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Asymptotic Properties of Neutral Differential Equations with Variable Coefficients

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Received: 24 April 2020; Accepted: 10 August 2020; Published: 12 August 2020

Abstract: The aim of this work is to study oscillatory behavior of solutions for even-order neutral nonlinear differential equations. By using the Riccati substitution, a new oscillation conditions is obtained which insures that all solutions to the studied equation are oscillatory. The obtained results complement the well-known oscillation results present in the literature. Some example are illustrated to show the applicability of the obtained results.

Keywords: even-order differential equations; neutral delay; oscillation

1. Introduction

Neutral differential equations appear in models concerning biological, physical and chemical phenomena, optimization, mathematics of networks, dynamical systems and their application in concerning materials and energy as well as problems of deformation of structures, elasticity or soil settlement, see [1].

Recently, there has been steady enthusiasm for acquiring adequate conditions for oscillatory and nonoscillatory behavior of differential equations of different orders; see [2–13]. Particular emphasize has been given to the study of oscillation and oscillatory behavior of these equations which have been under investigation by using different methods an various techniques; we refer to the papers [14–26]. In this paper we study the oscillatory behavior of the even-order nonlinear differential equation

$$\left(r\left(\varsigma\right)\left(z^{(n-1)}\left(\varsigma\right)\right)^{\alpha}\right)'+q\left(\varsigma\right)x^{\alpha}\left(\delta\left(\varsigma\right)\right)=0, \tag{1}$$

where $\varsigma\geq\varsigma_0$, n is an even natural number and $z\left(\varsigma\right):=x\left(\varsigma\right)+p\left(\varsigma\right)x\left(\tau\left(\varsigma\right)\right)$. Throughout this paper, we suppose that: $r\in C[\varsigma_0,\infty)$, $r\left(\varsigma\right)>0$, $r'\left(\varsigma\right)\geq0$, $p,q\in C\left([\varsigma_0,\infty)\right)$, $q\left(\varsigma\right)>0$, $0\leq p\left(\varsigma\right)<p_0<\infty$, q is not identically zero for large ς, $\tau\in C^1[\varsigma_0,\infty)$, $\delta\in C[\varsigma_0,\infty)$, $\tau'\left(\varsigma\right)>0$, $\tau\left(\varsigma\right)\leq\varsigma$, $\lim_{\varsigma\to\infty}\tau\left(\varsigma\right)=\lim_{\varsigma\to\infty}\delta\left(\varsigma\right)=\infty$, α is a quotient of odd positive integers and

$$\int_{\varsigma_0}^{\infty}r^{-1/\alpha}\left(s\right)\mathrm{d}s=\infty. \tag{2}$$

Definition 1. *Let x be a real function defined for all ς in a real interval $I := [\varsigma_x, \infty)$, $\varsigma_x \geq \varsigma_0$, and having an $(n-1)^{th}$ derivative for all $\varsigma \in I$. The function f is called a solution of the differential Equation (1) on I if it fulfills the following two requirements:*

$$\left(r(\varsigma) \left((x(t) + p(t)\, x(\tau(t)))^{(n-1)}(\varsigma) \right)^{\alpha} \right) \in C^1([\varsigma_x, \infty))$$

and

$$x(\varsigma) \text{ satisfies } (1) \text{ on } [\varsigma_x, \infty).$$

Definition 2. *A solution of (1) is called oscillatory if it has arbitrarily large zeros on $[\varsigma_x, \infty)$, and otherwise is called to be nonoscillatory.*

Definition 3. *The Equation (1) is said to be oscillatory if all its solutions are oscillatory.*

We collect some relevant facts and auxiliary results from the existing literature.

Bazighifan [2] using the Riccati transformation together with comparison method with second order equations, focuses on the oscillation of equations of the form

$$\left(r(\varsigma) \left(z^{(n-1)}(\varsigma) \right)^{\alpha} \right)' + q(\varsigma)\, f(x(\delta(\varsigma))) = 0, \tag{3}$$

where n is even.

Moaaz et al. [27] gives us some results providing informations on the asymptotic behavior of (1). This time, the authors used comparison method with first-order equations.

In [28] (Theorem 2), the authors considered Equation (1) and proved that (1) is oscillatory if

$$\int_{\varsigma_0}^{\infty} \left(\Psi(s) - \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{r(s)\,(\rho'(s))^{\alpha+1}}{\mu^{\alpha} s^{2\alpha} \rho^{\alpha}(s)} \right) \mathrm{d}s = \infty,$$

for some $\mu \in (0,1)$ and

$$\int_{\varsigma_0}^{\infty} \left(\vartheta(s) \left(\int_{\varsigma}^{\infty} (Q^*(v))^{\frac{1}{\alpha}}\, r^{\frac{-1}{\alpha}}(v)\, dv \right) - \frac{\vartheta_+'^2(s)}{4\vartheta(s)} \right) \mathrm{d}s = \infty,$$

where $\Psi(\varsigma) := \vartheta\rho(\varsigma)\, Q(\varsigma)\,(1 - p(g(\varsigma, a)))^{\alpha} (g(\varsigma, a)/\varsigma)^{3\alpha}$.

