

Article

The Split Various Variational Inequalities Problems for Three Hilbert Spaces

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Abstract: There are many methods for finding a common solution of a system of variational inequalities, a split equilibrium problem, and a hierarchical fixed-point problem in the setting of real Hilbert spaces. They proved the strong convergence theorem. Many split feasibility problems are generated in real Hilbert spaces. The open problem is proving a strong convergence theorem of three Hilbert spaces with different methods from the last method. In this research, a new split variational inequality in three Hilbert spaces is proposed. Important tools which are used to solve classical problems will be developed. The convergence theorem for finding a common element of the set of solution of such problems and the sets of fixed-points of discontinuous mappings has been proved.

Keywords: nonspreading; pseudo-nonspreading; new split variational inequalities

MSC: 47H10; 47J20; 49J40

1. Introduction

We use the following symbols throughout this paper: let H be a real Hilbert space and C be a nonempty closed convex subset of H . We also use the symbols $\rightharpoonup$ and $\rightharpoonup^*$, which represent strong and weak convergence, respectively. The variational inequality problem (VIP) is a well known problem. That is to find a point ω_* such that

$$\langle y - \omega_*, G\omega_* \rangle \geq 0, \text{ for all } y \in C, \quad (1)$$

where $G : C \rightarrow H$ is a mapping. The set of all solutions of (1) is denoted by $Var(C, G)$.

The variational inequality problem has been applied in various fields such as industry, finance, economics, social, ecology, regional, pure and applied sciences; see [1–3]. For every $i = 1, 2$, let H_i be a real Hilbert space and C, Q be nonempty closed convex subset of H_1 , and H_2 , respectively. Recently, Censor [4] has introduced a new variational problem called the split inequality problem (SIP). It entails finding a solution of one variational inequality problem (VIP), the image of which, under a given bounded linear transformation, is a solution of another VIP.

The split variational inequality problem is assigned to the following formula; find a point $\omega_* \in C$ such that

$$\langle f(\omega_*), x - \omega_* \rangle \geq 0, \text{ for all } x \in C \quad (2)$$

and a point $y^* = A\omega_*$ solves

$$\langle y - y^*, g(y^*) \rangle \geq 0, \text{ for all } y \in Q, \quad (3)$$

where $A : H_1 \rightarrow H_2$ is a bounded linear operator and $f : H_1 \rightarrow H_1, g : H_2 \rightarrow H_2$ are mappings. The set of all solutions of (2) and (3) is denoted by

$$\Omega = \{x \in \text{Var}(C, f) : \text{for all } x \in \text{Var}(Q, g)\}.$$

The split variational inequality problem can be applied to model in intensity-modulated radiation therapy (IMRT) treatment planning.

There are also a lot of authors who have introduced convergence theorem related to the split variational inequality and fixed point problems; see [5–7] for an example. In [8], they have studied the Mann implicit iterations for strongly accretive and strongly pseudo-contractive mappings and found that this implicit scheme gives a better convergence rate estimate.

The following definitions are important tools used in this research. A mapping ϑ of H into itself is called nonexpansive if $\|\vartheta x - \vartheta y\| \leq \|x - y\|$, for all $x, y \in H$. We denote by $F(\vartheta)$ the set of fixed points of ϑ (i.e., $F(\vartheta) = \{x \in C : \vartheta x = x\}$). A nonexpansive mapping ϑ is equivalent to the following inequality;

$$\langle (I - \vartheta)x - (I - \vartheta)y, \vartheta y - \vartheta x \rangle \leq \frac{1}{2} \|(I - \vartheta)x - (I - \vartheta)y\|^2,$$

for all $x, y \in H$. From the equation above if $y \in F(\vartheta)$ and $x \in H$, we can conclude that;

$$\langle (I - \vartheta)x, y - \vartheta x \rangle \leq \frac{1}{2} \|(I - \vartheta)x\|^2.$$

A mapping A of C into H is called inverse strongly monotonic, if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2,$$

for all $x, y \in C$. In [9], Kohsaka and Takahashi introduced the nonspreading mapping in Hilbert spaces H which is defined by the following inequality $2\|\vartheta x - \vartheta y\|^2 \leq \|\vartheta x - y\|^2 + \|x - \vartheta y\|^2$, for all $x, y \in C$.

Following the terminology of Browder and Petryshyn [10], in [11], Osilike and Isiogugu introduced the mapping $\vartheta : C \rightarrow C$, which is called κ -strictly pseudo-nonspreading mapping if there exists $\kappa \in [0, 1)$ such that

$$\|\vartheta x - \vartheta y\|^2 \leq \|x - y\|^2 + \kappa \|(I - \vartheta)x - (I - \vartheta)y\|^2 + 2\langle x - \vartheta x, y - \vartheta y \rangle,$$

for all $x, y \in C$. Clearly every nonspreading mapping is κ -strictly pseudo-nonspreading; see, for example, [11].

In [12], Bnouhachem modified a projection process for finding a common solution of a system of variational inequalities, a split equilibrium problem and a hierarchical fixed-point problem in the setting of real Hilbert spaces and also proved the strong convergence theorem of the sequence $\{x_n\}$ generated by

$$\begin{cases} u_n = \vartheta_{r_n}^{F_1} \left(x_n + \gamma A^* \left(T_{r_n}^{F_2} - I \right) Ax_n \right); \\ z_n = P_C [P_C [u_n - \alpha_1 B_2 u_n] - \alpha_1 B_1 P_C [u_n - \alpha_2 B_2 u_n]]; \\ y_n = \beta_n \varrho x_n + (1 - \beta_n) z_n; \\ x_{n+1} = P_C [\alpha_n \rho U(x_n) + (I - \alpha_n \mu F) \vartheta(y_n)], \text{ for all } n \geq 0, \end{cases}$$

where $A : H_1 \rightarrow H_2$ is a bounded linear operator. Assume that $F_1 : C \times C \rightarrow \mathbb{R}$ and $F_2 : Q \times Q \rightarrow \mathbb{R}$ are the bifunctions; $B_i : C \rightarrow H$ is a θ_i -inverse strongly monotonic mapping for each $i = 1, 2$ and $S, \vartheta : C \rightarrow C$ nonexpansive mapping; $F : C \rightarrow C$ is a k -Lipschitzian mapping and is η -strongly monotonic; $U : C \rightarrow C$ is a τ -Lipschitzian mapping; and the positive parameters are $r_n, \alpha_n, \alpha_1, \alpha_2, \rho, \mu$, for all $n \in \mathbb{N}$.

Let C and Q be nonempty closed convex subsets of the real Hilbert spaces H_1 and H_2 , respectively. The split feasibility problem (SFP) is formulated as :

$$\text{to find } x^* \in C \text{ such that } Ax^* \in Q, \quad (4)$$

where $A : H_1 \rightarrow H_2$ is a bounded linear operator. The SFP are also applied in [13,14].

