty} q_n$ exists.*

Lemma 5. [40] *Assume that $\{a_n\}, \{b_n\}$ are real numbers sequences such that $a_n \leq b_n, \forall n \in \mathcal{N}$. Let $\varrho, \sigma \in (0,1)$ and $\mu \in (0, \sigma)$. Then, there exists a sequence λ_n such that $\lambda_n a_n \leq \mu b_n$ and $\lambda_n \in (\varrho\mu, \sigma)$.*

Corollary 1. *Assume that f satisfies a Lipschitz-type condition on $\mathcal{K}$ with constants $c_1 > 0$ and $c_2 > 0$. Let $\varrho \in (0,1), 0 < \sigma < \min\{1, \frac{1}{2c_1}, \frac{1}{2c_2}\}$, and $\mu \in (0, \sigma)$. Then, there exists $\lambda > 0$ such that:*

$$\lambda\big(f(z_1, z_3) - f(z_1, z_2) - c_1\|z_1 - z_2\|^2 - c_2\|z_2 - z_3\|^2\big) \leq \mu f(z_2, z_3)$$

where $\varrho\mu < \lambda < \sigma$ with $z_1, z_2, z_3 \in \mathcal{K}$.

Assume that a bifunction f satisfies the following conditions:

(Ψ1) $f(z_2, z_2) = 0$, for all $z_2 \in \mathcal{K}$, and f is pseudomonotone on $\mathcal{K}$;
(Ψ2) f satisfies the Lipschitz-type condition on $\mathcal{E}$ through $c_1 > 0$ and $c_2 > 0$;
(Ψ3) $\limsup_{n\to\infty} f(z_n, v) \leq f(z^*, v)$ for each $v \in \mathcal{K}$ and $\{z_n\} \subset \mathcal{K}$ satisfy $z_n \rightharpoonup z^*$;
(Ψ4) $f(z_1, .)$ is subdifferentiable and convex on $\mathcal{K}$ for each $z_1 \in \mathcal{K}$.

3. An Accelerated Method for Pseudomonotone Equilibrium Problems and Its Convergence Analysis

Now, we present a method that consists of two strongly convex optimization problems with an inertial term and an explicit formula for stepsize evaluation. The detailed method is provided below:

Algorithm 1 (Accelerated method for pseudomonotone equilibrium problems)

Initialization: Choose $u_{-1}, u_0 \in \mathcal{E}$, $\varrho \in (0,1)$, $\sigma < \min\{1, \frac{1}{2c_1}, \frac{1}{2c_2}\}$, $\mu \in (0, \sigma)$, $\lambda_0 > 0$, $\theta \in [0,1)$, and a sequence $\{\epsilon_n\} \subset [0, +\infty)$ such that: $\sum_{n=0}^{+\infty} \epsilon_n < +\infty$.
Iterative steps: Let θ_n satisfy $0 \leq \theta_n \leq \bar{\theta_n}$ such that:

$$\bar{\theta_n} = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|u_n - u_{n-1}\|}\right\} & \text{if} \quad u_n \neq u_{n-1}, \\ \theta & \text{otherwise.} \end{cases} \tag{4}$$

Step 1: Compute:

$$v_n = \arg\min_{y \in \mathcal{K}}\{\lambda_n f(\rho_n, y) + \frac{1}{2}\|\rho_n - y\|^2\},$$

where $\rho_n = u_n + \theta_n(u_n - u_{n-1})$. If $\rho_n = v_n$; STOP. Otherwise, go to the next step.
Step 2: Set $0 < \beta \leq \beta_n \leq 1$, and compute

$$u_{n+1} = (1 - \beta_n)\rho_n + \beta_n z_n,$$

where $z_n = \arg\min\{\mu\lambda_n f(v_n, y) + \frac{1}{2}\|\rho_n - y\|^2 : y \in \mathcal{K}\}$.
Step 3: Revised the stepsize in the following way:

$$\lambda_{n+1} = \min\left\{\sigma, \frac{\mu f(v_n, z_n)}{f(\rho_n, z_n) - f(\rho_n, v_n) - c_1\|\rho_n - v_n\|^2 - c_2\|z_n - v_n\|^2 + 1}\right\}. \tag{5}$$

Set $n := n + 1$, and return back to **Iterative steps**.

Remark 1. *From Corollary 1, the definition of λ_{n+1} in (5) is well-defined such that:*

$$\lambda_{n+1}\Big[f(\rho_n, z_n) - f(\rho_n, v_n) - c_1\|\rho_n - v_n\|^2 - c_2\|v_n - z_n\|^2\Big] \le \mu f(v_n, z_n). \tag{6}$$

Remark 2. *Due to the summability of $\sum_{n=0}^{+\infty} \epsilon_n$, the expression (4) provides that:*

$$\sum_{n=1}^{\infty} \theta_n\|u_n - u_{n-1}\| \le \sum_{n=1}^{\infty} \bar{\theta_n}\|u_n - u_{n-1}\| \le \sum_{n=1}^{\infty} \theta\|u_n - u_{n-1}\| < \infty, \tag{7}$$

which implies that:

$$\lim_{n\to\infty} \theta\|u_n - u_{n-1}\| \ = \ 0. \tag{8}$$

Lemma 6. *If $v_n \ = \ \rho_n$ in Algorithm 1, then $\rho_n \in EP(f, \mathcal{K})$.*

Proof. From the value of v_n and Lemma 1, we have:

$$0 \in \partial_2\Big\{\lambda_n f(\rho_n, y) + \frac{1}{2}\|\rho_n - y\|^2\Big\}(v_n) + N_{\mathcal{K}}(v_n).$$

Thus, there exists $v_n \in \partial f(\rho_n, v_n)$ and $\overline{\omega} \in N_{\mathcal{K}}(v_n)$ such that:

$$\lambda_n v_n + v_n - \rho_n + \overline{\omega} \ = \ 0.$$

Thus, we have:

$$\langle \rho_n - v_n, y - v_n\rangle \ = \ \lambda_n\langle v_n, y - v_n\rangle + \langle \overline{\omega}, y - v_n\rangle, \ \forall y \in \mathcal{K}.$$

Given that $\rho_n \ = \ v_n, \overline{\omega} \in N_{\mathcal{K}}(v_n)$ implies that:

$$\lambda_n\langle v_n, y - v_n\rangle \ge 0. \tag{9}$$

Due to $v_n \in f(\rho_n, v_n)$ and using the subdifferential definition, we obtain:

$$f(\rho_n, y) - f(\rho_n, v_n) \ge \langle v_n, y - v_n, \rangle, \ \forall y \in \mathcal{K}. \tag{10}$$

Combining Expressions (9) and (10) and due to $\lambda_n > 0$, we get:

$$f(\rho_n, y) - f(\rho_n, v_n) \ge 0. \tag{11}$$

By $v_n \ = \ \rho_n$, the condition (Ψ1) implies that $f(\rho_n, y) \ge 0$, for all $y \in \mathcal{K}$. □

Lemma 7. *Suppose that $f : \mathcal{E} \times \mathcal{E} \to \mathcal{R}$ satisfies the conditions* (Ψ1)–(Ψ4). *For each $q^* \in EP(f, \mathcal{K}) \neq \emptyset$, we have:*

$$\begin{aligned}\|z_n - q^*\|^2 \le \|\rho_n - q^*\|^2 - (1 - \lambda_{n+1})\|z_n - \rho_n\|^2 \\ - \lambda_{n+1}(1 - 2c_1\lambda_n)\|\rho_n - v_n\|^2 - \lambda_{n+1}(1 - 2c_2\lambda_n)\|z_n - v_n\|^2.\end{aligned}$$