Xing et al. [29] proved that (1) is oscillatory if

$$\left(\delta^{-1}(\varsigma)\right)' \geq \delta_0 > 0,\ \tau'(\varsigma) \geq \tau_0 > 0,\ \tau^{-1}(\delta(\varsigma)) < \varsigma$$

and

$$\liminf_{\varsigma \to \infty} \int_{\tau^{-1}(\delta(\varsigma))}^{\varsigma} \frac{\widehat{q}(s)}{r(s)} \left(s^{n-1}\right)^{\alpha} \mathrm{d}s > \left(\frac{1}{\delta_0} + \frac{p_0^{\alpha}}{\delta_0 \tau_0} \right) \frac{((n-1)!)^{\alpha}}{\mathrm{e}},$$

where $\widehat{q}(\varsigma) := \min\left\{ q\left(\delta^{-1}(\varsigma)\right), q\left(\delta^{-1}(\tau(\varsigma))\right) \right\}$.

In this article, we establish some oscillation criteria for the Equation (1) which complements some of the results obtained in the literature. Some examples are presented to illustrate our main results.

To prove our main results we need the following lemmas:

Lemma 1 ([28]). *Let $\alpha \geq 1$ be a ratio of two odd numbers. Then*

$$Dw - Cw^{(\alpha+1)/\alpha} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{D^{\alpha+1}}{C^{\alpha}},\ C > 0.$$

Lemma 2 ([30]). *Let $h(\varsigma) \in C^n([\varsigma_0,\infty),(0,\infty))$. If $h^{(n-1)}(\varsigma)\,h^{(n)}(\varsigma) \le 0$ for all $\varsigma \ge \varsigma_x$, then for every $\theta \in (0,1)$, there exists a constant $M > 0$ such that*

$$h'(\theta\varsigma) \ge M\varsigma^{n-2}h^{(n-1)}(\varsigma),$$

for all sufficient large ς.

Lemma 3 ([31] Lemma 2.2.3). *Let $x \in C^n([\varsigma_0,\infty),(0,\infty))$. Assume that $x^{(n)}(\varsigma)$ is of fixed sign and not identically zero on $[\varsigma_0,\infty)$ and that there exists a $\varsigma_1 \ge \varsigma_0$ such that $x^{(n-1)}(\varsigma)\,x^{(n)}(\varsigma) \le 0$ for all $\varsigma \ge \varsigma_1$. If $\lim_{\varsigma\to\infty} x(\varsigma) \ne 0$, then for every $\mu \in (0,1)$ there exists $\varsigma_\mu \ge \varsigma_1$ such that*

$$x(\varsigma) \ge \frac{\mu}{(n-1)!}\varsigma^{n-1}\left|x^{(n-1)}(\varsigma)\right| \text{ for } \varsigma \ge \varsigma_\mu.$$

Lemma 4 ([32]). *Let $h \in C^n([\varsigma_0,\infty),(0,\infty))$. If $h^{(n)}(\varsigma)$ is eventually of one sign for all large ς, then there exists a $\varsigma_x > \varsigma_1$ for some $\varsigma_1 > \varsigma_0$ and an integer m, $0 \le m \le n$ with $n+m$ even for $h^{(n)}(\varsigma) \ge 0$ or $n+m$ odd for $h^{(n)}(\varsigma) \le 0$ such that $m > 0$ implies that $h^{(k)}(\varsigma) > 0$ for $\varsigma > \varsigma_x$, $k = 0,1,\ldots,m-1$ and $m \le n-1$ implies that $(-1)^{m+k}h^{(k)}(\varsigma) > 0$ for $\varsigma > \varsigma_x, k = m, m+1,\ldots,n-1$.*

2. One Condition Theorem

Notation 1. *Here, we define the next notation:*

$$\begin{aligned}
\Omega(s) &= \frac{\vartheta(s)}{\delta_0(\alpha+1)^{\alpha+1}(\lambda M)^{\alpha}}\left(\varphi(s)+\frac{\vartheta'(s)}{\vartheta(s)}\right)^{\alpha+1}, \\
\Theta(s) &= \frac{\vartheta(s)((n-2)!)^{\alpha}}{\mu^{\alpha}\delta_0(\alpha+1)^{\alpha+1}}\left(\varphi(s)+\frac{\vartheta'(s)}{\vartheta(s)}\right)^{\alpha+1}
\end{aligned}$$

and

$$Q(s) = \min\left\{q\left(\delta^{-1}(s)\right), q\left(\delta^{-1}(\tau(s))\right)\right\}.$$

Following [33], we say that a function $\Phi = \Phi(\varsigma,s,l)$ belongs to the function class Y if $\Phi \in (E,\mathbb{R})$ where $E = \{(\varsigma,s,l) : \varsigma_0 \le 1 \le s \le \varsigma\}$ which satisfies $\Phi(\varsigma,\varsigma,l) = 0$, $\Phi(\varsigma,l,l) = 0$ and $\Phi(\varsigma,s,l) > 0$, for $l < s < \varsigma$ and has the partial derivative $\partial\Phi/\partial s$ on E such that $\partial\Phi/\partial s$ is locally integrable with respect to s in E.