Recently, Moudafi [15] introduced the following new split feasibility problem, which is also called general split equality problem:

Let H_1, H_2, H_3 be real Hilbert spaces, $C \subset H_1, Q \subset H_2$ be two nonempty closed convex sets and $A : H_1 \rightarrow H_3, B : H_2 \rightarrow H_3$ be two bounded linear operators. Moudafi studied the convergence of a relaxed alternating CQ-algorithm for solving the new split feasibility problem, aiming to find

$$\omega_* \in C, y^* \in Q \text{ such that } A\omega_* = By^*. \quad (5)$$

In order to prove the weak convergence theory to solve general split equality problem (5), Moudafi defined the following iteration process $\{x_k\}$:

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \gamma_k B^*(Ax_{k+1} - By_k)), \end{cases}$$

where A^*, B^* are adjoint operators of A, B respectively, proper conditions of the positive parameter γ_k , for all $k \geq 1$. In order to avoid using the projection, Moudafi [16] introduced and studied the following problem: Let $T : H_1 \rightarrow H_1$ and $S : H_2 \rightarrow H_2$ be nonlinear operators such that $\text{Fix}(T) \neq \emptyset$ and $\text{Fix}(S) \neq \emptyset$, where $\text{Fix}(T)$ and $\text{Fix}(S)$ denote the sets of fixed points of T and S , respectively. If $C = \text{Fix}(T)$ and $Q = \text{Fix}(S)$; then the split equality problem reduces to

$$\text{to find } x \in \text{Fix}(T) \text{ and } y \in \text{Fix}(S) \text{ such that } Ax = By, \quad (6)$$

which is called a split equality fixed point problem (SEFPP).

Denote by Γ the solution set of split equality fixed point problem (6). There were recently SEFPP research articles in [17,18].

Question A. Can we prove a strong convergence theorem of three Hilbert spaces by different methods from Moudafi [15]?

For every $i = 1, 2, 3$, let H_i be a real Hilbert space and C_i be a nonempty closed convex subset of H_i . Let $B_i : C_i \rightarrow H_i$ be a mapping, for all $i = 1, 2, 3$, and let $A_2 : H_1 \rightarrow H_2$ and $A_3 : H_2 \rightarrow H_3$. The split various variational inequality is to find the points

$$\begin{cases} \omega_1^* \in C_1, \text{ such that } \langle B_1 \omega_1^*, x_1 - \omega_1^* \rangle \geq 0, \text{ for all } x_1 \in C_1, \text{ and} \\ \omega_2^* = A_2 \omega_1^* \in C_2, \text{ such that } \langle B_2 \omega_2^*, x_2 - \omega_2^* \rangle \geq 0, \text{ for all } x_2 \in C_2, \text{ and} \\ \omega_3^* = A_3 \omega_2^* \in C_3, \text{ such that } \langle B_3 \omega_3^*, x_3 - \omega_3^* \rangle \geq 0, \text{ for all } x_3 \in C_3. \end{cases} \quad (7)$$

The set of the solutions of (7) is denoted by $\Omega = \{\omega^* = (\omega_1^*, \omega_2^*, \omega_3^*) \in C_1 \times C_2 \times C_3 : \omega_i^* \in \text{Var}(C_i, B_i), \text{ for all } i = 1, 2, 3\}$.

To answer question A, we have created a new tool to prove a strong convergence theorem for three Hilbert spaces to be used for finding the solution of the problem (7) and the fixed points problem of nonspreading and pseudo-nonspreading mappings. Preliminaries In this section, we collect some definitions and lemmas in Hilbert space, which will be needed for proving our main results. More properties of Hilbert space can be found in [19].

Definition 1. The (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Lemmas 1 and 2 are properties of P_C .

Lemma 1. ([20]) For a given $x \in H$ and $y \in C$, $P_C x = y$ if and only if there holds the inequality $\langle y - x, z - y \rangle \geq 0$, for all $z \in C$.

Lemma 2. ([21]) Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H . Let $u \in C$. Then for $\lambda > 0$, $u = P_C(I - \lambda A)u$ if and only if $u \in \text{Var}(C, A)$, where P_C is the metric projection of H onto C .

Lemma 3. ([22]) Let $\{Y_n\}$ be a sequence of nonnegative real numbers satisfying $Y_{n+1} = (1 - \alpha_n)Y_n + \delta_n$, for all $n \geq 0$,

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (1) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;
- (2) $\limsup_{n \rightarrow +\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{+\infty} |\delta_n| < +\infty$.

Then $\lim_{n \rightarrow +\infty} Y_n = 0$.

Lemma 4. ([23]) Let $\{Y_n\}$ be a sequence of nonnegative real number satisfying,

$$Y_{n+1} = (1 - \alpha_n)Y_n + \alpha_n \beta_n, \text{ for all } n \geq 0$$

where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions

- (1) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^{+\infty} \alpha_n = +\infty$;
- (2) $\limsup_{n \rightarrow +\infty} \beta_n \leq 0$ or $\sum_{n=1}^{+\infty} |\alpha_n \beta_n| < +\infty$.

Then $\lim_{n \rightarrow +\infty} Y_n = 0$.

Lemma 5. For every $i = 1, 2, 3$, let H_i be a real Hilbert spaces and C_i be a nonempty closed convex subset of H_i . Let $B_i : C_i \rightarrow H_i$ be β_i -inverse strongly monotonic mappings with $\eta = \min_{i=1,2,3} \{\beta_i\}$ and let $A_2 : H_1 \rightarrow H_2$, $A_3 : H_2 \rightarrow H_3$ be bounded linear operators with the adjoint operator A_2^* and A_3^* , respectively. Assume that $\bar{x}_1 \in C_1$, $A_2 \bar{x}_1 = \bar{x}_2$, $A_3 \bar{x}_2 = \bar{x}_3$ and $\Omega \neq \emptyset$. The following are equivalent:

- (i) $\bar{x} \in \Omega$, where $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in C_1 \times C_2 \times C_3$.
(ii) $\bar{x}_1 = P_{C_1} (I_1 - \lambda_1 B_1) (\bar{x}_1 - \gamma_2 A_2^* ((I_2 - P_{C_2} (I_2 - \lambda_2 B_2)) \bar{x}_2 + \gamma_3 A_3^* (I_3 - P_{C_3} (I_3 - \lambda_3 B_3)) \bar{x}_3))$,

where $I_i : H_i \rightarrow H_i$ is an identity mapping, for all $i = 1, 2, 3$, $\gamma_2(1 + \gamma_3) \leq \frac{1}{L}$, $L = \max\{L_1, L_2\} \leq 1$ which L_1, L_2 are spectral radii of $A_2 A_2^*$ and $A_3 A_3^*$, respectively, $\lambda_i \in (0, 2\eta)$, for all $i = 1, 2, 3$ and $\gamma_2, \gamma_3 \geq 0$.

Proof. Let the conditions hold.

