Existence Results for Nonlocal Multi-Point and Multi-Term Fractional Order Boundary Value Problems

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Abstract: In this paper, we discuss the existence and uniqueness of solutions for a new class of multi-point and integral boundary value problems of multi-term fractional differential equations by using standard fixed point theorems. We also demonstrate the application of the obtained results with the aid of examples.

1. Introduction

Fractional differential equations are found to be of great utility in improving the mathematical modeling of many engineering and scientific disciplines such as physics [1] bioengineering [2], viscoelasticty [3], ecology [4], disease models [5–7], etc. For applications of differential equations containing more than one fractional order differential operators, we refer the reader to Bagley-Torvik [8], Basset equation [9] to name a few.

Fractional order boundary value problems equipped with a variety of classical and non-classical (nonlocal) boundary conditions have recently been investigated by many researchers and the literature on the topic is now much enriched, for instance, see [10–21] and the references cited therein. There has been a special focus on boundary value problems involving multi-term fractional differential equations [22–24].

The objective of the present work is to develop the existence theory for multi-term fractional differential equations equipped with nonlocal multi-point boundary conditions. Precisely, we investigate the following boundary value problem:

$$(q_2 \, {}^cD^{\sigma+2} + q_1 \, {}^cD^{\sigma+1} + q_0 \, {}^cD^{\sigma})x(t) = f(t, x(t)), \ 0 < \sigma < 1, \ 0 < t < 1, \tag{1}$$

$$x(0) = h(x), \ x(\xi) = \sum_{i=1}^{n} j_i x(\eta_i), \ x(1) = \lambda \int_0^{\delta} x(s)ds, \tag{2}$$

where ${}^cD^{\sigma}$ denote the Caputo fractional derivative of order σ, $0 < \sigma < 1$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, $h : C([0,1], \mathbb{R}) \to \mathbb{R}$ are given continuous functions, $0 < \delta < \xi < \eta_1 < \eta_2 < \ldots < \eta_n < 1$, $\lambda \in \mathbb{R}$, $q_0, q_1, and \ q_2$ are real constants with $q_2 \neq 0$. One can characterize the first and second conditions

in (2) as initial-nonlocal and nonlocl multi-point ones, while the last condition in (2) can be understood in the sense that the value of the unknown function x at the right-end point of the domain ($x(1)$) is proportional to the average value of x on the sub-domain $(0, \delta)$. Existence and uniqueness results are established by using the classical Banach and Krasnoselskii fixed point theorems and Leray–Schauder nonlinear alternative. Here, we emphasize that the results presented in this paper rely on the standard tools of the fixed point theory. However, their exposition to the given nonlocal problem for a multi-term (sequential) fractional differential equation produces new results which contributes to the related literature.

The rest of the paper is organized as follows: In Section 2 we recall some preliminary concepts of fractional calculus and prove a basic lemma, helping us to transform the boundary value problem (1) and (2) into a fixed point problem. The main existence and uniqueness results for the case $q_1^2 - 4q_0q_2 > 0$ are presented in details in Section 3. In Sections 4 and 5 we indicate the results for the cases $q_1^2 - 4q_0q_2 = 0$ and $q_1^2 - 4q_0q_2 < 0$ respectively. Examples illustrating the obtained results are also included.

2. Basic Results

Before presenting some auxiliary results, let us recall some preliminary concepts of fractional calculus [25,26].

Definition 1. *Let $y, y^{(m)} \in L_1[a, b]$. Then the Riemann–Liouville fractional derivative $D_a^\alpha y$ of order $\alpha \in (m - 1, m], m \in N$, existing almost everywhere on $[a, b]$, is defined as*

$$D_a^\alpha y(t) = \frac{d^m}{dt^m} J_a^{m-\alpha} y(t) = \frac{1}{\Gamma(m - \alpha)} \frac{d^m}{dt^m} \int_a^t (t - s)^{m-1-\alpha} y(s) ds.$$

The Caputo fractional derivative ${}^cD_a^\alpha y$ of order $\alpha \in (m - 1, m], m \in N$ is defined as

$${}^cD_a^\alpha y(t) = D_a^\alpha \left[y(t) - y(a) - y'(a) \frac{(t - a)}{1!} - \ldots - y^{(m-1)}(a) \frac{(t - a)^{m-1}}{(m - 1)!} \right].$$

Remark 1. *If $y \in AC^m[a, b]$, then the Caputo fractional derivative ${}^cD_a^\alpha y$ of order $\alpha \in (m - 1, m], m \in N$, existing almost everywhere on $[a, b]$, is defined as*

$${}^cD_a^\alpha y(t) = J_a^{m-\alpha} y^{(m)}(t) = \frac{1}{\Gamma(m - \alpha)} \int_a^t (t - s)^{m-1-\alpha} y^{(m)}(s) ds.$$

In the sequel, the Riemann–Liouville fractional integral I_a^α and the Caputo fractional derivative ${}^cD_a^\alpha$ with $a = 0$ are respectively denoted by I^α and ${}^cD^\alpha$.

Lemma 1. *[25] With the given notations, the following equality holds:*

$$I^\alpha({}^cD^\alpha y(t)) = y(t) - c_0 - c_1 t - \ldots - c_{n-1} t^{n-1}, \ t > 0, \ n - 1 < \alpha < n, \tag{3}$$

where c_i $(i = 1, \ldots, n - 1)$ are arbitrary constants.

The following lemmas associated with the linear variant of problem (1) and (2) plays an important role in the sequel.

Lemma 2. *For any* $\varphi \in C([0,1], \mathbb{R})$ *and* $q_1^2 - 4q_0q_2 > 0$, *the solution of linear multi-term fractional differential equation*

$$(q_2 \, {}^c D^{\sigma+2} + q_1 \, {}^c D^{\sigma+1} + q_0 \, {}^c D^{\sigma})x(t) = \varphi(t), \ \ 0 < \sigma < 1, \ 0 < t < 1, \tag{4}$$

supplemented with the boundary conditions (2) *is given by*

$$
\begin{aligned}
x(t) \ = \ & \frac{1}{q_2(m_2 - m_1)} \left\{ \int_0^t \int_0^s \mathcal{A}(t) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} \varphi(u) du \, ds \right. \\
& + \rho_1(t) \left[\int_0^\xi \int_0^s \mathcal{A}(\xi) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} \varphi(u) du \, ds \right. \\
& \left. - \sum_{i=1}^n j_i \int_0^{\eta_i} \int_0^s \mathcal{A}(\eta_i) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} \varphi(u) du \, ds \right] \\
& + \rho_2(t) \left[\int_0^1 \int_0^s \mathcal{A}(1) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} \varphi(u) du \, ds \right. \\
& \left. \left. - \lambda \int_0^\delta \int_0^s \left(\frac{(e^{m_2(\delta-s)} - 1)}{m_2} - \frac{(e^{m_1(\delta-s)} - 1)}{m_1} \right) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} \varphi(u) du \, ds \right] \right\} \\
& + h(x) \left[e^{m_2 t} + \rho_1(t) \left(e^{m_2 \xi} - \sum_{i=1}^n j_i + \sum_{i=1}^n j_i e^{m_2 \eta_i} \right) \right. \\
& \left. + \rho_2(t) \left(\frac{m_2 e^{m_2} - \lambda e^{m_2 \delta} - \lambda}{m_2} \right) \right],
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
\mathcal{A}(\kappa) \ = \ & e^{m_2(\kappa-s)} - e^{m_1(\kappa-s)}, \ \ \kappa = t, 1, \xi \ \text{and} \ \eta_i, \\
m_1 \ = \ & \frac{-q_1 - \sqrt{q_1^2 - 4q_0q_2}}{2q_2}, \ m_2 = \frac{-q_1 + \sqrt{q_1^2 - 4q_0q_2}}{2q_2}, \\
\rho_1(t) \ = \ & \frac{\omega_4 \varrho_1(t) - \omega_3 \varrho_2(t)}{\mu_1}, \ \rho_2(t) = \frac{\omega_1 \varrho_2(t) - \omega_2 \varrho_1(t)}{\mu_1}, \\
\varrho_1(t) \ = \ & \frac{m_1(1 - e^{m_2 t}) - m_2(1 - e^{m_1 t})}{m_1 m_2}, \\
\varrho_2(t) \ = \ & q_2(m_2 - m_1)(e^{m_2 t} - e^{m_1 t}), \\
\mu_1 \ = \ & \omega_1 \omega_4 - \omega_2 \omega_3 \neq 0, \\
\omega_1 \ = \ & \frac{1}{m_1 m_2} \left[m_2 \left(1 - \sum_{i=1}^n j_i - e^{m_1 \xi} + \sum_{i=1}^n j_i e^{m_1 \eta_i} \right) \right. \\
& \left. - m_1 \left(1 - \sum_{i=1}^n j_i - e^{m_2 \xi} + \sum_{i=1}^n j_i e^{m_2 \eta_i} \right) \right], \\
\omega_2 \ = \ & q_2 \left(m_2 - m_1 \right) \left(e^{m_1 \xi} - e^{m_2 \xi} - \sum_{i=1}^n j_i e^{m_1 \eta_i} + \sum_{i=1}^n j_i e^{m_2 \eta_i} \right), \\
\omega_3 \ = \ & \frac{1}{m_1 m_2} \left[m_2 \left(1 - e^{m_1} - \lambda \delta + \lambda/m_1 (e^{m_1 \delta} - 1) \right) \right. \\
& \left. - m_1 \left(1 - e^{m_2} - \lambda \delta + \lambda/m_2 (e^{m_2 \delta} - 1) \right) \right], \\
\omega_4 \ = \ & q_2 (m_2 - m_1) \left((e^{m_1} + \lambda/m_1 (1 - e^{m_1 \delta})) \right. \\
& \left. - (e^{m_2} + \lambda/m_2 (1 - e^{m_2 \delta})) \right).
\end{aligned}
\tag{6}
$$