Proof. From Lemma 1 and the value of z_n, we have:

$$0 \in \partial_2\Big\{\mu\lambda_n f(v_n, y) + \frac{1}{2}\|\rho_n - y\|^2\Big\}(z_n) + N_{\mathcal{K}}(z_n).$$

Thus, we have:

$$\mu\lambda_n\omega + z_n - \rho_n + \overline{\omega} \ = \ 0,$$

where $\omega \in \partial_2 f(v_n, z_n)$ and $\overline{\omega} \in N_{\mathcal{K}}(z_n)$. Thus, we have:

$$\langle \rho_n - z_n, y - z_n \rangle = \mu\lambda_n \langle \omega, y - z_n \rangle + \langle \overline{\omega}, y - z_n \rangle, \ \forall y \in \mathcal{K}.$$

Since $\overline{\omega} \in N_{\mathcal{K}}(z_n)$, then $\langle \overline{\omega}, y - z_n \rangle \leq 0$, $\forall y \in \mathcal{K}$. This implies that:

$$\mu\lambda_n \langle \omega, y - z_n \rangle \geq \langle \rho_n - z_n, y - z_n \rangle, \ \forall y \in \mathcal{K}. \tag{12}$$

Given that $\omega \in \partial_2 f(v_n, z_n)$ and due to the definition of the subdifferential, we have:

$$f(v_n, y) - f(v_n, z_n) \geq \langle \omega, y - z_n \rangle, \ \forall y \in \mathcal{K}. \tag{13}$$

Combining Expressions (12) and (13):

$$\mu\lambda_n f(v_n, y) - \mu\lambda_n f(v_n, z_n) \geq \langle \rho_n - z_n, y - z_n \rangle, \ \forall y \in \mathcal{K}. \tag{14}$$

By letting $y = q^*$ in Expression (14), we obtain:

$$\mu\lambda_n f(v_n, q^*) - \mu\lambda_n f(v_n, z_n) \geq \langle \rho_n - z_n, q^* - z_n \rangle, \ \forall y \in \mathcal{K}. \tag{15}$$

From hypothesis $q^* \in EP(f, \mathcal{K})$ such that $f(q^*, v_n) \geq 0$ and due to the condition (Ψ1) implying that $f(v_n, q^*) \leq 0$, we have:

$$\langle \rho_n - z_n, z_n - q^* \rangle \geq \mu\lambda_n f(v_n, z_n). \tag{16}$$

Combining Expressions (6) and (16), we obtain:

$$\begin{aligned} \langle \rho_n - z_n, z_n - q^* \rangle \geq \lambda_{n+1} \Big[&\lambda_n \{ f(\rho_n, z_n) - f(\rho_n, v_n) \} \\ &- c_1\lambda_n \|\rho_n - v_n\|^2 - c_2\lambda_n \|z_n - v_n\|^2 \Big]. \end{aligned} \tag{17}$$

In a similar way as Expression (14), we obtain:

$$\lambda_n f(\rho_n, y) - \lambda_n f(\rho_n, v_n) \geq \langle \rho_n - v_n, y - v_n \rangle, \ \forall y \in \mathcal{K}. \tag{18}$$

By substituting $y = z_n$, we obtain:

$$\lambda_n \{ f(\rho_n, z_n) - f(\rho_n, v_n) \} \geq \langle \rho_n - v_n, z_n - v_n \rangle. \tag{19}$$

The expressions (17) and (19) imply that:

$$\begin{aligned} 2\langle \rho_n - z_n, z_n - q^* \rangle \geq \lambda_{n+1} \Big[&2\langle \rho_n - v_n, z_n - v_n \rangle \\ &- 2c_1\lambda_n \|\rho_n - v_n\|^2 - 2c_2\lambda_n \|z_n - v_n\|^2 \Big]. \end{aligned} \tag{20}$$

We have the following formulas:

$$2\langle \rho_n - z_n, z_n - q^* \rangle = \|\rho_n - q^*\|^2 - \|z_n - \rho_n\|^2 - \|z_n - q^*\|^2.$$

$$2\langle \rho_n - v_n, z_n - v_n \rangle = \|\rho_n - v_n\|^2 + \|z_n - v_n\|^2 - \|\rho_n - z_n\|^2.$$

Combining the above equalities with Expression (20), we have:

$$\begin{aligned} \|z_n - q^*\|^2 \leq &\|\rho_n - q^*\|^2 - (1 - \lambda_{n+1}) \|z_n - \rho_n\|^2 \\ &- \lambda_{n+1}(1 - 2c_1\lambda_n) \|\rho_n - v_n\|^2 - \lambda_{n+1}(1 - 2c_2\lambda_n) \|z_n - v_n\|^2. \end{aligned}$$

□

Theorem 1. *Let $\{\rho_n\}$, $\{u_n\}$, and $\{v_n\}$ be the sequences generated by Algorithm 1 converging weakly to $q^* \in EP(f,\mathcal{K})$.*

Proof. From the value of u_{n+1} with Lemma 3, we have:

$$\begin{aligned}\|u_{n+1}-q^*\|^2 &= \|(1-\beta_n)\rho_n+\beta_n z_n-q^*\|^2 \\ &= \|(1-\beta_n)(\rho_n-q^*)+\beta_n(z_n-q^*)\|^2 \\ &= (1-\beta_n)\|\rho_n-q^*\|^2+\beta_n\|z_n-q^*\|^2-\beta_n(1-\beta_n)\|\rho_n-z_n\|^2 \\ &\le (1-\beta_n)\|\rho_n-q^*\|^2+\beta_n\|z_n-q^*\|^2. \end{aligned} \tag{21}$$

By Lemma 7 and Expression (21), we obtain:

$$\begin{aligned}\|u_{n+1}-q^*\|^2 \le{}& \|\rho_n-q^*\|^2-\beta_n(1-\lambda_{n+1})\|z_n-\rho_n\|^2 \\ &-\beta_n\lambda_{n+1}(1-2c_1\lambda_n)\|\rho_n-v_n\|^2-\beta_n\lambda_{n+1}(1-2c_2\lambda_n)\|z_n-v_n\|^2. \end{aligned} \tag{22}$$

From Corollary 1, we have $\varrho\mu < \lambda_n < \sigma$, for all $n \ge 1$. From Lemma 7, we have:

$$\|z_n-q^*\|^2 \le \|\rho_n-q^*\|^2, \ \forall n \ge 1. \tag{23}$$

Combining Expressions (21) and (23), we have:

$$\|u_{n+1}-q^*\|^2 \le \|\rho_n-q^*\|^2, \ \forall n \ge 1. \tag{24}$$

Due to definition of ρ_n, we have:

$$\begin{aligned}\|\rho_n-q^*\|^2 &= \|u_n+\theta_n(u_n-u_{n-1})-q^*\|^2 \\ &= \|(1+\theta_n)(u_n-q^*)-\theta_n(u_{n-1}-q^*)\|^2 \\ &= (1+\theta_n)\|u_n-q^*\|^2-\theta_n\|u_{n-1}-q^*\|^2+\theta_n(1+\theta_n)\|u_n-u_{n-1}\|^2 && (25)\\ &\le (1+\theta_n)\|u_n-q^*\|^2-\theta_n\|u_{n-1}-q^*\|^2+2\theta\|u_n-u_{n-1}\|^2. && (26)\end{aligned}$$

From the definition of ρ_n, we can obtain:

$$\|\rho_n-q^*\| = \|u_n+\theta_n(u_n-u_{n-1})-q^*\| \le \|u_n-q^*\|+\theta_n\|u_n-u_{n-1}\| \tag{27}$$