(i) $\Rightarrow$ (ii) Let $\bar{x} \in \Omega$ where $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in C_1 \times C_2 \times C_3$; we have

$$\bar{x}_i \in \text{Var}(C_i, B_i), \text{ for all } i = 1, 2, 3.$$

From Lemma 2, we have

$$\bar{x}_i \in F(P_{C_i}(I_i - \lambda_i B_i)), \text{ for all } i = 1, 2, 3.$$

From determining the definition of $\bar{x}$, we have

$$\bar{x}_1 = P_{C_1} (I_1 - \lambda_1 B_1) (\bar{x}_1 - \gamma_2 A_2^* ((I_2 - P_{C_2} (I_2 - \lambda_2 B_2)) \bar{x}_2 + \gamma_3 A_3^* (I_3 - P_{C_3} (I_3 - \lambda_3 B_3)) \bar{x}_3)).$$

(ii) $\Rightarrow$ (i) Let $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in C_1 \times C_2 \times C_3$, where $\bar{x}_2 = A_2 \bar{x}_1$, $\bar{x}_3 = A_3 \bar{x}_2$ and

$$\bar{x}_1 = P_{C_1} (I_1 - \lambda_1 B_1) (\bar{x}_1 - \gamma_2 A_2^* ((I_2 - P_{C_2} (I_2 - \lambda_2 B_2)) \bar{x}_2 + \gamma_3 A_3^* (I_3 - P_{C_3} (I_3 - \lambda_3 B_3)) \bar{x}_3)).$$

Since B_i is β_i -inverse strongly monotonic with $\lambda_i < 2\eta$, for all $i = 1, 2, 3$, we have $P_{C_i}(I_i - \lambda_i B_i)$ which is a nonexpansive mapping, for all $i = 1, 2, 3$.

Let $w \in \Omega$ where $w = (w_1, w_2, w_3) \in C_1 \times C_2 \times C_3$ where $w_2 = A_2 w_1$, $w_3 = A_3 w_2$. From (i) implies (ii), we have

$$w_1 = P_{C_1} (I_1 - \lambda_1 B_1) (w_1 - \gamma_2 A_2^* ((I_2 - P_{C_2} (I_2 - \lambda_2 B_2)) w_2 + \gamma_3 A_3^* (I_3 - P_{C_3} (I_3 - \lambda_3 B_3)) w_3)).$$

Put $M = (I_2 - P_{C_2} (I_2 - \lambda_2 B_2)) \bar{x}_2 + \gamma_3 A_3^* (I_3 - P_{C_3} (I_3 - \lambda_3 B_3)) \bar{x}_3$ and $N = (I_2 - P_{C_2} (I_2 - \lambda_2 B_2)) w_2 + \gamma_3 A_3^* (I_3 - P_{C_3} (I_3 - \lambda_3 B_3)) w_3$.

From determining the definition of $\bar{x}$ and w , we have

$$\begin{aligned}
 \|\bar{x}_1 - w_1\|^2 &\leq \|\bar{x}_1 - w_1 - \gamma_2 A_2^*(M - N)\|^2 \\
 &= \|\bar{x}_1 - w_1\|^2 - 2\gamma_2 \langle \bar{x}_1 - w_1, A_2^*(M - N) \rangle + \gamma_2^2 \|A_2^*(M - N)\|^2 \\
 &\leq \|\bar{x}_1 - w_1\|^2 - 2\gamma_2 \langle \bar{x}_2 - w_2, M - N \rangle + \gamma_2^2 L \|M - N\|^2 \\
 &\leq \|\bar{x}_1 - w_1\|^2 - 2\gamma_2 \langle \bar{x}_2 - w_2, (I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\bar{x}_2 + \gamma_3 A_3^*(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\bar{x}_3 \rangle \\
 &\quad + \gamma_2^2 L \|(I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\bar{x}_2 + \gamma_3 A_3^*(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\bar{x}_3\|^2 \\
 &= \|\bar{x}_1 - w_1\|^2 - 2\gamma_2 \langle \bar{x}_2 - w_2, (I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\bar{x}_2 \rangle + \gamma_3 \langle \bar{x}_3 - w_3, \\
 &\quad (I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\bar{x}_3 \rangle + \gamma_2^2 L \|(I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\bar{x}_2 \\
 &\quad + \gamma_3 A_3^*(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\bar{x}_3\|^2 \\
 &= \|\bar{x}_1 - w_1\|^2 + 2\gamma_2 \langle w_2 - \bar{x}_2, (I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\bar{x}_2 \rangle + 2\gamma_2 \gamma_3 \langle w_3 - \bar{x}_3, \\
 &\quad (I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\bar{x}_3 \rangle + \gamma_2^2 L \|(I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\bar{x}_2\|^2 \\
 &\quad + \gamma_3^2 L \|(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\bar{x}_3\|^2 + 2\gamma_3 \langle (I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\bar{x}_2, \\
 &\quad A_3^*(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\bar{x}_3 \rangle \\
 &\leq \|\bar{x}_1 - w_1\|^2 + 2\gamma_2 \langle w_2 - P_{C_2}(I_2 - \lambda_2 B_2)\bar{x}_2 + P_{C_2}(I_2 - \lambda_2 B_2)\bar{x}_2 - \bar{x}_2, \\
 &\quad (I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\bar{x}_2 \rangle + 2\gamma_2 \gamma_3 \langle w_3 - P_{C_3}(I_3 - \lambda_3 B_3)\bar{x}_3 + P_{C_3}(I_3 - \lambda_3 B_3)\bar{x}_3 - \bar{x}_3, \\
 &\quad (I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\bar{x}_3 \rangle + \gamma_2^2 L \|(I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\bar{x}_2\|^2 \\
 &\quad + \gamma_3^2 L \|(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\bar{x}_3\|^2 \\
 &\quad + \gamma_3 \|(I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\bar{x}_2\|^2 + \gamma_3 \|A_3^*(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\bar{x}_3\|^2 \\
 &\leq \|\bar{x}_1 - w_1\|^2 + 2\gamma_2 \left(\frac{1}{2} \|(I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\bar{x}_2\|^2 - \|(I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\bar{x}_2\|^2 \right) \\
 &\quad + 2\gamma_2 \gamma_3 \left(\frac{1}{2} \|(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\bar{x}_3\|^2 - \|(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\bar{x}_3\|^2 \right) \\
 &\quad + \gamma_2^2 L \|(I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\bar{x}_2\|^2 + \gamma_3^2 L \|(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\bar{x}_3\|^2 \\
 &\quad + \gamma_3 \|(I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\bar{x}_2\|^2 + \gamma_3 L \|(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\bar{x}_3\|^2 \\
 &= \|\bar{x}_1 - w_1\|^2 - \gamma_2 (1 - \gamma_2 L (1 + \gamma_3)) \|(I_2 - P_{C_2}(I_2 - \lambda_2 B_2))\bar{x}_2\|^2 \\
 &\quad - \gamma_2 \gamma_3 (1 - \gamma_2 L^2 (1 + \gamma_3)) \|(I_3 - P_{C_3}(I_3 - \lambda_3 B_3))\bar{x}_3\|^2.
 \end{aligned} \tag{8}$$

By applying above equation and Lemma 2, we have

$$\bar{x}_2 \in F(P_{C_2}(I - \lambda_2 B_2)) = \text{Var}(C_2, B_2) \text{ and } \bar{x}_3 \in F(P_{C_3}(I - \lambda_3 B_3)) = \text{Var}(C_3, B_3). \tag{9}$$