Proof. Applying the operator I^{σ} on (4) and using (3), we get

$$(q_2 D^2 + q_1 D + q_0)x(t) = \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} \varphi(s)ds + c_1, \tag{7}$$

where c_1 is an arbitrary constant. By the method of variation of parameters, the solution of (7) can be written as

$$
\begin{aligned}
x(t) \;=\; & c_1 \left[\frac{m_2(1 - e^{m_1 t}) - m_1(1 - e^{m_2 t})}{q_2 m_1 m_2 (m_2 - m_1)} \right] + c_2 e^{m_1 t} + c_3 e^{m_2 t} \\
& - \frac{1}{q_2(m_2 - m_1)} \int_0^t e^{m_1(t-s)} \left(\int_0^s \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} \varphi(u)du \right) ds \\
& + \frac{1}{q_2(m_2 - m_1)} \int_0^t e^{m_2(t-s)} \left(\int_0^s \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} \varphi(u)du \right) ds,
\end{aligned}
\tag{8}
$$

where m_1 and m_2 are given by (6). Using $x(0) = h(x)$ in (8), we get

$$
\begin{aligned}
x(t) \;=\; & c_1 \left[\frac{m_2(1 - e^{m_1 t}) - m_1(1 - e^{m_2 t})}{q_2 m_1 m_2 (m_2 - m_1)} \right] + c_2 \left(e^{m_1 t} - e^{m_2 t} \right) + h(x)e^{m_2 t} \\
& + \frac{1}{q_2(m_2 - m_1)} \left[\int_0^t \left(e^{m_2(t-s)} - e^{m_1(t-s)} \right) \left(\int_0^s \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} \varphi(u)du \right) ds \right],
\end{aligned}
\tag{9}
$$

which together with the conditions $x(\xi) = \sum_{i=1}^n j_i x(\eta_i)$ and $x(1) = \lambda \int_0^\delta x(s)ds$ yields the following system of equations in the unknown constants c_1 and c_2:

$$c_1 \omega_1 + c_2 \omega_2 \;=\; V_1, \tag{10}$$
$$c_1 \omega_3 + c_2 \omega_4 \;=\; V_2. \tag{11}$$

where

$$
\begin{aligned}
V_1 \;=\; & -\int_0^\xi \int_0^s \mathcal{A}(\xi) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} \varphi(u)du\, ds \\
& + \sum_{i=1}^n j_i \int_0^{\eta_i} \int_0^s \mathcal{A}(\eta_i) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} \varphi(u)du\, ds + h(x) \left(\sum_{i=1}^n j_i e^{m_2 \eta_i} - e^{m_2 \xi} \right), \\
V_2 \;=\; & -\int_0^1 \int_0^s \mathcal{A}(1) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} \varphi(u)du\, ds + h(x) \left(\frac{\lambda e^{m_2 \delta} - \lambda - m_2 e^{m_2}}{m_2} \right) \\
& + \lambda \int_0^\delta \int_0^s \left[\frac{(e^{m_1(\delta-s)} - 1)}{m_1} - \frac{(e^{m_2(\delta-s)} - 1)}{m_2} \right] \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} \varphi(u)du\, ds.
\end{aligned}
$$

Solving the system (10)–(11) together with the notations (6), we find that

$$c_1 = \frac{V_1 \omega_4 - V_2 \omega_2}{\mu_1}, \quad c_2 = \frac{V_2 \omega_1 - V_1 \omega_3}{\mu_1}.$$

Substituting the value of c_1 and c_2 in (9), we obtain the solution (5). The converse of the lemma follows by direct computation. This completes the proof. □

We do not provide the proofs of the following lemmas, as they are similar to that of Lemma 2.

Lemma 3. *For any* $\varphi \in C([0,1], \mathbb{R})$ *and* $q_1^2 - 4q_0q_2 = 0$, *the solution of linear multi-term fractional differential equation*

$$(q_2\,^cD^{\sigma+2} + q_1\,^cD^{\sigma+1} + q_0\,^cD^\sigma)x(t) = \varphi(t), \ 0 < \sigma < 1, \ 0 < t < 1, \tag{12}$$

supplemented with the boundary conditions (2) *is given by*

$$
\begin{aligned}
x(t) \ = \ & \frac{1}{q_2}\bigg\{ \int_0^t \int_0^s \mathcal{B}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}\varphi(u)du\,ds \\
& +\chi_1(t)\bigg[\int_0^\xi \int_0^s \mathcal{B}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}\varphi(u)du\,ds \\
& -\sum_{i=1}^n j_i \int_0^{\eta_i} \int_0^s \mathcal{B}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}\varphi(u)du\,ds \bigg] \\
& +\chi_2(t)\bigg[\int_0^1 \int_0^s \mathcal{B}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}\varphi(u)du\,ds \\
& -\lambda\int_0^\delta \int_0^s \bigg(\frac{m(\delta-s)e^{m(\delta-s)} - e^{m(\delta-s)} + 1}{m^2}\bigg)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}\varphi(u)du\,ds\bigg]\bigg\} \\
& +h(x)\bigg[e^{mt} + \chi_1(t)\Big(e^{m\xi} - \sum_{i=1}^n j_ie^{m\eta_i}\Big) + \chi_2(t)\Big(\frac{me^m - \lambda e^{m\delta} + \lambda}{m}\Big)\bigg],
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
\mathcal{B}(\kappa) \ = \ & (\kappa - s)e^{m(\kappa - s)}, \ \kappa = t, 1, \xi \ \text{and} \ \eta_i, \\
m \ = \ & \frac{-q_1}{2q_2}, \\
\chi_1(t) \ = \ & \frac{\omega_3 v_2(t) - \omega_4 v_1(t)}{\mu_2}, \ \chi_2(t) = \frac{\omega_2 v_1(t) - \omega_1 v_2(t)}{\mu_2}, \\
v_1(t) \ = \ & \frac{mte^{mt} - e^{mt} + 1}{m^2}, \ v_2(t) = q_2 te^{mt}, \\
\omega_1 \ = \ & \frac{m\xi e^{m\xi} - e^{m\xi} + 1 - \sum_{i=1}^n j_i(m\eta_i e^{m\eta_i} - e^{m\eta_i} + 1)}{m^2}, \\
\omega_2 \ = \ & q_2\Big(\xi e^{m\xi} - \sum_{i=1}^n j_i\eta_i e^{m\eta_i}\Big), \\
\omega_3 \ = \ & \frac{m^2 e^m - me^m + m - m\lambda\delta e^{m\delta} + 2\lambda e^{m\delta} - 2\lambda - m\lambda\delta}{m^3}, \\
\omega_4 \ = \ & q_2\Big(\frac{m^2 e^m - \lambda m\delta e^{m\delta} + \lambda e^{m\delta} - \lambda}{m^2}\Big), \\
\mu_2 \ = \ & \omega_1\omega_4 - \omega_2\omega_3 \neq 0.
\end{aligned}
\tag{14}
$$

Lemma 4. *For any* $\varphi \in C([0,1], \mathbb{R})$ *and* $q_1^2 - 4q_0q_2 < 0$, *the solution of linear multi-term fractional differential equation*

$$(q_2\,^cD^{\sigma+2} + q_1\,^cD^{\sigma+1} + q_0\,^cD^\sigma)x(t) = \varphi(t), \ 0 < \sigma < 1, \ 0 < t < 1, \tag{15}$$

supplemented with the boundary conditions (2) *is given by*

$$
x(t) \ = \ \frac{1}{q_2 b}\bigg\{ \int_0^t \int_0^s \mathcal{F}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}\varphi(u)du\,ds
$$

$$+\tau_1(t)\left[\int_0^\xi \int_0^s \mathcal{F}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}\varphi(u)du\,ds\right.$$

$$-\sum_{i=1}^n j_i \int_0^{\eta_i} \int_0^s \mathcal{F}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}\varphi(u)du\,ds\bigg]$$

$$+\tau_2(t)\left[\int_0^1 \int_0^s \mathcal{F}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}\varphi(u)du\,ds\right. \tag{16}$$

$$-\frac{\lambda}{a^2+b^2}\int_0^\delta \int_0^s \Big(b-be^{-a(\delta-s)}\cos b(\delta-s)$$

$$-ae^{-a(\delta-s)}\sin b(\delta-s)\Big)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}\varphi(u)du\,ds\bigg]\bigg\}$$

$$+h(x)\Big[e^{-at}\cos bt+\tau_1(t)\Big(e^{-a\xi}\cos b\xi-\sum_{i=1}^n j_i e^{-a\eta_i}\cos b\eta_i\Big)$$

$$+\tau_2(t)\Big(e^{-a}\cos b-\frac{\lambda}{a^2+b^2}(a-ae^{-a\delta}\cos b\delta+be^{-a\delta}\sin b\delta)\Big)\Big],$$

where

$$\mathcal{F}(\kappa) = e^{-a(\kappa-s)}\sin b(\kappa-s),\quad \kappa=t,\,1,\,\xi \text{ and } \eta_i,$$

$$m_{1,2} = -a\pm bi,\quad a=\frac{q_1}{2q_2},\quad b=\frac{\sqrt{4q_0q_2-q_1^2}}{2q_2},$$

$$\tau_1(t) = \frac{p_3 v_2(t)-p_4 v_1(t)}{\mu_3},\quad \tau_2(t)=\frac{p_2 v_1(t)-p_1 v_2(t)}{\mu_3},$$

$$v_1(t) = \frac{b-be^{-at}\cos bt-ae^{-at}\sin bt}{a^2+b^2},\quad v_2(t)=q_2 be^{-at}\sin bt$$

$$p_1 = \frac{1}{a^2+b^2}\Big[b-be^{-a\xi}\cos b\xi-ae^{-a\xi}\sin b\xi$$

$$-\sum_{i=1}^n j_i(b-be^{-a\eta_i}\cos b\eta_i-ae^{-a\eta_i}\sin b\eta_i)\Big],$$

$$p_2 = q_2 b\Big(e^{-a\xi}\sin b\xi-\sum_{i=1}^n j_i e^{-a\eta_i}\sin b\eta_i\Big), \tag{17}$$