Combining Expressions (24) and (27), we have:

$$\|u_{n+1}-q^*\| \le \|u_n-q^*\|+\theta\|u_n-u_{n-1}\|, \ \forall n \ge 1. \tag{28}$$

From Expressions (7) and (8), we deduce that:

$$\sum_{n=1}^{\infty}\|u_n-u_{n-1}\| < +\infty \text{ and } \lim_{n\to\infty}\|u_n-u_{n-1}\| = 0. \tag{29}$$

Using Lemma 4 with Expressions (28) and (29), we have:

$$\lim_{n\to\infty}\|u_n-q^*\| = l, \text{ for some finite } l \ge 0. \tag{30}$$

By using (29) and (30) and letting $n \to \infty$ in (25), it is implied that:

$$\lim_{n\to\infty}\|\rho_n-q^*\| = l. \tag{31}$$

Combining Expressions (22) and (26), we have:

$$\begin{aligned}\|u_{n+1}-q^*\|^2 \le\ & (1+\theta_n)\|u_n-q^*\|^2-\theta_n\|u_{n-1}-q^*\|^2+2\theta\|u_n-u_{n-1}\|^2\\ & -\beta_n(1-\lambda_{n+1})\|z_n-\rho_n\|^2\\ & -\beta_n\lambda_{n+1}(1-2c_1\lambda_n)\|\rho_n-v_n\|^2-\beta_n\lambda_{n+1}(1-2c_2\lambda_n)\|z_n-v_n\|^2, \end{aligned} \tag{32}$$

which further implies that (for $n \ge 1$):

$$\begin{aligned}0 &\le \beta(1-\sigma)\|z_n-\rho_n\|^2+\beta\sigma(1-2c_1\sigma)\|\rho_n-v_n\|^2\\ &\quad +\beta\sigma(1-2c_2\sigma)\|z_n-v_n\|^2\\ &\le \|u_n-q^*\|^2-\|u_{n+1}-q^*\|^2+\theta_n\big(\|u_n-q^*\|^2-\|u_{n-1}-q^*\|^2\big)\\ &\quad +2\theta\|u_n-u_{n-1}\|^2. \end{aligned} \tag{33}$$

By letting $n\to\infty$ in Expression (33), we obtain:

$$\lim_{n\to\infty}\|z_n-\rho_n\| \;=\; \lim_{n\to\infty}\|\rho_n-v_n\| \;=\; \lim_{n\to\infty}\|v_n-z_n\| \;=\; 0. \tag{34}$$

From Expressions (31) and (34), we obtain:

$$\lim_{n\to\infty}\|v_n-q^*\| \;=\; l. \tag{35}$$

It follows from Expressions (30), (31), and (35) that the sequences $\{\rho_n\}$, $\{u_n\}$ and $\{v_n\}$ are bounded, and for each $q^*\in EP(f,\mathcal{K})$, the limit of $\|\rho_n-q^*\|$, $\|u_n-q^*\|$ and $\|v_n-q^*\|$ exists. Next, for using Lemma 2, we need to show that any sequential weak limit point of the sequence $\{u_n\}$ belongs to the set $EP(f,\mathcal{K})$. Suppose z to be an arbitrary weak cluster point of $\{u_n\}$, i.e., a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ weakly converges to z. Due to $\|u_n-v_n\|\to 0$, then $\{v_{n_k}\}$ also weakly converges to z and $z\in\mathcal{K}$. Next, we need to show that $z\in EP(f,\mathcal{K})$. From (14) and the definition of λ_{n+1} and (19), we have:

$$\begin{aligned}\mu\lambda_{n_k}f(v_{n_k},y) &\ge \mu\lambda_{n_k}f(v_{n_k},z_{n_k})+\langle\rho_{n_k}-z_{n_k},y-z_{n_k}\rangle\\ &\ge \lambda_{n_k}\lambda_{n_k+1}f(\rho_{n_k},z_{n_k})-\lambda_{n_k}\lambda_{n_k+1}f(\rho_{n_k},v_{n_k})-c_1\lambda_{n_k}\lambda_{n_k+1}\|\rho_{n_k}-v_{n_k}\|^2\\ &\quad -c_2\lambda_{n_k}\lambda_{n_k+1}\|v_{n_k}-z_{n_k}\|^2+\langle\rho_{n_k}-z_{n_k},y-z_{n_k}\rangle\\ &\ge \lambda_{n_k+1}\langle\rho_{n_k}-v_{n_k},z_{n_k}-v_{n_k}\rangle-c_1\lambda_{n_k}\lambda_{n_k+1}\|\rho_{n_k}-v_{n_k}\|^2\\ &\quad -c_2\lambda_{n_k}\lambda_{n_k+1}\|v_{n_k}-z_{n_k}\|^2+\langle\rho_{n_k}-z_{n_k},y-z_{n_k}\rangle. \end{aligned} \tag{36}$$

By letting $n\to\infty$, in the above expression, we obtain:

$$0\le\limsup_{k\to\infty}f(v_{n_k},y)\le f(z,y),\;\;\forall y\in\mathcal{K}.$$

It is concluded that $z\in EP(f,\mathcal{K})$. Finally, by Lemma 2, $\{\rho_n\}$, $\{u_n\}$, and $\{v_n\}$ weakly converge to q^* as $n\to\infty$. □

4. Applications of the Main Results

We consider the implementation of our results to solve the variational inequality problems involving the pseudomonotone and Lipschitz-type continuous operator. A variational inequality problem is formulated in the following way:

$$\text{Find } q^*\in\mathcal{K} \text{ such that } \langle G(q^*),y-q^*\rangle\ge 0,\;\;\forall y\in\mathcal{K}.$$

An operator $G:\mathcal{E}\to\mathcal{E}$ is said to be:

(i) L-Lipschitz continuous on $\mathcal{K}$ if:

$$\|G(z_1) - G(z_2)\| \leq L\|z_1 - z_2\|, \ \forall z_1, z_2 \in \mathcal{K};$$

(ii) pseudomonotone on $\mathcal{K}$ if:

$$\langle G(z_1), z_2 - z_1 \rangle \geq 0 \Longrightarrow \langle G(z_2), z_1 - z_2 \rangle \leq 0, \ \forall z_1, z_2 \in \mathcal{K}.$$

Assume that G satisfies the following conditions:

(G1) G is pseudomonotone on $\mathcal{K}$ with solution set $VI(G, \mathcal{K}) \neq \emptyset$;
(G2) G is L-Lipschitz continuous on $\mathcal{K}$ with $L > 0$;
(G3) $\limsup_{n\to\infty} \langle G(u_n), y - u_n \rangle \leq \langle G(p), y - p \rangle$, $\forall y \in \mathcal{K}$ and $\{u_n\} \subset \mathcal{K}$ satisfy $u_n \rightharpoonup p$.