From determining the definitions of $\bar{x}$ and (9), we have

$$\bar{x}_1 \in F(P_{C_1}(I - \lambda_1 B_1)) = \text{Var}(C_1, B_1).$$

Hence $\bar{x} \in \Omega$. $\square$

Lemma 6. Let C be a nonempty closed convex subset of Hilbert space H . Let $\vartheta : C \rightarrow C$ be a nonspreading mapping and $q : C \rightarrow C$ be κ -pseudo-nonspreding mapping with $F(\vartheta) \cap F(q) \neq \emptyset$. Then $F(P_C(I - \gamma(a(I - \vartheta) + (1 - a)(I - q)))) = F(\vartheta) \cap F(q)$ for all $a \in (0, 1)$ and $\gamma > 0$. Moreover, if $\gamma < 1 - \kappa$, then

$$\|I - \gamma(a(I - \vartheta) + (1 - a)(I - q))x - \omega^*\| \leq \|\omega^* - x\|,$$

for all $x \in C$ and $\omega^* \in F(\varrho) \cap F(\vartheta)$.

Proof. Let $\omega_0 \in F(\varrho) \cap F(\vartheta)$, we have

$$P_C(I - \gamma(a(I - \vartheta) + (1 + a)(I - \varrho)))\omega_0 = \omega_0.$$

It follows that $\omega_0 \in F(P_C(I - \gamma(a(I - \vartheta) + (1 + a)(I - \varrho))))$. Therefore

$$F(\vartheta) \cap F(\varrho) \subseteq F(P_C(I - \gamma(a(I - \vartheta) + (1 + a)(I - \varrho))))$$

Let $\omega_0 \in F(P_C(I - \gamma(a(I - \vartheta) + (1 + a)(I - \varrho))))$ and $\omega^* \in F(\vartheta) \cap F(\varrho)$. From Lemma 2, we have

$$\langle y - \omega_0, a(I - \vartheta)\omega_0 + (1 - a)(I - \varrho)\omega_0 \rangle \geq 0,$$

for all $y \in C$.

From determining the definition of ϱ , we have

$$\begin{aligned} \|\omega_0 - \omega^*\|^2 + \kappa \|(I - \varrho)\omega_0\|^2 &\geq \|\varrho\omega_0 - \omega^*\|^2 \\ &= \|(I - \varrho)\omega_0 - (\omega_0 - \omega^*)\|^2 \\ &= \|(I - \varrho)\omega_0\|^2 - 2\langle (I - \varrho)\omega_0, \omega_0 - \omega^* \rangle + \|\omega_0 - \omega^*\|^2 \end{aligned} \quad (10)$$

From the result of the calculation from the inequality (10), we get

$$\langle (I - \varrho)\omega_0, \omega_0 - \omega^* \rangle \geq \left(\frac{1 - \kappa}{2}\right) \|(I - \varrho)\omega_0\|^2 \quad (11)$$

Assume that $\omega_0 \neq \vartheta\omega_0$; then we have $\|(I - \vartheta)\omega_0\| > 0$. Using the same method as (11) and definitions of ϑ , we get

$$\langle (I - \vartheta)\omega_0, \omega_0 - \omega^* \rangle \geq \frac{1}{2} \|(I - \vartheta)\omega_0\|^2 \quad (12)$$

From (11) and $a \in (0, 1)$, we obtain

$$\begin{aligned} \langle \omega^* - \omega_0, a(I - \vartheta)\omega_0 \rangle &= \langle \omega^* - \omega_0, a(I - \vartheta)\omega_0 + (1 - a)(I - \varrho)\omega_0 \rangle \\ &\quad - (1 - a)\langle \omega^* - \omega_0, (I - \varrho)\omega_0 \rangle \\ &\geq (1 - a)\langle \omega_0 - \omega^*, (I - \varrho)\omega_0 \rangle. \end{aligned} \quad (13)$$

From (13), we have

$$\langle \omega^* - \omega_0, (I - \vartheta)\omega_0 \rangle \geq 0.$$

From above and (12), we have

$$0 \leq \langle \omega^* - \omega_0, (I - \vartheta)\omega_0 \rangle \leq -\frac{1}{2} \|(I - \vartheta)\omega_0\|^2.$$

Thus, $\|(I - \vartheta)\omega_0\| \leq 0$. This is a contradiction.

Thus, we have $\omega_0 = \vartheta\omega_0$ and it implies that

$$\omega_0 \in F(\vartheta). \quad (14)$$

Similarly, by using the same technique as (14), we have

$$\omega_0 \in F(\varrho). \quad (15)$$

From (14) and (15), we have

$$F(P_C(I - \gamma(a(I - \vartheta) + (1 + a)(I - \varrho)))) \subseteq F(\varrho) \cap F(\vartheta).$$

Let $\omega^* \in F(\varrho) \cap F(\vartheta)$ and $x \in C$; we have

$$\begin{aligned} \|(I - \gamma(a(I - \vartheta) + (1 - a)(I - \varrho)))x - \omega^*\|^2 &= \|(I - \gamma(a(I - \vartheta) + (1 - a)(I - \varrho)))x \\ &\quad - (I - \gamma(a(I - \vartheta) + (1 - a)(I - \varrho)))\omega^*\|^2 \\ &= \|x - \omega^* - \gamma(a((I - \vartheta)x - (I - \vartheta)\omega^*) \\ &\quad + (1 - a)((I - \varrho)x - (I - \varrho)\omega^*))\|^2 \\ &= \|x - \omega^*\|^2 - 2\gamma\langle a((I - \vartheta)x - (I - \vartheta)\omega^*) \\ &\quad + (1 - a)((I - \varrho)x - (I - \varrho)\omega^*), x - \omega^* \rangle \\ &\quad + \gamma^2\|a((I - \vartheta)x - (I - \vartheta)\omega^*) \\ &\quad + (1 - a)((I - \varrho)x - (I - \varrho)\omega^*)\|^2 \\ &\leq \|x - \omega^*\|^2 - 2\gamma a\langle (I - \vartheta)x - (I - \vartheta)\omega^*, x - \omega^* \rangle \\ &\quad - 2\gamma(1 - a)\langle (I - \varrho)x - (I - \varrho)\omega^*, x - \omega^* \rangle \\ &\quad + \gamma^2 a\|(I - \vartheta)x - (I - \vartheta)\omega^*\|^2 \\ &\quad + (1 - a)\gamma^2\|(I - \varrho)x - (I - \varrho)\omega^*\|^2 \\ &\leq \|x - \omega^*\|^2 - 2\gamma a \frac{\|(I - T)x\|^2}{2} \\ &\quad - 2\gamma(1 - a)(1 - \kappa) \frac{\|(I - \varrho)x\|^2}{2} \\ &\quad + \gamma^2 a\|(I - \vartheta)x\|^2 + (1 - a)\gamma^2\|(I - \varrho)x\|^2 \\ &\leq \|x - \omega^*\|^2. \end{aligned}$$