$$p_3 = \frac{1}{a^2+b^2}\Big[b-be^{-a}\cos b-ae^{-a}\sin b-b\lambda\delta$$

$$+\frac{b\lambda}{a^2+b^2}(a-ae^{-a\delta}\cos b\delta+be^{-a\delta}\sin b\delta)$$

$$-\frac{a\lambda}{a^2+b^2}(b-be^{-a\delta}\cos b\delta-ae^{-a\delta}\sin b\delta)\Big],$$

$$p_4 = q_2 b\Big[e^{-a}\sin b-\frac{\lambda}{a^2+b^2}(b-be^{-a\delta}\cos b\delta-ae^{-a\delta}\sin b\delta)\Big],$$

$$\mu_3 = p_1 p_4-p_2 p_3 \neq 0.$$

3. Existence and Uniqueness Results

Denote by $\mathcal{C}=C([0,1],\mathbb{R})$ the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\|=\sup\{|x(t)|:t\in[0,1]\}$. In relation to the problem (1) and (2) with $q_1^2-4q_0q_2>0$, we define an operator $\mathcal{J}:\mathcal{C}\to\mathcal{C}$ by Lemma 2 as

$$(\mathcal{J}x)(t) = \frac{1}{q_2(m_2-m_1)}\left\{\int_0^t \int_0^s \mathcal{A}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}f(u,x(u))du\,ds\right.$$

$$+\rho_1(t)\left[\int_0^\xi \int_0^s \mathcal{A}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}f(u,x(u))du\,ds\right.$$

$$
-\sum_{i=1}^{n} j_i \int_0^{\eta_i} \int_0^s \mathcal{A}(\eta_i) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(u, x(u)) du\, ds \Big]
$$

$$
+\rho_2(t) \Big[\int_0^1 \int_0^s \mathcal{A}(1) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(u, x(u)) du\, ds \tag{18}
$$

$$
-\lambda \int_0^\delta \int_0^s \Big(\frac{(e^{m_2(\delta-s)}-1)}{m_2} - \frac{(e^{m_1(\delta-s)}-1)}{m_1} \Big) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(u, x(u)) du\, ds \Big] \Big\}
$$

$$
+h(x) \Big[e^{m_2 t} + \rho_1(t) \Big(e^{m_2 \xi} - \sum_{i=1}^{n} j_i + \sum_{i=1}^{n} j_i e^{m_2 \eta_i} \Big)
$$

$$
+\rho_2(t) \Big(\frac{m_2 e^{m_2} - \lambda e^{m_2 \delta} - \lambda}{m_2} \Big) \Big],
$$

where $\mathcal{A}(\cdot)$, $\rho_1(t)$ and $\rho_2(t)$ are defined by (6).

Observe that the problem (1) and (2) is equivalent to the operator equation

$$
x = \mathcal{J}x, \tag{19}
$$

In the sequel, for the sake of computational convenience, we set

$$
\widehat{\rho}_1 = \max_{t\in[0,1]} |\rho_1(t)|, \quad \widehat{\rho}_2 = \max_{t\in[0,1]} |\rho_2(t)|,
$$

$$
\varepsilon = \max_{t\in[0,1]} \Big| m_2(1 - e^{m_1 t}) - m_1(1 - e^{m_2 t}) \Big|,
$$

$$
\alpha = \frac{1}{|q_2 m_1 m_2 (m_2 - m_1)| \Gamma(\sigma+1)} \Big\{ \varepsilon + \widehat{\rho}_1 \Big[\xi^\sigma |m_2(1 - e^{m_1 \xi}) - m_1(1 - e^{m_2 \xi})|
$$

$$
+ \sum_{i=1}^{n} |j_i| \eta_i^\sigma |m_2(1 - e^{m_1 \eta_i}) - m_1(1 - e^{m_2 \eta_i})| \Big] \tag{20}
$$

$$
+ \widehat{\rho}_2 \Big[|m_2(1 - e^{m_1}) - m_1(1 - e^{m_2})|
$$

$$
+ \frac{\delta^\sigma |\lambda|}{|m_1 m_2|} |m_2^2(m_1 \delta - e^{m_1 \delta} + 1) - m_1^2(m_2 \delta - e^{m_2 \delta} + 1)| \Big] \Big\},
$$

$$
\Delta_1 = \max_{t\in[0,1]} |e^{m_2 t}| + \widehat{\rho}_1 \Big(|e^{m_2 \xi}| + \sum_{i=1}^{n} |j_i| |e^{m_2 \eta_i} + 1| \Big) + \widehat{\rho}_2 \Big(\frac{|m_2 e^{m_2}| + |\lambda| |e^{m_2 \delta} + 1|}{|m_2|} \Big).
$$

Now the platform is set to present our main results. In the first result, we use Krasnoselskii's fixed point theorem to prove the existence of solutions for the problem (1) and (2).

Theorem 1. (Krasnoselskii's fixed point theorem [27]). *Let Y be a bounded, closed, convex, and nonempty subset of a Banach space X. Let F_1 and F_2 be the operators satisfying the conditions: (i) $F_1 y_1 + F_2 y_2 \in Y$ whenever $y_1, y_2 \in Y$; (ii) F_1 is compact and continuous; (iii) F_2 is a contraction mapping. Then there exists $y \in Y$ such that $y = F_1 y + F_2 y$.*

In the forthcoming analysis, we need the following assumptions:

(G_1) $|f(t,x) - f(t,y)| \le \ell|x - y|$, for all $t \in [0,1]$, $x, y \in \mathbb{R}$, $\ell > 0$.
(G_2) $|h(x) - h(y)| \le L\|x - y\|$, for all $t \in [0,1]$, $x, y \in \mathcal{C}$, $L > 0$.
(G_3) $|f(t,x)| \le \vartheta(t)$, for all $t \in [0,1]$, $x \in \mathbb{R}$ and $\vartheta \in C([0,1], \mathbb{R}^+)$.

Theorem 2. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the conditions (G_1) and (G_3), $h : C([0,1], \mathbb{R}) \to \mathbb{R}$ be continuous function satisfying the conditions (G_2). Then the problem (1) and (2) with $q_1^2 - 4q_0 q_2 > 0$, has at least one solution on $[0,1]$ if*

$$
L\Delta_1 < 1, \tag{21}
$$

where Δ_1 is given by (20).

Proof. Setting $\sup_{t \in [0,1]} |\vartheta(t)| = \|\vartheta\|$, we can fix

$$
\begin{aligned}
r \geq{} & \frac{\|\vartheta\|}{|q_2 m_1 m_2 (m_2 - m_1)| \Gamma(\delta+1)} \Big\{ \varepsilon + \widehat{\rho}_1 \Big[\xi^\sigma |m_2(1 - e^{m_1 \xi}) - m_1(1 - e^{m_2 \xi})| \\
& + \sum_{i=1}^n |j_i| \eta_i^\sigma |m_2(1 - e^{m_1 \eta_i}) - m_1(1 - e^{m_2 \eta_i})| \Big] + \widehat{\rho}_2 \Big[|m_2(1 - e^{m_1}) - m_1(1 - e^{m_2})| \\
& + \frac{\delta^\sigma |\lambda|}{|m_1 m_2|} |m_2^2(m_1 \delta - e^{m_1 \delta} + 1) - m_1^2(m_2 \delta - e^{m_2 \delta} + 1)| \Big] \Big\} + \Delta_1 \|h\|,
\end{aligned}
\tag{22}
$$

and consider $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. Introduce the operators $\mathcal{J}_1$ and $\mathcal{J}_2$ on B_r as follows:

$$
\begin{aligned}
(\mathcal{J}_1 x)(t) ={} & \frac{1}{q_2(m_2 - m_1)} \int_0^t \int_0^s \mathcal{A}(t) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(u, x(u)) du\, ds \\
& + \frac{1}{q_2(m_2 - m_1)} \Big\{ \rho_1(t) \Big[\int_0^\xi \int_0^s \mathcal{A}(\xi) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(u, x(u)) du\, ds \\
& - \sum_{i=1}^n j_i \int_0^{\eta_i} \int_0^s \mathcal{A}(\eta_i) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(u, x(u)) du\, ds \Big] \\
& + \rho_2(t) \Big[\int_0^1 \int_0^s \mathcal{A}(1) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(u, x(u)) du\, ds \\
& - \lambda \int_0^\delta \int_0^s \Big(\frac{(e^{m_2(\delta-s)} - 1)}{m_2} - \frac{(e^{m_1(\delta-s)} - 1)}{m_1} \Big) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(u, x(u)) du\, ds \Big] \Big\},
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathcal{J}_2 x)(t) ={} & h(x) \Big[e^{m_2 t} + \rho_1(t) \Big(e^{m_2 \xi} - \sum_{i=1}^n j_i + \sum_{i=1}^n j_i e^{m_2 \eta_i} \Big) r \\
& + \rho_2(t) \Big(\frac{m_2 e^{m_2} - \lambda e^{m_2 \delta} - \lambda}{m_2} \Big) \Big].
\end{aligned}
\tag{23}
$$

