

Article

# Minimal Systems of Temporal Logic

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**Abstract:** The article discusses minimal temporal logic systems built on the basis of classical logic as well as intuitionistic logic. The constructions of these systems are discussed as well as their basic properties. The  $K_t$  system was discussed as the minimal temporal logic system built based on classical logic, while the  $IK_t$  system and its modification were discussed as the minimal temporal logic system built based on intuitionistic logic.

**Keywords:** temporal logic; intuitionistic logic; minimal system; knowledge

## 1. Temporal Logic

*Temporal logic* is the logic in which they appear, as logical constants, expressions whose meaning is determined by a reference to time. In its wide sense, temporal logic includes all logical problems of temporal representation of information. The task of temporal logic is to define and systematize inference rules for reasoning carried out in a language in which the same expression in terms of shape is used to pronounce sentences whose logical value may not be the same in different temporal contexts of their use.

The precursor of temporal logics was A. N. Prior. One of Prior's basic concepts was the temporal interpretation of modal operators. The enriched language of temporal logic was to enable formalization of reasoning regarding situations changing in time. Originally, temporal logic was to be a tool for formalizing philosophical, linguistic and semiotic considerations. Currently, apart from these applications, temporal logic is also widely used in computer science.

Among temporal logics, tense logic stands out, i.e., logic in a language whose only specific time operators are grammatical operators.

## 2. $K_t$ —Minimal Tense Logic

The basic deductive system of logic of time is the  $K_t$  system ( $K_t$  is a temporal analogue of the  $K$  system (minimal deductive system for modal logic).).  $K_t$  is a tense logic system built over classical propositional calculus by enriching this logic with specific axioms and rules. This is the minimal system. Therefore, the theses of this system are all and only those sentences that are true regardless of what properties time has (In fact, one assumption is made about the structure of time, namely it is assumed that a semantic time in the  $K_t$  has a point structure.).

The  $K_t$  system, as a minimal system, can be expanded by adding additional rules and specific axioms. In this sense, the minimality of  $K_t$  means that any other temporal logic system built above classical propositional logic is richer than the  $K_t$ . In the tense logics we have the tense operators:  $F, G, P, H$  understood as follows:

- $F\varphi$  - it will be that  $\varphi$ ,  
 $P\varphi$  - it was that  $\varphi$ ,  
 $G\varphi$  - it always will be that  $\varphi$ ,  
 $H\varphi$  - it always was that  $\varphi$ .

However, usually the operators  $F$  and  $G$  and operators  $P$  and  $H$  are mutually definable (The mutual definability of operators  $F$  and  $G$  as well as  $P$  and  $H$  occur in temporal logic systems based on classical logic. In temporal logic systems based on intuitionistic or multi-valued logic, the mutual definability of these operators usually does not take a place.).

**Definition 1** (The alphabet of the language  $\mathcal{L}_{K_t}$ ).

- countable set of propositional letters  $\mathcal{AP}$ ,
- connectives:  $\neg, \rightarrow$ ,
- temporal operators:  $G, H$  (In some tense logic systems, as a primary operators are assumed  $F$  and  $P$ .),
- brackets:  $), ($ .

A set of sentences is defined as follows:

**Definition 2** (A set of sentences ). The set of sentences is the smallest set  $FOR(\mathcal{L}_{K_t})$  such that:

- $\mathcal{AP} \subseteq FOR(\mathcal{L}_{K_t})$ ,
- if  $\varphi, \psi \in FOR(\mathcal{L}_{K_t})$ , then  $\neg\varphi, \varphi \rightarrow \psi, G\varphi, H\varphi \in FOR(\mathcal{L}_{K_t})$ .

In the language  $\mathcal{L}_{K_t}$ , all boolean symbols retain their meaning. However, there are additional specific operators in this language. Therefore, when we speak about the validity of propositions due to the meaning of classical propositional connectives, then we mean the sentences in which new operators occur.

We accept the following abbreviations:

- (a)  $\varphi \vee \psi \equiv \neg\varphi \rightarrow \psi$ ,  
(b)  $\varphi \wedge \psi \equiv \neg(\varphi \rightarrow \neg\psi)$ ,  
(c)  $\varphi \leftrightarrow \psi \equiv \neg[(\varphi \rightarrow \psi) \rightarrow \neg(\psi \rightarrow \varphi)]$ ,  
(d)  $F\varphi \equiv \neg G\neg\varphi$ ,  
(e)  $P\varphi \equiv \neg H\neg\varphi$ .

*Axioms*

The  $K_t$  system is axiomatizable (The axiomatic system is one of many possible forms of a deductive system. This approach to construction of a deductive system has many advantages when it comes to methodological research. However, in case of axiomatic systems, we have some problems when it comes to practical command. This is due to the unstructured axiomatic systems. The structure of the sentence does not indicate the method of proving this sentence. In the case of other approaches to construction of a deductive system, e.g., sequent calculus, natural deduction or semantic tables, it is different.). Various sets of axioms and rules of this system were proposed. These differences are primarily due to the decision on a set of specific primitive symbols. Usually, the set of these symbols consists of the symbols  $G$  and  $H$ , while  $F$  and  $P$  are defined. When building a set of axioms for invariant systems, i.e., systems without the rule of substitution for sentence letters, apart from specific axiom schemes, either all tautologies of

classical propositional logic or only selected tautology schemes are taken, but they are selected in such a way that all tautologies of classical propositional logic can be obtained. In this work, we used the second option and for the purposes of our considerations regarding  $\mathbf{K}_t$  we will adopt the following set of axioms:

#### Axioms:

For any sentences  $\varphi, \psi \in \mathcal{L}_{K_t}$  ( $\mathbf{K}_t$  can be axiomatizable in many ways. The completeness of the  $K_t$  with respect of these set of axioms was demonstrated by J. F. A. K. van Benthem [1].).

1. All tautologies of the classical propositional calculus of the language  $\mathcal{L}_{K_t}$ ,
2.  $G(\varphi \rightarrow \psi) \rightarrow (G\varphi \rightarrow G\psi)$ ,
3.  $H(\varphi \rightarrow \psi) \rightarrow (H\varphi \rightarrow H\psi)$ ,
4.  $\varphi \rightarrow GP\varphi$ ,
5.  $\varphi \rightarrow HF\varphi$ .

#### Rules

$$MP: \frac{\varphi \rightarrow \psi, \varphi}{\psi} \quad RG: \frac{\varphi}{G\varphi} \quad RH: \frac{\varphi}{H\varphi}.$$

The specific  $\mathbf{K}_t$  axioms are the 2–5 axioms. Axioms 2–3 are temporal equivalents of the  $\mathbf{K}$  axiom for modal logics. These axioms apply only to the properties of  $G$  and  $H$ , respectively. Axioms 4–5 bind the operators  $G$  and  $P$  as well as  $H$  and  $F$  respectively.

The proof in  $\mathbf{K}_t$  is understood in the usual way.

**Definition 3** (Proof in  $\mathbf{K}_t$ ). Let  $\Sigma$  be any set of sentences of the language  $\mathcal{L}_{K_t}$ . The sentence string  $\varphi_0, \varphi_1, \dots, \varphi_n$  is a proof of the sentence  $\varphi$  from the set  $\Sigma$ , (we write  $\Sigma \vdash_{K_t} \varphi$ ) if and only if  $\varphi = \varphi_n$  and for any  $i$  such that  $0 \leq i \leq n$  at least one of the following conditions holds:

1.  $\varphi_i$  is an element of the set  $\Sigma$ ,
2.  $\varphi_i$  is an axiom,
3.  $\varphi_i$  is obtained from their predecessors by MP, RG or RH, respectively.

The sentence  $\varphi$ , which is derived from the empty set  $\Sigma$ , or  $\emptyset \vdash_{K_t} \varphi$ , is the thesis of the system  $\mathbf{K}_t$ . Instead of writing  $\emptyset \vdash_{K_t} \varphi$ , we will write  $\vdash_{K_t} \varphi$ .

In the  $\mathbf{K}_t$  system, if a subsentences  $\varphi$  of the sentence  $\phi$  is equivalent to the sentence  $\psi$ , entering  $\phi$  in the place of the sentence  $\varphi$  as the inscription of the sentence  $\psi$ ,  $\phi(\psi/\varphi)$ , gives the sentence equivalent to  $\phi$ .

**Theorem 1.** If  $\Sigma \vdash_{K_t} \varphi \leftrightarrow \psi$ , then  $\Sigma \vdash_{K_t} \phi \leftrightarrow \phi(\psi/\varphi)$ . (This theorem is not just the  $\mathbf{K}_t$  theorem. It is the theorem of tense priorist logic.)

**Proof.** We will prove by induction due to the length of the sentence  $\phi$ . Let  $\Sigma \vdash_{K_t} \varphi \leftrightarrow \psi$ . Let  $\phi$  be a propositional letter  $p$ . The only subsentence of a sentence  $\phi$  is the propositional letter  $p$ . Then  $\varphi$  is equal  $p$ . Result of replacement  $\varphi$  in the  $\phi$  by  $\psi$  will be the sentence  $\psi$ . Because by assumption we have  $\Sigma \vdash_{K_t} \varphi \leftrightarrow \psi$ , then:

$$\Sigma \vdash_{K_t} \phi \leftrightarrow \phi(\psi/\varphi).$$

As an induction assumption, we assume that for any sentence  $\phi_i$  with length is not greater than  $k$  the thesis is true, i.e.,

$$\Sigma \vdash_{K_t} \phi_i \leftrightarrow \phi_i(\psi/\varphi).$$

We will show that this thesis is also true for sentences of length  $k + 1$ .

Let the string  $\varphi_1, \varphi_2, \dots, \varphi_n (= \phi_i \leftrightarrow \phi_i(\psi/\varphi))$  be a proof of the sentence:  $\phi_i \leftrightarrow \phi_i(\psi/\varphi)$ . We add the following sentences to this proof:

- n+1.**  $\neg\phi_i(\varphi/\psi) \leftrightarrow \neg\phi_i$  TRANS, n  
**n+2.**  $(\neg\phi_i(\varphi/\psi) \leftrightarrow \neg\phi_i) \rightarrow (\phi_i \leftrightarrow \phi_i(\varphi/\psi))$  axiom 1  
**n+3.**  $\phi_i \leftrightarrow \phi_i(\varphi/\psi)$  MP, n+1, n+2

The sentence  $\neg\phi_i(\varphi/\psi)$  is  $(\neg\phi_i)(\varphi/\psi)$ , then:

$$\Sigma \vdash_{K_t} \neg\phi_i \leftrightarrow (\neg\phi_i)(\varphi/\psi).$$

Let it now  $\phi$  will be according to the character  $\phi_i \rightarrow \phi_j$ , with sentences  $\phi_i$  and  $\phi_j$  meet the induction assumption, i.e.,

$$\Sigma \vdash_{K_t} \phi_i \leftrightarrow \phi_i(\varphi/\psi)$$

and

$$\Sigma \vdash_{K_t} \phi_j \leftrightarrow \phi_j(\varphi/\psi).$$

Let the string  $\varphi_1, \varphi_2, \dots, \varphi_k (= \phi_i \leftrightarrow \phi_i(\psi/\varphi))$  be a proof of the sentence  $\phi_i \leftrightarrow \phi_i(\psi/\varphi)$ , while the string  $\varphi_{k+1}, \varphi_{k+2}, \dots, \varphi_n (= \phi_j \leftrightarrow \phi_j(\psi/\varphi))$  be a proof of the sentence:  $\phi_j \leftrightarrow \phi_j(\psi/\varphi)$ . To the sequence of the sentences  $\varphi_1, \varphi_2, \dots, \varphi_k, \varphi_{k+1}, \varphi_{k+2}, \dots, \varphi_n$  we add sentences:

- n+1.**  $(\phi_i \leftrightarrow \phi_i(\psi/\varphi)) \rightarrow \{(\phi_j \leftrightarrow \phi_j(\psi/\varphi)) \rightarrow [(\phi_i \rightarrow \phi_j) \leftrightarrow (\phi_i(\psi/\varphi) \rightarrow \phi_j(\psi/\varphi))]\}$  axiom 1  
**n+2.**  $(\phi_j \leftrightarrow \phi_j(\psi/\varphi)) \rightarrow [(\phi_i \rightarrow \phi_j) \leftrightarrow (\phi_i(\psi/\varphi) \rightarrow \phi_j(\psi/\varphi))]$  MP, k, n+1  
**n+3.**  $[(\phi_i \rightarrow \phi_j) \leftrightarrow (\phi_i(\psi/\varphi) \rightarrow \phi_j(\psi/\varphi))]$  MP, n, n+2

$(\phi_i(\psi/\varphi) \rightarrow \phi_j(\psi/\varphi))$  is the sentence  $(\phi_i(\psi \rightarrow \phi_j(\psi/\varphi)))$ , so we received proof that

$$\Sigma \vdash_{K_t} (\phi_i \rightarrow \phi_j) \leftrightarrow (\phi_i \rightarrow \phi_j(\psi/\varphi)).$$

Now let us consider the case when the sentence  $\phi$  is the sentence of the form  $G\phi_i$ , with the sentence  $\phi_i$  is a sentence satisfying the induction assumption, i.e.,  $\Sigma \vdash_{K_t} \phi_i \leftrightarrow \phi_i(\varphi/\psi)$ . Let the string  $\varphi_1, \varphi_2, \dots, \varphi_n$  be a proof of the sentence  $\phi_i \leftrightarrow \phi_i(\varphi/\psi)$  from the sentence  $\Sigma$ . To the proof we add:

- n+1.**  $G\phi_i \leftrightarrow G\phi_i(\varphi/\psi)$ .