□

2. The Split Various Variational Inequality Theorem

Theorem 7. For every $i = 1, 2, 3$, let C_i be a closed convex subset of a real Hilbert space H_i . Let $B_i : C_i \rightarrow H_i$ be β_i -inverse strongly monotonic mappings with $\eta = \min_{i=1,2,3} \{\beta_i\}$ and let $A_2 : H_1 \rightarrow H_2$, $A_3 : H_2 \rightarrow H_3$ be a bounded linear operator with the adjoint operator A_2^* and A_3^* , respectively. Assume that $\bar{x}_1 \in H_1$, $\bar{x}_2 = A_2\bar{x}_1$, $\bar{x}_3 = A_3\bar{x}_2$ and $\Omega \neq \emptyset$. Let $\vartheta, \varrho : C \rightarrow C$ be nonspreading and κ -strictly pseudo-nonspreading mappings, respectively. Assume that $\Omega \cap F(\vartheta) \cap F(\varrho) \neq \emptyset$ and let the sequence $\{x_n\}$ generated by $u, x_1 \in C$, and

$$\begin{aligned} x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n P_{C_1}(I_1 - \lambda_n(a(I_1 - \vartheta) + (1 - a)(I_1 - \varrho)))x_n \\ &\quad + \delta_n P_{C_1}(I_1 - \lambda_1 B_1)(x_n^1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 + \gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3 B_3))x_n^3)), \end{aligned}$$

for all $n \geq 1$ and $a \in (0, 1)$, $I_i : H_i \rightarrow H_i$ are identity mappings, for all $i = 1, 2, 3$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and $x_n = x_n^1, x_n^2 = A_2 x_n^1, x_n^3 = A_3 x_n^2$, for all $n \in \mathbb{N}$, $0 < \lambda_i < 2\eta$, for all $i = 1, 2, 3$ and $\gamma_j > 0$, for all $j = 2, 3$. Suppose that the conditions (i)–(v) are true;

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{+\infty} \alpha_n = +\infty$;
- (ii) $\gamma_2(1 + \gamma_3) \leq \frac{1}{L}$, where $L = \max\{L_{A_1}, L_{A_2}\} \leq 1$ where L_{A_1}, L_{A_2} are spectral radius of $A_2A_2^*, A_3A_3^*$, respectively;
- (iii) $0 < a \leq \beta_n, \gamma_n, \delta_n \leq b < 1$, for some $a, b > 0$, for all $n \in \mathbb{N}$;
- (iv) $\sum_{n=1}^{+\infty} \lambda_n < +\infty$ and $0 < \lambda_n < 1 - \kappa$, for all $n \in \mathbb{N}$;
- (v) $\sum_{n=1}^{+\infty} |\alpha_{n+i} - \alpha_n|, \sum_{n=1}^{+\infty} |\beta_{n+i} - \beta_n|, \sum_{n=1}^{+\infty} |\gamma_{n+1} - \gamma_n| < +\infty$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\Omega \cap F(\vartheta) \cap F(\varrho)} u$.

Proof. Put $M = a(I - \vartheta) + (1 - a)(I - \varrho)$ and $u_n = P_{C_1}(I - \lambda_1 B_1)(x_n^1 - \gamma_2 A_2^*((I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 + \gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3 B_3))x_n^3))$. Thus, we can rewrite x_n as follows:

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_{C_1}(I - \lambda_n M)x_n + \delta_n u_n, \quad (16)$$

for all $n \geq 1$.

From determining the definition of u_n put $w_n = (I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 + \gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3 B_3))x_n^3$ and $z_n = (I_3 - P_{C_3}(I - \lambda_3 B_3))x_n^3$, we have

$$u_n = P_{C_1}(I_1 - \lambda_1 B_1)(x_n - \gamma_2 A_2^* w_n).$$

For every $n \in \mathbb{N}$, we have

$$\begin{aligned} \|u_n - u_{n-1}\|^2 &\leq \|x_n - \gamma_2 A_2^* w_n - x_{n-1} + \gamma_2 A_2^* w_{n-1}\|^2 \\ &= \|x_n - x_{n-1} - \gamma_2 A_2^*(w_n - w_{n-1})\|^2 \\ &= \|x_n - x_{n-1}\|^2 - 2\gamma_2 \langle A_2 x_n - A_2 x_{n-1}, w_n - w_{n-1} \rangle + \gamma_2^2 \|A_2^*(w_n - w_{n-1})\|^2 \\ &= \|x_n - x_{n-1}\|^2 - 2\gamma_2 \langle x_n^2 - x_{n-1}^2, (I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 \\ &\quad + \gamma_3 A_3^* z_n - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2 - \gamma_3 A_3^* z_{n-1} \rangle \\ &\quad + \gamma_2^2 \|A_2^*(w_n - w_{n-1})\|^2 \\ &= \|x_n - x_{n-1}\|^2 + 2\gamma_2 \langle x_{n-1}^2 - x_n^2, (I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 \\ &\quad - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2 \rangle + 2\gamma_2 \gamma_3 \langle x_{n-1}^3 - x_n^3, z_n \\ &\quad - z_{n-1} \rangle + \gamma_2^2 \|A_2^*(w_n - w_{n-1})\|^2 \\ &\leq \|x_n - x_{n-1}\|^2 + 2\gamma_2 \langle x_{n-1}^2 - x_n^2, (I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 \\ &\quad - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2 \rangle + 2\gamma_2 \gamma_3 \langle x_{n-1}^3 - x_n^3, z_n - z_{n-1} \rangle \\ &\quad + \gamma_2^2 L \left\| (I_2 - P_{C_2}(I_2 - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I_2 - \lambda_2 B_2))x_{n-1}^2 + \gamma_3 A_3^*(z_n - z_{n-1}) \right\|^2 \\ &\leq \|x_n - x_{n-1}\|^2 + 2\gamma_2 \langle x_{n-1}^2 - x_n^2, (I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 \\ &\quad - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2 \rangle + 2\gamma_2 \gamma_3 \langle x_{n-1}^3 - x_n^3, z_n - z_{n-1} \rangle \\ &\quad + \gamma_2^2 L \left\| (I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2 \right\|^2 \\ &\quad + \gamma_3^2 L \|z_n - z_{n-1}\|^2 + 2\gamma_3 \langle (I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 \\ &\quad - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2, A^*(z_n - z_{n-1}) \rangle \end{aligned}$$

