axioms

MDPI

Article

On a New Generalized Integral Operator and Certain Operating Properties

Paulo M. Guzman [1,2,†], Luciano M. Lugo [1,†], Juan E. Nápoles Valdés [1,†] and Miguel Vivas-Cortez [3,*,†]

1 FaCENA, UNNE, Av. Libertad 5450, Corrientes 3400, Argentina; paulomatiasguzman@hotmail.com (P.M.G.); lmlmb@yahoo.com.ar (L.M.L.); jnapoles@exa.unne.edu.ar (J.E.N.V.)
2 Facultad de Ingeniería, UNNE, Resistencia, Chaco 3500, Argentina
3 Facultad de Ciencias Exactas y Naturales, Escuela de Ciencias Físicas y Matemática, Pontificia Universidad Católica del Ecuador, Quito 170143, Ecuador
* Correspondence: mjvivas@puce.edu.ec
† These authors contributed equally to this work.

Received: 20 April 2020; Accepted: 22 May 2020; Published: 20 June 2020



Abstract: In this paper, we present a general definition of a generalized integral operator which contains as particular cases, many of the well-known, fractional and integer order integrals.

Keywords: integral operator; fractional calculus

1. Preliminars

Integral Calculus is a mathematical area with so many ramifications and applications, that the sole intention of enumerating them makes the task practically impossible. Suffice it to say that the simple procedure of calculating the area of an elementary figure is a simple case of this topic. If we refer only to the case of integral inequalities present in the literature, there are different types of these, which involve certain properties of the functions involved, from generalizations of the known Mean Value Theorem of classical Integral Calculus, to varied inequalities in norm.

Let $(U, \sum_U, v)$ and $(V, \sum_V, \mu)$ be σ-finite measure spaces, and let $(W, \sum_W, \lambda)$ be the product of these spaces, thus W = UxV and $\lambda = v$ x μ. In a general sense, an operator $I : A \to B$ is called integral if there exists a λ-measurable function $K(u, v)$ (with $u \in U, v \in V$) such that, for every $x \in A$ the image $y = I(t) \in B$ of x is $y(u) = \int_V K(u, v)x(v)d\mu(v)$ being U denotes an FS on $(V, \sum_V, \mu)$ and V is an FS on $(U, \sum_U, v)$. The function $K(u, v)$ is called the kernel of I. It is easy to see that the kernel is finite almost everywhere relative to λ.

In particular, we will deal with real integral operators defined on $\mathbb{R}$.

It is known that from the XVIIth century the study of problems dealing with derivatives and integrals of fractional order began. The first works that are registered deal with this subject from a theoretical point of view, however over time, until today, its applicability is undeniable. Important mathematicians, such as Euler, Laplace, Fourier, Abel, Liouville and Riemann, worked on this topic (see [1]).

Fractional derivative and fractional integral are generalizations of those always present in the ordinary calculation, considering derivatives of real or complex arbitrary order and a general form for multiple integrals. The principle used to find models for fractional derivatives has been to define, first, a fractional integral. Applicability in areas such as physics, engineering, biology, has managed to establish its usefulness and many important results have appeared in the literature. New definitions of differential and integral fractional operators have emerged in recent decades, and around this many researchers wonder what type of operator to choose from a possible problem considered. An attractive characteristic of this field is that there are numerous fractional operators, and this permits researchers to

choose the most appropriate operator for the sake of modeling the problem under investigation. In [2] a fairly complete classification of these fractional operators is presented, with abundant information, on the other hand, in the work [3] some reasons are presented why new operators linked to applications and developments theorists appear every day. These operators had been developed by numerous mathematicians with a barely specific formulation, for instance, the Riemann–Liouville (RL), the Weyl, Erdelyi–Kober, Hadamard integrals and the Liouville and Katugampola fractional operators and many authors have introduced new fractional operators generated from general classical local derivatives.

In addition, Section 1 of [4] presents a history of differential operators, both local and global, from Newton to Caputo and presents a definition of local derivative with new parameter, providing a large number of applications, with a difference qualitative between both types of operators, local and global. Most importantly, Section 1.4 (p. 24), dealing with limitations and strength of local and fractional derivatives, concludes: "We can therefore conclude that both the Riemann–Liouville and Caputo operators are not derivatives, and then they are not fractional derivatives, but fractional operators. We agree with the result that, the local fractional operator is not a fractional derivative". As we said before, they are new tools that have demonstrated their usefulness and potential in the modelling of different processes and phenomena.

In the literature many different types of fractional operators have been proposed, here we show that various of that different notions of derivatives can be considered particular cases of our definition and, even more relevant, that it is possible to establish a direct relationship between global (classical) and local derivatives, the latter not very accepted by the mathematical community, under two arguments: their local character and compliance with the Leibniz Rule. However, in the works [5–9], various results related to the existence and uniqueness of solutions of fractional differential equations and integral equations of the Volterra and Volterra–Fredholm type are investigated, within the framework of classical fractional derivatives, that is, using global operators, although they are important results because their applicability, they are not related to the operators used in our work that are of local type.

To facilitate the understanding of the scope of our definition, we present the best known definitions of integral operators and their corresponding differential operators (for more details you can consult [10]). Without many difficulties, we can extend these definitions, for any higher order.

We assume that the reader is familiar with the classic definition of the Riemann Integral, so we will not present it.

One of the first operators that can be called fractional is that of Riemann–Liouville fractional derivatives of order $\alpha \in \mathbb{C}, Re(\alpha) \geq 0$, defined by (see [11]).

Definition 1. *Let $f \in L^1((a,b);\mathbb{R}), (a,b) \in \mathbb{R}^2, a < b$. The right and left side Riemann–Liouville fractional integrals of order $\alpha > 0$ are defined by*

$$ {}^{RL}J^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, t > a \tag{1}$$

and

$$ {}^{RL}J^{\alpha}_{b^-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds, t < b. \tag{2}$$

and their corresponding differential operators are given by

$$ D^{\alpha}_{a^+} f(t) = \frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{a^+} f(t)\right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(t)}{(t-s)^{\alpha}} ds$$

$$ D^{\alpha}_{b^-} f(t) = -\frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{b^-} f(t)\right) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(t)}{(s-t)^{\alpha}} ds$$

Other definitions of fractional operators are as follows.