Definition 4. *Define the operator $B[\cdot;l,\varsigma]$ by*

$$B[h;l,\varsigma] = \int_l^{\varsigma}\Phi(\varsigma,s,l)\,h(s)\,\mathrm{d}s,$$

for $\varsigma_0 \le 1 \le s \le \varsigma$ and $h \in C([\varsigma_0,\infty),\mathbb{R})$. The function $\varphi = \varphi(\varsigma,s,l)$ is defined by

$$\frac{\partial\Phi(\varsigma,s,l)}{\partial s} = \varphi(\varsigma,s,l)\,\Phi(\varsigma,s,l).$$

Remark 1. *It is easy to verify that $B[\cdot;l,\varsigma]$ is a linear operator and that it satisfies*

$$B\left[h';l,\varsigma\right] = -B[h\varphi;l,\varsigma], \text{ for } h \in C^1([\varsigma_0,\infty),\mathbb{R}). \tag{4}$$

Lemma 5. *Assume that x is an eventually positive solution of (1) and*

$$\left(\delta^{-1}(\varsigma)\right)' \ge \delta_0 > 0,\ (\tau(\varsigma))' \ge \tau_0 > 0. \tag{5}$$

Then

$$\frac{1}{\delta_0}\left(r\left(\delta^{-1}(\varsigma)\right)\left(z^{(n-1)}\left(\delta^{-1}(\varsigma)\right)\right)^{\alpha}\right)' + \frac{p_0^{\alpha}}{\delta_0\tau_0}\left(r\left(\delta^{-1}(\tau(\varsigma))\right)\left(z^{(n-1)}\left(\delta^{-1}(\tau(\varsigma))\right)\right)^{\alpha}\right)' + Q(\varsigma)z^{\alpha}(\varsigma) \le 0. \tag{6}$$

Proof. Let x be an eventually positive solution of (1) on $[\varsigma_0,\infty)$. From (1), we see that

$$0 = \frac{1}{\left(\delta^{-1}(\varsigma)\right)'}\left(r\left(\delta^{-1}(\varsigma)\right)\left(z^{(n-1)}\left(\delta^{-1}(\varsigma)\right)\right)^{\alpha}\right)' + q\left(\delta^{-1}(\varsigma)\right)x^{\alpha}(\varsigma). \tag{7}$$

Thus, for all sufficiently large ς, we have

$$\begin{aligned} 0 \;=\; & \frac{1}{\left(\delta^{-1}(\varsigma)\right)'}\left(r\left(\delta^{-1}(\varsigma)\right)\left(z^{(n-1)}\left(\delta^{-1}(\varsigma)\right)\right)^{\alpha}\right)' \\ & + \frac{p_0^{\alpha}}{\left(\delta^{-1}(\tau(\varsigma))\right)'}\left(r\left(\delta^{-1}(\tau(\varsigma))\right)\left(z^{(n-1)}\left(\delta^{-1}(\tau(\varsigma))\right)\right)^{\alpha}\right)' \\ & + q\left(\delta^{-1}(\varsigma)\right)x^{\alpha}(\varsigma) + p_0^{\alpha}q\left(\delta^{-1}(\tau(\varsigma))\right)x^{\alpha}(\tau(\varsigma)). \end{aligned} \tag{8}$$

From (8) and the definition of z, we get

$$\begin{aligned} q\left(\delta^{-1}(\varsigma)\right)x^{\alpha}(\varsigma) + p_0^{\alpha}q\left(\delta^{-1}(\tau(\varsigma))\right)x^{\alpha}(\tau(\varsigma)) \;\ge\; & Q(\varsigma)\left(x(\varsigma) + p_0x(\tau(\varsigma))\right)^{\alpha} \\ \ge\; & Q(\varsigma)z^{\alpha}(\varsigma). \end{aligned} \tag{9}$$

Thus, by using (8) and (9), we obtain

$$\begin{aligned} 0 \;\ge\; & \frac{1}{\left(\delta^{-1}(\varsigma)\right)'}\left(r\left(\delta^{-1}(\varsigma)\right)\left(z^{(n-1)}\left(\delta^{-1}(\varsigma)\right)\right)^{\alpha}\right)' \\ & + \frac{p_0^{\alpha}}{\left(\delta^{-1}(\tau(\varsigma))\right)'}\left(r\left(\delta^{-1}(\tau(\varsigma))\right)\left(z^{(n-1)}\left(\delta^{-1}(\tau(\varsigma))\right)\right)^{\alpha}\right)' + Q(\varsigma)z^{\alpha}(\varsigma). \end{aligned} \tag{10}$$

From (5), we get

$$\begin{aligned} 0 \;\ge\; & \frac{1}{\delta_0}\left(r\left(\delta^{-1}(\varsigma)\right)\left(z^{(n-1)}\left(\delta^{-1}(\varsigma)\right)\right)^{\alpha}\right)' \\ & + \frac{p_0^{\alpha}}{\delta_0\tau_0}\left(r\left(\delta^{-1}(\tau(\varsigma))\right)\left(z^{(n-1)}\left(\delta^{-1}(\tau(\varsigma))\right)\right)^{\alpha}\right)' + Q(\varsigma)z^{\alpha}(\varsigma). \end{aligned}$$