$G\phi_i(\varphi/\psi)$  is the sentence  $(G\phi_i)(\varphi/\psi)$ . So we received proof that

$$\Sigma \vdash_{K_t} G\phi_i \leftrightarrow G\phi_i(\varphi/\psi).$$

The case where the sentence  $\phi$  is according to the form  $H\phi_i$  is similar to the case when  $\phi$  is the sentence  $G\phi_i$ .  $\square$

The Theorem 1 will be used in the proof of the next Theorem, which says that one of the  $K_t$  inference rules is the REQ replacement rule. This rule is a very useful rule in proving the theses of the  $K_t$  system.

**Theorem 2** (Rule REQ). If  $\Sigma \vdash_{K_t} \varphi \leftrightarrow \psi$ , then  $\frac{\phi}{\phi(\psi/\varphi)}$ .

**Proof.** Let  $\Sigma \vdash_{K_t} \varphi \leftrightarrow \psi$  and  $\Sigma \vdash_{K_t} \phi$ . According to the Theorem 1 there is a proof of the sentence  $\phi \leftrightarrow \phi(\varphi/\psi)$  from the set  $\Sigma$ . To this proof we add the proof of the sentence  $\phi$ . We add to the proof sequence the sentence  $\phi(\varphi/\psi)$ , which is a result from applying the Modus Ponens rule to sentences:  $\phi$  and  $\phi \leftrightarrow \phi(\varphi/\psi)$ .  $\square$

In addition to the three inference rules proposed in this version of the axiomatics of the  $\mathbf{K}_t$  system can be used to derive in this system the rules corresponding to the regularity rule for modal logics.

**Theorem 3.** The RRG rule :  $\frac{\varphi \rightarrow \psi}{G\varphi \rightarrow G\psi}$  is a rule of  $\mathbf{K}_t$ .

**Proof.** To demonstrate that RRG is a secondary rule  $\mathbf{K}_t$ , it must be demonstrated that

$$\text{if } \Sigma \vdash_{K_t} \varphi \rightarrow \psi, \text{ then } \Sigma \vdash_{K_t} G\varphi \rightarrow G\psi.$$

Let  $\Sigma \vdash_{K_t} \varphi \rightarrow \psi$ . Let the sequence  $\varphi_1, \dots, \varphi_n$  will prove the sentence  $\varphi \rightarrow \psi$  from the set  $\Sigma$ . To this we add the following sentences:

- n+1.  $G(\varphi \rightarrow \psi)$  RG,n
- n+2.  $G(\varphi \rightarrow \psi) \rightarrow (G\varphi \rightarrow G\psi)$  axiom 2
- n+3.  $G\varphi \rightarrow G\psi$  MP,n+1,n+2.

The resulting sequence is a proof of the sentence  $G\varphi \rightarrow G\psi$  from the set  $\Sigma$ .  $\square$

**Theorem 4.** The RRH rule :  $\frac{\varphi \rightarrow \psi}{H\varphi \rightarrow H\psi}$  is a secondary rule of  $\mathbf{K}_t$ .

**Proof.** Analogical to the proof of the previous theorem (using the axiom 3 and the rule RH).  $\square$

Based on Theorems 3 and 4 two further inference rules can be derived in  $\mathbf{K}_t$ .

**Theorem 5.** The RF rule :  $\frac{\varphi \rightarrow \psi}{F\varphi \rightarrow F\psi}$  is a secondary rule of  $\mathbf{K}_t$ .

**Proof.** Let  $\Sigma \vdash_{K_t} \varphi \rightarrow \psi$ . Let the sequence:  $\varphi_1, \dots, \varphi_n$  will prove the sentence  $\varphi \rightarrow \psi$  from the set  $\Sigma$ . To this we add the following sentences:

- n+1.  $\neg\psi \rightarrow \neg\varphi$  TRANS,n
- n+2.  $G\neg\psi \rightarrow G\neg\varphi$  RRG,n+1
- n+3.  $\neg G\neg\varphi \rightarrow \neg G\neg\psi$  TRANS,n+2
- n+4.  $F\varphi \rightarrow F\psi$  REQ( $\neg G\neg\varphi / F\varphi$ ), REQ( $\neg G\neg\psi / F\psi$ ).

The resulting sequence is proof of the sentence  $F\varphi \rightarrow F\psi$  from the set  $\Sigma$ .  $\square$

**Theorem 6.** The RP rule :  $\frac{\varphi \rightarrow \psi}{P\varphi \rightarrow P\psi}$  is a secondary rule of  $\mathbf{K}_t$ .

**Proof.** Analogical to the proof of the Theorem 5.  $\square$

Operators  $H, P$  and  $G, F$  have the Mirror Image Property.

**Definition 4** (Mirror Image Property). The mirror image of the  $\varphi$  formula is created by simultaneously replacing each instance of the  $H$  operator with the  $G$  operator and the  $G$  operator with the  $H$  operator in the  $\varphi$  formula, and simultaneously replacing each instance of the  $P$  operator with the  $F$  operator and the  $F$  operator with the  $P$  operator.

The Mirror Image of the  $\varphi$  we will mean by  $MI(\varphi)$ . E.g:  $MI(\varphi \rightarrow GP\varphi) = \varphi \rightarrow HF\varphi$ . The mirror image of the set of  $\Sigma$  is the mirror image set of the  $\Sigma$  elements. We mean the mirror image of  $\Sigma$  by  $MI(\Sigma)$  and define as follows:

**Definition 5** (A mirror image of a set of formulas).  $M(\Sigma) = \{MI(\varphi) : \varphi \in \Sigma\}$ .

If  $\varphi$  is derivable from  $\Sigma$ , then mirror image of  $\varphi$  is derivable from mirror image of the  $\Sigma$ .

**Theorem 7.** For any  $\Sigma (\subset FOR(\mathfrak{L}_{K_t}))$ : if  $\Sigma \vdash_{K_t} \varphi$ , then  $MI(\Sigma) \vdash_{K_t} MI(\varphi)$ .

**Proof.** Let  $\Sigma \vdash_{K_t} \varphi$ . Let the sequence  $\varphi_1, \varphi_2, \dots, \varphi_n$  will be a proof of  $\varphi$  from the  $\Sigma$ . We will show that the sequence  $MI(\varphi_1), MI(\varphi_2), \dots, MI(\varphi_n)$  is a proof of the sentence  $MI(\varphi)$  from the  $MI(\Sigma)$ ,  $MI(\Sigma) \vdash_{K_t} MI(\varphi)$ . We will carry out the proof by induction due to the length of the proof of the sentence  $\varphi$ .

If  $\varphi_1$  is an axiom, then  $MI(\varphi_1)$  is also an axiom. If  $\varphi_1$  is an element of  $\Sigma$ , then  $MI(\varphi_1)$  is also an element of  $MI(\Sigma)$ . Then if  $\Sigma \vdash_{K_t} \varphi_1$ , then  $MI(\Sigma) \vdash_{K_t} MI(\varphi_1)$ .

Let us assume that for  $i, i \leq k$ :

$$\text{if } \Sigma \vdash_{K_t} \varphi_i, \text{ then } MI(\Sigma) \vdash_{K_t} MI(\varphi_i).$$

We will show that if  $\Sigma \vdash_{K_t} \varphi_{k+1}$ , then  $MI(\Sigma) \vdash_{K_t} MI(\varphi_{k+1})$ . Let  $\Sigma \vdash_{K_t} \varphi_{k+1}$ . The sentence  $\varphi_{k+1}$  can be an axiom or an element of a set  $\Sigma$ . There are cases discussed for the sentence  $\varphi_1$ . Now let us consider the cases where the sentence  $\varphi_{k+1}$  was obtained using one of the inference rules. Let them  $\varphi_{k+1}$  will be a sentence derived from sentences  $\varphi_m$  and  $\varphi_m \rightarrow \varphi_{k+1}$  by applying the rule  $MP$ . By induction, we have that

$$MI(\Sigma) \vdash_{K_t} MI(\varphi_m)$$

and

$$MI(\Sigma) \vdash_{K_t} MI(\varphi_m \rightarrow \varphi_{k+1}).$$

Because  $MI(\varphi_m \rightarrow \varphi_{k+1})$  has the form  $MI(\varphi_m) \rightarrow MI(\varphi_{k+1})$ , so applying the rule  $MP$  to the sentences  $MI(\varphi_m) \rightarrow MI(\varphi_{k+1})$  and  $MI(\varphi_m)$ , we obtain  $MI(\varphi_{k+1})$ . Let it now  $\varphi_{k+1}$  will be the sentence derived from the sentence  $\varphi_m$  by applying the rule  $RG$ . By induction, we have that  $MI(\Sigma) \vdash_{K_t} MI(\varphi_m)$ . After applying the rule  $RH$  to the sentence  $MI(\varphi_m)$  we obtain  $HMI(\varphi_m)$ . However, this sentence is equal to the sentence  $MI(G\varphi_m)$ . Then  $MI(\Sigma) \vdash_{K_t} MI(G\varphi_m)$ . The case when the sentence  $\varphi_{k+1}$  was obtained by applying the  $RH$  rule to the sentence  $\varphi_k$  is similar to the previous case.  $\square$

**Corollary 1.** Let  $MI(\Sigma) \subseteq \Sigma$ .

$$\text{If } \Sigma \vdash_{K_t} \varphi, \text{ then } \Sigma \vdash_{K_t} MI(\varphi)$$

or

$$\frac{\varphi}{MI(\varphi)}$$

is a secondary rule.

**Corollary 2.** Let  $MI(\Sigma) \subseteq \{\varphi : \Sigma \vdash_{K_t} \varphi\}$ .

$$\text{If } \Sigma \vdash_{K_t} \varphi, \text{ then } \Sigma \vdash_{K_t} MI(\varphi)$$

or

$$\frac{\varphi}{MI(\varphi)}$$

is a secondary rule.

### 3. $\mathbf{IK}_t$ —Minimal Intuitionistic Temporal Logic

Now we will discuss a system of temporal logic over intuitionistic propositional logic. It is a system of minimal intuitionistic temporal logic  $\mathbf{IK}_t$  ( $\mathbf{IK}_t$  is the intuitionistic analogue of the system  $\mathbf{K}_t$  - minimal temporal logic built over classical propositional logic.).

This system can be used to formally describe knowledge that changes over time, although there are no explicit epistemic operators in the language of this system. Knowledge representation is not implemented at the syntactic level, but because of the properties of intuitionistic logic, knowledge is represented at the semantic level. This is the result of semantics proposed for intuitionistic logic, using terms such as *proof* (It was proposed by Kolmogorov.), *information*, or *knowledge* (Kripke-style semantics.).

Kripke-style semantics are proposed for intuitionistic temporal logic. Thus, in Kripke models we have a set of worlds  $W$  and the relationship  $R$ . In the case of intuitionistic logic, we do not speak about elements of the  $W$  set as possible worlds, but rather as information states, states of knowledge, etc. The reachability relationship between the elements  $w$  and  $v$  (i.e.,  $wRv$ ) is interpreted as *w has access to v*, which means that the  $v$  information state is available from the  $w$  information state. The key difference between Kripke models for intuitionistic logic and Kripke models for modal logic built over classical logic lies in the fact that in the case of modal logic built over classical logic, the  $R$  relation is only used to interpret modal operators, and in the case of intuitionistic logic, this relation is used to interpret the intuitionistic negation and implication.