Definition 2. *Let $f \in L^1((a,b);\mathbb{R}), (a,b) \in \mathbb{R}^2, a < b$. The right and left side Hadamard fractional integrals of order α with $Re(\alpha) > 0$ are defined by*

$$H^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (log\frac{t}{s})^{\alpha-1} \frac{f(s)}{s} ds, \quad a < t < b, \tag{3}$$

and

$$H^{\alpha}_{b^-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (log\frac{s}{t})^{\alpha-1} \frac{f(s)}{s} ds, \quad a < t < b. \tag{4}$$

Hadamard differential operators are given by the following expressions.

$$({}^H D^{\alpha}_{a^+} f)(t) = t\frac{d}{dt}\left(H^{\alpha}_{a^+} f(t)\right) = \frac{-\Gamma(\alpha+1)}{B(\alpha,1-\alpha)} \int_a^t (log\frac{t}{s})^{-\alpha-1} \frac{f(s)}{s} ds, \quad a < t < b$$
$$({}^H D^{\alpha}_{b^-} f)(t) = -t\frac{d}{dt}\left(H^{\alpha}_{b^-} f(t)\right) = -\frac{\Gamma(\alpha+1)}{B(\alpha,1-\alpha)} \int_t^b (log\frac{s}{t})^{-\alpha-1} \frac{f(s)}{s} ds, \quad a < t < b$$

In [12], the author introduced new fractional integral operators, called the Katugampola fractional integrals, in the following way:

Definition 3. *Let $0 < a < b < +\infty$, $f : [a,b] \to \mathbb{R}$ is an integrable function, and $\alpha \in (0,1)$ and $\rho > 0$ two fixed real numbers. The right and left side Katugampola fractional integrals of order α are defined by*

$$K^{\alpha,\rho}_{a^+} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} f(s) ds, \quad a < t \tag{5}$$

and

$$K^{\alpha,\rho}_{b^-} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{t^{\rho-1}}{(s^\rho - t^\rho)^{1-\alpha}} f(s) ds, \quad t < b. \tag{6}$$

In [13], it appeared a generalization to the Riemann–Liouville and Hadamard fractional derivatives, called the Katugampola fractional derivatives:

$$(D^{\alpha}_{a^+} f)(t) = \frac{\rho^{\alpha}}{\Gamma(1-\alpha)} t^{1-\rho} \frac{d}{dt} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{\alpha}} f(s) ds, \quad a < t,$$
$$(D^{\alpha,\rho}_{b^-} f)(t) = \frac{-\rho^{\alpha}}{\Gamma(1-\alpha)} t^{1-\rho} \frac{d}{dt} \int_a^t \frac{s^{\rho-1}}{(s^\rho - t^\rho)^{\alpha}} f(s) ds, \quad t < b.$$

The relation between these two fractional operators is the following:

$$(D^{\alpha,\rho}_{a^+} f)(t) = t^{1-\rho} \frac{d}{dt} K^{1-\alpha,\rho}_{a^+} f(t), (D^{\alpha,\rho}_{b^-} f)(t) = -t^{1-\rho} \frac{d}{dt} K^{1-\alpha,\rho}_{b^-} f(t).$$

There are other definitions of integral operators in the global case, but they can be slight modifications of the previous ones, some include non-singular kernel and others incorporate different terms. In the local case, there are different types of operators, but their definition is much more obvious in the case of our definition.

In our work we are interested in presenting a generalization of these integral operators and applying to different known inequalities.

2. A New Fractional Integral Operator

In [14] a generalized fractional derivative was defined in the following way.

Definition 4. *Given a function $f : [0, +\infty) \to \mathbb{R}$. Then the N-derivative of f of order α is defined by*

$$N_F^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon F(t, \alpha)) - f(t)}{\varepsilon} \tag{7}$$

for all $t > 0$, $\alpha \in (0, 1)$ being $F(\alpha, t)$ is some function.

Remark 1. *Here we will use some cases of F defined in function of $E_{a,b}(.)$ the classic definition of Mittag-Leffler function with $Re(a), Re(b) > 0$. In addition, we consider $E_{a,b}(.)_k$ is the k-nth term of $E_{a,b}(.)$.*

If f is α-differentiable in some $(0, \alpha)$, and $\lim_{t \to 0^+} N_F^{\alpha} f(t)$ exists, then define $N_F^{\alpha} f(0) = \lim_{t \to 0^+} N_F^{\alpha} f(t)$, note that if f is differentiable, then $N_F^{\alpha} f(t) = F(t, \alpha) f'(t)$ where $f'(t)$ is the ordinary derivative.

The classic Mittag-Leffler function plays an active role in fractional calculus, and is defined by

$$\mathbf{E}_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \alpha k)}, \quad \alpha \in C, \quad Re(\alpha) > 0, \quad z \in \mathbb{C},$$

where Γ is the well known Gamma function. The original function $E_{\alpha,1}(z) = E_{\alpha}(z)$ was defined and studied by Mittag-Leffler in the year 1903, that is, a uniparameter function, see [15,16]. It is a direct generalization of the exponential function. Wiman proposed and studied a generalization of the role of Mittag-Leffler, who will call the Mittag–Leffler function with two parameters $E_{\alpha,\beta}(z)$, (see [17]). Agarwal in 1953 [18] and Humbert and Agarwal in 1953 [19], also made contributions to the final formalization of this function, see also [20]. In 1971, Prabhakar in [21] introduced the function $E_{\alpha,\beta}^{\gamma}(z)$ in the form of

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \quad \alpha, \beta, \gamma \in C; \quad Re(\alpha), Re(\beta), Re(\gamma) > 0, \quad z \in C,$$

where $(\gamma)_k$ is the Pochhammer symbol (see [22]).

Gorenflo et al. (see [11,23]) and Kilbas and Saigo (see [24,25]) investigated several properties and applications of the original Mittag-Leffler function and its generalizations.