This completes the proof. □

Theorem 1. *Let (2) hold. Assume that there exist positive functions $\vartheta \in C^1([\varsigma_0,\infty),\mathbb{R})$ such that for all $M > 0$*

$$\limsup_{\varsigma\to\infty} B\left[\vartheta(s)Q(s) - \Omega(s)\left(\frac{r\left(\delta^{-1}(s)\right)}{\left(\left(\delta^{-1}(s)\right)^{n-2}\right)^{\alpha}} + \frac{p_0^{\alpha}r\left(\delta^{-1}(\tau(s))\right)}{\tau_0\left(\left(\delta^{-1}(\tau(s))\right)^{n-2}\right)^{\alpha}}\right); l,\varsigma\right] > 0, \tag{11}$$

for some $\lambda \in (0,1)$, then (1) is oscillatory.

Proof. Suppose that (1) has a nonoscillatory solution in $[\varsigma_0,\infty)$. Without loss of generality, we let x be an eventually positive solution of (1). Then, there exists a $\varsigma_1 \ge \varsigma_0$ such that $x(\varsigma) > 0$, $x(\tau(\varsigma)) > 0$ and $x(\delta(\varsigma)) > 0$ for $\varsigma \ge \varsigma_1$. Thus, we have

$$z(\varsigma) > 0,\ z'(\varsigma) > 0,\ z^{(n-1)}(\varsigma) > 0,\ z^{(n)}(\varsigma) < 0. \tag{12}$$

By Lemma 2, we get

$$z'(\lambda\varsigma) \geq M\varsigma^{n-2} z^{(n-1)}(\varsigma), \tag{13}$$

where M is positive constant. Now, we define a function ψ by

$$\psi(\varsigma) = \vartheta(\varsigma) \frac{r\left(\delta^{-1}(\varsigma)\right)\left(z^{(n-1)}\left(\delta^{-1}(\varsigma)\right)\right)^{\alpha}}{z^{\alpha}(\lambda\varsigma)}. \tag{14}$$

Then we obtain $\psi(\varsigma) > 0$ for $\varsigma \geq \varsigma_1$, and

$$\begin{aligned}\psi'(\varsigma) &= \vartheta'(\varsigma) \frac{r\left(\delta^{-1}(\varsigma)\right)\left(z^{(n-1)}\left(\delta^{-1}(\varsigma)\right)\right)^{\alpha}}{z^{\alpha}(\lambda\varsigma)} + \vartheta(\varsigma) \frac{\left(r\left(\delta^{-1}(\varsigma)\right)\left(z^{(n-1)}\left(\delta^{-1}(\varsigma)\right)\right)^{\alpha}\right)'}{z^{\alpha}(\lambda\varsigma)} \\ &\quad -\alpha\lambda\vartheta(\varsigma) \frac{r\left(\delta^{-1}(\varsigma)\right)\left(z^{(n-1)}\left(\delta^{-1}(\varsigma)\right)\right)^{\alpha} z'(\lambda\varsigma)}{z^{\alpha+1}(\lambda\varsigma)}.\end{aligned} \tag{15}$$

Combining (13) and (14) in (15), we obtain

$$\begin{aligned}\psi'(\varsigma) &\leq \frac{\vartheta'(\varsigma)}{\vartheta(\varsigma)}\psi(\varsigma) + \vartheta(\varsigma) \frac{\left(r\left(\delta^{-1}(\varsigma)\right)\left(z^{(n-1)}\left(\delta^{-1}(\varsigma)\right)\right)^{\alpha}\right)'}{z^{\alpha}(\lambda\varsigma)} \\ &\quad -\alpha\lambda M\left(\delta^{-1}(\varsigma)\right)^{n-2} \frac{(\psi(\varsigma))^{\alpha+1/\alpha}}{\left(\vartheta(\varsigma) r\left(\delta^{-1}(\varsigma)\right)\right)^{1/\alpha}}.\end{aligned} \tag{16}$$

Similarly, define

$$\tilde{\psi}(\varsigma) = \vartheta(\varsigma) \frac{r\left(\delta^{-1}(\tau(\varsigma))\right)\left(z^{(n-1)}\left(\delta^{-1}(\tau(\varsigma))\right)\right)^{\alpha}}{z^{\alpha}(\lambda\varsigma)}. \tag{17}$$

Then we obtain $\tilde{\psi}(\varsigma) > 0$ for $\varsigma \geq \varsigma_1$, and

$$\begin{aligned}\tilde{\psi}'(\varsigma) &\leq \frac{\vartheta'(\varsigma)}{\vartheta(\varsigma)}\tilde{\psi}(\varsigma) + \vartheta(\varsigma) \frac{\left(r\left(\delta^{-1}(\tau(\varsigma))\right)\left(z^{(n-1)}\left(\delta^{-1}(\tau(\varsigma))\right)\right)^{\alpha}\right)'}{z^{\alpha}(\lambda\varsigma)} \\ &\quad -\alpha\lambda M\left(\delta^{-1}(\tau(\varsigma))\right)^{n-2} \frac{(\tilde{\psi}(\varsigma))^{\alpha+1/\alpha}}{\left(\vartheta(\varsigma) r\left(\delta^{-1}(\tau(\varsigma))\right)\right)^{1/\alpha}}.\end{aligned} \tag{18}$$