Observe that $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$. For $x, y \in B_r$, we have

$$
\begin{aligned}
& \|\mathcal{J}_1 x + \mathcal{J}_2 y\| \\
={} & \sup_{t \in [0,1]} |(\mathcal{J}_1 x)(t) + (\mathcal{J}_2 y)(t)| \\
\leq{} & \frac{1}{|q_2(m_2 - m_1)|} \sup_{t \in [0,1]} \Big\{ \int_0^t \int_0^s \mathcal{A}(t) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} |f(u, x(u))| du\, ds \\
& + |\rho_1(t)| \Big[\int_0^\xi \int_0^s \mathcal{A}(\xi) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} |f(u, x(u))| du\, ds \\
& + \sum_{i=1}^n |j_i| \int_0^{\eta_i} \int_0^s \mathcal{A}(\eta_i) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(u, x(u)) du\, ds \Big] \\
& + |\rho_2(t)| \Big[\int_0^1 \int_0^s \mathcal{A}(1) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} |f(u, x(u))| du\, ds \\
& + |\lambda| \int_0^\delta \int_0^s \Big(\frac{(e^{m_2(\delta-s)} - 1)}{m_2} - \frac{(e^{m_1(\delta-s)} - 1)}{m_1} \Big) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} |f(u, x(u))| du\, ds \Big] \Big\} \\
& + |h(y)| \Big[|e^{m_2 t}| + \rho_1(t) \Big(|e^{m_2 \xi}| + \sum_{i=1}^n |j_i| |e^{m_2 \eta_i} + 1| \Big) + \rho_2(t) \Big(\frac{|m_2 e^{m_2}| + |\lambda| |e^{m_2 \delta} + 1|}{|m_2|} \Big) \Big] \\
\leq{} & \frac{\|\vartheta\|}{|q_2(m_2 - m_1)| \Gamma(\sigma+1)} \sup_{t \in [0,1]} \Big\{ t^\sigma \int_0^t |e^{m_2(t-s)} - e^{m_1(t-s)}| ds \\
& + |\rho_1(t)| \Big[\xi^\sigma \int_0^\xi |e^{m_2(\xi-s)} - e^{m_1(\xi-s)}| ds + \sum_{i=1}^n |j_i| \eta_i^\sigma \int_0^{\eta_i} |e^{m_2(\eta_i-s)} - e^{m_1(\eta_i-s)}| ds
\end{aligned}
$$

$$
+|\rho_2(t)|\left[\int_0^1\left|e^{m_2(1-s)}-e^{m_1(1-s)}\right|ds+|\lambda|\delta^\sigma\int_0^\delta\left|\frac{(e^{m_2(\delta-s)}-1)}{m_2}-\frac{(e^{m_1(\delta-s)}-1)}{m_1}\right|ds\right]\Bigg\}
$$

$$
+\Delta_1\|h\|
$$

$$
\leq\ \frac{\|\vartheta\|}{|q_2m_1m_2(m_2-m_1)|\Gamma(\sigma+1)}\Big\{\varepsilon+\widehat{\rho}_1[\xi^\sigma|m_2(1-e^{m_1\xi})-m_1(1-e^{m_2\xi})|
$$

$$
+\sum_{i=1}^n|j_i|\eta_i^\sigma|m_2(1-e^{m_1\eta_i})-m_1(1-e^{m_2\eta_i})|]+\widehat{\rho}_2[|m_2(1-e^{m_1})-m_1(1-e^{m_2})|
$$

$$
+\frac{\delta^\sigma|\lambda|}{|m_1m_2|}|m_2^2(m_1\delta-e^{m_1\delta}+1)-m_1^2(m_2\delta-e^{m_2\delta}+1)|]\Big\}+\Delta_1\|h\|\leq r,
$$

where we used (22). Thus $\mathcal{J}_1x+\mathcal{J}_2y\in B_r$. Using the assumptions $(G_1)-(G_3)$ together with (21), we show that $\mathcal{J}_2$ is a contraction as follows:

$$
\|\mathcal{J}_2x-\mathcal{J}_2y\|
$$

$$
=\ \sup_{t\in[0,1]}|(\mathcal{J}_2x)(t)-(\mathcal{J}_2y)(t)|
$$

$$
\leq\ |h(x)-h(y)|\left[|e^{m_2t}|+\rho_1(t)\left(|e^{m_2\xi}|+\sum_{i=1}^n|j_i||e^{m_2\eta_i}+1|\right)+\rho_2(t)\left(\frac{|m_2e^{m_2}|+|\lambda||e^{m_2\delta}+1|}{|m_2|}\right)\right]
$$

$$
\leq\ L\Delta_1\|x-y\|.
$$

Note that continuity of f implies that the operator $\mathcal{J}_1$ is continuous. Also, $\mathcal{J}_1$ is uniformly bounded on B_r as

$$
\|\mathcal{J}_1x\|\ =\ \sup_{t\in[0,1]}|(\mathcal{J}_1x)(t)|
$$

$$
\leq\ \frac{\|\vartheta\|}{|q_2m_1m_2(m_2-m_1)|\Gamma(\sigma+1)}\Big\{\varepsilon+\widehat{\rho}_1[\xi^\sigma|m_2(1-e^{m_1\xi})-m_1(1-e^{m_2\xi})|
$$

$$
+\sum_{i=1}^n|j_i|\eta_i^\sigma|m_2(1-e^{m_1\eta_i})-m_1(1-e^{m_2\eta_i})|]+\widehat{\rho}_2[|m_2(1-e^{m_1})-m_1(1-e^{m_2})|
$$

$$
+\frac{\delta^\sigma|\lambda|}{|m_1m_2|}|m_2^2(m_1\delta-e^{m_1\delta}+1)-m_1^2(m_2\delta-e^{m_2\delta}+1)|]\Big\}.
$$

Now we prove the compactness of operator $\mathcal{J}_1$. We define $\sup_{(t,x)\in[0,1]\times B_r}|f(t,x)|=\overline{f}$. Thus, for $0<t_1<t_2<1$, we have

$$
|(\mathcal{J}x)(t_2)-(\mathcal{J}x)(t_1)|
$$

$$
\leq\ \frac{1}{|q_2(m_2-m_1)|}\Bigg\{\left|\int_0^{t_1}\int_0^s[\mathcal{A}(t_2)-\mathcal{A}(t_1)]\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}f(u,x(u))du\,ds\right.
$$

$$
+\int_{t_1}^{t_2}\int_0^s\mathcal{A}(t_2)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}f(u,x(u))du\,ds\Bigg|
$$

$$
+|\rho_1(t_2)-\rho_1(t_1)|\left[\int_0^\xi\int_0^s\mathcal{A}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|f(u,x(u))|du\,ds\right.
$$

$$
+\sum_{i=1}^n|j_i|\int_0^{\eta_i}\int_0^s\mathcal{A}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|f(u,x(u))|du\,ds\right]
$$

$$
+|\rho_2(t_2)-\rho_2(t_1)|\left[\int_0^1\int_0^s\mathcal{A}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|f(u,x(u))|du\,ds\right.
$$

$$
+|\lambda|\int_0^\delta\int_0^s\left(\frac{(e^{m_2(\delta-s)}-1)}{m_2}-\frac{(e^{m_1(\delta-s)}-1)}{m_1}\right)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|f(u,x(u))|du\,ds\right]\Bigg\}
$$

$$
\leq \quad \frac{\overline{f}}{|q_2 m_1 m_2 (m_2 - m_1)| \Gamma(\sigma + 1)} \left\{ \left(t_1^\sigma - t_2^\sigma \right) \left| m_1 (1 - e^{m_2(t_2 - t_1)}) - m_2 (1 - e^{m_1(t_2 - t_1)}) \right| \right.
$$
$$
+ t_1^\sigma \left| m_1 (e^{m_2 t_2} - e^{m_2 t_1}) - m_2 (e^{m_1 t_2} - e^{m_1 t_1}) \right|
$$
$$
+ |\rho_1(t_2) - \rho_1(t_1)| [\xi^\sigma | m_2 (1 - e^{m_1 \xi}) - m_1 (1 - e^{m_2 \xi})|
$$
$$
+ \sum_{i=1}^{n} |j_i| \eta_i^\sigma | m_2 (1 - e^{m_1 \eta_i}) - m_1 (1 - e^{m_2 \eta_i})|]
$$
$$
+ |\rho_2(t_2) - \rho_2(t_1)| [| m_2 (1 - e^{m_1}) - m_1 (1 - e^{m_2})|
$$
$$
\left. + \frac{\delta^\sigma |\lambda|}{|m_1 m_2|} | m_1^2 (m_2 \delta - e^{m_2 \delta} + 1) - m_2^2 (m_1 \delta - e^{m_1 \delta} + 1)|] \right\} \to 0, \text{ as } t_1 \to t_2,
$$

independent of x. Thus, $\mathcal{J}_1$ is relatively compact on B_r. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{J}_1$ is compact on B_r. Thus all the assumption of Theorem 1 are satisfied. So, by the conclusion of Theorem 1, the problem (1) and (2) has at least one solution on $[0, 1]$. The proof is completed. □

Remark 2. *In the above theorem we can interchange the roles of the operators $\mathcal{J}_1$ and $\mathcal{J}_2$ to obtain a second result by replacing (21) by the following condition:*

$$
\ell \alpha < 1.
$$

Now we apply Banach's contraction mapping principle to prove existence and uniqueness of solutions for the problem (1) and (2).

Theorem 3. *Assume that $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that (G_1) and (G_2) are satisfied. Then there exists a unique solution for the problem (1) and (2) on $[0, 1]$ if $\ell \alpha + L \Delta_1 < 1$, where α and Δ_1 are given by (20).*

Proof. Let us define $\sup_{t \in [0,1]} |f(t, 0)| = M$, $\sup_{t \in [0,1]} |h(0)| = L_0$ and select $\bar{r} \geq \dfrac{\alpha M + L_0 \Delta_1}{1 - (\ell \alpha + L \Delta_1)}$ to show that $\mathcal{J} B_{\bar{r}} \subset B_{\bar{r}}$, where $B_{\bar{r}} = \{ x \in \mathcal{C} : \|x\| \leq \bar{r} \}$ and $\mathcal{J}$ is defined by (18). Using the condition (G_1) and (G_2), we have

$$
\begin{aligned}
|f(t, x)| &= |f(t, x) - f(t, 0) + f(t, 0)| \leq |f(t, x) - f(t, 0)| + |f(x, 0)| \\
&\leq \ell \|x\| + M \leq \ell \bar{r} + M,
\end{aligned}
\tag{24}
$$

$$
|h(x)| = |h(x) - h(0) + h(0)| \leq |h(x) - h(0)| + |h(0)| \leq L \|x\| + L_0 \leq L \bar{r} + L_0.
\tag{25}
$$