Assigning some particular values to the parameter α, one obtains some interesting cases of Mittag-Leffler function $E_{\alpha}(z)$:

1. $E_0(z) = \frac{1}{1-z}, \ |z| < 1$
2. $E_1(z) = e^z$
3. $E_1(iz) = e^{iz}$
4. $E_2(z) = \cosh(\sqrt{z}), \ z \in C$
5. $E_2(-z^2) = \cos z, \ z \in C$
6. $E_3(z) = \frac{1}{2}\left[e^{z^{\frac{1}{3}}} + 2e^{-\frac{1}{2}z^{\frac{1}{3}}} \cos\left(\frac{\sqrt{3}}{2} z^{\frac{1}{3}}\right)\right]$
7. $E_4(z) = \frac{1}{2}\left[\cos(z^{\frac{1}{4}}) + \cosh(z^{\frac{1}{4}})\right]$

Taking into consideration the Mittag–Leffler function of two parameters $E_{\alpha,\beta}$, we have the following examples:

(I) $F(t, \alpha) \equiv 1$, in this case we have the ordinary derivative.

(II) $F(t, \alpha) = E_{1,1}(t^{-\alpha})$. In this case we obtain, from Definition 4, the non conformable derivative $N_1^{\alpha} f(t)$ defined in [26] (see also [27]).

(III) $F(t, \alpha) = E_{1,1}((1 - \alpha)t) = e^{(\alpha-1)t}$, this kernel satisfies that $F(t, \alpha) \to 1$ as $\alpha \to 1$, a conformable derivative used in [28].

(IV) $F(t, \alpha) = E_{1,1}(t^{1-\alpha})_1 = t^{1-\alpha}$, with this kernel we have $F(t, \alpha) \to 0$ as $\alpha \to 1$ (see [29]), a conformable derivative.

(V) $F(t,\alpha) = E_{1,1}(t^{-\alpha})_1 = t^{\alpha}$, with this kernel we have $F(t,\alpha) \to t$ as $\alpha \to 1$ (see [30]). It is clear that since it is a non-conformable derivative, the results will differ from those obtained previously, which enhances the study of these cases.

(VI) $F(t,\alpha) = E_{1,1}(t^{-\alpha})_1 = t^{-\alpha}$, with this kernel we have $F(t,\alpha) \to t^{-1}$ as $\alpha \to 1$. This is the derivative N_3^{α} studied in [14]. As in the previous case, the results obtained have not been reported in the literature.

Now, we give the definition of a general fractional integral. Throughout the work we will consider that the integral operator kernel T defined below is an absolutely continuous function.

Definition 5. *Let I be an interval $I \subseteq \mathbb{R}$, $a, t \in I$ and $\alpha \in \mathbb{R}$. The integral operator $J^a_{T,a+}$, right and left, is defined for every locally integrable function f on I as*

$$J^a_{T,a+}(f)(t) = \int_a^t \frac{f(s)}{T(t-s,\alpha)} ds, t > a. \tag{8}$$

$$J^a_{T,b-}(f)(t) = \int_t^b \frac{f(s)}{T(s-t,\alpha)} ds, b > t. \tag{9}$$

Remark 2. *Sometimes, the kernel of the integral operator may not be the same as the derivative operator, from the theoretical point of view it does not affect it; in fact what it does is complicate expressions and some elementary properties.*

Remark 3. *It is easy to see that the case of the J^{α}_{T} operator defined above contains, as particular cases, the integral operators obtained from conformable and non-conformable local derivatives. However, we will see that it goes much further by containing the cases listed at the beginning of the work. We have*
(1) If $F(t,\alpha) = t^{1-\alpha}$, $T(t,\alpha) = \Gamma(\alpha)F(t-s,\alpha)$, from Equation (8) we have the right side Riemann–Liouville fractional integrals $(R^{\alpha}_{a+}f)(t)$, similarly from Equation (9) we obtain the left derivative of Riemann–Liouville. Then its corresponding right differential operator is

$$({}^{RL}D^{\alpha}_{a^+}f)(t) = \frac{d}{dt}(R^{1-\alpha}_{a+}f)(t),$$

analogously we obtain the left.
(2) With $F(t,\alpha) = t^{1-\alpha}$, $T(t-s,\alpha) = \Gamma(\alpha)F(\log(t) - \log(s),\alpha)t$, we obtain the right Hadamard integral from Equation (8), the left Hadamard integral is obtained similarly from Equation (9). The right derivative is

$$({}^{H}D^{\alpha}_{a^+}f)(t) = t\frac{d}{dt}(H^{1-\alpha}_{a+}f)(t),$$

in a similar way we can obtain the left.
(3) The right Katugampola integral is obtained from Equation (8) making

$$F(t,\alpha) = t^{1-\alpha}, \quad e(t) = t^{\varrho}, \quad T(t,\alpha) = \frac{\Gamma(\alpha)}{F(\rho,\alpha)}\frac{F(e(t)-e(s),\alpha)}{e'(s)},$$

analogously for the left fractional integral. In this case, the right derivative is

$$({}^{K}D^{\alpha,\rho}_{a^+}f)(t) = t^{1-\rho}\frac{d}{dt}K^{1-\alpha,\rho}_{a^+}f(t) = F(t,\rho)\frac{d}{dt}K^{1-\alpha,\rho}_{a^+}f(t),$$

and we can obtain the left derivative in the same way.
(4) The solution of equation $(-\Delta)^{-\frac{\alpha}{2}}\phi(u) = -f(u)$ called Riesz potential, is given by the expression $\phi = C^{\alpha}_{n}\int_{R^n}\frac{f(v)}{|u-v|^{n-\alpha}}dv$, where C^{α}_{n} is a constant (see [31–33]). Obviously, this solution can be expressed in terms of the operator in Equation (8) very easily.

(5) Obviously, we can define the lateral derivative operators (right and left) in the case of our generalized derivative, for this it is sufficient to consider them from the corresponding integral operator. To do this, just make use of the fact that if f is differentiable, then $N_F^\alpha f(t) = F(t,\alpha)f'(t)$ where $f'(t)$ is the ordinary derivative. For the right derivative we have $\left(N_{F,a+}^\alpha f\right)(t) = N_F^\alpha\left[J_{T,a+}^\alpha(f)(t)\right] = \frac{d}{dx}\left[J_{T,a+}^\alpha(f)(t)\right]F(x,\alpha)$, similarly to the left.
(6) It is clear then, that from our definition, new extensions and generalizations of known integral operators can be defined. For example, in [34] they presented the definition of the fractional integral of f with respect to g of the following way. Let $g:[a,b]\to\mathbb{R}$ be an increasing and positive monotone function on $(a,b]$ having a continuous derivative $g'(t)$ on (a,b). The left-sided fractional integral of f with respect to the function g on $[a,b]$ of order $\alpha>0$ is defined by

$$I_{g,a+}^\alpha(f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{g'(s)f(s)}{[g(t)-g(s)]^{1-\alpha}}ds, \quad t>a, \tag{10}$$

similarly, the right lateral derivative is defined as well

$$I_{g,b-}^\alpha(f)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \frac{g'(s)f(s)}{[g(s)-g(t)]^{1-\alpha}}ds, \quad t<b. \tag{11}$$

It will be very easy for the reader to build the kernel T in this case.