Let $f(x, y) := \langle G(x), y - x \rangle$, $\forall x, y \in \mathcal{K}$. Then, the problem (1) translates into the variational inequality problem with $L = 2c_1 = 2c_2$. From the value of v_n, we have:

$$\begin{aligned} v_n &= \arg\min_{y\in\mathcal{K}} \Big\{ \lambda_n f(\rho_n, y) + \frac{1}{2}\|\rho_n - y\|^2 \Big\} \\ &= \arg\min_{y\in\mathcal{K}} \Big\{ \lambda_n \langle G(\rho_n), y - \rho_n \rangle + \frac{1}{2}\|\rho_n - y\|^2 + \frac{\lambda_n^2}{2}\|G(\rho_n)\|^2 - \frac{\lambda_n^2}{2}\|G(\rho_n)\|^2 \Big\} \\ &= \arg\min_{y\in\mathcal{K}} \Big\{ \frac{1}{2}\|y - (\rho_n - \lambda_n G(\rho_n))\|^2 \Big\} \\ &= P_{\mathcal{K}}(\rho_n - \lambda_n G(\rho_n)). \end{aligned} \tag{37}$$

In a similar way as Expression (37), the value of u_{n+1} is written as:

$$z_n = P_{\mathcal{K}}(\rho_n - \mu\lambda_n G(v_n)).$$

Corollary 2. *Let $G : \mathcal{K} \to \mathcal{E}$ be an operator satisfying the conditions (G1)–(G3). Assume that $\{\rho_n\}$, $\{u_n\}$, and $\{v_n\}$ are the sequences generated in the following way:*

(i) *Choose $u_{-1}, u_0 \in \mathcal{E}$, $\varrho \in (0, 1)$, $\sigma < \min\{1, \frac{1}{L}\}$, $\mu \in (0, \sigma)$, $\lambda_0 > 0$, $\theta \in [0, 1)$, and a sequence $\{\epsilon_n\} \subset [0, +\infty)$ such that:*

$$\sum_{n=0}^{+\infty} \epsilon_n < +\infty. \tag{38}$$

(ii) *Let θ_n satisfy $0 \leq \theta_n \leq \bar{\theta_n}$ such that:*

$$\bar{\theta_n} = \begin{cases} \min\Big\{\theta, \frac{\epsilon_n}{\|u_n - u_{n-1}\|}\Big\} & \textit{if} \quad u_n \neq u_{n-1}, \\ \theta & \textit{otherwise.} \end{cases} \tag{39}$$

(iii) *Set $\rho_n = u_n + \theta_n(u_n - u_{n-1})$, and compute:*

$$\begin{cases} v_n = P_{\mathcal{K}}(\rho_n - \lambda_n G(\rho_n)), \\ z_n = P_{\mathcal{K}}(\rho_n - \mu\lambda_n G(v_n)), \\ u_{n+1} = (1 - \beta_n)\rho_n + \beta_n z_n, \end{cases} \tag{40}$$

where $0 < \beta \leq \beta_n \leq 1$.

(iv) *Next, stepsize λ_{n+1} is obtained as follows:*

$$\lambda_{n+1} = \min\Big\{\sigma, \frac{\mu\langle G(v_n), z_n - v_n\rangle}{\langle G(\rho_n), z_n - v_n\rangle - \frac{L}{2}\|\rho_n - v_n\|^2 - \frac{L}{2}\|z_n - v_n\|^2 + 1}\Big\}. \tag{41}$$

Then, $\{\rho_n\}$, $\{u_n\}$ and $\{v_n\}$ are weakly convergent to $q^* \in VI(G, \mathcal{K})$.

Next, consider the applications of our results that are discussed in Section 3 to solve fixed point problems involving κ-strict pseudocontraction. A mapping $T : \mathcal{K} \to \mathcal{K}$ is said to be:

(i) a κ-strict pseudo-contraction [47] on $\mathcal{K}$ if:

$$\|Tz_1 - Tz_2\|^2 \leq \|z_1 - z_2\|^2 + \kappa\|(z_1 - Tz_1) - (z_2 - Tz_2)\|^2,\ \forall z_1, z_2 \in \mathcal{K}; \tag{42}$$

that is equivalent to:

$$\big\langle Tz_1 - Tz_2, z_1 - z_2\big\rangle \leq \|z_1 - z_2\|^2 - \frac{1-\kappa}{2}\|(z_1 - Tz_1) - (z_2 - Tz_2)\|^2,\ \forall z_1, z_2 \in \mathcal{K}; \tag{43}$$

(ii) sequentially weakly continuous on $\mathcal{K}$ if:

$$T(u_n) \rightharpoonup T(p) \text{ for every sequence in } \mathcal{K} \text{ satisfying } u_n \rightharpoonup p \text{ (weakly converges).}$$

A fixed point problem is formulated in the following way:

$$\text{Find } q^* \in \mathcal{K} \text{ such that } T(q^*) = q^*.$$

Let $f(x, y) = \langle x - Tx, y - x\rangle$, $\forall x, y \in \mathcal{K}$. Then, the problem (1) translates into the fixed point problem with $2c_1 = 2c_2 = \frac{3-2\kappa}{1-\kappa}$. The value of v_n in Algorithm 1 converts into the followings:

$$\begin{aligned}
v_n &= \arg\min_{y\in\mathcal{K}}\{\lambda_n f(\rho_n, y) + \frac{1}{2}\|\rho_n - y\|^2\} \\
&= \arg\min_{y\in\mathcal{K}}\{\lambda_n\langle\rho_n - T(\rho_n), y - \rho_n\rangle + \frac{1}{2}\|\rho_n - y\|^2\} \\
&= \arg\min_{y\in\mathcal{K}}\{\lambda_n\langle\rho_n - T(\rho_n), y - \rho_n\rangle + \frac{1}{2}\|\rho_n - y\|^2 + \frac{\lambda_n^2}{2}\|\rho_n - T(\rho_n)\|^2\} \\
&= \arg\min_{y\in\mathcal{K}}\{\frac{1}{2}\|y - \rho_n + \lambda_n(\rho_n - T(\rho_n))\|^2\} \\
&= P_{\mathcal{K}}[\rho_n - \lambda_n(\rho_n - T(\rho_n))].
\end{aligned} \tag{44}$$

Corollary 3. *Assume that $\mathcal{K}$ is a nonempty, closed, and convex subset of a Hilbert space $\mathcal{E}$ and $T : \mathcal{K} \to \mathcal{K}$ is a κ-strict pseudocontraction and weakly continuous with $Fix(T) \neq \emptyset$.*

(i) *Choose $u_{-1}, u_0 \in \mathcal{E}$, $\varrho \in (0, 1)$, $\sigma < \min\{1, \frac{1-\kappa}{3-2\kappa}\}$, $\mu \in (0, \sigma)$, $\lambda_0 > 0$, $\theta \in [0, 1)$, and a sequence $\{\epsilon_n\} \subset [0, +\infty)$ such that:*

$$\sum_{n=0}^{+\infty} \epsilon_n < +\infty. \tag{45}$$

(ii) *Let θ_n satisfy $0 \leq \theta_n \leq \bar{\theta_n}$ such that:*

$$\bar{\theta_n} = \begin{cases} \min\Big\{\theta, \frac{\epsilon_n}{\|u_n - u_{n-1}\|}\Big\} & \text{if} \quad u_n \neq u_{n-1}, \\ \theta & \text{otherwise.} \end{cases} \tag{46}$$

(iii) *Set* $\rho_n = u_n + \theta_n(u_n - u_{n-1})$, *and compute:*

$$\begin{cases} v_n = P_{\mathcal{K}}[\rho_n - \lambda_n(\rho_n - T(\rho_n))], \\ z_n = P_{\mathcal{K}}[\rho_n - \mu\lambda_n(v_n - T(v_n))], \\ u_{n+1} = (1-\beta_n)\rho_n + \beta_n z_n, \end{cases} \tag{47}$$

where $0 < \beta \leq \beta_n \leq 1$.

(iv) *Next, stepsize* λ_{n+1} *is obtained as follows:*

$$\lambda_{n+1} = \min\left\{\sigma, \frac{\mu\langle v_n - Tv_n, z_n - v_n\rangle}{\langle \rho_n - T\rho_n, z_n - v_n\rangle - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|\rho_n - v_n\|^2 - \left(\frac{3-2\kappa}{2-2\kappa}\right)\|z_n - v_n\|^2 + 1}\right\}. \tag{48}$$

Then, the $\{\rho_n\}$, $\{u_n\}$, *and* $\{v_n\}$ *sequences are weakly convergent to* $q^* \in Fix(T)$.