Then, for $x \in B_{\bar{r}}$, we obtain

$$
\begin{aligned}
\|\mathcal{J}(x)\| &= \sup_{t \in [0,1]} |\mathcal{J}(x)(t)| \\
&\leq \frac{1}{|q_2(m_2 - m_1)|} \sup_{t \in [0,1]} \left\{ \int_0^t \int_0^s \mathcal{A}(t) \frac{(s - u)^{\sigma - 1}}{\Gamma(\sigma)} |f(u, x(u))| du\, ds \right. \\
&\quad + |\rho_1(t)| \left[\int_0^\xi \int_0^s \mathcal{A}(\xi) \frac{(s - u)^{\sigma - 1}}{\Gamma(\sigma)} |f(u, x(u))| du\, ds \right. \\
&\quad + \sum_{i=1}^{n} |j_i| \int_0^{\eta_i} \int_0^s \mathcal{A}(\eta_i) \frac{(s - u)^{\sigma - 1}}{\Gamma(\sigma)} |f(u, x(u))| du\, ds \Big] \\
&\quad + |\rho_2(t)| \left[\int_0^1 \int_0^s \mathcal{A}(1) \frac{(s - u)^{\sigma - 1}}{\Gamma(\sigma)} |f(u, x(u))| du\, ds \right.
\end{aligned}
$$

$$
+|\lambda|\int_0^\delta\int_0^s\left(\frac{(e^{m_2(\delta-s)}-1)}{m_2}-\frac{(e^{m_1(\delta-s)}-1)}{m_1}\right)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|f(u,x(u))|du\,ds\bigg]\bigg\}
$$

$$
+|h(x)|\bigg[|e^{m_2t}|+\rho_1(t)(|e^{m_2\xi}|+\sum_{i=1}^n|j_i||e^{m_2\eta_i}+1|)
$$

$$
+\rho_2(t)\Big(\frac{|m_2e^{m_2}|+|\lambda||e^{m_2\delta}+1|}{|m_2|}\Big)\bigg]
$$

$$
\leq\quad\frac{(\ell\bar r+M)}{|q_2(m_2-m_1)|}\sup_{t\in[0,1]}\bigg\{\int_0^t\Big|e^{m_2(t-s)}-e^{m_1(t-s)}\Big|\frac{s^\sigma}{\Gamma(\sigma+1)}ds
$$

$$
+|\rho_1(t)|\bigg[\int_0^\xi\Big|e^{m_2(\xi-s)}-e^{m_1(\xi-s)}\Big|\frac{s^\sigma}{\Gamma(\sigma+1)}ds
$$

$$
+\sum_{i=1}^n|j_i|\int_0^{\eta_i}\Big|e^{m_2(\eta_i-s)}-e^{m_1(\eta_i-s)}\Big|\frac{s^\sigma}{\Gamma(\sigma+1)}ds\bigg]
$$

$$
+\quad|\rho_2(t)|\bigg[\int_0^1\Big|e^{m_2(1-s)}-e^{m_1(1-s)}\Big|\frac{s^\sigma}{\Gamma(\sigma+1)}ds
$$

$$
+|\lambda|\int_0^\delta\Big|\frac{(e^{m_2(\delta-s)}-1)}{m_2}-\frac{(e^{m_1(\delta-s)}-1)}{m_1}\Big|\frac{s^\sigma}{\Gamma(\sigma+1)}ds\bigg]\bigg\}+(L\bar r+L_0)\Delta_1
$$

$$
\leq\quad\frac{(\ell\bar r+M)}{|q_2m_1m_2(m_2-m_1)|\Gamma(\sigma+1)}\Big\{\varepsilon+\widehat\rho_1[\xi^\sigma|m_2(1-e^{m_1\xi})-m_1(1-e^{m_2\xi})|
$$

$$
+\sum_{i=1}^n|j_i|\eta_i^\sigma|m_2(1-e^{m_1\eta_i})-m_1(1-e^{m_2\eta_i})|]+\widehat\rho_2[|m_2(1-e^{m_1})-m_1(1-e^{m_2})|
$$

$$
+\frac{\delta^\sigma|\lambda|}{|m_1m_2|}|m_2^2(m_1\delta-e^{m_1\delta}+1)-m_1^2(m_2\delta-e^{m_2\delta}+1)|]\Big\}+(L\bar r+L_0)\Delta_1
$$

$$
=\quad(\ell\bar r+M)\alpha+(L\bar r+L_0)\Delta_1\leq\bar r,
$$

which clearly shows that $\mathcal{J}x\in B_{\bar r}$ for any $x\in B_{\bar r}$. Thus $\mathcal{J}B_{\bar r}\subset B_{\bar r}$. Now, for $x,y\in\mathcal{C}$ and for each $t\in[0,1]$, we have

$$
\|(\mathcal{J}x)-(\mathcal{J}y)\|
$$

$$
\leq\quad\frac{1}{|q_2(m_2-m_1)|}\sup_{t\in[0,1]}\bigg\{\int_0^t\int_0^s\mathcal{A}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|f(u,x(u))-f(u,y(u))|du\,ds
$$

$$
+|\rho_1(t)|\bigg[\int_0^\xi\int_0^s\mathcal{A}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|f(u,x(u))-f(u,y(u))|du\,ds
$$

$$
+\sum_{i=1}^n|j_i|\int_0^{\eta_i}\int_0^s\mathcal{A}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|f(u,x(u))-f(u,y(u))|du\,ds\bigg]
$$

$$
+|\rho_2(t)|\bigg[\int_0^1\int_0^s\mathcal{A}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|f(u,x(u))-f(u,y(u))|du\,ds
$$

$$
+|\lambda|\int_0^\delta\int_0^s\left(\frac{(e^{m_2(\delta-s)}-1)}{m_2}-\frac{(e^{m_1(\delta-s)}-1)}{m_1}\right)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|f(u,x(u))-f(u,y(u))|du\,ds\bigg]\bigg\}
$$

$$
+|h(x)-h(y)|\bigg[|e^{m_2t}|+\rho_1(t)(|e^{m_2\xi}|+\sum_{i=1}^n|j_i||e^{m_2\eta_i}+1|)
$$

$$
+\rho_2(t)\Big(\frac{|m_2e^{m_2}|+|\lambda||e^{m_2\delta}+1|}{|m_2|}\Big)\bigg]
$$

$$
\leq\quad\frac{\ell}{|q_2(m_2-m_1)|}\sup_{t\in[0,1]}\bigg\{\int_0^t\Big|e^{m_2(t-s)}-e^{m_1(t-s)}\Big|\frac{s^\sigma}{\Gamma(\sigma+1)}ds
$$

$$
+|\rho_1(t)| \left[\int_0^\xi \left| e^{m_2(\xi-s)} - e^{m_1(\xi-s)} \right| \frac{s^\sigma}{\Gamma(\sigma+1)} ds \right.
$$

$$
+ \sum_{i=1}^n |j_i| \int_0^{\eta_i} \left| e^{m_2(\eta_i-s)} - e^{m_1(\eta_i-s)} \right| \frac{s^\sigma}{\Gamma(\sigma+1)} ds \right]
$$

$$
+|\rho_2(t)| \left[\int_0^1 \left| e^{m_2(1-s)} - e^{m_1(1-s)} \right| \frac{s^\sigma}{\Gamma(\sigma+1)} ds \right.
$$

$$
+|\lambda| \int_0^\delta \left| \frac{(e^{m_2(\delta-s)}-1)}{m_2} - \frac{(e^{m_1(\delta-s)}-1)}{m_1} \right| \frac{s^\sigma}{\Gamma(\sigma+1)} ds \right] \Bigg\} \|x-y\| + L\Delta_1 \|x-y\|
$$

$$
\leq \frac{\ell}{|q_2 m_1 m_2 (m_2-m_1)|\Gamma(\sigma+1)} \Big\{ \varepsilon + \widehat{\rho}_1 [\xi^\sigma |m_2(1-e^{m_1\xi}) - m_1(1-e^{m_2\xi})|
$$

$$
+ \sum_{i=1}^n |j_i|\eta_i^\sigma |m_2(1-e^{m_1\eta_i}) - m_1(1-e^{m_2\eta_i})|] + \widehat{\rho}_2 [|m_2(1-e^{m_1}) - m_1(1-e^{m_2})|
$$

$$
+ \frac{\delta^\sigma|\lambda|}{|m_1 m_2|} |m_2^2(m_1\delta - e^{m_1\delta} + 1) - m_1^2(m_2\delta - e^{m_2\delta} + 1)|] \Big\} \|x-y\| + L\Delta_1 \|x-y\|
$$

$$
= (\ell\alpha + L\Delta_1)\|x-y\|,
$$

α and Δ_1 are given by (20) and depend only on the parameters involved in the problem. In view of the condition $\ell\alpha + L\Delta_1 < 1$, it follows that $\mathcal{J}$ is a contraction. Thus, by the contraction mapping principle (Banach fixed point theorem), the problem (1) and (2) has a unique solution on $[0,1]$. This completes the proof. $\square$

The next existence result is based on Leray–Schauder nonlinear alternative.

Theorem 4. *(Nonlinear alternative for single valued maps [28]). Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i)   *F has a fixed point in $\overline{U}$, or*
(ii)   *there is a $u \in \partial U$ (the boundary of U in C) and $\epsilon \in (0,1)$ with $u = \epsilon F(u)$.*

We need the following assumptions:

(H_1)   There exist a function $g \in C([0,1], \mathbb{R}^+)$, and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t,y)| \leq g(t)\psi(\|y\|), \ \forall (t,y) \in [0,1] \times \mathbb{R}$.