(7) A k-analogue of the above definition is defined in [35] (also see [36]), under the same assumptions on function g

$$I_{g,a+}^{\alpha,k}(f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{g'(s)f(s)}{[g(t)-g(s)]^{1-\frac{\alpha}{k}}}ds, \quad t>a, \tag{12}$$

similarly, the right lateral derivative is defined as well

$$I_{g,b-}^{\alpha,k}(f)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \frac{g'(s)f(s)}{[g(s)-g(t)]^{1-\frac{\alpha}{k}}}ds, \quad t<b. \tag{13}$$

The corresponding differential operator is also very easy to obtain.
(8) We can define the function space $L_\alpha^p[a,b]$ as the set of functions over $[a,b]$ such that $(J_{T,a+}^\alpha[f(t)]^p(b))<+\infty$.

Proposition 1. *Let I be an interval $I\subseteq\mathbb{R}$, $a\in I$, $0<\alpha\leq 1$ and f an α-differentiable function on I such that f' is a locally integrable function on I. Then, we have for all $t\in I$*

$$J_{F,a+}^\alpha\left(N_F^\alpha(f)\right)(t) = f(t)-f(a).$$

Proof. Since f' is a locally integrable function on I, from [27] we have

$$J_{F,a+}^\alpha\left(N_F^\alpha(f)\right)(t) = \int_a^t \frac{N_F^\alpha(f)(s)}{F(s,\alpha)}ds = \int_a^t f'(s)ds = f(t)-f(a),$$

which is the desired result. □

Proposition 2. *Let I be an interval $I\subseteq\mathbb{R}$, $a\in I$ and $\alpha\in(0,1]$.*

$$N_F^\alpha\left(J_{F,a+}^\alpha(f)\right)(t) = f(t),$$

for every continuous function f on I and $a,t\in I$.

Proof. Let f be a continuous function f on I. Proposition 1 gives for every $a, t \in I$

$$\left(J^{\alpha}_{F,a}(f)(t)\right)' = \left(\int_a^t \frac{f(s)}{F(s,\alpha)}\, ds\right)' = \frac{f(t)}{F(t,\alpha)},$$
$$N^{\alpha}_F J^{\alpha}_{F,a}(f)(t) = F(t,\alpha)\left(J^{\alpha}_{F,a}(f)(t)\right)' = F(t,\alpha)\,\frac{f(t)}{F(t,\alpha)} = f(t).$$

□

In [29] it is defined the integral operator $J^{\alpha}_{F,a}$ for the choice of the function F given by $F(t,\alpha) = t^{1-\alpha}$, and Theorem 3.1, in the same work, shows

$$N^{\alpha} J^{\alpha}_{t^{1-\alpha},a}(f)(t) = f(t),$$

for every continuous function f on I, $a, t \in I$ and $\alpha \in (0,1]$. Hence, Proposition 2 extends to any F this important equality.

The following result summarizes some elementary properties of the integral operator $J^{\alpha}_{T,a+}$.

Theorem 1. *Let I be an interval $I \subseteq \mathbb{R}$, $a, b \in I$ and $\alpha \in \mathbb{R}$. Suppose that f, g are locally integrable functions on I, and $k_1, k_2 \in \mathbb{R}$. Then we have*

(1) $J^{\alpha}_{T,a+}\left(k_1 f + k_2 g\right)(t) = k_1 J^{\alpha}_{T,a+} f(t) + k_2 J^{\alpha}_{T,a+} g(t)$,
(2) *if $f \geq g$, then $J^{\alpha}_{T,a+} f(t) \geq J^{\alpha}_{T,a+} g(t)$ for every $t \in I$ with $t \geq a$,*
(3) $\left|J^{\alpha}_{T,a+} f(t)\right| \leq J^{\alpha}_{T,a+} |f|\,(t)$ *for every $t \in I$ with $t \geq a$,*
(4) $\int_a^b \frac{f(s)}{T(s,\alpha)} ds = J^{\alpha}_{T,a+} f(t) - J^{\alpha}_{T,b-} f(t) = J^{\alpha}_{T,a+} f(t)(b)$ *for every $t \in I$.*

Proof. (1). Let us note that

$$\begin{aligned} J^{\alpha}_{F,a+}(k_1 f + k_2 g)(t) &= \int_a^t \frac{(k_1 f + k_2 g)(s)}{F\,(s,\alpha)} ds \\ &= \int_a^t \frac{k_1 f(s) + k_2 g(s)}{F\,(s,\alpha)} ds \\ &= k_1 \int_a^t \frac{f(s)}{F\,(s,\alpha)} ds + k_2 \int_a^t \frac{g(s)}{F\,(s,\alpha)} ds \\ &= k_1 J^{\alpha}_{F,a+} f(t) + k_2 J^{\alpha}_{F,a+} g(t). \end{aligned}$$

(2) For all $t \in I$ and $f(t) \geq g(t)$, as $F(t,\alpha) > 0$ then,

$$\begin{aligned} \frac{f(t)}{F(t,\alpha)} &\geq \frac{g(t)}{F(t,\alpha)} \\ \int_a^t \frac{f(s)}{F(s,\alpha)} ds &\geq \int_a^t \frac{g(s)}{F(s,\alpha)} ds \\ J^{\alpha}_{F,a+} f(t) &\geq J^{\alpha}_{F,a+} g(t) \end{aligned}$$

(3) Additionally,

$$\begin{aligned} \left|J^{\alpha}_{F,a+} f(t)\right| &= \left|\int_a^t \frac{f(s)}{F\,(s,\alpha)} ds\right| \\ &\leq \int_a^t \left|\frac{f(s)}{F\,(s,\alpha)}\right| ds \\ &\leq J^{\alpha}_{F,a+} |f(t)| \end{aligned}$$