$$\begin{aligned}
&\leq \|x_n - x_{n-1}\|^2 + 2\gamma_2 \langle x_{n-1}^2 - x_n^2, (I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2 \rangle \\
&\quad + 2\gamma_2 \gamma_3 \langle x_{n-1}^3 - x_n^3, (I_3 - P_{C_3}(I - \lambda_3 B_3))x_n^3 - (I_3 - P_{C_3}(I - \lambda_3 B_3))x_{n-1}^3 \rangle \\
&\quad + \gamma_2^2 L (\|(I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2\|^2 + \gamma_3^2 L \|z_n - z_{n-1}\|^2 \\
&\quad + \gamma_3 \|(I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2\|^2 + \gamma_3 L \|z_n - z_{n-1}\|^2) \\
&\leq \|x_n - x_{n-1}\|^2 + 2\gamma_2 (-\|(I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2\|^2 \\
&\quad + \frac{1}{2} \|(I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2\|^2) \\
&\quad + 2\gamma_2 \gamma_3 (-\|z_n - z_{n-1}\|^2 + \frac{1}{2} \|z_n - z_{n-1}\|^2) \\
&\quad + \gamma_2^2 L (\|(I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2\|^2 + \gamma_3^2 L \|z_n - z_{n-1}\|^2 \\
&\quad + \gamma_3 \|(I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2\|^2 + \gamma_3 L \|z_n - z_{n-1}\|) \\
&= \|x_n - x_{n-1}\|^2 - \gamma_2 \|(I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2\|^2 \\
&\quad - \gamma_2 \gamma_3 \|z_n - z_{n-1}\|^2 + \gamma_2^2 L (\|(I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2\|^2 \\
&\quad + \gamma_2^2 \gamma_3^2 L^2 \|z_n - z_{n-1}\|^2 + \gamma_2^2 \gamma_3 L (\|(I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2\|^2 \\
&\quad + \gamma_2^2 \gamma_3 L^2 \|z_n - z_{n-1}\|) \\
&= \|x_n - x_{n-1}\|^2 - \gamma_2 (1 - \gamma_2 L (1 + \gamma_3)) \|(I_2 - P_{C_2}(I - \lambda_2 B_2))x_n^2 \\
&\quad - (I_2 - P_{C_2}(I - \lambda_2 B_2))x_{n-1}^2\|^2 - \gamma_2 \gamma_3 (1 - \gamma_2 L^2 (1 + \gamma_3)) \|z_n - z_{n-1}\|^2 \\
&\leq \|x_n - x_{n-1}\|^2.
\end{aligned} \tag{17}$$

Let $\omega^* \in \Omega \cap F(\varrho) \cap F(\vartheta)$. From Lemma 6 and utilization of (8), we have

$$\begin{aligned}
\|x_{n+1} - \omega^*\| &\leq \alpha_n \|u - \omega^*\| + \beta_n \|x_n - \omega^*\| + \gamma_n \|P_{C_1}(I - \lambda_n M)x_n - \omega^*\| \\
&\quad + \delta_n \|u_n - \omega^*\| \\
&\leq \alpha_n \|u - \omega^*\| + (1 - \alpha_n) \|x_n - \omega^*\| \\
&\leq \tilde{M},
\end{aligned} \tag{18}$$

where $\tilde{M} = \max\{\|u - \omega^*\|, \|x_1 - \omega^*\|\}$. By induction we can conclude that the sequence $\{x_n\}$ is bounded and so are $\{u_n\}$ and $\{P_{C_1}(I - \lambda_n M)x_n\}$.

From determining the definition of x_n and (17), we have

$$\begin{aligned}
 \|x_{n+1} - x_n\| &= \|\alpha_n u + \beta_n x_n + \gamma_n P_{C_1}(I - \lambda_n M)x_n + \delta_n u_n \\
 &\quad - \alpha_{n-1} u - \beta_{n-1} x_{n-1} - \gamma_{n-1} P_{C_1}(I - \lambda_{n-1} M)x_{n-1} - \delta_{n-1} u_{n-1}\| \\
 &\leq |\alpha_n - \alpha_{n-1}| \|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \delta_n \|u_n - u_{n-1}\| \\
 &\quad + |\delta_n - \delta_{n-1}| \|u_{n-1}\| + \gamma_n \|P_{C_1}(I - \lambda_n M)x_n - P_{C_1}(I - \lambda_{n-1} M)x_{n-1}\| \\
 &\quad + |\gamma_n - \gamma_{n-1}| \|P_{C_1}(I_1 - \lambda_{n-1} M)x_{n-1}\| \\
 &\leq |\alpha_n - \alpha_{n-1}| \|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \delta_n \|u_n - u_{n-1}\| \\
 &\quad + |\delta_n - \delta_{n-1}| \|u_{n-1}\| + \gamma_n \|x_n - x_{n-1}\| + \|\lambda_{n-1} M x_{n-1} - \lambda_n M x_n\| \\
 &\quad + |\gamma_n - \gamma_{n-1}| \|P_{C_1}(I_1 - \lambda_{n-1} M)x_{n-1}\| \\
 &\leq |\alpha_n - \alpha_{n-1}| \|u\| + \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + \delta_n \|x_n - x_{n-1}\| \\
 &\quad + |\delta_n - \delta_{n-1}| \|u_{n-1}\| + \gamma_n \|x_n - x_{n-1}\| + \lambda_{n-1} \|M x_{n-1}\| + \lambda_n \|M x_n\| \\
 &\quad + |\gamma_n - \gamma_{n-1}| \|P_{C_1}(I_1 - \lambda_{n-1} M)x_{n-1}\| \\
 &= (1 - \alpha_n) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|u\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| \\
 &\quad + |\delta_n - \delta_{n-1}| \|u_{n-1}\| + \lambda_{n-1} \|M x_{n-1}\| + \lambda_n \|M x_n\| \\
 &\quad + |\gamma_n - \gamma_{n-1}| \|P_{C_1}(I_1 - \lambda_{n-1} M)x_{n-1}\|.
 \end{aligned}$$

From the conditions (i), (iv), (v) and Lemma 3, we have

$$\lim_{n \rightarrow +\infty} \|x_{n+1} - x_n\| = 0 \quad (19)$$

Applying (8) and the definition of x_n , we have

$$\begin{aligned}
 \|x_{n+1} - \omega^*\|^2 &\leq \alpha_n \|u - \omega^*\|^2 + \beta_n \|x_n - \omega^*\|^2 + \gamma_n \|P_{C_1}(I - \lambda_n M)x_n - \omega^*\|^2 \\
 &\quad + \delta_n \|u_n - \omega^*\|^2 - \gamma_n \beta_n \|P_{C_1}(I - \lambda_n M)x_n - x_n\|^2 - \delta_n \beta_n \|u_n - x_n\|^2 \\
 &\leq \alpha_n \|u - \omega^*\|^2 + \|x_n - \omega^*\|^2 - \gamma_n \beta_n \|P_{C_1}(I - \lambda_n M)x_n - x_n\|^2 \\
 &\quad - \delta_n \beta_n \|u_n - x_n\|^2,
 \end{aligned}$$

which implies that

$$\begin{aligned}
 \gamma_n \beta_n \|P_{C_1}(I - \lambda_n M)x_n - x_n\|^2 + \delta_n \beta_n \|u_n - x_n\|^2 &\leq (\|x_{n+1} - \omega^*\| + \|x_n - \omega^*\|) \|x_{n+1} - x_n\| \\
 &\quad + \alpha_n \|u - \omega^*\|^2.
 \end{aligned}$$

From the conditions (i), (iii) and (19) we can conclude the following results

$$\lim_{n \rightarrow +\infty} \|u_n - x_n\| = \lim_{n \rightarrow +\infty} \|P_{C_1}(I - \lambda_n M)x_n - x_n\| = 0. \quad (20)$$