(H_2) $h : C([0,1], \mathbb{R}) \to \mathbb{R}$, is continuous function with $h(0) = 0$ and there exist constant $L_1 > 0$ with $L_1 < \Delta_1^{-1}$, such that
$$
|h(x)| \leq L_1 \|x\| \quad \forall \ x \in C.
$$

(H_3)   There exists a constant $K > 0$ such that
$$
\frac{(1 - L_1\Delta_1)K}{\|g\|\psi(K)\alpha} > 1.
$$

Theorem 5. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the problem (1) and (2) has at least one solution on $[0,1]$, if (H_1)–(H_3) are satisfied.*

Proof. Consider the operator $\mathcal{J} : \mathcal{C} \to \mathcal{C}$ defined by (18). We show that $\mathcal{J}$ maps bounded sets into bounded sets in $\mathcal{C}$. For a positive number ζ, let $\mathcal{E}_\zeta = \{x \in \mathcal{C} : \|x\| \leq \zeta\}$ be a bounded set in $\mathcal{C}$. Then we have

$$
\|\mathcal{J}(x)\| = \sup_{t \in [0,1]} |\mathcal{J}(x)(t)|
$$

$$\leq \frac{1}{|q_2(m_2-m_1)|} \sup_{t\in[0,1]} \left\{ \int_0^t \int_0^s \mathcal{A}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|f(u,x(u))|du\,ds \right.$$

$$+|\rho_1(t)|\Big[\int_0^\xi \int_0^s \mathcal{A}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|f(u,x(u))|du\,ds$$

$$+\sum_{i=1}^n |j_i|\int_0^{\eta_i}\int_0^s \mathcal{A}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|f(u,x(u))|du\,ds\Big]$$

$$+|\rho_2(t)|\Big[\int_0^1\int_0^s \mathcal{A}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|f(u,x(u))|du\,ds$$

$$+|\lambda|\int_0^\delta\int_0^s \Big(\frac{(e^{m_2(\delta-s)}-1)}{m_2}-\frac{(e^{m_1(\delta-s)}-1)}{m_1}\Big)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}|f(u,x(u))|du\,ds\Big]\Big\}$$

$$+|h(x)|\Big[|e^{m_2 t}|+\rho_1(t)(|e^{m_2\xi}|+\sum_{i=1}^n|j_i||e^{m_2\eta_i}+1|)$$

$$+\rho_2(t)\Big(\frac{|m_2 e^{m_2}|+|\lambda||e^{m_2\delta}+1|}{|m_2|}\Big)\Big]$$

$$\leq \frac{\|g\|\psi(\zeta)}{|q_2(m_2-m_1)|}\sup_{t\in[0,1]}\left\{\int_0^t \Big|e^{m_2(t-s)}-e^{m_1(t-s)}\Big|\frac{s^\sigma}{\Gamma(\sigma+1)}ds\right.$$

$$+|\rho_1(t)|\Big[\int_0^\xi\Big|e^{m_2(\xi-s)}-e^{m_1(\xi-s)}\Big|\frac{s^\sigma}{\Gamma(\sigma+1)}ds$$

$$+\sum_{i=1}^n|j_i|\int_0^{\eta_i}\Big|e^{m_2(\eta_i-s)}-e^{m_1(\eta_i-s)}\Big|\frac{s^\sigma}{\Gamma(\sigma+1)}ds\Big]$$

$$+|\rho_2(t)|\Big[\int_0^1\Big|e^{m_2(1-s)}-e^{m_1(1-s)}\Big|\frac{s^\sigma}{\Gamma(\sigma+1)}ds$$

$$+|\lambda|\int_0^\delta\Big|\frac{(e^{m_2(\delta-s)}-1)}{m_2}-\frac{(e^{m_1(\delta-s)}-1)}{m_1}\Big|\frac{s^\sigma}{\Gamma(\sigma+1)}ds\Big]\right\}+L_1\Delta_1\zeta$$

$$\leq \frac{\|g\|\psi(\zeta)}{|q_2 m_1 m_2(m_2-m_1)|\Gamma(\sigma+1)}\Big\{\varepsilon+\widehat{\rho}_1[\xi^\sigma|m_2(1-e^{m_1\xi})-m_1(1-e^{m_2\xi})|$$

$$+\sum_{i=1}^n|j_i|\eta_i^\sigma|m_2(1-e^{m_1\eta_i})-m_1(1-e^{m_2\eta_i})|]+\widehat{\rho}_2[|m_2(1-e^{m_1})-m_1(1-e^{m_2})|$$

$$+\frac{\delta^\sigma|\lambda|}{|m_1 m_2|}|m_2^2(m_1\delta-e^{m_1\delta}+1)-m_1^2(m_2\delta-e^{m_2\delta}+1)|]\Big\}+L_1\Delta_1\zeta,$$

which yields

$$\|\mathcal{J}x\| \leq \frac{\|g\|\psi(\zeta)}{|q_2 m_1 m_2(m_2-m_1)|\Gamma(\sigma+1)}\Big\{\varepsilon+\widehat{\rho}_1\xi^\sigma|m_2(1-e^{m_1\xi})-m_1(1-e^{m_2\xi})|$$

$$+\sum_{i=1}^n|j_i|\eta_i^\sigma|m_2(1-e^{m_1\eta_i})-m_1(1-e^{m_2\eta_i})|]+\widehat{\rho}_2[|m_2(1-e^{m_1})-m_1(1-e^{m_2})|$$

$$+\frac{\delta^\sigma|\lambda|}{|m_1 m_2|}|m_2^2(m_1\delta-e^{m_1\delta}+1)-m_1^2(m_2\delta-e^{m_2\delta}+1)|]\Big\}+L_1\Delta_1\zeta.$$

Next we show that $\mathcal{J}$ maps bounded sets into equicontniuous sets of $\mathcal{C}$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $y \in \mathcal{E}_\zeta$, where $\mathcal{E}_\zeta$ is a bounded set of $\mathcal{C}$. Then we obtain

$$|(\mathcal{J}x)(t_2)-(\mathcal{J}x)(t_1)|$$

$$\leq \frac{1}{|q_2(m_2-m_1)|}\left\{\Big|\int_0^{t_1}\int_0^s\Big[\mathcal{A}(t_2)-\mathcal{A}(t_1)\Big]\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}f(u,x(u))du\,ds\right.$$

$$+ \int_{t_1}^{t_2} \int_0^s \mathcal{A}(t_2) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(u, x(u)) du\, ds \bigg|$$

$$+ |\rho_1(t_2) - \rho_1(t_1)| \bigg[\int_0^\xi \int_0^s \mathcal{A}(\xi) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} |f(u, x(u))| du\, ds$$

$$+ \sum_{i=1}^n |j_i| \int_0^{\eta_i} \int_0^s \mathcal{A}(\eta_i) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} |f(u, x(u))| du\, ds \bigg]$$

$$+ |\rho_2(t_2) - \rho_2(t_1)| \bigg[\int_0^1 \int_0^s \mathcal{A}(1) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} |f(u, x(u))| du\, ds$$

$$+ |\lambda| \int_0^\delta \int_0^s \bigg(\frac{(e^{m_2(\delta-s)} - 1)}{m_2} - \frac{(e^{m_1(\delta-s)} - 1)}{m_1} \bigg) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} |f(u, x(u))| du\, ds \bigg] \bigg\}$$

$$+ |h(x)| \bigg[|e^{m_2 t_2} - e^{m_2 t_1}| + (\rho_1(t_2) - \rho_1(t_1))(|e^{m_2 \xi}| + \sum_{i=1}^n |j_i| |e^{m_2 \eta_i} + 1|)$$

$$+ (\rho_2(t_2) - \rho_2(t_1)) \bigg(\frac{|m_2 e^{m_2}| + |\lambda| |e^{m_2 \delta} + 1|}{|m_2|} \bigg) \bigg]$$

$$\leq \frac{\overline{f}}{|q_2 m_1 m_2 (m_2 - m_1)| \Gamma(\sigma+1)} \bigg\{ \big(t_1^\sigma - t_2^\sigma \big) \big| m_1 (1 - e^{m_2(t_2 - t_1)}) - m_2 (1 - e^{m_1(t_2 - t_1)}) \big|$$

$$+ t_1^\sigma \big| m_1 (e^{m_2 t_2} - e^{m_2 t_1}) - m_2 (e^{m_1 t_2} - e^{m_1 t_1}) \big|$$

$$+ |\rho_1(t_2) - \rho_1(t_1)| [\xi^\sigma | m_2 (1 - e^{m_1 \xi}) - m_1 (1 - e^{m_2 \xi})|$$

$$+ \sum_{i=1}^n |j_i| \eta_i^\sigma |m_2 (1 - e^{m_1 \eta_i}) - m_1 (1 - e^{m_2 \eta_i})|]$$

$$+ |\rho_2(t_2) - \rho_2(t_1)| [|m_2 (1 - e^{m_1}) - m_1 (1 - e^{m_2})|$$

$$+ \frac{\delta^\sigma |\lambda|}{|m_1 m_2|} |m_1^2 (m_2 \delta - e^{m_2 \delta} + 1) - m_2^2 (m_1 \delta - e^{m_1 \delta} + 1)|] \bigg\} + |h(x)| \bigg[|e^{m_2 t_2} - e^{m_2 t_1}|$$

$$+ (\rho_1(t_2) - \rho_1(t_1)) \bigg(|e^{m_2 \xi}| + \sum_{i=1}^n |j_i| |e^{m_2 \eta_i} + 1| \bigg)$$

$$+ (\rho_2(t_2) - \rho_2(t_1)) \bigg(\frac{|m_2 e^{m_2}| + |\lambda| |e^{m_2 \delta} + 1|}{|m_2|} \bigg) \bigg],$$

which tends to zero independently of $x \in \mathcal{E}_\zeta$ as $t_2 - t_1 \to 0$. As $\mathcal{J}$ satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{J} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

The result will follow from the Leray–Schauder nonlinear alternative once it is shown that there exists $\mathcal{U} \subseteq \mathcal{C}$ with $x \neq \theta \mathcal{J} x$ for $\theta \in (0, 1)$ and $x \in \partial \mathcal{U}$.