Therefore, from (16) and (18), we obtain

$$\begin{aligned}\frac{1}{\delta_0}\psi'(\varsigma) + \frac{p_0^{\alpha}}{\delta_0\tau_0}\tilde{\psi}'(\varsigma) &\leq \frac{\vartheta(\varsigma)}{\delta_0} \frac{\left(r\left(\delta^{-1}(\varsigma)\right)\left(z^{(n-1)}\left(\delta^{-1}(\varsigma)\right)\right)^{\alpha}\right)'}{z^{\alpha}(\lambda\varsigma)} \\ &\quad + \frac{p_0^{\alpha}}{\delta_0\tau_0}\vartheta(\varsigma) \frac{\left(r\left(\delta^{-1}(\tau(\varsigma))\right)\left(z^{(n-1)}\left(\delta^{-1}(\tau(\varsigma))\right)\right)^{\alpha}\right)'}{z^{\alpha}(\lambda\varsigma)} \\ &\quad + \frac{\vartheta'(\varsigma)}{\delta_0\vartheta(\varsigma)}\psi(\varsigma) + \frac{p_0^{\alpha}}{\delta_0\tau_0}\frac{\vartheta'(\varsigma)}{\vartheta(\varsigma)}\tilde{\psi}(\varsigma) \\ &\quad - \frac{1}{\delta_0}\alpha\lambda M\left(\delta^{-1}(\varsigma)\right)^{n-2} \frac{(\psi(\varsigma))^{\alpha+1/\alpha}}{\left(\vartheta(\varsigma) r\left(\delta^{-1}(\varsigma)\right)\right)^{1/\alpha}} \\ &\quad -\alpha\lambda M\frac{p_0^{\alpha}}{\delta_0\tau_0}\left(\delta^{-1}(\tau(\varsigma))\right)^{n-2} \frac{(\tilde{\psi}(\varsigma))^{\alpha+1/\alpha}}{\left(\vartheta(\varsigma) r\left(\delta^{-1}(\tau(\varsigma))\right)\right)^{1/\alpha}}.\end{aligned} \tag{19}$$

From (16), we obtain

$$\begin{aligned}\frac{1}{\delta_0}\psi'(\varsigma)+\frac{p_0^\alpha}{\delta_0\tau_0}\tilde{\psi}'(\varsigma) &\leq -\vartheta(\varsigma)Q(\varsigma)+\frac{\vartheta'(\varsigma)}{\delta_0\vartheta(\varsigma)}\psi(\varsigma)+\frac{p_0^\alpha}{\delta_0\tau_0}\frac{\vartheta'(\varsigma)}{\vartheta(\varsigma)}\tilde{\psi}(\varsigma)\\ &\quad-\frac{1}{\delta_0}\alpha\lambda M\left(\delta^{-1}(\varsigma)\right)^{n-2}\frac{(\psi(\varsigma))^{\alpha+1/\alpha}}{\left(\vartheta(\varsigma)r\left(\delta^{-1}(\varsigma)\right)\right)^{1/\alpha}}\\ &\quad-\alpha\lambda M\frac{p_0^\alpha}{\delta_0\tau_0}\left(\delta^{-1}(\tau(\varsigma))\right)^{n-2}\frac{(\tilde{\psi}(\varsigma))^{\alpha+1/\alpha}}{\left(\vartheta(\varsigma)r\left(\delta^{-1}(\tau(\varsigma))\right)\right)^{1/\alpha}}. \end{aligned} \tag{20}$$

Applying $B[\cdot;l,\varsigma]$ to (20), we obtain

$$\begin{aligned}B\left[\tfrac{1}{\delta_0}\psi'(\varsigma)+\tfrac{p_0^\alpha}{\delta_0\tau_0}\tilde{\psi}'(\varsigma);l,\varsigma\right] &\leq B[-\vartheta(s)Q(s)+\tfrac{\vartheta'(s)}{\delta_0\vartheta(s)}\psi(s)+\tfrac{p_0^\alpha}{\delta_0\tau_0}\tfrac{\vartheta'(s)}{\vartheta(s)}\tilde{\psi}(s)\\ &\quad-\tfrac{1}{\delta_0}\alpha\lambda M\left(\delta^{-1}(s)\right)^{n-2}\tfrac{(\psi(s))^{\alpha+1/\alpha}}{\left(\vartheta(s)r\left(\delta^{-1}(s)\right)\right)^{1/\alpha}}\\ &\quad-\alpha\lambda M\tfrac{p_0^\alpha}{\delta_0\tau_0}\left(\delta^{-1}(\tau(s))\right)^{n-2}\tfrac{(\tilde{\psi}(s))^{\alpha+1/\alpha}}{\left(\vartheta(s)r\left(\delta^{-1}(\tau(s))\right)\right)^{1/\alpha}};l,\varsigma]. \end{aligned} \tag{21}$$