5. Numerical Experiments

MATLAB Version 9.5 (R2018b) was run on a PC (with Intel(R) Core(TM)i3-4010U CPU @ 1.70 GHz 1.70 GHz, RAM 4.00 GB). We used the built-in MATLAB fmincon function to solve the minimization problems. In this section, we discuss a number of test problems and explain the effectiveness of the proposed methodology.

Example 1. *Let* $f : \mathcal{K} \times \mathcal{K} \to \mathcal{R}$ *be defined by:*

$$f(u,v) = \sum_{i=2}^{5}(v_i - u_i)\|u\|, \ \forall u, v \in \mathcal{R}^5$$

where $\mathcal{K} = \{(u_1, \cdots, u_5) : u_1 \geq -1, u_i \geq 1, i = 2, \cdots, 5\}$. *The bifunction* f *is Lipschitz-type continuous with constants* $c_1 = c_2 = 2$, *satisfying the conditions* (Ψ1)–(Ψ4). *The solution set is* $EP(f, \mathcal{K}) = \{(u_1, 1, 1, 1, 1) : u_1 > 1\}$ *(see [48] for more details). In addition, to estimate the optimal values of the control parameters, two experiments are performed by assuming the variation of the control parameters* λ, λ_0 *and inertial factor* θ. *The values of the control parameters for Algorithm 1 (Alg2) are* $\sigma = \frac{1}{2.3c_1}$, $\mu = \frac{1}{2.4c_1}$, $u_0 = u_{-1} = (2,3,2,5,5)$, $\beta_n = 0.80$, *and* $\epsilon_n = \frac{1}{n^2}$; *for Algorithm 1 (Alg1) in [19], they are* $u_0 = u_{-1} = (2,3,2,5,5)$ *and* $\epsilon_n = \frac{1}{n^2}$. *Numerical results are shown in Tables 1–2 by using the stopping criterion* $D_n = \|\rho_n - v_n\| \leq \epsilon = 10^{-4}$.

Table 1. Example 1: Algorithm 1's (Alg2) numerical comparison with Algorithm 1 in [19].

			Number of Iterations		CPU Time in Seconds	
θ	λ	λ_0	Alg1	Alg2	Alg1	Alg2
0.40	0.22	0.45	14	2	0.9134	0.5424
0.40	0.17	0.35	18	2	0.8615	0.5223
0.40	0.12	0.32	20	2	1.0815	0.5112
0.40	0.07	0.25	22	2	1.4219	0.5367
0.40	0.02	0.05	26	2	1.7329	0.5181

Table 2. Example 1: Algorithm 1's numerical comparison with Algorithm 1 in [19].

			Number of Iterations		CPU Time in Seconds	
θ	λ	λ_0	Alg1	Alg2	Alg1	Alg2
0.80	0.22	0.16	21	2	1.0482	0.0811
0.60	0.22	0.16	15	2	0.8676	0.0626
0.40	0.22	0.16	12	2	1.0545	0.0791
0.20	0.22	0.16	11	2	0.09923	0.0892
0.05	0.22	0.16	19	2	1.09151	0.0788

Example 2. *Let us consider the Nash–Cournot equilibrium model that appeared in the paper [5,49]. A bifunction f can be written in the following manner:*

$$f(u,v) = \langle Pu + Qv + c, v - u\rangle,$$

where $c \in \mathcal{R}^5$, and the matrices P, Q are:

$$P = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \quad Q = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad c = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

where the Lipschitz constants are $c_1 = c_2 = \frac{1}{2}\|P - Q\|$. The set $\mathcal{K} \subset \mathcal{R}^5$ is considered as follows:

$$\mathcal{K} := \{u \in \mathcal{R}^5 : -5 \leq u_i \leq 5\}.$$

To see the suitable values of the control parameters, different tests are performed by assuming the variation of the control parameters' inertial factor θ. The values of the control parameters for Algorithm 1 are $\sigma = \frac{1}{2.3c_1}$, $\mu = \frac{1}{2.4c_1}$, $u_0 = u_{-1} = (1,1,1,1,1)$, $\lambda = \frac{1}{3c_1}$, $\beta_n = 0.80$, and $\epsilon_n = \frac{1}{n^2}$; for Algorithm 1 in [19], they are $u_0 = u_{-1} = (1,1,1,1,1)$, $\epsilon_n = \frac{1}{n^2}$, and $\lambda = 0.20$. Table 3 reports the numerical results by using the stopping criterion $D_n = \|\rho_n - v_n\| \leq \epsilon = 10^{-6}$.

Table 3. Example 2: Algorithm 1's numerical comparison with Algorithm 1 in [19].

	Number of Iterations		CPU Time in Seconds	
θ	Alg1	Alg2	Alg1	Alg2
0.10	42	8	1.5851	0.2822
0.15	29	8	1.3148	0.2433
0.40	28	7	1.1278	0.2662
0.55	37	8	1.2211	0.2745
0.70	47	8	1.7188	0.2279
0.85	49	8	1.6188	0.2179

To determine the suitable values of the control parameters, different tests are presented by assuming the different initial points. The control parameter values for Algorithm 1 are $\sigma = \frac{1}{2.1c_1}$, $\mu = \frac{1}{2.2c_1}$, $u_0 = u_{-1}$, $\lambda = \frac{1}{3c_1}$, $\theta = 0.45$, $\beta_n = 0.80$, and $\epsilon_n = \frac{1}{n^2}$; for Algorithm 1 in [19], they are $u_0 = u_{-1}$, $\theta = 0.45$, $\epsilon_n = \frac{1}{n^2}$, and $\lambda = 0.18$. Table 4 reports the numerical results by using the stopping criterion $D_n = \|\rho_n - v_n\| \leq \epsilon = 10^{-6}$.

Table 4. Example 2: Algorithm 1's numerical comparison with Algorithm 1 in [19].

	Number of Iterations		CPU Time in Seconds	
$u_0 = u_{-1}$	Alg1	Alg2	Alg1	Alg2
$(0,0,1,1,0)$	22	7	1.0321	0.1634
$(1,1,1,1,0)$	24	7	1.0945	0.1858
$(1,2,0,3,0)$	25	7	1.0328	0.2012
$(2,2,1,-4,5)$	30	7	1.0517	0.2020
$(2,-2,2,-4,5)$	35	8	1.0919	0.1428

Example 3. *Assume that* $\mathcal{E} = L^2([0,1])$ *through an inner product* $\langle u,v\rangle = \int_0^1 u(t)v(t)dt$ *and induced norm* $\|u\| = \sqrt{\int_0^1 u^2(t)dt}$, $\forall u,v \in \mathcal{E}$. *The set* $\mathcal{K} := \{u \in L^2([0,1]) : \int_0^1 tu(t)dt = 2\}$. *Assume that* $f : \mathcal{E} \times \mathcal{E} \to \mathcal{R}$ *is defined by:*

$$f(u,v) = \langle G(u), v-u\rangle,$$

where $G(u(t)) = \int_0^t u(s)ds$ *for every* $u \in L^2([0,1])$ *and* $t \in [0,1]$. *It is observed that* f *is monotone, and Lipschitz-type constants* $c_1 = c_2 = \frac{1}{\pi}$ *(see [45]). The projection on set* $\mathcal{K}$ *is computed in the following manner:*

$$P_{\mathcal{K}}(u)(t) := u(t) - \frac{\int_0^1 tu(t)dt - 2}{\int_0^1 t^2dt}t, \;\; t \in [0,1].$$

The values of the control parameters for Algorithm 1 (Alg2) are $\sigma = \frac{1}{3.4c_1}$, $\mu = \frac{1}{3.6c_1}$, $\theta = 0.45$, $u_0 = u_{-1}$, $\lambda_0 = \frac{1}{3c_1}$, $\beta_n = 0.90$, *and* $\epsilon_n = \frac{1}{n^2}$; *for Algorithm 1 (Alg1) in [19], they are* $\theta = 0.45$, $u_0 = u_{-1}$, $\epsilon_n = \frac{1}{n^2}$, *and* $\lambda = 0.12$. *Numerical results are reported in Table 5 by using the stopping criterion* $D_n = \|\rho_n - v_n\| \leq \epsilon = 10^{-4}$.