(4) Finally, for $a \leq t \leq b$, we have

$$\begin{aligned}\int_a^b \frac{f(s)}{F(s,\alpha)}ds &= \int_a^t \frac{f(s)}{F(s,\alpha)}ds + \int_t^b \frac{f(s)}{F(s,\alpha)}ds \\ &= \int_a^t \frac{f(s)}{F(s,\alpha)}ds - \int_b^t \frac{f(s)}{F(s,\alpha)}ds \\ &= J^{\alpha}_{F,a+}f(t) - J^{\alpha}_{F,b-}f(t)\end{aligned}$$

□

Let $C^1[a,b]$ be the set of functions f with first ordinary derivative continuous on $[a,b]$, we consider the following norms on $C^1[a,b]$:

$$\|F\|_C = \max_{[a,b]} |f(t)|, \quad \|F\|_{C^1} = \left\{ \max_{[a,b]} |f(t)| + \max_{[a,b]} \left|f'(t)\right| \right\}$$

The Propositions 1 and 2 were obtained under the case that the kernel of both operators coincide (as is the case with local operators), we will give some results in the event that this does not happen.

Theorem 2. *For a function $f \in C^1[a,b]$ and $x \in [a,b]$, we have*

$$\left|N^{\alpha}_{F,a+}f(t)\right| \leq K(\alpha)\|F\|_C \max_{t\in[a,x]} |f(t)|. \tag{14}$$

$$\left|N^{\alpha}_{F,b-}f(t)\right| \leq K(\alpha)\|F\|_C \max_{t\in[x,b]} |f(t)|. \tag{15}$$

Remark 4. *The constant $K(\alpha)$ of the theorem can depend on other parameters, as in the case of the Katugampola operator, where ρ will appear.*

Proof. It is easily obtained from the previous definitions. □

Theorem 3. *The fractional derivatives $N^{\alpha}_{F,a+}f(t)$ and $N^{\alpha}_{F,b-}f(t)$ are bounded operators from $C^1[a,b]$ to $C[a,b]$ with*

$$\left|N^{\alpha}_{F,a+}f(t)\right| \leq K\|F\|_C\|f\|_{C^1}, \tag{16}$$

$$\left|N^{\alpha}_{F,b-}f(t)\right| \leq K\|F\|_C\|f\|_{C^1}, \tag{17}$$

where the constant K may be dependent on the derivative frame considered.

Proof. Given $x \in [a,b]$ and $f \in C^1[a,b]$, using simple properties of norm and previous theorem, the result follows. □

Remark 5. *From previous results we obtain that the derivatives $N^{\alpha}_{F,a+}f(t)$ and $N^{\alpha}_{F,b-}f(t)$ are well defined.*

Theorem 4. *Let $\alpha, \beta \in (0,1]$, $f : [a,b] \to \mathbb{R}$ is a α-differentiable function. Then we have*

$$N^{\alpha}_{F,a+}(N^{\beta}_{F,a+}f(t)) = \frac{F(x,\alpha)\,F(x,\beta)}{T(x,\alpha)\,T(x,\beta)}f(x). \tag{18}$$

Proof.

$$N^{\alpha}_{F,a+}(N^{\beta}_{F,a+}f(t)) = N_{F,a+}\left[F(t,\beta)\frac{d}{dt}\int_a^t \frac{f(s)}{T(s,\beta)}ds\right](x) =$$
$$= N_{F,a+}\left[\frac{F(t,\beta)f(t)}{T(t,\beta)}\right](x) = F(x,\alpha)\frac{d}{dx}\int_a^x \frac{\frac{F(t,\beta)f(t)}{T(t,\beta)}}{T(t,\alpha)}dt =$$
$$= \frac{F(x,\alpha)F(x,\beta)}{T(x,\alpha)T(x,\beta)}f(x).$$

□

Corollary 1. *Under assumptions of the previous Theorem, if $\alpha \equiv \beta$ we obtain*

$$N^{\alpha}_{F,a+}(N^{\alpha}_{F,a+}f(t)) = \frac{F(x,\alpha)^2}{T(x,\alpha)^2}f(x). \tag{19}$$

Theorem 5. *(Integration by parts) Let $f, g : [a,b] \to \mathbb{R}$ differentiable functions and $\alpha \in (0,1]$. Then, the following property holds*

$$J^{\alpha}_{F,a+}((f)(N^{\alpha}_{F,a+}g(t))) = [f(t)g(t)]_a^b - J^{\alpha}_{F,a+}((g)(N^{\alpha}_{F,a+}f(t))). \tag{20}$$

Theorem 6. *If $f : [a,b] \to \mathbb{R}$ is a continuous function and $\alpha \in (0,1]$ then, the following inequality is fulfilled*

$$\left|J^{\alpha}_{F,a+}(f)(t)\right| \leq J^{\alpha}_{F,a+}|f|(t). \tag{21}$$

Proof.

$$\left|J^{\alpha}_{F,a+}(f)(t)\right| = \left|\int_a^x \frac{f(t)}{F(t,\alpha)}dt\right| \leq \int_a^x \frac{|f(t)|}{|F(t,\alpha)|}dt = \int_a^x \frac{|f(t)|}{F(t,\alpha)}dt = J^{\alpha}_{F,a+}|f|(t) \tag{22}$$

□

Theorem 7. *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ is a integrable function such that $M = \sup_{[a,b]}|f(t)|$. Then, on $[a,b]$ we have*

$$\left|J^{\alpha}_{F,a+}(f)(t)\right| \leq \frac{M}{m}(b-a). \tag{23}$$

Proof.

$$\left|J^{\alpha}_{F,a+}(f)(t)\right| \leq \int_a^x \frac{|f(t)|}{F(t,\alpha)}dt \leq M\int_a^x \frac{1}{F(t,\alpha)}dt \leq \frac{M}{m}(b-a)$$

where $m = \inf_{[a,b]}|F(t,\alpha)|$ □

Theorem 8. *Suppose that functions f and g satisfy the following assumptions on $[a,b]$:*

(1) f, g are integrable functions on $[a,b]$.
(2) Let g be a non-negative (or non-positive) function on $[a,b]$.
(3) Let $m = \inf_{[a,b]}|f(t)|$ and $M = \sup_{[a,b]}|f(t)|$.