By (4) and the inequality above, we find

$$\begin{aligned}\varsigma[\vartheta(s)Q(s);l,\varsigma] &\leq \varsigma[\frac{1}{\delta_0}\left(\varphi(s)+\frac{\vartheta'(s)}{\vartheta(s)}\right)\psi(s)+\frac{p_0^\alpha}{\delta_0\tau_0}\left(\varphi(s)+\frac{\vartheta'(s)}{\vartheta(s)}\right)\tilde{\psi}(s)\\ &\quad-\frac{1}{\delta_0}\alpha\lambda M\left(\delta^{-1}(s)\right)^{n-2}\frac{(\psi(s))^{\alpha+1/\alpha}}{\left(\vartheta(s)r\left(\delta^{-1}(s)\right)\right)^{1/\alpha}}\\ &\quad-\alpha\lambda M\frac{p_0^\alpha}{\delta_0\tau_0}\left(\delta^{-1}(\tau(s))\right)^{n-2}\frac{(\tilde{\psi}(s))^{\alpha+1/\alpha}}{\left(\vartheta(s)r\left(\delta^{-1}(\tau(s))\right)\right)^{1/\alpha}};l,\varsigma]. \end{aligned} \tag{22}$$

Using Lemma 1, we set

$$D=\frac{1}{\delta_0}\left(\varphi(s)+\frac{\vartheta'(s)}{\vartheta(s)}\right),\quad C=\frac{\frac{1}{\delta_0}\alpha\lambda M\left(\delta^{-1}(s)\right)^{n-2}}{\left(\vartheta(s)r\left(\delta^{-1}(s)\right)\right)^{1/\alpha}}\quad\text{and}\quad w=\psi,$$

we have

$$\begin{aligned}&\frac{1}{\delta_0}\left(\varphi(s)+\frac{\vartheta'(s)}{\vartheta(s)}\right)\psi(s)-\frac{1}{\delta_0}\alpha\lambda M\left(\delta^{-1}(s)\right)^{n-2}\frac{(\psi(s))^{\alpha+1/\alpha}}{\left(\vartheta(\varsigma)r\left(\delta^{-1}(s)\right)\right)^{1/\alpha}}\\ <\ &\frac{1}{(\alpha+1)^{\alpha+1}\delta_0}\left(\varphi(\varsigma)+\frac{\vartheta'(s)}{\vartheta(s)}\right)^{\alpha+1}\frac{\vartheta(\varsigma)r\left(\delta^{-1}(\tau(s))\right)}{\left(\lambda M\left(\delta^{-1}(s)\right)^{n-2}\right)^{\alpha}}. \end{aligned} \tag{23}$$

Hence, from (22) and (23), we have

$$\begin{aligned}B[\vartheta(s)Q(s);l,\varsigma] &\leq B[\left(\varphi(s)+\frac{\vartheta'(s)}{\vartheta(s)}\right)^{\alpha+1}\frac{\vartheta(s)r\left(\delta^{-1}(s)\right)}{(\alpha+1)^{\alpha+1}\delta_0\left(\lambda M\left(\delta^{-1}(s)\right)^{n-2}\right)^{\alpha}}\\ &\quad+\left(\varphi(s)+\frac{\vartheta'(s)}{\vartheta(s)}\right)^{\alpha+1}\frac{p_0^\alpha\vartheta(s)r\left(\delta^{-1}(\tau(s))\right)}{(\alpha+1)^{\alpha+1}\delta_0\tau_0\left(\lambda M\left(\delta^{-1}(\tau(s))\right)^{n-2}\right)^{\alpha}};l,\varsigma]. \end{aligned}$$

Easily, we find that

$$B\left[\vartheta(s)Q(s);l,\varsigma\right] \leq B\left[\frac{\vartheta(s)}{\delta_0(\alpha+1)^{\alpha+1}(\lambda M)^{\alpha}}\left(\varphi(s)+\frac{\vartheta'(s)}{\vartheta(s)}\right)^{\alpha+1} \times\left(\frac{r\left(\delta^{-1}(s)\right)}{\left(\left(\delta^{-1}(s)\right)^{n-2}\right)^{\alpha}}+\frac{p_0^{\alpha}r\left(\delta^{-1}(\tau(s))\right)}{\tau_0\left(\left(\delta^{-1}(\tau(s))\right)^{n-2}\right)^{\alpha}}\right);l,\varsigma\right].$$

That is,

$$B\left[\vartheta(s)Q(s)-\Omega(\varsigma)\left(\frac{r\left(\delta^{-1}(s)\right)}{\left(\left(\delta^{-1}(s)\right)^{n-2}\right)^{\alpha}}+\frac{p_0^{\alpha}r\left(\delta^{-1}(\tau(s))\right)}{\tau_0\left(\left(\delta^{-1}(\tau(s))\right)^{n-2}\right)^{\alpha}}\right);l,\varsigma\right]\leq 0.$$