Next, we show that

$$\limsup_{n \rightarrow +\infty} \langle u - z_0, z_0 - x_n \rangle \leq 0, \quad (21)$$

where $z_0 = P_{\Omega \cap F(\varrho) \cap F(\vartheta)} u$. In order to prove this we may assume that

$$\limsup_{n \rightarrow +\infty} \langle u - z_0, x_n - z_0 \rangle = \lim_{k \rightarrow +\infty} \langle u - z_0, x_{n_k} - z_0 \rangle, \quad (22)$$

where $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. Since $\{x_n\}$ is bounded, we may assume that $x_{n_k} \rightharpoonup q$ as $k \rightarrow +\infty$. Assume that $q \notin F(\varrho) \cap F(\vartheta)$. From Lemma 6, we have $q \notin F(P_{C_1}(I - \lambda_{n_k}M))$. By using properties of Opial's condition and (20), we have

$$\begin{aligned} \liminf_{k \rightarrow +\infty} \|x_{n_k} - q\| &< \liminf_{k \rightarrow +\infty} \|x_{n_k} - P_{C_1}(I - \lambda_{n_k}M)q\| \\ &\leq \liminf_{k \rightarrow +\infty} (\|x_{n_k} - P_{C_1}(I - \lambda_{n_k}M)x_{n_k}\| \\ &\quad + \|P_{C_1}(I - \lambda_{n_k}M)x_{n_k} - P_{C_1}(I - \lambda_{n_k}M)q\|) \\ &\leq \liminf_{k \rightarrow +\infty} (\|x_{n_k} - q\| + \lambda_{n_k} \|Mx_{n_k} - Mq\|) \\ &\leq \liminf_{k \rightarrow +\infty} \|x_{n_k} - q\|. \end{aligned}$$

This is a contradiction. Therefore $q \in F(\varrho) \cap F(\vartheta)$.

Assume $q \notin \Omega$. From Lemma 5, we have

$$q \neq P_{C_1}(I - \lambda_1 B_1)(q - \gamma_2 A_2^*((I_2 - P_{C_2}(I - \lambda_2 B_2))A_2 q + \gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3 B_3))A_3 A_2 q)).$$

By using properties of Opial's condition, and the definitions of u_n and (20), we have

$$\begin{aligned} \liminf_{k \rightarrow +\infty} \|x_{n_k} - q\| &< \liminf_{k \rightarrow +\infty} \|x_{n_k} - P_{C_1}(I - \lambda_1 B_1)(q - \gamma_2 A_2^*((I_2 - P_{C_2}(I - \lambda_2 B_2))A_2 q \\ &\quad + \gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3 B_3))A_3 A_2 q))\| \\ &\leq \liminf_{k \rightarrow +\infty} (\|x_{n_k} - u_{n_k}\| \\ &\quad + \|u_{n_k} - P_{C_1}(I - \lambda_1 B_1)(q - \gamma_2 A_2^*((I_2 - P_{C_2}(I - \lambda_2 B_2))A_2 q \\ &\quad + \gamma_3 A_3^*(I_3 - P_{C_3}(I - \lambda_3 B_3))A_3 A_2 q))\|) \\ &\leq \liminf_{k \rightarrow +\infty} \|x_{n_k} - q\|. \end{aligned}$$

This is a contradiction. Then $q \in \Omega$. Therefore $q \in \Omega \cap F(\varrho) \cap F(\vartheta)$.

From (22) and the well-known properties of metric projection, we have

$$\limsup_{n \rightarrow +\infty} \langle u - z_0, x_n - z_0 \rangle \leq 0.$$

From determining the definition of x_n , we can conclude that

$$\|x_{n+1} - z_0\|^2 \leq (1 - \alpha_n) \|x_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle,$$

where $z_0 = P_{\Omega \cap F(\varrho) \cap F(\vartheta)} u$. From Lemma 4, we can conclude that the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\Omega \cap F(\varrho) \cap F(\vartheta)} u$. $\square$

The following results were obtained directly from the main theorem.

Corollary 8. For every $i = 1, 2, 3$, let C_i be a closed convex subset of a real Hilbert space H_i . Let $B_i : C_i \rightarrow H_i$ be β_i -inverse strongly monotonic mappings with $\eta = \min_{i=1,2,3} \{\beta_i\}$ and let $A_2 : H_1 \rightarrow H_2, A_3 : H_2 \rightarrow H_3$ be a bounded linear operator with the adjoint operator A_2^* and A_3^* , respectively. Assume that $\bar{x}_1 \in H_1, \bar{x}_2 = A_2 \bar{x}_1, \bar{x}_3 = A_3 \bar{x}_2$ and $\Omega \neq \emptyset$. Let $\vartheta, \varrho : C \rightarrow C$ be nonspreading mappings, respectively. Assume that $\Omega \cap F(\vartheta) \cap F(\varrho) \neq \emptyset$ and let the sequence $\{x_n\}$ generated by $u, x_1 \in C$, and

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_{C_1} (I_1 - \lambda_n (a (I_1 - \vartheta) + (1 - a) (I_1 - \varrho))) x_n + \delta_n P_{C_1} (I_1 - \lambda_1 B_1) (x_n^1 - \gamma_2 A_2^* ((I_2 - P_{C_2} (I - \lambda_2 B_2)) x_n^2 + \gamma_3 A_3^* (I_3 - P_{C_3} (I - \lambda_3 B_3)) x_n^3)),$$

for all $n \geq 1$ and $a \in (0, 1)$, $I_i : H_i \rightarrow H_i$ are identity mappings, for all $i = 1, 2, 3$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and $x_n = x_n^1, x_n^2 = A_2 x_n^1, x_n^3 = A_3 x_n^2$, for all $n \in \mathbb{N}, 0 < \lambda_i < 2\eta$, for all $i = 1, 2, 3$ and $\gamma_j > 0$, for all $j = 2, 3$. Suppose that the conditions (i) to (v) are true;

- (i) $\lim_{n \rightarrow +\infty} \alpha_n = 0, \sum_{n=1}^{+\infty} \alpha_n = +\infty$;
- (ii) $\gamma_2 (1 + \gamma_3) \leq \frac{1}{L}$, where $L = \max \{L_{A_1}, L_{A_2}\} \leq 1$, where L_{A_1}, L_{A_2} are spectral radius of $A_2 A_2^*, A_3 A_3^*$, respectively;
- (iii) $0 < a \leq \beta_n, \gamma_n, \delta_n \leq b < 1$, for some $a, b > 0$, for all $n \in \mathbb{N}$;
- (iv) $\sum_{n=1}^{+\infty} \lambda_n < +\infty$ and $0 < \lambda_n < 1 - \kappa$, for all $n \in \mathbb{N}$;
- (v) $\sum_{n=1}^{+\infty} |\alpha_{n+i} - \alpha_n|, \sum_{n=1}^{+\infty} |\beta_{n+i} - \beta_n|, \sum_{n=1}^{+\infty} |\gamma_{n+1} - \gamma_n| < +\infty$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\Omega \cap F(\vartheta) \cap F(\varrho)} u$.