The formula  $\neg\varphi$  is true (In intuitionistic logic the term *forced* is also used.) in some information state  $w$  if and only if there is no information state available from  $w$  in which  $\varphi$  is true. In other words, the formula  $\neg\varphi$  is true in the state  $w$  if there is no possibility that  $\varphi$  is true in any information state accessible from the state  $w$ .

The same is true with the intuitionistic implication. The formula  $\varphi \rightarrow \psi$  is true in the information state  $w$ , if and only if, in any information state available from the state  $w$ , the truth of  $\varphi$  implies the truth of  $\psi$ . In addition, Kripke models assume monotonicity for intuitionistic logic. The formula fulfilled in a given information state remains fulfilled in any extension of this state.

Modality in intuitionistic logic can be seen on the example of the syntactic definition of intuitionistic negation. The  $\neg\varphi$  formula is equivalent to the  $\varphi \rightarrow \perp$  formula. Intuitionistic negation can therefore be seen as a kind of impossibility operator.

Kripke's intuitionistic model is a triangle  $\mathfrak{M} = \langle W, R, V \rangle$ , where  $V : \mathcal{AP} \rightarrow 2^W$ . The formula  $\varphi$  is satisfied in the model  $\mathfrak{M}$ , in the state  $w$ , when:

$$\begin{aligned} \mathfrak{M}, w \models \varphi &\equiv w \in V(\varphi), \text{ when } \varphi \in \mathcal{AP}, \\ \mathfrak{M}, w \models \neg\varphi &\equiv \text{for any } wRw' : \mathfrak{M}, w' \not\models \varphi, \\ \mathfrak{M}, w \models \varphi \wedge \psi &\equiv \mathfrak{M}, w \models \varphi \text{ and } \mathfrak{M}, w \models \psi, \\ \mathfrak{M}, w \models \varphi \vee \psi &\equiv \mathfrak{M}, w \models \varphi \text{ or } \mathfrak{M}, w \models \psi, \\ \mathfrak{M}, w \models \varphi \rightarrow \psi &\equiv \text{for any } w' \text{ such that } wRw', \text{ if } \mathfrak{M}, w' \models \varphi, \text{ then } \mathfrak{M}, w' \models \psi. \end{aligned}$$

In intuitionistic logic from the truth of the  $\neg\varphi$  formula in the current information state, we do not only know that  $\varphi$  is not true in the current information state (such information is obtained in the case of classical logic), but we also know that the formula  $\varphi$  will never be true, and our *never* applies to all available extensions of the current information state. In addition to the information provided explicitly, we therefore have an additional *information* in intuitionistic logic. This feature of intuitionistic logic van Benthem calls *knowledge implicate* [2]. No additional specific operators are needed to express it in intuitionistic logic. Despite similar semantics, this feature definitely distinguishes intuitionistic logic from epistemic logic built on classical logic. The language of epistemic logic is used to represent *knowledge explicitly*, and to

represent it, in addition to classical sentence connectives, the epistemic operator  $K$  is used. The language of intuitionistic logic allows expressing certain concepts without explicitly referring to epistemic operators. For example, based on the truth of the formula  $\neg\neg\varphi$ , we say that for each information state there is such an extension in which  $\varphi$  is true. Apart from details, it is very close to that *we know that  $\varphi$  must be true*.

In Kripke semantics for epistemic logic built over classical propositional calculus, the formula  $K\varphi$  in the  $\mathfrak{M}$  model, in the  $w$  information state, was defined as follows:

$$\mathfrak{M}, w \models K\varphi \quad \equiv \quad \text{for any } wRw' : \mathfrak{M}, w' \models \varphi.$$

Let us consider the truth of the formula  $K\neg\varphi$  in the model  $\mathfrak{M}$ , in the state  $w$ . In accordance with the condition of satisfy with the operator  $K$  we have:

$$\mathfrak{M}, w \models K\neg\varphi \quad \equiv \quad \text{for any } wRw' : \mathfrak{M}, w' \models \neg\varphi.$$

Taking into account the condition of fulfilling of the negation in epistemic logic built over classical logic, we have:

$$\mathfrak{M}, w \models K\neg\varphi \quad \equiv \quad \text{for any } wRw' : \mathfrak{M}, w' \not\models \varphi.$$

The condition of fulfilling of the intuitionistic negation, i.e.,

$$\mathfrak{M}, w \models \neg\varphi \quad \equiv \quad \text{for any } wRw' : \mathfrak{M}, w' \not\models \varphi$$

Indicates that intuitionistic negation ( $\neg$ ) can be seen as a combination of the  $K$  operator and classical negation ( $K\neg$ ). Similarly, it can be shown that the intuitionistic formula  $\varphi \Rightarrow \psi$  can be seen, aside from the details, as *modalized implication*  $K(\varphi \rightarrow \psi)$ , i.e., a combination of the  $K$  epistemic operator and the classic implication.

**IK<sub>t</sub>** (The construction of the **IK<sub>t</sub>** system and proof of the system's completeness with respect to the proposed semantics was provided by W.B. Ewald [3].) is a system of temporal logic built over intuitionistic propositional calculus. The language  $\mathcal{L}_{\mathbf{IK}_t}$  is the language of intuitionistic propositional logic enriched with temporal operators:  $G, H, F, P$ .

**Definition 6.** The set of sentences  $FOR(\mathcal{L}_{\mathbf{IK}_t})$  is the smallest set of finite sequences of elements of the language alphabet  $\mathcal{L}_{\mathbf{IK}_t}$  such that:

1. if  $\varphi \in \mathcal{AP}$ , then  $\varphi \in FOR(\mathcal{L}_{\mathbf{IK}_t})$ ,
2. if  $\varphi, \psi \in FOR(\mathcal{L}_{\mathbf{IK}_t})$ , then  $\neg\varphi, G\varphi, F\varphi, H\varphi, P\varphi, (\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi), (\varphi \leftrightarrow \psi) \in FOR(\mathcal{L}_{\mathbf{IK}_t})$ .

In the **IK<sub>t</sub>** system, the operators  $G$  and  $F$  as well as  $H$  and  $P$ , unlike systems built over classical logic, are not mutually definable.

#### 4. Semantics for **IK<sub>t</sub>** Proposed by Ewald

The construction of semantics for **IK<sub>t</sub>** is based on a partially ordered set of states of knowledge, which is considered by the cognitive subject. Each state of knowledge is assigned a set of time moments and temporal order. When the cognitive subject reaches a greater state of knowledge (According to Ewald [3], the cognitive subject moves to a greater states of knowledge.), retains all the information that he had in lower states of knowledge. To define semantics for this system, Ewald constructs an intuitionistic temporal structure.



**Definition 7** (intuitionistic temporal structure [3]). An intuitionistic temporal structure  $\mathfrak{M}$  is an ordered quintuple

$$\langle S, \leq, \{T_s\}_{s \in S}, \{\mu_s\}_{s \in S}, \{R_t^s\}_{s \in S, t \in T_s} \rangle$$

where:

- $(S, \leq)$  is a partially ordered set,
- $T_s$  is a non-empty set,
- $\mu_s$  is a binary relation to  $T_s$ ,
- $R_t^s$  is a formula relation that satisfies the conditions:

1.  $R_t^s(\varphi) \equiv R_t^{s'}(\varphi)$ , when  $\varphi \in \mathcal{AP}$  and  $s \leq s'$ ,
2.  $R_t^s(\varphi \wedge \psi) \equiv R_t^s(\varphi) \text{ and } R_t^s(\psi)$ ,
3.  $R_t^s(\varphi \vee \psi) \equiv R_t^s(\varphi) \text{ or } R_t^s(\psi)$ ,
4.  $R_t^s(\neg\varphi) \equiv \text{for any } s \leq s' \text{ it is not true that } R_t^{s'}(\varphi)$ ,
5.  $R_t^s(\varphi \rightarrow \psi) \equiv \text{for any } s \leq s' \text{ (if } R_t^{s'}(\varphi), \text{ then } R_t^{s'}(\psi))$ ,
6.  $R_t^s(F\varphi) \equiv \text{there is } t', t\mu_s t' : R_{t'}^{s'}(\varphi)$ ,
7.  $R_t^s(P\varphi) \equiv \text{there is } t', t'\mu_s t : R_{t'}^{s'}(\varphi)$ ,
8.  $R_t^s(G\varphi) \equiv \text{for any } s', t' \text{ such that: } s \leq s', t' \in T_{s'}, t\mu_{s'} t' : R_{t'}^{s'}(\varphi)$ ,
9.  $R_t^s(H\varphi) \equiv \text{for any } s', t' \text{ such that: } s \leq s', t' \in T_{s'}, t'\mu_{s'} t : R_{t'}^{s'}(\varphi)$ ,

We will now give intuitions related to individual elements of the above structure. The  $(S, \leq)$  pair is a partially ordered set of states of knowledge.  $T_s$  is a set of time moments in the state  $s$ .  $\mu_s$  is a binary relation on the set  $T_s$ . In addition, to fulfill the postulate that the cognitive entity, achieving a greater state of knowledge, retains all information from smaller states, it is required that for  $s \leq s'$  the following conditions holds:  $T_s \subseteq T_{s'}$  and  $\mu_s \subseteq \mu_{s'}$ . In other words, a cognitive subject achieving a higher state of knowledge maintains a set of time moments and temporal order from smaller states of knowledge.

The truth of a formula in an intuitionistic temporal structure and the truth of the formula are defined as follows:

**Definition 8** (the truth in an intuitionistic temporal structure).  $\mathfrak{M} \models \varphi$ , the formula  $\varphi$  is true in the intuitionistic temporal structure  $\mathfrak{M}$ , if and only if for any  $s \in S$  and any  $t \in T_s : R_t^s(\varphi)$ .

**Definition 9** (the truth of the formula).  $\models \varphi$ , formula  $\varphi$  is true if and only if, for any  $\mathfrak{M} : \mathfrak{M} \models \varphi$ .

## 5. Axioms $\text{IK}_t$

- (1)  $\varphi$ , if  $\varphi$  is a tautology of the intuitionistic logic of the language  $\mathcal{L}_{\text{IK}_t}$ .

- |   |  |
|---|--|
| (2) $G(\varphi \rightarrow \psi) \rightarrow (G\varphi \rightarrow G\psi)$  | (2') $H(\varphi \rightarrow \psi) \rightarrow (H\varphi \rightarrow H\psi)$  |
| (3) $G(\varphi \wedge \psi) \leftrightarrow (G\varphi \wedge G\psi)$        | (3') $H(\varphi \wedge \psi) \leftrightarrow (H\varphi \wedge H\psi)$        |
| (4) $F(\varphi \vee \psi) \leftrightarrow (F\varphi \vee F\psi)$            | (4') $P(\varphi \vee \psi) \leftrightarrow (P\varphi \vee P\psi)$            |
| (5) $G(\varphi \rightarrow \psi) \rightarrow (F\varphi \rightarrow F\psi)$  | (5') $H(\varphi \rightarrow \psi) \rightarrow (P\varphi \rightarrow P\psi)$  |
| (6) $(G\varphi \wedge F\psi) \rightarrow F(\varphi \wedge \psi)$            | (6') $(H\varphi \wedge P\psi) \rightarrow P(\varphi \wedge \psi)$            |
| (7) $G\neg\varphi \rightarrow \neg F\varphi$                                | (7') $H\neg\varphi \rightarrow \neg P\varphi$                                |
| (8) $FH\varphi \rightarrow \varphi$   | (8') $PG\varphi \rightarrow \varphi$   |
| (9) $\varphi \rightarrow GP\varphi$   | (9') $\varphi \rightarrow HF\varphi$   |
| (10) $(F\varphi \rightarrow G\psi) \rightarrow G(\varphi \rightarrow \psi)$ | (10') $(P\varphi \rightarrow H\psi) \rightarrow H(\varphi \rightarrow \psi)$ |
| (11) $F(\varphi \rightarrow \psi) \rightarrow (G\varphi \rightarrow F\psi)$ | (11') $P(\varphi \rightarrow \psi) \rightarrow (H\varphi \rightarrow P\psi)$ |

Rules: MP, RH, RG.