Taking the super limit in the inequality above, we obtain

$$\limsup_{\varsigma\to\infty}B\left[\vartheta(s)Q(s)-\Omega(s)\left(\frac{r\left(\delta^{-1}(s)\right)}{\left(\left(\delta^{-1}(s)\right)^{n-2}\right)^{\alpha}}+\frac{p_0^{\alpha}r\left(\delta^{-1}(\tau(s))\right)}{\tau_0\left(\left(\delta^{-1}(\tau(s))\right)^{n-2}\right)^{\alpha}}\right);l,\varsigma\right]\leq 0, \tag{24}$$

which is a contradiction. The proof is complete. □

3. Tow Conditions Theorem

Lemma 6 ([22]). (Lemma 1.2) *Assume that x is an eventually positive solution of (1). Then, there exists two possible cases:*

$$\begin{array}{ll}(\mathbf{I}_1) & z(\varsigma)>0,\ z'(\varsigma)>0, z''(\varsigma)>0,\ \ z^{(n-1)}(\varsigma)>0,\ \ z^{(n)}(\varsigma)<0,\\ (\mathbf{I}_2) & z(\varsigma)>0, z^{(j)}(\varsigma)>0, z^{(j+1)}(\varsigma)<0 \text{ for all odd integer}\\ & j\in\{1,2,\ldots,n-3\}, z^{(n-1)}(\varsigma)>0, z^{(n)}(\varsigma)<0,\end{array}$$

for $\varsigma\geq\varsigma_1$, where $\varsigma_1\geq\varsigma_0$ is sufficiently large.

Lemma 7 ([22]). (Lemma 1.2) *Assume that x is an eventually positive solution of (1) and*

$$\int_{\varsigma_0}^{\infty}\left(\Psi(s)-\frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(s)\left(\rho'(s)\right)^{\alpha+1}}{\mu^{\alpha}s^{2\alpha}\rho^{\alpha}(s)}\right)\mathrm{d}s=\infty, \tag{25}$$

where

$$\Psi(\varsigma)=\vartheta\rho(\varsigma)q(\varsigma)\left(1-p(\delta(\varsigma))\right)^{\alpha}\left(\delta(\varsigma)\backslash\varsigma\right)^{3\alpha},$$

where $\rho\in C^1([\varsigma_0,\infty),(0,\infty))$, then it will be z does not satisfy case $(\mathbf{I}_1)$.

Lemma 8. *Let (2) holds and assume that x is an eventually positive solution of (1). If there exists positive functions $\vartheta\in C^1([\varsigma_0,\infty),\mathbb{R})$ such that for all $M>0$*

$$\limsup_{\varsigma\to\infty}B\left[\vartheta(s)Q(s)-\Theta(s)\left(\frac{r\left(\delta^{-1}(s)\right)}{\left(\left(\delta^{-1}(s)\right)^{n-2}\right)^{\alpha}}+\frac{p_0^{\alpha}r\left(\delta^{-1}(\tau(s))\right)}{\tau_0\left(\left(\delta^{-1}(\tau(s))\right)^{n-2}\right)^{\alpha}}\right);l,\varsigma\right]>0, \tag{26}$$

for some $\mu\in(0,1)$, then z not satisfies case $(\mathbf{I}_2)$.

Proof. Assume to the contrary that (1) has a nonoscillatory solution in $[\varsigma_0,\infty)$. Without loss of generality, we let x be an eventually positive solution of (1). From Lemma 3, we obtain

$$z'(\varsigma)\geq\frac{\mu}{(n-2)!}\varsigma^{n-2}z^{(n-1)}(\varsigma). \tag{27}$$

Now, we define a function ω by

$$\omega(\varsigma) = \vartheta(\varsigma) \frac{r\left(\delta^{-1}(\varsigma)\right)\left(z^{(n-1)}\left(\delta^{-1}(\varsigma)\right)\right)^{\alpha}}{z^{\alpha}(\varsigma)}. \tag{28}$$

Then we see that $\omega(\varsigma) > 0$ for $\varsigma \geq \varsigma_1$, and

$$\begin{aligned} \omega'(\varsigma) &\leq \frac{\vartheta'(\varsigma)}{\vartheta(\varsigma)}\omega(\varsigma) + \vartheta(\varsigma)\frac{\left(r\left(\delta^{-1}(\varsigma)\right)\left(z^{(n-1)}\left(\delta^{-1}(\varsigma)\right)\right)^{\alpha}\right)'}{z^{\alpha}(\lambda\varsigma)} \\ &\quad -\alpha\frac{\mu}{(n-2)!}\left(\delta^{-1}(\varsigma)\right)^{n-2}\frac{(\omega(\varsigma))^{\alpha+1/\alpha}}{\left(\vartheta(\varsigma)\, r\left(\delta^{-1}(\varsigma)\right)\right)^{1/\alpha}}. \end{aligned} \tag{29}$$

Similarly, define

$$\tilde{\omega}(\varsigma) = \vartheta(\varsigma) \frac{r\left(\delta^{-1}(\tau(\varsigma))\right)\left(z^{(n-1)}\left(\delta^{-1}(\tau(\varsigma))\right)\right)^{\alpha}}{z^{\alpha}(\varsigma)}. \tag{30}$$