Then, there exists a number $x_0 \in [a,b]$ such that $f(x_0) \in [m,M]$ and

$$J^{\alpha}_{F,a+}(fg)(t) = f(x_0)J^{\alpha}_{F,a+}(g)(t) \tag{24}$$

Theorem 9. *Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a given function that satisfies:*

(i) f is continuous on $[a,b]$,

(ii) *f is N-differentiable for some $\alpha \in (0,1)$.*

Then, we have that if $N_F^\alpha f(t) \geq 0$ (≤ 0) then f is a non-decreasing (increasing) function.

Analogously we have the following result

Theorem 10 (Racetrack Type Principle)**.** *Let $a > 0$ and $f, g : [a,b] \to \mathbb{R}$ be given functions satisfying:*

(i) *f and g are continuous on $[a,b]$,*
(ii) *f and g are N-differentiable for some $\alpha \in (0,1)$,*
(iii) *$N_F^\alpha f(t) \geq N_F^\alpha g(t)$ for all $t \in (a,b)$.*

Then, we have that following:

(I) *If $f(a) = g(a)$, then $f(t) \geq g(t)$ for all $t \in (a,b)$.*
(II) *If $f(b) = g(b)$, then $f(t) \leq g(t)$ for all $t \in (a,b)$.*

Proof. Consider the auxiliary function $h(t) = f(t) - g(t)$. Then h is continuous on $[a,b]$ and N-differentiable for some $\alpha \in (0,1)$. From here we obtain that $N_F^\alpha h(t) \geq 0$ for all $t \in (a,b)$, so by Theorem 9 h is a non increasing function. Hence, for any $t \in [a,b]$ we have that $h(a) \leq h(t)$ and since $h(a) = f(a) - g(a) = 0$ by assumption, the result follows. In a similar way the second part is proved. This concludes the proof. □

We will discuss the occurrence of local maxima and local minima of a function. In fact, these points are crucial to many questions related to application problems.

Definition 6. *A function f is said to have a local maximum at c iff there exists an interval I around c such that $f(c) \geq f(t)$ for all $x \in I$. Analogously, f is said to have a local minimum at c iff there exists an interval I around c such that $f(c) \leq f(t)$ for all $x \in I$. A local extremum is a local maximum or a local minimum.*

Remark 6. *As in the classic Calculus, if the function f is N-differentiable at a point c where it reaches an extreme, then $N_F^\alpha f(c) = 0$.*

Theorem 11 (Rolle's Theorem)**.** *Let $a > 0$, $f : [a,b] \to \mathbb{R}$ be a given function that satisfies*

(i) *$f \in C[a,b]$*
(ii) *f is N-differentiable on (a,b) for some $\alpha \in [0,1]$*
(iii) *$f(a) = f(b)$*

Then, there exists $c \in (a,b)$ such that $N_F^\alpha f(c) = 0$.

Proof. We prove this using contradiction. From assumptions, since f is continuous in $[a,b]$, and $f(a) = f(b)$, there is $c \in (a,b)$, at least one, which is a point of local extreme. By other hand, how f is N-differentiable in (a,b) for some α we have

$$\begin{aligned} N_F^\alpha f(c) &= N_F^\alpha f(c^+) = \lim_{h \to 0^+} \frac{f(c + hF(t,\alpha)) - f(c)}{h} \\ &= N_F^\alpha f(c^-) = \lim_{h \to 0^-} \frac{f(c + hF(t,\alpha)) - f(c)}{h} \end{aligned}$$

but $N_F^\alpha f(c^+)$ and $N_F^\alpha f(c^-)$ have opposite signs. Hence $N_F^\alpha f(c) = 0$. If $N_F^\alpha f(c^+)$ and $N_F^\alpha f(c^-)$ they have the same sign then as $f(a) = f(b)$, we have that f is constant and the result is trivially followed. This concludes the proof. □

Theorem 12 (Mean Value Theorem)**.** *Let $a > 0$, and $f : [a,b] \to \mathbb{R}$ be a function that satisfies*

(i) f *is continuous in* $[a,b]$

(ii) f *is N-differentiable on* (a,b)*, for some* $\alpha \in (0,1]$

Then, exists $c \in (a,b)$ *such that*

$$N_F^\alpha f(c) = \left[\frac{f(b)-f(a)}{b-\ a}\right] F(c,\alpha).$$

Proof. Consider the function

$$g(t) = f(t) - f(a) - \left[\frac{f(b)-f(a)}{b-\ a}\right](\ t-\ a).$$

The auxiliary function g satisfies all the conditions of Theorem 11 and, therefore, exists c in (a,b) such that $N_F^\alpha g(c) = 0$. Then, we have

$$N_F^\alpha g(t) = N_F^\alpha\left(f(t) - f(a)\right) - \frac{f(b)-f(a)}{b-a} N_F^\alpha\left(t-a\right)$$

and from here it follows that

$$N_F^\alpha g(c) = N_F^\alpha f(c) - \frac{f(b)-f(a)}{b-a} F(c,\alpha) = 0$$

from where

$$N_\alpha[f(c)] = \frac{f(b)-f(a)}{b-a} F(c,\alpha).$$

This concludes the proof. □

Theorem 13. *Let* $a > 0$ *and* $f : [a,b] \to \mathbb{R}$ *be a given function that satisfies:*

(i) f *is continuous on* $[a,b]$*,*

(ii) f *is N-differentiable for some* $\alpha \in (0,1)$*.*

If $N_F^\alpha f(t) = 0$ *for all* $t \in (a,b)$*, then* f *is a constant on* $[a,b]$*.*

Proof. It is sufficient to apply the Theorem 12 to the function f over any non-degenerate interval contained in $[a,b]$. □

As a consequence of the previous theorem we have

Corollary 2. *Let* $a > 0$ *and* $F, G : [a,b] \to \mathbb{R}$ *be functions such that for all* $\alpha \in (0,1)$*,* $N_F^\alpha F(t) = N_F^\alpha G(t)$ *for all* $t \in (a,b)$*. Then there exists a constant* C *such that* $F(t) = G(t) + C$*.*

Along the same lines of classic calculus, one can use the previous results to prove the following result.