Ewald [3] proves the adequacy of the  $\mathbf{IK}_t$  system with respect to the class of intuitionistic temporal structures. For the purposes of proof of adequacy, the concept of consistent pair of sets is introduced.

**Definition 10** (consistent pair of sets). *The  $(X, Y)$  pair of set of sentences is consistent if and only if such finite subsets do not exist  $X_0 (= \{\varphi_1, \varphi_2, \dots, \varphi_m\}) \subseteq X$  and  $Y_0 (= \{\psi_1, \psi_2, \dots, \psi_n\}) \subseteq Y$  such that  $\vdash (\varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_m) \rightarrow (\psi_1 \vee \psi_2 \vee \dots \vee \psi_n)$*

In the  $\mathbf{IK}_t$  we can to prove the intuitionistic equivalent of the Lindenbaum lemma, namely:

**Theorem 8.** *If the pair  $(X, Y)$  is consistent, then there is the consistent pair of  $(X', Y')$  such that:*

1.  $X \subseteq X'$  and  $Y \subseteq Y'$ ,
2.  $X' \cap Y' = \emptyset$ ,
3. for any formula  $\varphi$  :  $\varphi \in X'$  or  $\varphi \in Y'$ .

The pair that fulfills these conditions is *maximum consistent pair*. Each  $(X, Y)$  maximum consistent pair can be represented by a valuation  $v : v : \text{FOR}(\mathbf{IK}_t) \rightarrow \{0, 1\}$ , such that  $v(\varphi) = 1$  iff  $\varphi \in X$ . Ewald proves for the  $\mathbf{IK}_t$  system the strong completeness Theorem in the following version:

**Theorem 9** (Adequacy  $\mathbf{IK}_t$  [3]). *For any  $\mathbf{IK}_t$ - valuation  $v$  there is an intuitionistic structure  $\mathfrak{M} = \langle S, \leq, \{T_s\}_{s \in S}, \{u_s\}_{s \in S}, \{R_t^s\}_{s \in S, t \in T_s}\rangle$ , state on knowledge  $s \in S$  and moment  $t \in T_s$  such that for any formula  $\varphi \in \text{FOR}(\mathbf{IK}_t)$  holds  $R_t^s(\varphi)$  iff  $v(\varphi) = 1$ .*

In the semantic of the  $\mathbf{IK}_t$  system, we did not impose any conditions on the temporal order in intuitionistic temporal structures. The  $\mathbf{IK}_t$  system is therefore an analogue of the  $\mathbf{K}_t$  system, i.e., it is a minimal system of intuitionistic temporal logic.

## 6. Modified Semantics for $\mathbf{IK}_t$

We will consider the modified semantics for  $\mathbf{IK}_t$  and examine its basic properties.  $\mathbf{IK}_t$  is used to describe states of knowledge that change as knowledge gains. Acquiring knowledge in  $\mathbf{IK}_t$  is understood as moving to states of knowledge; however, as in the  $\mathbf{IK}_t$  system, it is assumed that all knowledge from a given state of knowledge is available in any state of knowledge not lesser than contemplated. Therefore,

the monotonicity of the knowledge acquisition process is assumed. We achieve knowledge by enriching our knowledge with new facts. This can occur in several cases.

We can enrich our knowledge when by research we describe events from the past that took place at times that were not known in a given state of knowledge. We did not have any information about these events in this state of knowledge. In this case, the temporal structure in not lesser state of knowledge expands into the past and is a superset of the temporal structure of a given state of knowledge. For the same reasons, the time structure of the state of knowledge may expand into the future.

The expansion of the temporal structure (regardless of whether it takes place in the past or in the future) causes a change in the domain of the relationship. Therefore, in the new state of knowledge, the changed relation between moments of time should be considered.

Another possible option to achieve knowledge is the situation when the set of moments of time does not change, but the powers of sets of formulas increase, which we can determine if they are fulfilled in given time moments. Therefore, in this case there is no expansion of the time structure, neither into the past nor into the future, but by getting to know the present, past or future better within the known temporal structure, we attribute to moments more numerous sets of formulas fulfilled in these moments.

In the proposed semantics, the state of knowledge consists of a set of facts, which are semantic correlates of formulas, a set of moments of time, and the relationship at the set of moments of time. A subset of the set of facts assigned to a specific moment is understood as the set of facts known at that moment.

Achievable states of knowledge are different in their level of knowledge. The level of knowledge is determined by its constituent elements, namely: a set of moments of time, the temporal order relation and sets of formulas fulfilled at individual time moments. We will say that the state of knowledge of  $m''$  has not lesser level of knowledge than the state of knowledge of  $m'$ , if and only if the following conditions are satisfied:

1. The set of moments of time in the state  $m'$  is included in the set of moments of time in the state  $m''$ . (Changing the number of moments of time causes a change in the level of knowledge.)
2. In the  $m''$ , there are - occurring between moments of time - *earlier-later* relationships that existed in the  $m'$  state of knowledge. Also, in the  $m''$ , such relationships can occur that did not take place in the state  $m'$ .
3. All events that are known in the state of knowledge  $m'$  are also known in the state of knowledge  $m''$ . (What is known does not cease to be known also when new known events occur.) In addition at the moments of time of the state of knowledge  $m''$ , may be known some events that are not known in the equivalents of these moments in the state of knowledge  $m'$ .

There are specific relationships between conditions 1, 2 and 3. Fulfillment of condition 1 implies fulfillment of condition 2, because we skip situations in which *new* moments of time are not in any relationship *earlier-later* with other moments. A change in the set of moments of time therefore entails a change in the relationship between the moments of time. It is not the other way round. Changing the relationship between the moments of time does not have to involve changing the set of time moments. In the state of knowledge with no less level of knowledge, new relationships *earlier-later* can occur between time moments in the state of knowledge with a lower level of knowledge. Therefore, fulfillment of condition 2 does not entail fulfillment of condition 1. Similarly, fulfillment of condition 3 does not entail fulfillment of condition 1 or 2, because new facts may be known without new time moments or new relationships *earlier-later*.

Each moment is assigned a non-empty set of known events. If there are new moments, there are also new facts known. The fulfillment of condition 1 implies the fulfillment of condition 3.

The existence of new relationships *earlier-later*, on the other hand, entails the existence of new facts known at the times in which new relationships *earlier-later* take place. Thus, as in the case of condition 1, the fulfillment of condition 2 implies the fulfillment of condition 3.

We have two types of time. The first is the time that is assigned to the state of knowledge. It is a structure consisting of a set of moments of time and relationship *earlier-later* of a given state of knowledge. The other is time that is not relativized to any state of knowledge. This time is the sum of the times assigned to all possible states of knowledge.

We write these intuitions in a formal way.

- $I$  is a non-empty set (indexes of state of knowledge).
- $T_i$  ( $i \in I$ ) is a non-empty set (of moments in the state of knowledge indexed by  $i$ ).
- $R_i (\subseteq T_i \times T_i)$  is a binary relation defined on a set of moments of time in the state of knowledge indexed by  $i$ . Relation  $R_i$  is understood as the relation *earlier-later* on the set of moments of time of state of knowledge indexed by  $i$ .
- $\mathfrak{T}_i = \langle T_i, R_i \rangle$ . It is a time in the state of knowledge indexed by  $i$ .
- $T = \bigcup_{i \in I} T_i$  is a set of all time moments existing in any state of knowledge.
- $R = \bigcup_{i \in I} R_i$  is a binary relation on the set  $T$ . This relation is understood as the *earlier-later* relation for a time not relativized to any state of knowledge. We note that  $R \subseteq T \times T$ .
- $\mathfrak{T} = \langle T, R \rangle$  it is a time not relativized to any state of knowledge.
- $V_i \subseteq T_i \times 2^{\mathcal{AP}}$ , where  $i \in I$ .  $V_i$  is a function that assigns  $t \in T_i$  subsets  $V_i(t)$  to a set of sentence letters.
- $\mathcal{F} = \{V_i : i \in I\}$  is a set of valuations.
- $m_i = \langle T_i, R_i, V_i \rangle$  where  $i \in I$ . ( $m_i$  is the state of knowledge indexed by  $i$ .)
- $\mathfrak{M} = \{\langle T_i, R_i, V_i \rangle : V_i \in \mathcal{F}, i \in I\}$ , or  $\mathfrak{M} = \{m_i : i \in I\}$ .  $\mathfrak{M}$  is a model based on the  $\mathfrak{T}$  and class  $\mathcal{F}$  function.

We define the relationship  $\leq (\subseteq \mathfrak{M} \times \mathfrak{M})$

**Definition 11.** For any  $i, j \in I$  :

$$m_i \leq m_j \text{ iff } (T_i \subseteq T_j \text{ and } R_i \subseteq R_j \text{ and for any } t \in T_i : V_i(t) \subseteq V_j(t)).$$

That for the states of knowledge  $m_i, m_j$  the relation  $\leq (m_i \leq m_j)$  is understood as follows: state of knowledge  $m_j$  has no lower level of knowledge than the state of knowledge  $m_i$ .

The relationship  $\leq$  is determined by the inclusions of a set of moments of time, the relationship between the moments of time and sets of events known at particular moments of time. The  $\leq$  relation is therefore reflexive and transitive.

**Theorem 10** ([4]). For any  $m_i (\in \mathfrak{M}) : m_i \leq m_i$ .

**Theorem 11** ([4]). For any  $m_i, m_j, m_k (\in \mathfrak{M})$  :

$$\text{if } (m_i \leq m_j \text{ and } m_j \leq m_k), \text{ then } m_i \leq m_k.$$

The relationship  $\leq$  partially organizes the set of states of knowledge. In the states of knowledge, various relationships may occur between sets of time moments, earlier-later relations and valuations. Let us consider some of them.

The first possible situation is:

$$T_i = T_j \text{ and } R_i = R_j \text{ and } \forall_{t \in T_i} (V_i(t) \subseteq V_j(t)).$$

This situation occurs when sets of time moments of states of knowledge  $m_i$  and  $m_j$  are the same ( $T_i = T_j$ ). The relations ( $R_i = R_j$ ) are the same in both states of knowledge. The state of knowledge  $m_j$ , as a state of knowledge with no lower level of knowledge than the state of knowledge  $m_i$ , is created by changing the value of the function  $V_i$  that assigns moments to subsets of the set  $\mathcal{AP}$ . In other words, in this case, the state of knowledge about a not lower level of knowledge is created by increasing the amount of facts known at particular times.

The second possible situation may be as follows:

$$T_i \subseteq T_j, R_i \subseteq R_j \text{ and } \forall_{t \in T_i} (V_i(t) = V_j(t)).$$

In this case, the  $m_j$ , as a state of knowledge with not lesser level of knowledge than the  $m_i$ , is created by adding to the structure of the state of knowledge  $m_i$  new moments of time. For any time  $t (\in T_i)$  does not change the set  $V_i(t)$ . The change in the level of knowledge is that in the state of knowledge  $m_j$  new time moments appear (in the future or in the past). Due to the new time moments, in the state of knowledge  $m_j$  all the components change. The set of time moments changes. The relation *earlier-later* is changing, because certain time moments of the state of knowledge  $m_i$  will be in relation *earlier-later* with new time moments. The evaluating function is also changing, assigning subsets of the sentence letter set to moments of time because its domain is changing (subsets of the set of sentence letters will be assigned new time moments).

Yet another option is:

$$T_i = T_j, R_i \subseteq R_j \text{ and } \forall_{t \in T_i} (V_i(t) \subseteq V_j(t)).$$

It may also be that the change in the level of knowledge of the state of knowledge does not consist of changing the set of time moments known in the state of knowledge  $m_i$  but on the change of the property of time in the state of knowledge  $m_i$ . In other words, the change of ownership of the relationship in this state of knowledge. Such a change, however, entails a change in the number of facts known at these times.

Further states of knowledge - with an increasingly higher level of knowledge—can arise by increasing the level of knowledge regarding the various components of the state of knowledge.

To shorten the entries we will introduce the designation:

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$m_i^* (= \langle T_i^*, R_i^*, V_i^* \rangle)$  (where  $i \in I$ ) is any  $m_j (\in \mathfrak{M})$  such that  $m_i \leq m_j$ .