Table 5. Example 3: Algorithm 1's numerical comparison with Algorithm 1 in [19].

	Number of Iterations		CPU Time in Seconds	
$u_{-1} = u_0$	Alg1	Alg2	Alg1	Alg2
$2t$	37	4	4.9566	0.5460
$2t^2$	41	2	5.2378	0.4331
$2\sin(t)$	48	3	6.4556	0.3945
$2\cos(t)$	51	6	6.6756	0.4945
$2\exp(t)$	56	6	6.8713	0.5108

Example 4. *Assume that an operator* $G : \mathcal{R}^2 \to \mathcal{R}^2$ *is defined by:*

$$G(u) = \begin{pmatrix} 0.5u_1u_2 - 2u_2 - 10^7 \\ -4u_1 - 0.1u_2^2 - 10^7 \end{pmatrix}$$

on $\mathcal{K} = \{u \in \mathcal{R}^2 : (u_1-2)^2 + (u_2-2)^2 \leq 1\}$. *We can easily see that* G *is Lipschitz continuous with* $L = 5$ *and pseudomonotone. Let the bifunction* $f(u,v) = \langle G(u), v-u\rangle$ *and* $c_1 = c_2 = \frac{5}{2}$. *For these tests, we used the same initial values as seen in the table below and stepsize* $\lambda = 0.001$, $u_{-1} = u_0$, $\theta = 0.45$, *and* $\epsilon_n = \frac{1}{n^2}$ *for Algorithm 1 in [19]. For Algorithm 1, we used* $\sigma = \frac{1}{3.2c_1}$, $\mu = \frac{1}{3.4c_1}$, $\lambda_0 = \frac{1}{3c_1}$, $u_{-1} = u_0$, $\theta = 0.45$, *and* $\epsilon_n = \frac{1}{n^2}$. *Figures 1–4 and Table 6 report the numerical results by letting the stopping criterion* $D_n = \|\rho_n - v_n\| \leq \epsilon = 10^{-6}$.

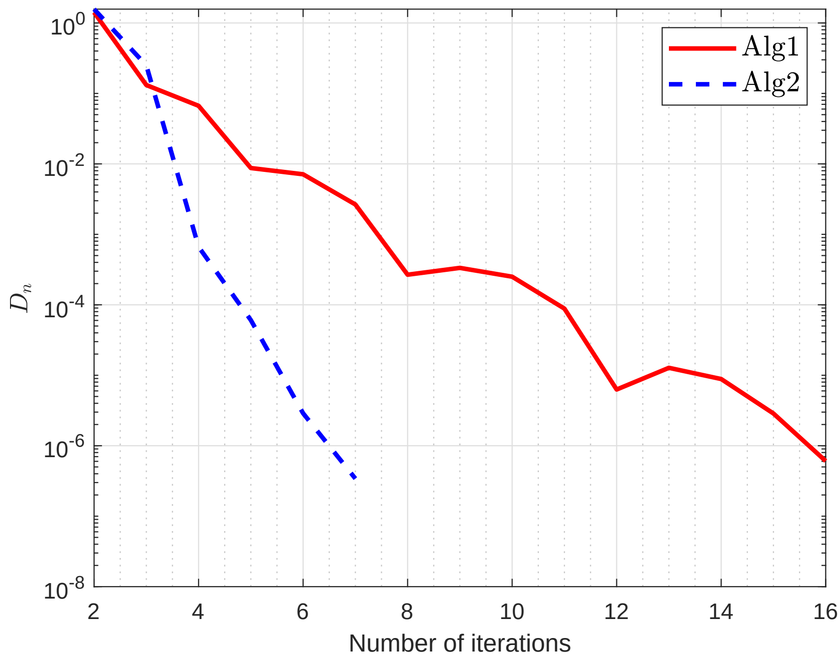

Figure 1. Example 4: Algorithm 1's numerical comparison with Algorithm 1 in [19] by letting $u_{-1} = u_0 = (1.7, 1.9)$.

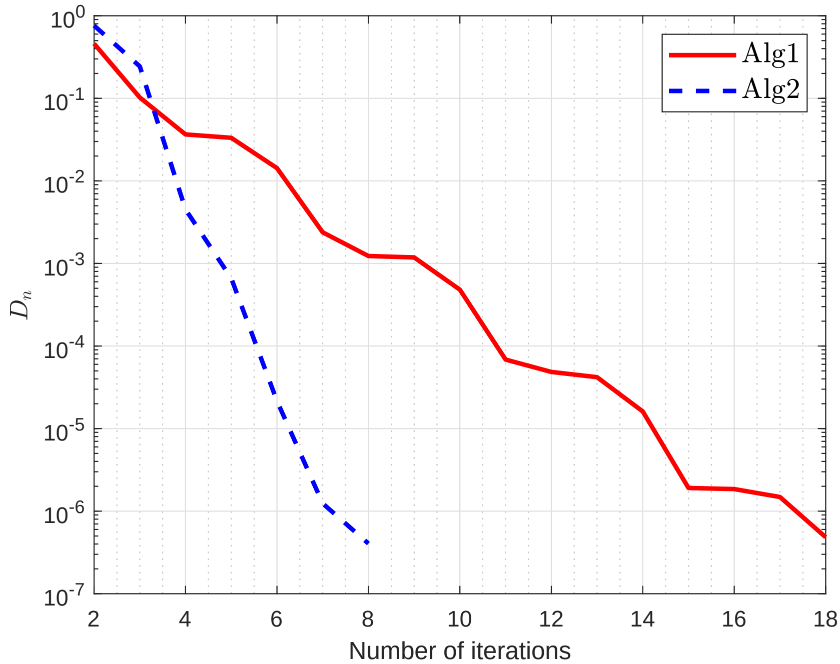

Figure 2. Example 4: Algorithm 1's numerical comparison with Algorithm 1 in [19] by letting $u_{-1} = u_0 = (2.5, 3.5)$.

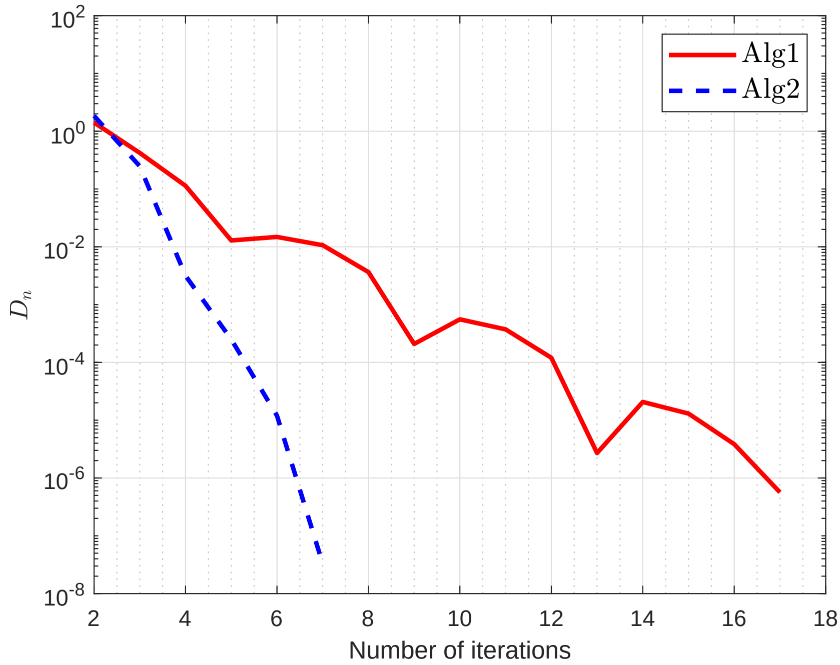

Figure 3. Example 4: Algorithm 1's numerical comparison with Algorithm 1 in [19] by letting $u_{-1} = u_0 = (1.5, 2.5)$.