Then we see that $\tilde{\omega}(\varsigma) > 0$ for $\varsigma \geq \varsigma_1$, and

$$\begin{aligned} \tilde{\omega}'(\varsigma) &\leq \frac{\vartheta'(\varsigma)}{\vartheta(\varsigma)}\tilde{\omega}(\varsigma) + \vartheta(\varsigma)\frac{\left(r\left(\delta^{-1}(\tau(\varsigma))\right)\left(z^{(n-1)}\left(\delta^{-1}(\tau(\varsigma))\right)\right)^{\alpha}\right)'}{z^{\alpha}(\varsigma)} \\ &\quad -\alpha\frac{\mu}{(n-2)!}\left(\delta^{-1}(\tau(\varsigma))\right)^{n-2}\frac{(\tilde{\omega}(\varsigma))^{\alpha+1/\alpha}}{\left(\vartheta(\varsigma)\, r\left(\delta^{-1}(\tau(\varsigma))\right)\right)^{1/\alpha}}. \end{aligned}$$

Thus, we get

$$\limsup_{\varsigma\to\infty} B\left[\vartheta(s)\,Q(s) - \Theta(s)\left(\frac{r\left(\delta^{-1}(s)\right)}{\left(\left(\delta^{-1}(s)\right)^{n-2}\right)^{\alpha}} + \frac{p_0^{\alpha} r\left(\delta^{-1}(\tau(s))\right)}{\tau_0\left(\left(\delta^{-1}(\tau(s))\right)^{n-2}\right)^{\alpha}}\right); l, \varsigma\right] \leq 0,$$

which is a contradiction. The proof is complete. □

Theorem 2. *Assume that (25) and (26) hold for some $\mu \in (0,1)$. Then every solution of (1) is oscillatory.*

Example 1. *Consider the equation*

$$(x(\varsigma) + 2x(\varsigma - 5\pi))'' + q_0 x(\varsigma - \pi) = 0. \tag{31}$$

We note that $r(\varsigma) = 1$, $p(\varsigma) = 2$, $\tau(\varsigma) = \varsigma - 5\pi$, $\delta(\varsigma) = \varsigma - \pi$, $\delta^{-1}(s) = \varsigma + \pi$ and $q(\varsigma) = Q(\varsigma) = q_0$. Thus, if we choose $\Phi(\varsigma) = (\varsigma - s)(s - l)$, then it is easy to see that

$$\varphi(\varsigma, s, l) = \frac{(\varsigma - s) - (s - l)}{(\varsigma - s)(s - l)}$$

and

$$\begin{aligned} \Omega(s) &= \frac{\vartheta(s)}{\delta_0(\alpha+1)^{\alpha+1}(\lambda M)^{\alpha}}\left(\varphi(s) + \frac{\vartheta'(s)}{\vartheta(s)}\right)^{\alpha+1} \\ &= \frac{1}{4\lambda M}\left(\frac{(\varsigma - s) - (s - l)}{(\varsigma - s)(s - l)}\right)^2. \end{aligned}$$

Thus,

$$\begin{aligned} & \limsup_{\varsigma\to\infty} B\left[\vartheta(s)\,Q(s)-\Omega(s)\left(\frac{r\left(\delta^{-1}(s)\right)}{\left(\left(\delta^{-1}(s)\right)^{n-2}\right)^{\alpha}}+\frac{p_0^{\alpha} r\left(\delta^{-1}(\tau(s))\right)}{\tau_0\left(\left(\delta^{-1}(\tau(s))\right)^{n-2}\right)^{\alpha}}\right);l,\varsigma\right] \\ = & \limsup_{\varsigma\to\infty} B\left[q_0-\frac{3}{4\lambda M}\left(\frac{(\varsigma-s)-(s-l)}{(\varsigma-s)(s-l)}\right)^2;l,\varsigma\right]>0. \end{aligned}$$

Therefore, by Theorem 1, every solution of Equation (31) is oscillatory.

Example 2. *Consider the equation*

$$(x(\varsigma)+p_0 x(\varsigma-5\pi))^{(4)}+q_0 x(\varsigma-\pi), \tag{32}$$

where $q_0>0$. *Let* $r(\varsigma)=1$, $p(\varsigma)=p_0$, $\tau(\varsigma)=\varsigma-5\pi$, $\delta(\varsigma)=\varsigma-\pi$, $\delta^{-1}(s)=\varsigma+\pi$ *and* $q(\varsigma)=Q(\varsigma)=q_0$, *then we have*

$$\int_{\varsigma_0}^{\infty} r^{-1/\alpha}(s)\,\mathrm{d}s=\infty.$$

Next, if we choose $\varphi(\varsigma)=(\varsigma-s)(s-l)$, *then we conclude that the conditions (25) and (26) are satisfied. Thus, using Theorem 2, Equation (32) is oscillatory.*

4. Conclusions

In this work, by using the generalized Riccati transformations technique, we provided new oscillation criteria for (1). Furthermore, in future work, by using the comparison method, we find some new Hille and Nehari types and Philos type oscillation criteria of (1).

Author Contributions: Writing original draft, formal analysis, writing review and editing, O.B. and O.M.; writing review and editing, funding and supervision, R.A.E.-N. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: The authors thank the reviewers for for their useful comments, which led to the improvement of the content of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

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