**Definition 12** (the truth of a formula in the state of knowledge at some moment of time). *The truth of the formula  $\varphi (\in \text{FOR}(\mathcal{L}_{\mathbf{IK}_i}))$  in the model  $\mathfrak{M}$ , state of knowledge  $m_i (= \langle T_i, R_i, V_i \rangle)$ , at the moment  $t (\in T_i)$  we define as follows:*

1.  $\mathfrak{M}, m_i, t \models \varphi \quad \equiv \quad \varphi \in V_i(t), \text{ if } \varphi \in \mathcal{AP},$
2.  $\mathfrak{M}, m_i, t \models \neg \varphi \quad \equiv \quad \text{for any } m_i^* \in \mathfrak{M} : \mathfrak{M}, m_i^*, t \not\models \varphi$
3.  $\mathfrak{M}, m_i, t \models \varphi \vee \psi \quad \equiv \quad \mathfrak{M}, m_i, t \models \varphi \text{ or } \mathfrak{M}, m_i, t \models \psi,$
4.  $\mathfrak{M}, m_i, t \models \varphi \wedge \psi \quad \equiv \quad \mathfrak{M}, m_i, t \models \varphi \text{ and } \mathfrak{M}, m_i, t \models \psi,$
5.  $\mathfrak{M}, m_i, t \models \varphi \rightarrow \psi \quad \equiv \quad \text{for any } m_i^* \in \mathfrak{M} : (\mathfrak{M}, m_i^*, t \not\models \varphi \text{ or } \mathfrak{M}, m_i^*, t \models \psi),$
6.  $\mathfrak{M}, m_i, t \models F\varphi \quad \equiv \quad \text{there exists } t' \in T_i, tR_it' : \mathfrak{M}, m_i, t' \models \varphi,$
7.  $\mathfrak{M}, m_i, t \models G\varphi \quad \equiv \quad \text{for any } m_i^* (\in \mathfrak{M}), \text{ for any } t' (\in T_i^*) \text{ such that } tR_i^*t' : \mathfrak{M} \models m_i^*, t' \varphi,$
8.  $\mathfrak{M}, m_i, t \models P\varphi \quad \equiv \quad \text{there exists } t' \in T_i, t'R_it : \mathfrak{M}, m_i, t' \models \varphi,$
9.  $\mathfrak{M}, m_i, t \models H\varphi \quad \equiv \quad \text{for any } m_i^* (\in \mathfrak{M}), \text{ for any } t' (\in T_i^*) \text{ such that } t'R_i^*t : \mathfrak{M} \models m_i^*, t' \varphi,$

The necessary condition for the sentence  $F\varphi$  to be true in the state of knowledge  $m_i$ , at the time of  $t$  ( $\in T_i$ ) is the existence in the time structure of the state of knowledge  $m_i$  the moment  $t' (\in T_i)$ , later than  $t$  ( $tR_it'$ ), in which the sentence  $\varphi$  is true. If such a moment exists in the structure of time of  $m_i$ , then from the definition of the relationship  $\leq$  and the theory of multiplicative properties of inclusions it follows that such a moment also exists in the structure of time of each state of knowledge with a level of knowledge not less than the level of state of knowledge  $m_i$ . Hence verification of the truth of the sentence  $F\varphi$  in the state of knowledge  $m_i$  can be limited to the state of knowledge  $m_i$ . Please note that if the sentence  $F\varphi$  is not true at the time  $t$  it does not mean that in  $t$  the sentence  $F\neg\varphi$  is true.

For the  $G$  operator the situation is different. According to understanding the  $G$  operator, the sentence  $G\varphi$  reads: *it will always be in the future that  $\varphi$* . For the sentence  $G\varphi$  to be true in the state of knowledge  $m_i$  at  $t (\in T_i)$ , it is necessary that the sentence  $\varphi$  is true in any state of knowledge  $m_i^*$  at any time  $t' (\in T_i^*)$  later than  $t$  ( $tR_i^*t'$ ). The truth of the sentence  $G\varphi$  cannot be considered only within the temporal limits of a given state of knowledge. Just because the sentence  $\varphi$  is always true in the future means that  $\varphi$  is true at any point in the future. Since the state of knowledge  $m_i$  is assigned only a certain fragment of the time structure, when defining the concept of the truth for a sentence built using the operator  $G$ , all states of knowledge with a level of knowledge not lower than the level of knowledge of state  $m_i$ .

If the definition of the truth of the sentence  $G\varphi$  were in the form that was adopted in the system, e.g., in the system  $T_m$  [5] (intuitionistic temporal logic of unchanging time (By *unchanging time* (in accepted terminology) is understood a time such that for any  $i, j \in I$ : ( $T_i = T_j$  and  $R_i = R_j$ )).), i.e.,

$$\mathfrak{M}, m_i, t \models G\varphi \text{ iff for any } t' \in T_i, \text{ such that } tR_it' : \mathfrak{M}, m_i, t' \models \varphi$$

this would lead to contradictions. It would be possible that in some state of knowledge  $m_i$  would occur at the moment  $t$

$$\mathfrak{M}, m_i, t \models G\varphi. \quad (1)$$

and at some level of knowledge  $m_j$ , with a level of knowledge not lesser than the level of knowledge of the state of knowledge  $m_i$ , i.e.,  $m_i \leq m_j$ , there would be a moment  $t_1 (\in T_j)$  such that:  $t_1 \notin T_i$ ,  $tR_jt_1$  and  $\mathfrak{M}, m_j, t_1 \not\models \varphi$ . Therefore, we have:

$$\mathfrak{M}, m_j, t \not\models G\varphi. \quad (2)$$

What is known does not cease to be known when the level of knowledge increases. Since the state of knowledge of  $m_j$  is a state of knowledge with a level of knowledge of not less than the level of knowledge of the state of  $m_i$ , so that  $\mathfrak{M}, m_i, t \models G\varphi$  we conclude that  $\mathfrak{M}, m_j, t \models G\varphi$ . This is contrary to (2).

The understanding of the truth of the formula  $G\varphi$ , in the state of knowledge  $m_i$ , at the moment  $t$  excludes the situation described above.

We will now give some basic definitions.

**Definition 13.**  $\mathfrak{M} \models \varphi$ ,  $\varphi$  is true in the model  $\mathfrak{M}$ , iff for any state of knowledge  $m_i (\in \mathfrak{M})$  and for any  $t (\in T_i) : \mathfrak{M}, m_i, t \models \varphi$ .

**Definition 14.**  $\mathfrak{T} \models \varphi$ ,  $\varphi$  is true in time  $\mathfrak{T}$ , iff  $\varphi$  is true in the model  $\mathfrak{M}$  for any non-empty class  $\mathcal{F} (= \{V_i : i \in I\})$  of function.

**Definition 15.**  $\models \varphi$ ,  $\varphi$  is true iff for any  $\mathfrak{T} : \mathfrak{T} \models \varphi$ .

In some sciences (e.g., empirical sciences) it happens that sentences considered to be true at some time, with the development of scientific theories, turn out to be false. It happens that certain laws of empirical sciences in force in a given period are subject to verification and are changed, and sometimes even rejected, as laws that inaccurately or even misrepresent the state of the world. Such verification is possible due to the increase in the level of knowledge. In our terminology, we would write this fact as follows: the sentence true in some state of knowledge  $m_i$ , in some state of knowledge which level of knowledge is not lesser than the level of knowledge of  $m_i$  may not be true. In the  $\mathbf{IK}_t$  system, this is not possible. What is true in the state of knowledge  $m_i$  is also true in any state of knowledge, with a level of knowledge not lesser than the level of knowledge of  $m_i$ .

There are many differences between temporal logic systems based on classical logic and temporal logic systems based on intuitionistic logic. One of them is that failing to the truth of  $\varphi$  does not entail the truth of  $\neg\varphi$ .

Let us consider the following situation. The sentence  $\varphi$  is not known in the state of knowledge  $m_i$  at the moment  $t (\in T_i)$ , while is known at this moment in a state of knowledge  $m_j$ , whose level knowledge is not lesser than the level of knowledge in the state  $m_i$ . If the sentence  $\varphi$  is not known at the time  $t$  in the state  $m_i$ , it would be considered that at the time  $t$  the sentence  $\neg\varphi$  is known, then—according to the accepted condition of fulfilling  $\neg\varphi$  - the sentence  $\varphi$  could not be known at the time of  $t$  in any state of knowledge with a level of knowledge not lesser than the level of knowledge of  $m_i$ . In particular, the sentence  $\varphi$  could not be known at the time  $t$ , in the state of knowledge  $m_j$ . This leads to a contradiction, since we get that  $\varphi$  is known at the time of  $t$ , in the state  $m_j$ , and we conclude that it is known and unknown at the same time. When the sentence  $\varphi$  is known at some moment of time, in some state of knowledge  $m_i$ , then in any state of knowledge with the level of knowledge not lesser than the level of knowledge of state  $m_i$  at this moment the sentence  $\varphi$  is known. However, when  $\neg\varphi$  is not known at some moment of time, it does not mean that at this moment, in any state of knowledge with a level of knowledge no lesser than the level of knowledge of  $m_i$ , is known  $\varphi$ . It only means that it is not true that in every state of knowledge in which the level of knowledge is not lesser than the level of knowledge of  $m_i$ ,  $\varphi$  is currently unknown.

We will prove a lemma that expresses the monotonicity of knowledge in the  $\mathbf{IK}_t$  system. What is known in the state of knowledge  $m_i$  is also known in every state of knowledge whose level of knowledge is not lesser than the level of knowledge of the state  $m_i$ .

**Lemma 1.** For any formula  $\varphi (\in \text{FOR}(\mathcal{L}_{\mathbf{IK}_t}))$ , for any  $m_i, m_j (\in \mathfrak{M})$  :

$$\text{if } (m_i \leq m_j \text{ and } \mathfrak{M}, m_i, t \models \varphi), \text{ then } \mathfrak{M}, m_j, t \models \varphi.$$

**Proof.** We will prove by induction, due to the length of the formula  $\varphi$ . Suppose that  $m_i \leq m_j$ .

( $\varphi \in \mathcal{AP}$ ) Let us first consider the case when  $\varphi$  is a sentence letter.

By Definition 11 if  $m_i \leq m_j$ , then for any  $t \in T_i$  holds

$$V_i(t) \subseteq V_j(t). \quad (3)$$

If  $\mathfrak{M}, m_i, t \models \varphi$ , then from the Definition 12

$$\varphi \in V_i(t). \quad (4)$$

From (3) and (4) we receive

$$\varphi \in V_j(t). \quad (5)$$

Because  $\varphi$  is a sentence letter, so from (5) and the definition of 12 we have  $\mathfrak{M}, m_j, t \models \varphi$ .

**Induction assumption:** Let  $\varphi, \psi$  be such that :

- (a) if  $\mathfrak{M}, m_i, t \models \varphi$ , then  $\mathfrak{M}, m_j, t \models \varphi$ ,  
and
- (b) if  $\mathfrak{M}, m_i, t \models \psi$ , then  $\mathfrak{M}, m_j, t \models \psi$ .

We will consider complex formulas built from the formulas  $\varphi, \psi$  using sentence connectives and temporal operators.

( $\neg\varphi$ ) Let us assume that  $\mathfrak{M}, m_i, t \models \neg\varphi$ .

From the definition of the condition for negation (Definition 12) we have:

$$\text{for any } m_k, \text{ such that } m_i \leq m_k : \mathfrak{M}, m_k, t \not\models \varphi. \quad (6)$$

Let us consider any state of knowledge  $m_l$  with a level of knowledge not lesser than the level of  $m_j$ , i.e.,

$$m_j \leq m_l. \quad (7)$$

From (7), the assumption that  $m_i \leq m_j$  and the transitivity of the  $\leq$ , we have that  $m_i \leq m_l$ . Therefore, from (6) we have:  $\mathfrak{M}, m_l, t \not\models \varphi$ . Because  $m_l$  is any state of knowledge whose level of knowledge is not lesser than the level of knowledge of  $m_j$ , we get:

$$\text{for any } m_l \text{ such that } m_j \leq m_l \text{ we have: } \mathfrak{M}, m_l, t \not\models \varphi. \quad (8)$$

From (8) and the condition for negation (Definition 12) we have:  $\mathfrak{M}, m_j, t \models \neg\varphi$ .