3. Application

We have applied the problem (7) for the various fixed point problems in three Hilbert spaces as follows:

For every $i = 1, 2, 3$, let H_i be a real Hilbert space and C_i be a nonempty closed convex subset of H_i . Let $\vartheta_i : C_i \rightarrow C_i$ be a mapping, for all $i = 1, 2, 3$ and let $A_2 : H_1 \rightarrow H_2$ and $A_3 : H_2 \rightarrow H_3$. The fixed points problem in three Hilbert spaces is meant to find the point

$$\begin{cases} \omega_1^* \in C_1, \text{ such that } \omega_1^* \in F(\vartheta_1) \text{ and} \\ \omega_2^* = A_2 \omega_1^* \in C_2, \text{ such that } \omega_2^* \in F(\vartheta_2) \text{ and} \\ \omega_3^* = A_3 \omega_2^* \in C_3, \text{ such that } \omega_3^* \in F(\vartheta_3). \end{cases} \quad (23)$$

The set of the solutions of (23) is denoted by $\underline{\Omega} = \{\omega^* = (\omega_1^*, \omega_2^*, \omega_3^*) \in C_1 \times C_2 \times C_3 : \omega_i^* \in F(\vartheta_i), \text{ for all } i = 1, 2, 3\}$. It is clear that $\text{Var}(C, I - T) = F(\vartheta)$, where $\vartheta : C \rightarrow C$ is a nonexpansive mapping with $F(\vartheta) \neq \emptyset$. By leveraging Lemma 5 and such knowledge, we have the following results:

Lemma 9. For every $i = 1, 2, 3$, let H_i be a real Hilbert spaces and C_i be a nonempty closed convex subset of H_i . Let $\vartheta_i : C_i \rightarrow C_i$ be nonexpansive mappings and let $A_2 : H_1 \rightarrow H_2, A_3 : H_2 \rightarrow H_3$ be a bounded linear operator with the adjoint operator A_2^* and A_3^* , respectively. Assume that $\bar{x}_1 \in C_1, A_2 \bar{x}_1 = \bar{x}_2, A_3 \bar{x}_2 = \bar{x}_3$ and $\underline{\Omega} \neq \emptyset$. The following are equivalent:

- (i) $\bar{x} \in \underline{\Omega}$, where $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in C_1 \times C_2 \times C_3$.
- (ii) $\bar{x}_1 = P_{C_1} (I_1 - \lambda_1 (I_1 - \vartheta_1)) (\bar{x}_1 - \gamma_2 A_2^* ((I_2 - P_{C_2} (I - \lambda_2 (I_2 - \vartheta_2))) \bar{x}_2 + \gamma_3 A_3^* (I_3 - P_{C_3} (I - \lambda_3 (I_3 - \vartheta_3))) \bar{x}_3)),$

where $I_i : H_i \rightarrow H_i$ is an identity mapping, for all $i = 1, 2, 3, \gamma_2 (1 + \gamma_3) \leq \frac{1}{L}, L = \max \{L_1, L_2\} \leq 1$ which L_1, L_2 are spectral radii of $A_2 A_2^*$ and $A_3 A_3^*$, respectively, $\lambda_i \in (0, 1)$, for all $i = 1, 2, 3$ and $\gamma_2, \gamma_3 \geq 0$

Proof. Given $F(\vartheta_i) = \text{Var}(C, I_i - \vartheta_i)$ for all $i = 1, 2, 3$; $(I_i - \vartheta_i)$ —a $\frac{1}{2}$ -inverse strongly monotonic; and Lemma 5, we can summarize the result of Lemma 9. $\square$

As the direct benefits of Lemma 9, we get Corollary 10.

Corollary 10. For every $i = 1, 2, 3$, let C_i be a closed convex subset of a real Hilbert space H_i . Let $\vartheta_i : C_i \rightarrow C_i$ be a nonexpansive mapping and let $A_2 : H_1 \rightarrow H_2, A_3 : H_2 \rightarrow H_3$ be a bounded linear operator with the adjoint operator A_2^* and A_3^* , respectively. Assume that $\bar{x}_1 \in H_1, \bar{x}_2 = A_2\bar{x}_1, \bar{x}_3 = A_3\bar{x}_2$. Let $\vartheta, \varrho : C \rightarrow C$ be nonspreading and κ -strictly pseudo-nonspreading mappings, respectively. Assume that $\underline{\Omega} \cap F(\vartheta) \cap F(\varrho) \neq \emptyset$ and let the sequence $\{x_n\}$ generated by $u, x_1 \in C$, and

$$\begin{aligned} x_{n+1} = & \alpha_n u + \beta_n x_n + \gamma_n P_{C_1} (I_1 - \lambda_n (a(I_1 - \vartheta) + (1-a)(I_1 - \varrho))) x_n \\ & + \delta_n P_{C_1} (I_1 - \lambda_1 (I_1 - \vartheta_1)) (x_n^1 - \gamma_2 A_2^* ((I_2 - P_{C_2} (I - \lambda_2 (I_2 - \vartheta_2))) x_n^2 \\ & + \gamma_3 A_3^* (I_3 - P_{C_3} (I - \lambda_3 (I_3 - \vartheta_3))) x_n^3)), \end{aligned}$$

for all $n \geq 1$ and $a \in (0, 1)$, $I_i : H_i \rightarrow H_i$ is an identity mapping, for all $i = 1, 2, 3$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0, 1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and $x_n = x_n^1, x_n^2 = A_2 x_n^1, x_n^3 = A_3 x_n^2$, for all $n \in \mathbb{N}, 0 < \lambda_i < 1$, for all $i = 1, 2, 3$ and $\gamma_j > 0$, for all $j = 2, 3$. Suppose that the conditions (i) to (v) are true;

- (i) $\lim_{n \rightarrow +\infty} \alpha_n = 0, \sum_{n=1}^{+\infty} \alpha_n = +\infty$;
- (ii) $\gamma_2 (1 + \gamma_3) \leq \frac{1}{L}$, where $L = \max \{L_{A_1}, L_{A_2}\} \leq 1$, where L_{A_1}, L_{A_2} are spectral radii of $A_2 A_2^*, A_3 A_3^*$, respectively;
- (iii) $0 < a \leq \beta_n, \gamma_n, \delta_n \leq b < 1$, for some $a, b > 0$, for all $n \in \mathbb{N}$;
- (iv) $\sum_{n=1}^{+\infty} \lambda_n < +\infty$ and $0 < \lambda_n < 1 - \kappa$, for all $n \in \mathbb{N}$;
- (v) $\sum_{n=1}^{+\infty} |\alpha_{n+i} - \alpha_n|, \sum_{n=1}^{+\infty} |\beta_{n+i} - \beta_n|, \sum_{n=1}^{+\infty} |\gamma_{n+1} - \gamma_n| < +\infty$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\underline{\Omega} \cap F(T) \cap F(S)} u$.

4. Conclusions

We have proposed a new split variational inequality in three Hilbert spaces. The convergence theorem for finding a common element of the set of solutions of such problems and the sets of fixed-points of discontinuous mappings are proved.

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