Let $x \in \mathcal{C}$ be such that $x = \theta \mathcal{J} x$ for $\theta \in [0, 1]$. Then, for $t \in [0, 1]$, we have

$$|x(t)| = |\theta \mathcal{J} x(t)|$$

$$\leq \frac{1}{|q_2(m_2 - m_1)|} \sup_{t \in [0,1]} \bigg\{ \int_0^t \int_0^s \mathcal{A}(t) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} |f(u, x(u))| du\, ds$$

$$+ |\rho_1(t)| \bigg[\int_0^\xi \int_0^s \mathcal{A}(\xi) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} |f(u, x(u))| du\, ds$$

$$+ \sum_{i=1}^n |j_i| \int_0^{\eta_i} \int_0^s \mathcal{A}(\eta_i) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} |f(u, x(u))| du\, ds \bigg]$$

$$+ |\rho_2(t)| \bigg[\int_0^1 \int_0^s \mathcal{A}(1) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} |f(u, x(u))| du\, ds$$

$$+ |\lambda - \int_0^\delta \int_0^s \bigg(\frac{(e^{m_2(\delta-s)} - 1)}{m_2} - \frac{(e^{m_1(\delta-s)} - 1)}{m_1} \bigg) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} |f(u, x(u))| du\, ds \bigg] \bigg\}$$

$$+ |h(x)| \left[\left| e^{m_2 t} \right| + \rho_1(t) \left(\left| e^{m_2 \xi} \right| + \sum_{i=1}^{n} |j_i| |e^{m_2 \eta_i} + 1| \right) \right.$$

$$\left. + \rho_2(t) \left(\frac{|m_2 e^{m_2}| + |\lambda| |e^{m_2 \delta} + 1|}{|m_2|} \right) \right]$$

$$\leq \frac{\|g\| \psi(\|x\|)}{|q_2(m_2 - m_1)|} \sup_{t \in [0,1]} \left\{ \int_0^t \left| e^{m_2(t-s)} - e^{m_1(t-s)} \right| \frac{s^\sigma}{\Gamma(\sigma+1)} ds \right.$$

$$+ |\rho_1(t)| \left[\int_0^\xi \left| e^{m_2(\xi-s)} - e^{m_1(\xi-s)} \right| \frac{s^\sigma}{\Gamma(\sigma+1)} ds \right.$$

$$+ \sum_{i=1}^{n} |j_i| \int_0^{\eta_i} \left| e^{m_2(\eta_i-s)} - e^{m_1(\eta_i-s)} \right| \frac{s^\sigma}{\Gamma(\sigma+1)} ds \right]$$

$$+ |\rho_2(t)| \left[\int_0^1 \left| e^{m_2(1-s)} - e^{m_1(1-s)} \right| \frac{s^\sigma}{\Gamma(\sigma+1)} ds \right.$$

$$\left. + |\lambda| \int_0^\delta \left| \frac{(e^{m_2(\delta-s)} - 1)}{m_2} - \frac{(e^{m_1(\delta-s)} - 1)}{m_1} \right| \frac{s^\sigma}{\Gamma(\sigma+1)} ds \right] \right\} + |h(x)| \Delta_1$$

$$\leq \frac{\|g\| \psi(\|x\|)}{|q_2 m_1 m_2 (m_2 - m_1)| \Gamma(\sigma+1)} \left\{ \varepsilon + \widehat{\rho}_1 [\xi^\sigma |m_2(1 - e^{m_1 \xi}) - m_1(1 - e^{m_2 \xi})| \right.$$

$$+ \sum_{i=1}^{n} |j_i| \eta_i^\sigma |m_2(1 - e^{m_1 \eta_i}) - m_1(1 - e^{m_2 \eta_i})|] + \widehat{\rho}_2 [|m_2(1 - e^{m_1}) - m_1(1 - e^{m_2})|$$

$$\left. + \frac{\delta^\sigma |\lambda|}{|m_1 m_2|} |m_1^2(m_2 \delta - e^{m_2 \delta} + 1) - m_2^2(m_1 \delta - e^{m_1 \delta} + 1)|] \right\} + L_1 \Delta_1 \|x\|$$

$$= \|g\| \psi(\|x\|) \alpha + L_1 \Delta_1 \|x\|,$$

which implies that

$$\frac{(1 - L_1 \Delta_1) \|x\|}{\|g\| \psi(\|x\|) \alpha} \leq 1.$$

In view of (H_3), there is no solution x such that $\|x\| \neq K$. Let us set

$$U = \{x \in \mathcal{C} : \|x\| < K\}.$$

The operator $\mathcal{J} : \overline{U} \to \mathcal{C}$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \theta \mathcal{J}(u)$ for some $\theta \in (0,1)$. Consequently, by the nonlinear alternative of Leray–Schauder type [28], we deduce that $\mathcal{J}$ has a fixed point $u \in \overline{U}$ which is a solution of the problem (1) and (2). $\square$

Example 1. *Let us consider the following boundary value problem*

$$(2 \, {}^c D^{12/5} + 3 \, {}^c D^{7/5} + {}^c D^{2/5}) x(t) = \frac{e^{-t}}{4\sqrt{4 + t^2}} \tan^{-1} x + \cos t, \quad 0 < t < 1, \tag{26}$$

subject the boundary condition

$$x(0) = \frac{1}{9} \sin x(\hat{t}), \quad x(1/5) = x(1/4) + 2x(1/3) + x(1/2), \quad x(1) = 2 \int_0^{1/6} x(s) ds. \tag{27}$$

Here, $q_2 = 2$, $q_1 = 3$, $q_0 = 1$, $\sigma = 2/5$, $\xi = 1/5$, $\eta_1 = 1/4$, $\eta_2 = 1/3$, $\eta_3 = 1/2$, $\delta = 1/6$, $j_1 = 1$, $j_2 = 2$, $j_3 = 1$, $\lambda = 2$, $\hat{t}$ is a fixed value in $[0, 1]$ and

$$f(t, x) = \frac{e^{-t}}{4\sqrt{4 + t^2}} \tan^{-1} x + \cos t.$$

Clearly $q_1^2 - 4q_0q_2 = 1 > 0$, and

$$|f(t,x) - f(t,y)| \leq \frac{1}{8}|x - y|,$$

$$|h(x) - h(y)| \leq \frac{1}{9}\|x - y\|.$$

where $\ell = 1/8$, $L = 1/9$. Using the given values, we found $\alpha \approx 0.095961$, $\Delta_1 \approx 6.9171$.

It is easy to check that $|f(t,x)| \leq \frac{\pi e^{-t}}{8\sqrt{4 + t^2}} + \cos t = \vartheta(t)$ and $L\Delta_1 < 1$. As all the condition of Theorem 2 are satisfied the problem (26) and (27) has at least one solution on $[0,1]$. On the other hand, $\ell\alpha + L\Delta_1 < 1$ and thus there exists a unique solution for the problem (26) and (27) on $[0,1]$ by Theorem 3.

Example 2. *Consider the following fractional differential equation*

$$(2\,^cD^{12/5} + 3\,^cD^{7/5} + \,^cD^{2/5})x(t) = \frac{1}{\pi\sqrt{9 + t^2}}\left(x \tan^{-1} x + \pi/2\right), \quad 0 < t < 1, \tag{28}$$

subject the boundary conditions (27).

Here

$$f(t,x) = \frac{1}{\pi\sqrt{9 + t^2}}(x \tan^{-1} x + \pi/2).$$

Clearly

$$|f(t,x)| \leq \frac{1}{2\sqrt{9 + t^2}}\left(\|x\| + 1\right),$$

with $g(t) = \frac{1}{2\sqrt{9+t^2}}$, $\psi(\|x\|) = \|x\| + 1$.

Then by using the condition (H_3), we find that $K > 0.241877$ (we have used $\alpha = 0.27045$). Thus, the conclusion of Theorem 5 applies to problem (28) and (27).

4. Existence Results for Problem (1) and (2) with $q_1^2 - 4q_0q_2 = 0$

In view of Lemma 3, we can transform problem (1) and (2) into equivalent fixed point problem as follows:

$$x = \mathcal{H}x, \tag{29}$$

where the operator $\mathcal{H} : \mathcal{C} \to \mathcal{C}$ is defined by

$$
\begin{aligned}
(\mathcal{H}x)(t) = \frac{1}{q_2}\Bigg\{ &\int_0^t \int_0^s \mathcal{B}(t)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}f(u,x(u))du\,ds \\
&+\chi_1(t)\Big[\int_0^\xi \int_0^s \mathcal{B}(\xi)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}f(u,x(u))du\,ds \\
&\quad - \sum_{i=1}^n j_i \int_0^{\eta_i} \int_0^s \mathcal{B}(\eta_i)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}f(u,x(u))du\,ds\Big] \\
&+\chi_2(t)\Big[\int_0^1 \int_0^s \mathcal{B}(1)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}f(u,x(u))du\,ds \\
&\quad -\lambda \int_0^\delta \int_0^s \Big(\frac{m(\delta-s)e^{m(\delta-s)} - e^{m(\delta-s)}+1}{m^2}\Big)\frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)}f(u,x(u))du\,ds\Big]\Bigg\} \\
&+h(x)\Big[e^{mt} + \chi_1(t)\Big(e^{m\xi} - \sum_{i=1}^n j_i e^{m\eta_i}\Big) + \chi_2(t)\Big(\frac{me^m - \lambda e^{m\delta} + \lambda}{m}\Big)\Big],
\end{aligned}
\tag{30}
$$

where $\mathcal{B}(\cdot)$, $\chi_1(t)$ and $\chi_2(t)$ are defined by (14). We set

$$\widehat{\chi}_1 = \max_{t \in [0,1]} |\chi_1(t)|, \quad \widehat{\chi}_2 = \max_{t \in [0,1]} |\chi_2(t)|,$$

$$\beta = \frac{1}{|q_2||m^2\Gamma(\sigma+1)}\left\{(1+\widehat{\chi}_2)|me^m - e^m + 1| + \widehat{\chi}_1\left[\xi^\sigma|m\xi e^{m\xi} - e^{m\xi} + 1|\right.\right.$$

$$\left.\left. + \sum_{i=1}^{n}|j_i||\eta_i^\sigma||m\eta_i e^{m\eta_i} - e^{m\eta_i} + 1|\right] + \frac{|\lambda||\delta^\sigma\widehat{\chi}_2|}{|m|}\left|m\delta(e^{m\delta}+1) + 2(1 - e^{m\delta})\right|\right\}, \tag{31}$$

$$\Delta_2 = \max_{t\in[0,1]}|e^{mt}| + \widehat{\chi}_1(|e^{m\xi}| + \sum_{i=1}^{n}|j_i||e^{m\eta_i}|) + \widehat{\chi}_2\left(\frac{|me^m| + |\lambda||e^{m\delta} + 1|}{|m|}\right).$$

Now we present our main results for problem (1) and (2) with $q_1^2 - 4q_0q_2 = 0$. Since the methods for proof of these results are similar to the ones obtained in Section 3, so we omit the proofs.