( $\varphi \wedge \psi$ ) Let us assume that  $\mathfrak{M}, m_i, t \models \varphi \wedge \psi$ .

So from the condition for the conjunction (Definition 12) we have:

$$\mathfrak{M}, m_i, t \models \varphi, \quad (9)$$

and

$$\mathfrak{M}, m_i, t \models \psi. \quad (10)$$

From (9) and point a) of the induction assumption we get:

$$\mathfrak{M}, m_j, t \models \varphi. \quad (11)$$



Similarly, from (10) and point b) of the induction assumption we get:

$$\mathfrak{M}, m_j, t \models \psi. \quad (12)$$

From (11), (12) and the condition for the conjunction (Definition 12) we get  $\mathfrak{M}, m_j, t \models \varphi \wedge \psi$ .

( $\varphi \vee \psi$ ) Reasoning analogous to conjunction.

( $\varphi \rightarrow \psi$ ) Let us assume that  $\mathfrak{M}, m_i, t \models \varphi \rightarrow \psi$ .

From the condition for the implication (Definition 12) we have:

$$\text{for any } m_i^* (\in \mathfrak{M}) : (\mathfrak{M}, m_i^*, t \not\models \varphi \text{ or } \mathfrak{M}, m_i^*, t \models \psi), \quad (13)$$

Let us consider the state of knowledge  $m_l$  with a level of knowledge not lesser than the level of knowledge of  $m_j$ , i.e.,

$$m_j \leq m_l. \quad (14)$$

From (14), the assumption that  $m_i \leq m_j$  and the transitivity of the relationship  $\leq$  we get that  $m_i \leq m_l$ . From (13) we have:  $\mathfrak{M}, m_l, t \not\models \varphi$  or  $\mathfrak{M}, m_l, t \models \psi$ . Because  $m_l$  is any state of knowledge in which the level of knowledge is not lesser than the level of knowledge in the state  $m_j$ , we get:

$$\text{for any } m_l \text{ such that } m_j \leq m_l : \mathfrak{M}, m_l, t \not\models \varphi \text{ or } \mathfrak{M}, m_l, t \models \psi. \quad (15)$$

From (15) and the condition for the implications (Definition 12) we get  $\mathfrak{M}, m_j, t \models \varphi \rightarrow \psi$ .

( $G\varphi$ ) Suppose  $\mathfrak{M}, m_i, t \models G\varphi$ . From the condition for the  $G$  operator (Definition 12) we have:

$$\text{for any } m_i^* (\in \mathfrak{M}), \text{ for any } t_1 (\in T_i^*) \text{ such that } tR_i^* t_1 : \mathfrak{M}, m_i^*, t_1 \models \varphi, \quad (16)$$

Let us consider any state of knowledge  $m_l$  with a level of knowledge is not lesser than the level of knowledge of the state  $m_j$ , i.e.,

$$m_j \leq m_l. \quad (17)$$

From (17), the assumption that  $m_i \leq m_j$  and the transitivity of the relationship  $\leq$ , we get that  $m_i \leq m_l$ . Som from (16) we get :

$$\text{for any } t_1 (\in T_l) \text{ such that } tR_l t_1 \text{ holds: } \mathfrak{M}, m_l, t_1 \models \varphi. \quad (18)$$

Because the state of knowledge  $m_l$  is a state of knowledge with a level of knowledge not lower than the level of knowledge in the state  $m_j$  we have:

$$\text{for any } m_l, \text{ for any } t_1 (\in T_l) \text{ if } (m_j \leq m_l \text{ and } tR_l t_1), \text{ then } \mathfrak{M}, m_l, t_1 \models \varphi. \quad (19)$$

From (19) and the condition for the  $G$  operator (Definition 12) we obtain:  $\mathfrak{M}, m_j, t \models G\varphi$

( $H\varphi$ ) Reasoning similar to the  $G$  operator.

( $F\varphi$ ) Let us assume that  $\mathfrak{M}, m_i, t \models F\varphi$ . From the condition for the operator  $F$  (Definition 12) there is the moment  $t_1 (\in T_i)$ ,  $tR_i t_1$ , such that:

$$\mathfrak{M}, m_i, t_1 \models \varphi. \quad (20)$$

From (2) and point a) of the induction assumption we have:

$$\mathfrak{M}, m_j, t_1 \models \varphi. \quad (21)$$

Assuming that  $m_i \leq m_j$  and the definition of 11 we get that:

$$t \in T_j, t_1 \in T_j, tR_j t_1. \quad (22)$$

From (21), (22) and the condition for the  $F$  operator (Definition 12) we obtain  $\mathfrak{M}, m_j, t \models F\varphi$ .  
( $P\varphi$ ) Reasoning similar to the  $F$  operator.

□

We have therefore shown that what is true in a given state of knowledge  $m_i$  it is also true in any state of knowledge in which the level of knowledge is not lesser than the level of knowledge in the state  $m_i$ .

## 7. Simplified Axiomatics $\mathbf{IK}_t$

The axioms proposed by Ewald  $\mathbf{IK}_t$  are dependent axioms. Some axioms can be derived from other axioms. Proofs of dependencies of selected axioms were provided by Surowik [6]. We offer a simplified set of axioms for  $\mathbf{IK}_t$ :

- |  |  |
|--|--|
| A1) $\varphi$ , if $\varphi$ is a tautology of the intuitionistic logic of the language $\mathfrak{L}_{\mathbf{IK}_t}$ . |  |
| (A2) $G(\varphi \rightarrow \psi) \rightarrow (G\varphi \rightarrow G\psi)$  | (A2') $H(\varphi \rightarrow \psi) \rightarrow (H\varphi \rightarrow H\psi)$ |
| (A3) $F(\varphi \vee \psi) \rightarrow (F\varphi \vee F\psi)$  | (A3') $P(\varphi \vee \psi) \rightarrow (P\varphi \vee P\psi)$               |
| (A4) $G(\varphi \rightarrow \psi) \rightarrow (F\varphi \rightarrow F\psi)$  | (A4') $H(\varphi \rightarrow \psi) \rightarrow (P\varphi \rightarrow P\psi)$ |
| (A5) $F\varphi \rightarrow \neg G\neg\varphi$  | (A5') $P\varphi \rightarrow \neg H\neg\varphi$                               |
| (A6) $FH\varphi \rightarrow \varphi$   | (A6') $PG\varphi \rightarrow \varphi$  |
| 9A7) $\varphi \rightarrow GP\varphi$   | (A7') $\varphi \rightarrow HF\varphi$  |
| (A8) $(F\varphi \rightarrow G\psi) \rightarrow G(\varphi \rightarrow \psi)$  | (A8') $(P\varphi \rightarrow H\psi) \rightarrow H(\varphi \rightarrow \psi)$ |

Rules: MP, RH, RG.

We will prove that this axiomatics is equivalent to the axiomatics proposed by Ewald. To demonstrate the derivability of some  $\mathbf{IK}_t$  axioms with the other axioms of this system, the following Theorems will be useful.

### Theorem 12.

- (a) The RRG rule :  $\frac{\varphi \rightarrow \psi}{G\varphi \rightarrow G\psi}$  is a rule of  $\mathbf{IK}_t$ .
- (b) The RRH rule :  $\frac{\varphi \rightarrow \psi}{H\varphi \rightarrow H\psi}$  is a rule of  $\mathbf{IK}_t$ .

**Proof.** We will prove only (a). Proof (b) is analogous.

(a)

- |  |            |
|--|------------|
| 1. $\vdash_{\mathbf{IK}_t} \varphi \rightarrow \psi$   | assumption |
| 2. $\vdash_{\mathbf{IK}_t} G(\varphi \rightarrow \psi)$  | 1, RG      |
| 3. $\vdash_{\mathbf{IK}_t} G(\varphi \rightarrow \psi) \rightarrow (G\varphi \rightarrow G\psi)$ | A2         |
| 4. $\vdash_{\mathbf{IK}_t} G\varphi \rightarrow G\psi$   | 2, 3, MP   |

□

**Theorem 13.**

- (a) The RF rule :  $\frac{\varphi \rightarrow \psi}{F\varphi \rightarrow F\psi}$  is a rule of  $\mathbf{IK}_t$ .
- (b) The RP rule :  $\frac{\varphi \rightarrow \psi}{P\varphi \rightarrow P\psi}$  is a rule of  $\mathbf{IK}_t$ .

The proof of this theorem is obtained in a manner analogous to the proof of the theorem of the previous one, with the difference that instead of the axiom A2 (A2') we use the A4 (A4') axiom.

We will show that in  $\mathbf{IK}_t$  "old" axioms 3, 3', 6, 6', 7, 7', 11, 11' are inferable. The implications of the "old" 4 and 4' axioms are also inferable.

**Lemma 2.**  $\vdash_{\mathbf{IK}_t} G(\varphi \wedge \psi) \leftrightarrow (G\varphi \wedge G\psi)$

**Proof.**

(A)  $\vdash_{\mathbf{IK}_t} G(\varphi \wedge \psi) \rightarrow (G\varphi \wedge G\psi)$

1.  $\vdash_{\mathbf{IK}_t} (\varphi \wedge \psi) \rightarrow \varphi$  A1
2.  $\vdash_{\mathbf{IK}_t} (\varphi \wedge \psi) \rightarrow \psi$  A1
3.  $\vdash_{\mathbf{IK}_t} G(\varphi \wedge \psi) \rightarrow G\varphi$  1, RRG
4.  $\vdash_{\mathbf{IK}_t} G(\varphi \wedge \psi) \rightarrow G\psi$  2, RRG
5.  $\vdash_{\mathbf{IK}_t} (G(\varphi \wedge \psi) \rightarrow G\varphi) \rightarrow ((G(\varphi \wedge \psi) \rightarrow G\psi) \rightarrow (G(\varphi \wedge \psi) \rightarrow (G\varphi \wedge G\psi)))$  A1
6.  $\vdash_{\mathbf{IK}_t} (G(\varphi \wedge \psi) \rightarrow G\psi) \rightarrow (G(\varphi \wedge \psi) \rightarrow (G\varphi \wedge G\psi))$  3, 5, MP
7.  $\vdash_{\mathbf{IK}_t} G(\varphi \wedge \psi) \rightarrow (G\varphi \wedge G\psi)$  4, 6, MP

(B)  $\vdash_{\mathbf{IK}_t} (G\varphi \wedge G\psi) \rightarrow G(\varphi \wedge \psi)$

1.  $\vdash_{\mathbf{IK}_t} \varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$  A1
2.  $\vdash_{\mathbf{IK}_t} G\varphi \rightarrow G(\psi \rightarrow (\varphi \wedge \psi))$  1, RRG
3.  $\vdash_{\mathbf{IK}_t} G(\psi \rightarrow (\varphi \wedge \psi)) \rightarrow (G\psi \rightarrow G(\varphi \wedge \psi))$  A2
4.  $\vdash_{\mathbf{IK}_t} G\varphi \rightarrow (G\psi \rightarrow G(\varphi \wedge \psi))$  2, 3, SYLL
5.  $\vdash_{\mathbf{IK}_t} (G\varphi \rightarrow (G\psi \rightarrow G(\varphi \wedge \psi))) \rightarrow ((G\varphi \wedge G\psi) \rightarrow G(\varphi \wedge \psi))$  A1
6.  $\vdash_{\mathbf{IK}_t} (G\varphi \wedge G\psi) \rightarrow G(\varphi \wedge \psi)$  4, 5, MP

With (A) and (B) we get a thesis.  $\square$

The next lemma is proved similarly.