Theorem 14. *Let* $f : [a,b] \to \mathbb{R}$ *be N-differentiable for some* $\alpha \in (0,1)$*. If*

(i) $N_F^\alpha f(t)$ *bounded on* $[a,b]$ *where* $a > 0$*, then* f *is uniformly continuous on* $[a,b]$ *and hence* f *is bounded.*

(ii) $N_F^\alpha f(t)$ *bounded on* $[a,b]$ *and continuous at* a *where* $a > 0$*, then* f *is uniformly continuous on* $[a,b]$ *and hence* f *is bounded.*

Theorem 15 (Extended Mean Value Theorem)**.** *Let* $\alpha \in (0,1]$ *and* $a > 0$*. If* $f, g : [a,b] \to \mathbb{R}$ *they are functions that satisfy*

(i) f, g *are continuous in* $[a,b]$*.*

(ii) f, g *are N-differentiable on* (a, b)*, for some* $\alpha \in (0, 1]$

(iii) $N_F^\alpha g(t) \neq 0$ *for all* $t \in (a, b)$.

Then, exists $c \in (a, b)$ *such that*

$$\frac{N_F^\alpha f(c)}{N_F^\alpha g(c)} = \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right].$$

Remark 7. *If* $g(t) = t$ *then this is just the statement of the Theorem 12.*

Proof. Let us now define a new function as follows

$$F(t) = f(t) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(t) - g(a))$$

Then the auxiliary function F satisfies the assumptions of Theorem 11. Thus, there exists $c \in (a, b)$ such that $N_F^\alpha F(c) = 0$ for some $\alpha \in (0, 1)$. From here we have

$$N_F^\alpha F(c) = N_F^\alpha f(c) - \frac{f(b) - f(a)}{g(b) - g(a)} N_F^\alpha g(c) = 0.$$

where the desired result is obtained. □

Taking into account the ideas of [37] we can define the generalized partial derivatives as follows.

Definition 7. *Given a real valued function* $f : \mathbb{R}^n \to \mathbb{R}$ *and* $\overrightarrow{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ *a point whose ith component is positive. Then the generalized partial N-derivative of* f *of order* α *in the point* $\overrightarrow{a} = (a_1, \ldots, a_n)$ *is defined by*

$$N_{F_i,t_i}^\alpha f(\overrightarrow{a}) = \lim_{\varepsilon \to 0} \frac{f(a_1, .., a_i + \varepsilon F_i(a_i, \alpha), \ldots, a_n) - f(a_1, \ldots, a_i, ..., a_n))}{\varepsilon} \tag{25}$$

if it exists, it is denoted $N_{F_i,t_i}^\alpha f(\overrightarrow{a})$*, and called the ith generalized partial derivative of* f *of the order* $\alpha \in (0, 1]$ *at* $\overrightarrow{a}$.

Remark 8. *If a real valued function* f *with* n *variables has all generalized partial derivatives of the order* $\alpha \in (0, 1]$ *at* $\overrightarrow{a}$*, each* $a_i > 0$*, then the generalized* α*-gradient of* f *of the order* $\alpha \in (0, 1]$ *at* $\overrightarrow{a}$ *is*

$$\nabla_N^\alpha f(\overrightarrow{a}) = (N_{t_1}^\alpha f(\overrightarrow{a}), \ldots, N_{t_n}^\alpha f(\overrightarrow{a})) \tag{26}$$

Taking into account the above definitions, it is not difficult to demonstrate the following result, on the equality of mixed partial derivatives.

Theorem 16. *Under assumptions of Definition 7, assume that* $f(t_1, t_2)$ *is a function for which mixed generalized partial derivatives exist and are continuous,* $N_{F_{1,2},t_1,t_2}^{\alpha+\beta}(f(t_1, t_2))$ *and* $N_{F_{2,1},t_2,t_1}^{\beta+\alpha}(f(t_1, t_2))$ *over some domain of* $\mathbb{R}^2$ *then*

$$N_{F_{1,2},t_1,t_2}^{\alpha+\beta}(f(t_1, t_2)) = N_{F_{2,1},t_2,t_1}^{\beta+\alpha}(f(t_1, t_2)) \tag{27}$$

3. Extensions

It is clear that, under the Definitions 4 and 5 many of the results reported in the literature, for the derivatives and integrals presented above as particular cases, can be extended without much difficulty. For example, in [38] the existence of solutions of a non-local initial value problem involving generalized Katugampola fractional derivative is studied. A more general formulation, in terms of Definitions 4 and 5, can also be obtained by including some cases not reported in the literature. For the above definitions, the reciprocal action between them and some properties such as linearity and

monotonous behavior have been established. Similarly, the integration rule has been established in parts, a version corresponding to Rolle's theorem and the mean value theorem. In addition, it was defined a generalized partial derivative (Definition 7) following the proposed idea in this work. The authors hope that this work will motivate for future work in the area.

Author Contributions: All authors contributed equally in the preparation of the present work: the theorems and corollaries P.M.G., L.M.L., J.E.N.V. and M.V.-C., the review of the articles and books cited P.M.G., L.M.L., J.E.N.V. and M.V.-C.; formal analysis P.M.G., L.M.L., J.E.N.V. and M.V.-C.; writing—original draft preparation and writing—review and editing P.M.G., L.M.L., J.E.N.V. and M.V.-C. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Dirección de Investigación from Pontificia Universidad Católica del Ecuador in the research project entitled: Some integrals inequalities for generalized convex functions and applications (Algunas desigualdades integrales para funciones convexas generalizadas y aplicaciones).