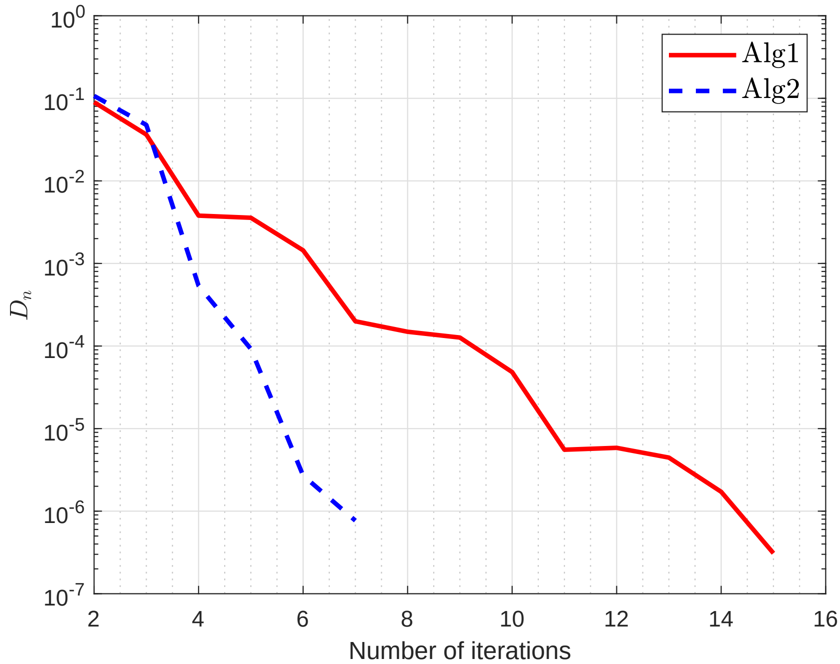

Figure 4. Example 4: Algorithm 1's numerical comparison with Algorithm 1 in [19] by letting $u_{-1} = u_0 = (2.7, 2.0)$.

Table 6. Figures 1–4: Algorithm 1's numerical comparison with Algorithm 1 in [19].

	Number of Iterations		CPU Time in Seconds	
$u_{-1} = u_0$	Alg1	Alg2	Alg1	Alg2
$(1.7, 1.9)$	17	6	0.482301	0.3473423
$(2.5, 3.5)$	18	8	0.789869	0.5659991
$(1.5, 2.5)$	17	7	0.709038	0.3791612
$(2.7, 2.0)$	15	7	0.585472	0.4401882

Author Contributions: Conceptualization, H.u.R., N.P. and I.K.A.; Writing—Original Draft Preparation, N.W., N.P. and H.u.R.; Writing—Review & Editing, N.W., N.P., H.u.R. and I.K.A.; Methodology, N.P. and H.u.R.; Visualization, N.W. and N.P.; Software, H.u.R.; Funding Acquisition, I.K.A.; Supervision, I.K.A. and H.u.R.; Project Administration; I.K.A.; Resources; I.A.K. and H.u.R. All authors have read and agreed to the published version of this manuscript.

Funding: This research work was financially supported by King Mongkut's University of Technology Thonburi through the "KMUTT 55th Anniversary Commemorative Fund". In particular, Nuttapol Pakkaranang was supported by the Thailand Research Fund (TRF) through the Royal Golden Jubilee Ph.D. (RGJ-PHD) Program (Grant No. PHD/0205/2561). Habib ur Rehman was supported by the Petchra Pra Jom Doctoral Academic Scholarship for a Ph.D. Program at KMUTT (Grant Number 39/2560).

Acknowledgments: We are very grateful to the Editor and the anonymous referees for their valuable and useful comments, which helped improve the quality of this work. The second author would like to thank the "Petchra Pra Jom Klao PhD Research Scholarship from the King Mongkut's University of Technology Thonburi".

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Blum, E. From optimization and variational inequalities to equilibrium problems. *Math. Stud.* **1994**, *63*, 123–145.
2. Muu, L.D.; Oettli, W. Convergence of an adaptive penalty scheme for finding constrained equilibria. *Nonlinear Anal. Theory Methods Appl.* **1992**, *18*, 1159–1166. [CrossRef]
3. Facchinei, F.; Pang, J.S. *Finite-Dimensional Variational Inequalities and Complementarity Problems*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2007.
4. Konnov, I. *Equilibrium Models and Variational Inequalities*; Elsevier: Amsterdam, The Netherlands, 2007; Volume 210.
5. Quoc Tran, D.; Le Dung, M.N.V.H. Extragradient algorithms extended to equilibrium problems. *Optimization* **2008**, *57*, 749–776. [CrossRef]
6. Quoc, T.D.; Anh, P.N.; Muu, L.D. Dual extragradient algorithms extended to equilibrium problems. *J. Glob. Optim.* **2011**, *52*, 139–159. [CrossRef]
7. ur Rehman, H.; Kumam, P.; Cho, Y.J.; Yordsorn, P. Weak convergence of explicit extragradient algorithms for solving equilibirum problems. *J. Inequal. Appl.* **2019**, *2019*. [CrossRef]
8. Lyashko, S.I.; Semenov, V.V. A New Two-Step Proximal Algorithm of Solving the Problem of Equilibrium Programming. In *Optimization and Its Applications in Control and Data Sciences*; Springer International Publishing: Berlin/Heidelberg, Germany, 2016; pp. 315–325. [CrossRef]
9. ur Rehman, H.; Kumam, P.; Je Cho, Y.; Suleiman, Y.I.; Kumam, W. Modified Popov's explicit iterative algorithms for solving pseudomonotone equilibrium problems. *Optim. Methods Softw.* **2020**, 1–32. [CrossRef]
10. Takahashi, S.; Takahashi, W. Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces. *J. Math. Anal. Appl.* **2007**, *331*, 506–515. [CrossRef]
11. Anh, P.N.; Hai, T.N.; Tuan, P.M. On ergodic algorithms for equilibrium problems. *J. Glob. Optim.* **2015**, *64*, 179–195. [CrossRef]
12. Hieu, D.V.; Quy, P.K.; Vy, L.V. Explicit iterative algorithms for solving equilibrium problems. *Calcolo* **2019**, *56*. [CrossRef]
13. Hieu, D.V. New extragradient method for a class of equilibrium problems in Hilbert spaces. *Appl. Anal.* **2017**, *97*, 811–824. [CrossRef]
14. ur Rehman, H.; Kumam, P.; Abubakar, A.B.; Cho, Y.J. The extragradient algorithm with inertial effects extended to equilibrium problems. *Comput. Appl. Math.* **2020**, *39*. [CrossRef]