**Lemma 3.**  $\vdash_{\mathbf{IK}_t} H(\varphi \wedge \psi) \leftrightarrow (H\varphi \wedge H\psi)$

**Lemma 4.**  $\vdash_{\mathbf{IK}_t} (F\varphi \vee F\psi) \rightarrow F(\varphi \vee \psi)$

**Proof.**

1.  $\vdash_{\mathbf{IK}_t} \varphi \rightarrow (\varphi \vee \psi)$  A1
2.  $\vdash_{\mathbf{IK}_t} \psi \rightarrow (\varphi \vee \psi)$  A1
3.  $\vdash_{\mathbf{IK}_t} F\varphi \rightarrow F(\varphi \vee \psi)$  1, RF

4.  $\vdash_{\mathbf{IK}_t} F\psi \rightarrow F(\varphi \vee \psi)$  2,RF
  5.  $\vdash_{\mathbf{IK}_t} (F\varphi \rightarrow F(\varphi \vee \psi)) \rightarrow ((F\psi \rightarrow F(\varphi \vee \psi)) \rightarrow ((F\varphi \vee F\psi) \rightarrow F(\varphi \vee \psi)))$  A1
  6.  $\vdash_{\mathbf{IK}_t} (F\psi \rightarrow F(\varphi \vee \psi)) \rightarrow ((F\varphi \vee F\psi) \rightarrow F(\varphi \vee \psi))$  3,5,MP
  7.  $\vdash_{\mathbf{IK}_t} (F\varphi \vee F\psi) \rightarrow F(\varphi \vee \psi)$  4,6,MP
- 

**Lemma 5.**  $\vdash_{\mathbf{IK}_t} (P\varphi \vee P\psi) \rightarrow P(\varphi \vee \psi)$

Proof analogous to the proof of the previous lemma.

**Lemma 6.**  $\vdash_{\mathbf{IK}_t} (G\varphi \wedge F\psi) \rightarrow F(\varphi \wedge \psi)$

**Proof.**

1.  $\vdash_{\mathbf{IK}_t} \varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$  A1
  2.  $\vdash_{\mathbf{IK}_t} G\varphi \rightarrow G(\psi \rightarrow (\varphi \wedge \psi))$  1, RRG
  3.  $\vdash_{\mathbf{IK}_t} G(\psi \rightarrow (\varphi \wedge \psi)) \rightarrow (F\psi \rightarrow F(\varphi \wedge \psi))$  A4
  4.  $\vdash_{\mathbf{IK}_t} G\varphi \rightarrow (F\psi \rightarrow F(\varphi \wedge \psi))$  2,3, SYLL
  5.  $\vdash_{\mathbf{IK}_t} (G\varphi \rightarrow (F\psi \rightarrow F(\varphi \wedge \psi))) \rightarrow ((G\varphi \wedge F\psi) \rightarrow F(\varphi \wedge \psi))$  A1
  6.  $\vdash_{\mathbf{IK}_t} (G\varphi \wedge F\psi) \rightarrow F(\varphi \wedge \psi)$  4,5, MP
- 

**Lemma 7.**  $\vdash_{\mathbf{IK}_t} (H\varphi \wedge P\psi) \rightarrow P(\varphi \wedge \psi)$

Proof analogous to the proof of the previous lemma.

**Lemma 8.**  $\vdash_{\mathbf{IK}_t} G\neg\varphi \rightarrow F\neg\varphi$

**Proof.**

1.  $\vdash_{\mathbf{IK}_t} F\varphi \rightarrow \neg G\neg\varphi$  A5
  2.  $\vdash_{\mathbf{IK}_t} (F\varphi \rightarrow \neg G\neg\varphi) \rightarrow (G\neg\varphi \rightarrow \neg F\varphi)$  A1
  3.  $\vdash_{\mathbf{IK}_t} (G\neg\varphi \rightarrow \neg F\varphi)$  1,2,MP
- 

**Lemma 9.**  $\vdash_{\mathbf{IK}_t} H\neg\varphi \rightarrow \neg P\varphi$

Proof analogous to the proof of the previous lemma.

**Lemma 10.**  $\vdash_{\mathbf{IK}_t} F(\varphi \rightarrow \psi) \rightarrow (G\varphi \rightarrow F\psi)$

**Proof.**

1.  $\vdash_{\mathbf{IK}_t} \varphi \rightarrow ((\varphi \rightarrow \psi) \rightarrow \psi)$  A1
2.  $\vdash_{\mathbf{IK}_t} G\varphi \rightarrow G((\varphi \rightarrow \psi) \rightarrow \psi)$  1,RRG
3.  $\vdash_{\mathbf{IK}_t} G((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow (F(\varphi \rightarrow \psi) \rightarrow F\psi)$  A4
4.  $\vdash_{\mathbf{IK}_t} G\varphi \rightarrow (F(\varphi \rightarrow \psi) \rightarrow F\psi)$  2,3, SYLL

5.  $\vdash_{\mathbf{IK}_t} (G\varphi \rightarrow (F(\varphi \rightarrow \psi) \rightarrow F\psi)) \rightarrow (F(\varphi \rightarrow \psi) \rightarrow (G\varphi \rightarrow F\psi))$  A1
  6.  $\vdash_{\mathbf{IK}_t} F(\varphi \rightarrow \psi) \rightarrow (G\varphi \rightarrow F\psi)$  4,5, MP
- 

**Lemma 11.**  $\vdash_{\mathbf{IK}_t} P(\varphi \rightarrow \psi) \rightarrow (H\varphi \rightarrow P\psi)$

Proof analogous to the proof of the previous lemma.

We will show that the "new" A5 and A5' axioms are we can derive from the 'old' '8 and 8' axioms.

**Lemma 12.**  $G\neg\varphi \rightarrow \neg F\varphi \vdash_{\mathbf{IK}_t} F\varphi \rightarrow \neg G\neg\varphi$

**Proof.**

1.  $\vdash_{\mathbf{IK}_t} G\neg\varphi \rightarrow \neg F\varphi$  assumption
  2.  $\vdash_{\mathbf{IK}_t} (G\neg\varphi \rightarrow \neg F\varphi) \rightarrow (F\varphi \rightarrow \neg G\neg\varphi)$  axiom 1
  3.  $\vdash_{\mathbf{IK}_t} (F\varphi \rightarrow \neg G\neg\varphi)$  1,2,MP
- 

It is likewise proved that:

**Lemma 13.**  $H\neg\varphi \rightarrow \neg P\varphi \vdash_{\mathbf{IK}_t} P\varphi \rightarrow \neg H\neg\varphi$

Thus, we have shown that the given axioms are equivalent. In further considerations we will use "new" axiomatics of  $\mathbf{IK}_t$ .

## 8. The Adequacy of $\mathbf{IK}_t$ Relative to Modified Semantics

The natural question is the question about the relationship between modified semantics and the assumed set of axioms for  $\mathbf{IK}_t$ .

**Theorem 14.** *The  $\mathbf{IK}_t$  axioms are true in any model, and the  $\mathbf{IK}_t$  inference rules are infallible.*

**Proof.** We will prove only A2', A4' axioms and RH rule. Proofs for the other rules and axioms is carried out in analogous manner.

**A2'** For any  $\mathfrak{M}, m_i (\in \mathfrak{M})$ , and  $t (\in T_i)$ :  $\mathfrak{M}, m_i, t \models H(\varphi \rightarrow \psi) \rightarrow (H\varphi \rightarrow H\psi)$ .

Suppose for some  $\mathfrak{M}, m_i (\in \mathfrak{M})$  and  $t (\in T_i)$ :  $\mathfrak{M}, m_i, t \not\models H(\varphi \rightarrow \psi) \rightarrow (H\varphi \rightarrow H\psi)$ .

Therefore, from the condition of the truth for the implications, there is a state of knowledge  $m_j$ ,  $m_i \leq m_j$ , such that:

$$\mathfrak{M}, m_j, t \models H(\varphi \rightarrow \psi), \quad (23)$$

$$\mathfrak{M}, m_j, t \not\models H\varphi \rightarrow H\psi. \quad (24)$$

From (24) and the condition of the truth for the implications, in a certain state of knowledge  $m_k$ , with a level of knowledge not lesser than the level of knowledge of the state  $m_j$ , i.e., such that  $m_j \leq m_k$ :

$$\mathfrak{M}, m_k, t \models H\varphi, \quad (25)$$

$$\mathfrak{M}, m_k, t \not\models H\psi. \quad (26)$$

From (25) and the condition of the truth for the  $H$  operator we get:

$$\begin{aligned} & \text{for any state of knowledge } m_l \text{ such that } m_k \leq m_l \text{ and} \\ & \text{for any } t_1 \in T_l \text{ such that } t_1 R_l t \text{ holds: } \mathfrak{M}, m_l, t_1 \models \varphi. \end{aligned} \quad (27)$$

From (26) and the condition of the truth for the  $H$  operator, there is a state  $m_p$  such that  $m_k \leq m_p$  and there is a moment  $t_2 \in T_p$  such that  $t_2 R_p t$ , in which:

$$\mathfrak{M}, m_p, t_2 \not\models \psi. \quad (28)$$

Because  $m_k \leq m_p$  and  $t_2 R_p t$  therefore from (27) we have that at the moment  $t_2$  holds  $\mathfrak{M}, m_p, t_2 \models \varphi$ . Hence, from (28) and the condition of the truth of the implications we get:

$$\mathfrak{M}, m_p, t_2 \not\models \varphi \rightarrow \psi. \quad (29)$$

From (23) and the condition of the truth of the operator  $H$  we have:

$$\begin{aligned} & \text{for any } m_r \text{ such that } m_j \leq m_r \text{ and} \\ & \text{for any } t_3 \in T_r \text{ such that } t_3 R_r t \text{ holds: } \mathfrak{M}, m_r, t_3 \models \varphi \rightarrow \psi. \end{aligned} \quad (30)$$

Because:  $m_j \leq m_k$ ,  $m_k \leq m_p$ , so from the transitivity of the relationship  $\leq$  we get  $m_j \leq m_p$ . The moment  $t_2$  is such that  $t_2 R_p t$ . Therefore, from (30) we have:

$$\mathfrak{M}, m_p, t_2 \models \varphi \rightarrow \psi.$$

This is contrary to 29.

**A4'** For any  $\mathfrak{M}, m_i (\in \mathfrak{M})$  and  $t (\in T_i)$ :  $\mathfrak{M}, m_i, t \models H(\varphi \rightarrow \psi) \rightarrow (P\varphi \rightarrow P\psi)$ .

Suppose for some  $\mathfrak{M}, m_i (\in \mathfrak{M})$  and  $t (\in T_i)$   $\mathfrak{M}, m_i, t \not\models H(\varphi \rightarrow \psi) \rightarrow (P\varphi \rightarrow P\psi)$ .

Thus, from the condition of the truth of the implications, in a certain state of knowledge  $m_j$ , such that  $m_i \leq m_j$  we have:

$$\mathfrak{M}, m_j, t \models H(\varphi \rightarrow \psi), \quad (31)$$

$$\mathfrak{M}, m_j, t \not\models P\varphi \rightarrow P\psi. \quad (32)$$

From (32) and the condition of the truth of the implications, in some state of knowledge  $m_k$ , such that  $m_j \leq m_k$ :

$$\mathfrak{M}, m_k, t \models P\varphi, \quad (33)$$

and

$$\mathfrak{M}, m_k, t \not\models P\psi. \quad (34)$$

From (33) and the condition of the truth of the  $P$  operator we have:

$$\text{there exists } t_1 (\in T_k), t_1 R_k t \text{ such that } \mathfrak{M}, m_k, t_1 \models \varphi. \quad (35)$$

From (34) and the condition of the truth of the  $P$  operator we obtain:

$$\text{does not exist moment of time } t_2 (\in T_k), t_2 R_k t, \text{ such that } \mathfrak{M}, m_k, t_2 \models \psi. \quad (36)$$

Let us consider the moment  $t_1$  satisfying (35). Because  $t_1 R_k t$ , so from (36) we have:

$$\mathfrak{M}, m_k, t_1 \not\models \psi. \quad (37)$$

If  $\mathfrak{M}, m_k, t_1 \models \psi$ , it would be against (36).

From (35), (37) and the condition of the truth of the implications, we get that  $\mathfrak{M}, m_k, t_1 \not\models \varphi \rightarrow \psi$ . From (31) and condition the truth of the operator  $H$  we have:

$$\text{for any } m_l \text{ such that } m_j \leq m_l \text{ and for any } t_3 (\in T_l) \text{ such that } t_3 R_l t : \mathfrak{M}, m_l, t_3 \models \varphi \rightarrow \psi. \quad (38)$$

Because  $m_j \leq m_k, t_1 R_k t$  and  $\mathfrak{M}, m_k, t_1 \not\models (\varphi \rightarrow \psi)$ , so we get a contradiction with (38).

**RH** If  $\mathfrak{M} \models \varphi$ , then  $\mathfrak{M} \models H\varphi$ .

Let us assume that  $\mathfrak{M} \models \varphi$ . So for any  $m_i$  and for any  $t (\in T_i)$  holds  $\mathfrak{M}, m_i, t \models \varphi$ . So especially for any  $t_1 (\in T_i)$  such that  $t_1 R_i t : \mathfrak{M}, m_i, t_1 \models \varphi$ . So for any  $t (\in T_i)$  holds  $\mathfrak{M}, m_i, t \models H\varphi$ . Because we were considering any  $m_i$ , therefore  $\mathfrak{M} \models H\varphi$ .