Theorem 6. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the conditions (G_1)-(G_3). Then the problem (1) and (2) with $q_1^2 - 4q_0q_2 = 0$, has at least one solution on $[0,1]$ if*

$$L\Delta_2 < 1, \tag{32}$$

where Δ_2 is given by (31).

Theorem 7. *Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that (G_1) is satisfied. Then there exists a unique solution for problem (1) and (2) with $q_1^2 - 4q_0q_2 = 0$, on $[0,1]$ if $\ell\beta + L\Delta_2 < 1$, where β and Δ_2 are given by (31).*

Theorem 8. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the problem (1) and (2) with $q_1^2 - 4q_0q_2 = 0$, has at least one solution on $[0,1]$, if (H_1), (H_2) and the following condition hold:*

(H_3') *There exists a constant $K_1 > 0$ such that*

$$\frac{(1 - L_1\Delta_2)K_1}{\|g\|\psi(K_1)\beta} > 1,$$

    where β is defined by (31).

Example 3. *Consider the sequential fractional differential equation*

$$(2\,{}^cD^{12/5} + 4\,{}^cD^{7/5} + 2\,{}^cD^{2/5})x(t) = \frac{|x|}{(t+6)(|x|+1)} + e^{-t},\ 0 < t < 1, \tag{33}$$

subject the boundary conditions (27).

Here

$$f(t,x) = \frac{|x|}{(t+6)(|x|+1)} + e^{-t}.$$

Clearly $q_1^2 - 4q_0q_2 = 0$, and

$$|f(t,x) - f(t,y)| \le \frac{1}{6}|x-y|,$$

$$|h(x) - h(y)| \le \frac{1}{9}\|x-y\|.$$

where $\ell = 1/6$, $L = 1/9$. Using the given values, we find that $\beta \approx 0.29913$, $\beta_1 \approx 0.15022$ and $\Delta_2 \approx 5.135$.

It is easy to check that $|f(t,x)| \le \frac{B}{t+6} + e^{-t} = \vartheta(t)$ and $L\Delta_2 < 1$. As all the conditions of Theorem 6 are satisfied, the problem (27)–(33) has at least one solution on $[0,1]$. On the other hand, $\ell\beta + L\Delta_2 < 1$ and thus there exists a unique solution for the problem (27)–(33) on $[0,1]$ by Theorem 7.

5. Existence Results for Problem (1) and (2) with $q_1^2 - 4q_0q_2 < 0$

In view of Lemma 4, we can transform problem (1) and (2) into equivalent fixed point problem as follows:

$$x = \mathcal{K}x, \tag{34}$$

where the operator $\mathcal{K} : \mathcal{C} \to \mathcal{C}$ is defined by

$$
\begin{aligned}
(\mathcal{K}x)(t) = {} & \frac{1}{q_2 b} \Bigg\{ \int_0^t \int_0^s \mathcal{F}(t) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(u, x(u)) du\, ds \\
& + \tau_1(t) \Big[\int_0^\xi \int_0^s \mathcal{F}(\xi) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} y(u) du\, ds \\
& - \sum_{i=1}^n j_i \int_0^{\eta_i} \int_0^s \mathcal{F}(\eta_i) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(u, x(u)) du\, ds \Big] \\
& + \tau_2(t) \Big[\int_0^1 \int_0^s \mathcal{F}(1) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(u, x(u)) du\, ds \\
& - \frac{\lambda}{a^2 + b^2} \int_0^\delta \int_0^s \Big(b - b e^{-a(\delta-s)} \cos b(\delta - s) \\
& - a e^{-a(\delta-s)} \sin b(\delta - s) \Big) \frac{(s-u)^{\sigma-1}}{\Gamma(\sigma)} f(u, x(u)) du\, ds \Big] \Bigg\} \\
& + h(x) \Big[e^{-at} \cos bt + \tau_1(t) (e^{-a\xi} \cos b\xi - \sum_{i=1}^n j_i e^{-a\eta_i} \cos b\eta_i) \\
& + \tau_2(t) (e^{-a} \cos b - \frac{\lambda}{a^2 + b^2} (a - a e^{-a\delta} \cos b\delta + b e^{-a\delta} \sin b\delta)) \Big],
\end{aligned}
$$

where $\mathcal{F}(\cdot)$, $\tau_1(t)$ and $\tau_2(t)$ are defined by (17). We set

$$
\begin{aligned}
\widehat{\tau}_1 = {} & \max_{t \in [0,1]} |\tau_1(t)|, \qquad \widehat{\tau}_2 = \max_{t \in [0,1]} |\tau_2(t)| \\
\gamma = {} & \frac{1}{|q_2 b(a^2 + b^2)| \Gamma(\sigma + 1)} \Big\{ (1 + \widehat{\tau}_2) \Big[|b - b e^{-a} \cos b - a e^{-a} \sin b| \Big] \\
& + \widehat{\tau}_1 \Big[\xi^\sigma |b - b e^{-a\xi} \cos b\xi - a e^{-a\xi} \sin b\xi| + \sum_{i=1}^n |j_i| \eta_i^\sigma |b - b e^{-a\eta_i} \cos b\eta_i \\
& - a e^{-a\eta_i} \sin b\eta_i| \Big] + |\lambda| \delta^\sigma \widehat{\tau}_2 \Big[|b\delta - e^{-a\delta} \sin b\delta| \Big] \Big\}, \\
\Delta_3 = {} & \max_{t \in [0,1]} |e^{-at} \cos bt| + \widehat{\tau}_1 (|e^{-a\xi} \cos b\xi| + \sum_{i=1}^n |j_i| |e^{-a\eta_i} \cos b\eta_i|) \\
& + \widehat{\tau}_2 \Big(|e^{-a} \cos b| + \frac{|\lambda|}{a^2 + b^2} (|a - a e^{-a\delta} \cos b\delta + b e^{-a\delta} \sin b\delta|) \Big).
\end{aligned} \tag{35}
$$

Here are the existence and uniqueness results for problem (1) and (2) with $q_1^2 - 4q_0q_2 < 0$. As argued in the last section, we do not provide the proofs for these results.

Theorem 9. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the conditions (G_1)–(G_3). Then the problem (1) and (2) with $p_1^2 - 4p_0p_2 < 0$, has at least one solution on $[0, 1]$ if*

$$L\Delta_3 < 1, \tag{36}$$

where γ_1 and Δ_3 are given by (35).

Theorem 10. *Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that (G_1) and (G_2) are satisfied. Then there exists a unique solution for the problem (1) and (2) with $q_1^2 - 4q_0q_2 < 0$, on $[0,1]$ if $\ell\gamma + L\Delta_3 < 1$, where γ and Δ_3 are given by (35).*

Theorem 11. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the problem (1) and (2) with $q_1^2 - 4q_0q_2 < 0$, has at least one solution on $[0,1]$, if (H_1), (H_2) and the following condition are satisfied:*

(H_3'') *There exists a constant $K_2 > 0$ such that*

$$\frac{(1 - L_1\Delta_3)K_2}{\|g\|\psi(K_2)\gamma} > 1,$$

where γ and Δ_3 are defined by (35).

Example 4. *Consider the following boundary value problem*

$$(2\,{}^cD^{12/5} + 3\,{}^cD^{7/5} + 2\,{}^cD^{2/5})x(t) = \frac{1}{(t+4)^2}\cos x + \frac{e^{-2t}}{13}, \quad 0 < t < 1, \tag{37}$$

subject the boundary condition

$$x(0) = \frac{1}{8}x(\hat{t}), \quad x(1/5) = x(1/4) + 2x(1/3) + x(1/2), \quad x(1) = 2\int_0^{1/6} x(s)ds. \tag{38}$$

Here, $\sigma = 2/5$, $\xi = 1/5$, $\eta_1 = 1/4$, $\eta_2 = 1/3$, $\eta_3 = 1/2$, $\delta = 1/6$, $j_1 = 1$, $j_2 = 2$, $j_3 = 1$, $\lambda = 2$, $\hat{t}$ is a fixed value in $[0,1]$ and

$$f(t,x) = \frac{1}{(t+4)^2}\cos x + \frac{e^{-2t}}{13}.$$

Clearly $q_1^2 - 4q_0q_2 = -7 < 0$, and

$$|f(t,x) - f(t,y)| \le \frac{1}{16}|x - y|,$$

$$|h(x) - h(y)| \le \frac{1}{8}\|x - y\|,$$

where $\ell = 1/16$, $L = 1/8$. Using the given values, it is found that $\gamma \approx 0.34744$, $\gamma_1 \approx 0.17937$ and $\Delta_3 \approx 1.8499$.

Obviously $|f(t,x)| \le \dfrac{1}{(t+4)^2} + \dfrac{e^{-2t}}{13} = \vartheta(t)$ and $L\Delta_3 < 1$. As the hypothesis of Theorem 9 holds true, the problem (37) and (38) has at least one solution on $[0,1]$. Furthermore, we have $\ell\gamma + L\Delta_3 < 1$, which implies that there exists a unique solution for the problem (37) and (38) on $[0,1]$ by Theorem 10.

6. Conclusions

We have presented a detailed analysis for a multi-term fractional differential equation supplemented with nonlocal multi-point integral boundary conditions. The existence and uniqueness results are given for all three cases depending on the coefficients of the multi-term fractional differential equation: (i) $q_1^2 - 4q_0q_2 > 0$, (ii) $q_1^2 - 4q_0q_2 = 0$ and (iii) $q_1^2 - 4q_0q_2 < 0$. Existence results are proved by means of Krasnoselskii fixed point theorem and Leray–Schauder nonlinear alternative, while Banach contraction mapping principle is applied to establish the uniqueness of solutions for the given problem. The obtained results are well-illustrated with examples. Our results are new and enrich the literature on nonlocal integro-multipoint boundary problems for multi-term Caputo type fractional differential equations.

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