Acknowledgments: Miguel J. Vivas-Cortez thanks to Dirección de Investigación from Pontificia Universidad Católica del Ecuador for the technical support given to the research project entitled: Some integrals inequalities for generalized convex functions and applications (Algunas desigualdades integrales para funciones convexas generalizadas y aplicaciones).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Machado, J.T.; Kiryakova, V.; Mainardi, F. Recent history of fractional calculus. *Commun. Nonlinear Sci. Numer. Simul.* **2011**, *16*, 1140–1153. [CrossRef]
2. Baleanu, D.; Fernandez, A. On fractional operators and their classifications. *Mathematics* **2019**, *7*, 830. [CrossRef]
3. Baleanu, D. Comments on: The failure of certain fractional calculus operators in two physical models. *Fract. Calc. Appl. Anal.* **2020**, *23*, 292–297. [CrossRef]
4. Atangana, A. *Derivative with a New Parameter Theory, Methods and Applications*; Academic Press: New York, NY, USA, 2016.
5. Abdeljawad, T.; Agarwal, R.P.; Karapınar, E.; Sumati Kumari, P. Solutions of the Nonlinear Integral Equation and Fractional Differential Equation Using the Technique of a Fixed Point with a Numerical Experiment in Extended b-Metric. *Symmetry* **2019**, *11*, 686 [CrossRef]
6. Afshari, H.; Kalantari, S.; Karapinar, E. Solution of fractional differential equations via coupled fixed point. *Electron. J. Differ. Equ.* **2015**, *2015*, 1–12.
7. Alqahtani, B.; Aydi, H.; Karapınar, E.; Rakočević, V. A Solution for Volterra Fractional Integral Equations by Hybrid Contractions. *Mathematics* **2019**, *7*, 694. [CrossRef]
8. Karapınar, E.; Abdeljawad, T.; Jarad, F. Applying new fixed point theorems on fractional and ordinary differential equations. *Adv. Differ. Eq.* **2019**, *421*. [CrossRef]
9. Karapınar, E.; Fulga, A.; Rashid, M.; Shahid, l.; Aydi, H. Large Contractions on Quasi-Metric Spaces with an Application to Nonlinear Fractional Differential Equations. *Mathematics* **2019**, *7*, 444 [CrossRef]
10. Capelas de Oliveira, E.; Tenreiro Machado, J.A. A Review of Definitions for Fractional Derivatives and Integral. *Math. Probl. Eng.* **2014**, *2014*, 1–6. [CrossRef]
11. Gorenflo, R.; Mainardi, F. *Fractional Calculus: Integral and Differential Equations of Fractional Order*; Springer: Wien, Austria, 1997.
12. Katugampola, U.N. New approach to a generalized fractional integral. *Appl. Math. Comput.* **2011**, *218*, 860–865. [CrossRef]
13. Katugampola, U.N. A new approach to generalized fractional derivatives. *Bull. Math. Anal. App.* **2014**, *6*, 1–15.
14. Martínez, F.; Mohammed, P.O.; Nápoles Valdés, J.E. Non Conformable Fractional Laplace Transform. *Kragujev. J. Math.* **2022**, *46*, 341–354.
15. Mittag-Leffler, G.M. Sur la nouvelle fonction. *C. R. Acad. Sci.* **1903**, *137*, 554–558.
16. Mittag-Leffler, G.M. Sur la representation analytique dâ une branche uniforme d'une fonction monogane. *Acta Math.* **1904**, *29*, 101–181. [CrossRef]

17. Wiman, A. Über der Fundamentalsatz in der Theorie der Funktionen $E_\alpha(x)$. *Acta Math.* **1905**, *29*, 191–201. [CrossRef]
18. Agarwal, R.P. A propos d'une note de M. Pierre Humbert. *C. R. Acad. Sci. Paris* **1953**, *236*, 2031–2032.
19. Humbert, P.; Agarwal, R.P. Sur la fonction de Mittag-Leffler et quelques-unes de ses généralisations. *Bull. Sci. Math. Ser. II* **1953**, *77*, 180–185.
20. Erdelyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions*; McGraw-Hill: New York, NY, USA, 1955.
21. Prabhakar, T.R. A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Math. J.* **1971**, *19*, 7–15.
22. Rainville, E.D. *Special Functions*; Macmillan: New York, NY, USA, 1960.
23. Gorenflo, R.; Kilbas, A.A.; Rogosin, S.V. On the generalised Mittag-Leffler type function. *Integral Transform. Spec. Funct.* **1998**, *7*, 215–224. [CrossRef]
24. Kilbas, A.A.; Saigo, M. On Mittag-Leffler type function, fractional calculus operators and solution of integral equations. *Integral Transform. Spec. Funct.* **1996**, *4*, 355–370. [CrossRef]
25. Saigo, M.; Kilbas, A.A. On Mittag-Leffler type function and applications. *Integral Transform. Spec. Funct.* **1998**, *7*, 97–112. [CrossRef]
26. Guzman, P.M.; Langton, G.; Lugo, L.M.; Medina, J.; Nápoles Valdés, J.E. A new definition of a fractional derivative of local type. *J. Math. Anal.* **2018**, *9*, 88–98.
27. Nápoles Valdés, J.E.; Guzman, P.M.; Lugo, L.M. Some New Results on Nonconformable Fractional Calculus. *Adv. Dyn. Syst. Appl.* **2018**, *13*, 167–175.
28. Fleitas, A.; Méndez, J.A.; Nápoles, J.E.; Sigarreta, J.M. On the some classical systems of Liénard in general context. *Rev. Mex. Física* **2019**, *65*, 618–625. [CrossRef]
29. Khalil, R.; Al Horani, M.; Yousef, A.; Sababheh, M. A new definition of fractional derivative. *J. Comput. Appl. Math.* **2014**, *264*, 65–70. [CrossRef]
30. Nápoles Valdés, J.E.; Rodríguez, J.M.; Sigarreta, J.M. On Hermite-Hadamard type inequalities for non-conformable integral operators. *Symmetry* **2019**, *11*, 1108. [CrossRef]
31. Cınar, I. On Some Properties of Generalized Riesz Potentials. *Int. Math. J.* **2003**, *3*, 1393–1397
32. Martínez, C.; Sanz, M.; Periogo, F. Distributional Fractional Powers of Laplacian Riesz Potential. *Stud. Math.* **1999**, *135, 253–271.*.
33. Helms, L.L. *Introduction To Potential Theory*; Wiley-Interscience: New York, NY, USA, 1969.
34. Kilbas, A.A.; Marichev, O.I.; Samko, S.G. *Fractional Integrals and Derivatives. Theory and Applications*; Gordon & Breach: Montreux, Switzerland, 1993.
35. Kwun, Y.C.; Farid, G.; Nazeer, W.; Ullah, S.; Kang, S.M. Generalized Riemann-Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities. *IEEE Access* **2018**, *6*, 64946–64953. [CrossRef]
36. Farid, G. Study of a generalized Riemann-Liouville fractional integral via convex functions. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **2020**, *69*, 37–48 [CrossRef]
37. Martínez, F.; Nápoles Valdés, J.E. Towards a Non-Conformable Fractional Calculus of N-variables. *J. Math. Appl.* to appear.
38. Sandeep, P.; Bhairat, S.P.; Dnyanoba-Bhaurao Dhaigude, D.B. Existence of solutions of generalized fractional differential equation with nonlocal initial condition. *Math. Bohem.* **2019**, *144*, 203–220.



© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).