15. ur Rehman, H.; Kumam, P.; Kumam, W.; Shutaywi, M.; Jirakitpuwapat, W. The Inertial Sub-Gradient Extra-Gradient Method for a Class of Pseudo-Monotone Equilibrium Problems. *Symmetry* **2020**, *12*, 463. [CrossRef]
16. ur Rehman, H.; Kumam, P.; Argyros, I.K.; Deebani, W.; Kumam, W. Inertial Extra-Gradient Method for Solving a Family of Strongly Pseudomonotone Equilibrium Problems in Real Hilbert Spaces with Application in Variational Inequality Problem. *Symmetry* **2020**, *12*, 503. [CrossRef]
17. Hieu, D.V.; Cho, Y.J.; Xiao, Y.B. Modified extragradient algorithms for solving equilibrium problems. *Optimization* **2018**, *67*, 2003–2029. [CrossRef]
18. Yordsorn, P.; Kumam, P.; ur Rehman, H.; Ibrahim, A.H. A Weak Convergence Self-Adaptive Method for Solving Pseudomonotone Equilibrium Problems in a Real Hilbert Space. *Mathematics* **2020**, *8*, 1165. [CrossRef]
19. Vinh, N.T.; Muu, L.D. Inertial Extragradient Algorithms for Solving Equilibrium Problems. *Acta Math. Vietnam.* **2019**, *44*, 639–663. [CrossRef]
20. ur Rehman, H.; Kumam, P.; Argyros, I.K.; Alreshidi, N.A.; Kumam, W.; Jirakitpuwapat, W. A Self-Adaptive Extra-Gradient Methods for a Family of Pseudomonotone Equilibrium Programming with Application in Different Classes of Variational Inequality Problems. *Symmetry* **2020**, *12*, 523. [CrossRef]
21. ur Rehman, H.; Gopal, D.; Kumam, P. Generalizations of Darbo's fixed point theorem for new condensing operators with application to a functional integral equation. *Demonstr. Math.* **2019**, *52*, 166–182. [CrossRef]
22. Gibali, A.; Hieu, D.V. A new inertial double-projection method for solving variational inequalities. *J. Fixed Point Theory Appl.* **2019**, *21*. [CrossRef]
23. Censor, Y.; Gibali, A.; Reich, S. Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space. *Optimization* **2012**, *61*, 1119–1132. [CrossRef]
24. ur Rehman, H.; Kumam, P.; Dhompongsa, S. Existence of tripled fixed points and solution of functional integral equations through a measure of noncompactness. *Carpath. J. Math.* **2019**, *35*, 193–208.
25. Dong, Q.L.; Jiang, D.; Gibali, A. A modified subgradient extragradient method for solving the variational inequality problem. *Numer. Algorithms* **2018**, *79*, 927–940. [CrossRef]
26. Abubakar, J.; Kumam, P.; ur Rehman, H.; Ibrahim, A.H. Inertial Iterative Schemes with Variable Step Sizes for Variational Inequality Problem Involving Pseudomonotone Operator. *Mathematics* **2020**, *8*, 609. [CrossRef]
27. Abubakar, J.; Sombut, K.; ur Rehman, H.; Ibrahim, A.H. An Accelerated Subgradient Extragradient Algorithm for Strongly Pseudomonotone Variational Inequality Problems. *Thai J. Math.* **2019**, *18*, 166–187.
28. Hammad, H.A.; ur Rehman, H.; la Sen, M.D. Advanced Algorithms and Common Solutions to Variational Inequalities. *Symmetry* **2020**, *12*, 1198. [CrossRef]
29. Santos, P.; Scheimberg, S. An inexact subgradient algorithm for equilibrium problems. *Comput. Appl. Math.* **2011**, *30*, 91–107.
30. ur Rehman, H.; Kumam, P.; Argyros, I.K.; Shutaywi, M.; Shah, Z. Optimization Based Methods for Solving the Equilibrium Problems with Applications in Variational Inequality Problems and Solution of Nash Equilibrium Models. *Mathematics* **2020**, *8*, 822. [CrossRef]
31. Hieu, D.V. Halpern subgradient extragradient method extended to equilibrium problems. *Rev. Real Acad. Cienc. Exactas Físicas Nat. Ser. Matemáticas* **2016**, *111*, 823–840. [CrossRef]
32. Yordsorn, P.; Kumam, P.; Rehman, H.U. Modified two-step extragradient method for solving the pseudomonotone equilibrium programming in a real Hilbert space. *Carpath. J. Math.* **2020**, *36*, 313–330.
33. Anh, P.N.; An, L.T.H. The subgradient extragradient method extended to equilibrium problems. *Optimization* **2012**, *64*, 225–248. [CrossRef]
34. ur Rehman, H.; Kumam, P.; Shutaywi, M.; Alreshidi, N.A.; Kumam, W. Inertial Optimization Based Two-Step Methods for Solving Equilibrium Problems with Applications in Variational Inequality Problems and Growth Control Equilibrium Models. *Energies* **2020**, *13*, 3292. [CrossRef]
35. Muu, L.D.; Quoc, T.D. Regularization Algorithms for Solving Monotone Ky Fan Inequalities with Application to a Nash-Cournot Equilibrium Model. *J. Optim. Theory Appl.* **2009**, *142*, 185–204. [CrossRef]
36. Argyros, I.K.; Magreñán, Á. *Iterative Methods and Their Dynamics with Applications*; CRC Press: Boca Raton, FL, USA, 2017. [CrossRef]
37. Argyros, I.K.; d Hilout, S. *Computational Methods in Nonlinear Analysis: Efficient Algorithms, Fixed Point Theory and Applications*; World Scientific, Singapore : 2013.

38. Flåm, S.D.; Antipin, A.S. Equilibrium programming using proximal-like algorithms. *Math. Program.* **1996**, *78*, 29–41. [CrossRef]
39. Korpelevich, G. The extragradient method for finding saddle points and other problems. *Matecon* **1976**, *12*, 747–756.
40. Dadashi, V.; Iyiola, O.S.; Shehu, Y. The subgradient extragradient method for pseudomonotone equilibrium problems. *Optimization* **2019**, 1–23. [CrossRef]
41. Bianchi, M.; Schaible, S. Generalized monotone bifunctions and equilibrium problems. *J. Optim. Theory Appl.* **1996**, *90*, 31–43. [CrossRef]
42. Mastroeni, G. On Auxiliary Principle for Equilibrium Problems. In *Nonconvex Optimization and Its Applications*; Springer US: New York, NY, USA, 2003; pp. 289–298. [CrossRef]
43. Tiel, J.V. *Convex Analysis: An Introductory Text*, 1st ed.; Wiley: New York, NY, USA, 1984.
44. Opial, Z. Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bull. Am. Math. Soc.* **1967**, *73*, 591–598. [CrossRef]
45. Bauschke, H.H.; Combettes, P.L. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*; Springer: Berlin/Heidelberg, Germany, 2011; Volume 408.
46. Tan, K.; Xu, H. Approximating Fixed Points of Nonexpansive Mappings by the Ishikawa Iteration Process. *J. Math. Anal. Appl.* **1993**, *178*, 301–308. [CrossRef]
47. Browder, F.; Petryshyn, W. Construction of fixed points of nonlinear mappings in Hilbert space. *J. Math. Anal. Appl.* **1967**, *20*, 197–228. [CrossRef]
48. Wang, S.; Zhang, Y.; Ping, P.; Cho, Y.; Guo, H. New extragradient methods with non-convex combination for pseudomonotone equilibrium problems with applications in Hilbert spaces. *Filomat* **2019**, *33*, 1677–1693. [CrossRef]
49. ur Rehman, H.; Pakkaranang, N.; Hussain, A.; Wairojjana, N. A modified extra-gradient method for a family of strongly pseudomonotone equilibrium problems in real Hilbert spaces. *J. Math. Comput. Sci.* **2020**, *22*, 38–48. [CrossRef]