□

Adequacy  $\mathbf{IK}_t$  with respect to modified semantics was demonstrated by Surowik [4].

**Theorem 15.**  $\Sigma \vdash_{\mathbf{IK}_t} \varphi$  iff  $\Sigma \models_{\mathbf{IK}_t} \varphi$ .

The proof of this theorem is similar to the proof of the adequacy theorem demonstrated by Ewald in [3].

## 9. Mutual Undefinability in $\mathbf{IK}_t$ Operators $H, P$ and $G, F$

We will now prove theorems that show some special properties of the  $\mathbf{IK}_t$  system, essentially distinguishing this system from systems built on the basis of classical logic. For the formula to be the tautology of the  $\mathbf{IK}_t$  system, it needs to be true at any time, in any state of knowledge. To show that a formula is not true, it is enough to indicate the state of knowledge and the moment in which this formula is not true.

We will show that some relationships between the operators  $H$  and  $P$  and  $G$  and  $F$  holds in the system  $\mathbf{K}_t$  but do not occur between the equivalents of these operators in the system  $\mathbf{IK}_t$ .

**Theorem 16.**

- (a)  $\not\models_{\mathbf{IK}_t} \neg P \neg p \rightarrow Hp$ ,
- (b)  $\not\models_{\mathbf{IK}_t} \neg H \neg p \rightarrow Pp$
- (c)  $\not\models_{\mathbf{IK}_t} \neg Hp \rightarrow P \neg p$
- (d)  $\not\models_{\mathbf{IK}_t} \neg F \neg p \rightarrow Gp$
- (e)  $\not\models_{\mathbf{IK}_t} \neg G \neg p \rightarrow Fp$

**Proof.**

- (a) Let  $T = \{t_1, t_2\}$ ,  $R = \{(t_1, t_2)\}$ . Let  $I$  be a set of indexes. For any  $i$ :  $T_i = T$ ,  $R_i = R$ . Let  $k, k > 1$ , be a certain index of state of knowledge. Let  $\mathcal{F} = \{V_i\}_{i \in I}$  be a class of functions satisfied the following conditions:

$$\text{for any } i \text{ such that } i \leq k \text{ holds } p \notin V_i(t_1), \quad (39)$$

and

$$\text{for any } i \text{ such that } k < i \text{ holds } p \in V_i(t_1). \quad (40)$$

The  $V_i$  valuations are therefore selected so that the sentence  $p$  is true at the time of  $t_1$  in the states of knowledge with index not greater than  $k$  and at the same time it was not true at the time of  $t_1$  in the states of knowledge with index greater than  $k$ .

Let  $\mathcal{T} = \langle T, R \rangle$ . Let  $\mathfrak{M} = \{m_i : i \in I\}$ . From the construction of the  $\mathfrak{M}$  model, we get that there are states of knowledge in the  $\mathfrak{M}$  which level of knowledge is not lesser than the level of knowledge of  $m_1$  in which at the moment  $t_1$   $p$  is true and there are states of knowledge with a level of knowledge not lesser than the level of knowledge of  $m_1$ , in which at the moment  $t_1$  is not true that  $p$ . Therefore, it is not true that in any state of knowledge  $m_1^* (\in \mathfrak{M})$  holds  $\mathfrak{M}, m_1^*, t_1 \models p$ . Therefore, by Definition 12 we get  $\mathfrak{M}, m_1, t_1 \models \neg p$ . From the construction of the  $\mathfrak{M}$  model we get that in any state of knowledge  $m_1^* (\in \mathfrak{M})$  holds  $\mathfrak{M}, m_1^*, t_1 \models \neg p$ . Because  $t_1 R_1^* t_2$ , therefore, by the Definition 12 we have  $\mathfrak{M}, m_1^*, t_2 \models P\neg p$ . By the Definition 12 we get

$$\mathfrak{M}, m_1, t_2 \models \neg P\neg p. \quad (41)$$

Because the moment  $t_1$  is such that  $t_1 R_1 t_2$  and  $\mathfrak{M}, m_1, t_1 \models p$  so by the Definition 12

$$\mathfrak{M}, m_1, t_2 \models Hp. \quad (42)$$

From (41), (42) and the Definition 12: we have  $\mathfrak{M}, m_1, t_2 \models \neg P\neg p \rightarrow Hp$ . Therefore  $\not\models_{\mathbf{IK}_t} \neg P\neg p \rightarrow Hp$ .

- (b) The  $\mathfrak{M}$  model proposed in the proof of a) will be used to prove that  $\neg H\neg p \rightarrow Pp$  is not a tautology of  $\mathbf{IK}_t$ . Please note that from the construction of the model and by Definition 12 we have  $\mathfrak{M}, m_1^*, t_1 \models \neg p$ . Because in any state of knowledge  $m_1^* (\in \mathfrak{M})$  the only time before  $t_2$  is the time  $t_1$ , so by the Definition 12 for any  $m_1^*$  holds  $\mathfrak{M}, m_1^*, t_2 \models H\neg p$ . Hence, by the Definition 12

$$\mathfrak{M}, m_1, t_2 \models \neg H\neg p. \quad (43)$$

From the construction of the model  $\mathfrak{M}$  we have  $\mathfrak{M}, m_1, t_1 \models p$ . Because  $t_1 R_1 t_2$ , therefore by the Definition 12

$$\mathfrak{M}, m_1, t_2 \models Pp. \quad (44)$$

From (43), (44) and the Definition of 12 we obtain:  $\mathfrak{M}, m_1, t_2 \models \neg H\neg p \rightarrow Pp$ . Therefore  $\not\models_{\mathbf{IK}_t} \neg H\neg p \rightarrow Pp$ .

- (c) We will now show that  $\neg Hp \rightarrow P\neg p$  is not a tautology of  $\mathbf{IK}_t$ . Let  $T_1 = \{t_1, t_2\}$ ,  $R_1 = \{(t_2, t_1)\}$ . Let the function  $V_1$  be such that  $p \notin V_1(t_2)$ . States of knowledge in which the level of knowledge is not lower than the level of  $m_1$  we construct as follows:

$$T_{i+1} = T_i \cup \{t_{i+2}\}, \quad (45)$$

$$R_{i+1} = R_i \cup \{(t_{i+2}, t_1)\}. \quad (46)$$

$$V_{i+1} \text{ is such that for } t \neq t_{i+2} : p \in V_{i+1}(t), \text{ and for } t = t_{i+2} : p \notin V_{i+1}(t). \quad (47)$$



State of knowledge  $m_{i+1}$  is an ordered triple  $\langle T_{i+1}, R_{i+1}, V_{i+1} \rangle$ . Let  $\mathcal{F} = \{V_i\}_{i \in I}$  will be a class of functions satisfying the condition (47),  $\mathcal{T} = \langle \bigcup_{i \in I} T_i, \bigcup_{i \in I} R_i \rangle$ ,  $\mathfrak{M} = \{m_i : i \in I\}$ , the states of knowledge  $m_i$  are constructed in accordance with conditions (45), (46) and (47). From the construction of the  $\mathfrak{M}$  model we get that in every state of knowledge  $m_1^* (\in \mathfrak{M})$ , there is a moment  $t$ , earlier than  $t_1$  such that  $tR_1^*t_1$  in which such that  $\mathfrak{M}, m_1^*, t \not\models p$ . So by the Definition 12 for any state of knowledge  $m_1^*$  we have  $\mathfrak{M}, m_1^*, t_1 \not\models Hp$ . From the definition of 12 we have that:

$$\mathfrak{M}, m_1, t_2 \models \neg Hp. \quad (48)$$

From the construction of the model we have that if at some moment of time  $t$ , in any state of knowledge  $m_i (\in \mathfrak{M})$  is that  $\mathfrak{M}, m_i, t \not\models p$ , then in every state of knowledge  $m_i^* (\in \mathfrak{M})$  holds  $\mathfrak{M}, m_i^*, t \models p$ . In the state of knowledge  $m_1 (\in \mathfrak{M})$  the only time before  $t_1$  is the moment  $t_2$ . The moment  $t_2$  is such that  $p \notin V_1(t_2)$ . In the classical model, this would suffice to say that  $\mathfrak{M}, m_1, t_2 \models \neg p$ . This is not the case in the temporal logic model built upon intuitionistic logic. From the way of constructing states of knowledge with no lower level of knowledge than the level of knowledge in the state  $m_1$  we have  $p \in V_2(t_2)$ . Therefore, by the Definition 12

$$\mathfrak{M}, m_1, t_2 \not\models \neg p. \quad (49)$$

By the Definition 12 we have  $\mathfrak{M}, m_1, t_1 \not\models \neg Hp \rightarrow P\neg p$ .

We construct counter-examples for d), e) and f) in an analogous way.  $\square$

In the  $\mathbf{IK}_t$  system, between the  $G$  and  $F$  and  $H$  and  $P$  operators there are no relationships usually found in temporal logic systems that are based on classical logic. However, the above conclusion is not sufficient to state that the operators  $G$  and  $F$  as well as  $H$  and  $P$  are not mutually definable in  $\mathbf{IK}_t$ . The conclusion is only that they do not occur between these operators definition relationships the same as those in *classical* tense logics. We will show that in intuitionistic temporal logic, temporal operators are not definable as they are in temporal logics based on classical propositional logic. We will show that intuitionistic temporal operators are not definable in any other way using sentence connectives and other intuitionistic temporal operators.

To show that a temporal operator is not definable in the  $\mathbf{IK}_t$ , two structures should be indicated such that the sentence with the considered operator at a moment  $t$  in one structure is true, and it is false in the other. On the other hand, all sentences in which the operator does not appear have the same logical value in both structures at the moment  $t$ .

**Theorem 17 ([4]).** *The intuitionistic temporal operators  $F$  and  $G$  as well as  $P$  and  $H$  are not each other definable in the  $\mathbf{IK}_t$ .*

**Proof.** We will show first that the operator  $F$  is not definable if we use of intuitionistic sentence connectives and other temporal operators. We will show that  $Fp$  is not equivalent to any temporal formula in which the  $F$  operator does not occur.

**F:** Let  $T_1 = \{t_1, t_2\}$ ,  $T_2 = \{t_1, t_2, t_3\}$ ,  $R_1 = \{(t_1, t_2)\}$ ,  $R_2 = \{(t_1, t_2), (t_1, t_3)\}$ ,  $T = T_1 \cup T_2$ . Let  $V_1 : T_1 \rightarrow 2^{\mathcal{A}^P}$  be such that  $p \notin V_1(t_2)$  while  $V_2 : T_2 \rightarrow 2^{\mathcal{A}^P}$  will be such a function that:  $V_2(t_1) = V_1(t_1)$ ,  $V_2(t_2) = V_1(t_2) \cup \{p\}$ ,  $V_2(t_3) = V_1(t_2)$ . Let  $\mathcal{F} = \{V_1, V_2\}$ ,  $m_1 = \langle T_1, R_1, V_1 \rangle$ ,  $m_2 = \langle T_2, R_2, V_2 \rangle$ ,  $\mathfrak{M} = \{m_1, m_2\}$ .

By means of structural induction, it can be shown that for any  $\varphi$  without the  $F$  operator we have

$$\mathfrak{M}, m_1, t_1 \models \varphi \text{ iff } \mathfrak{M}, m_2, t_1 \models \varphi. \quad (50)$$

At the same time,  $\mathfrak{M}, m_2, t_1 \models Fp$  and  $\mathfrak{M}, m_1, t_1 \not\models Fp$ . Therefore, the  $F$  operator is not definable in  $\mathbf{IK}_t$ .

G: We will now show that the operator  $G$  is not definable if we use of intuitionistic sentence connectives and other temporal operators. We will show that  $Gp$  is not equivalent to any temporal formula in which the  $G$  operator is not present.

Let  $T_1 = \{t_1, t_2, t_3\}$ ,  $T_2 = \{t_1, t_2, t_3\}$ ,  $R_1 = \{(t_1, t_2)\}$ ,  $R_2 = \{(t_1, t_2), (t_1, t_3)\}$ ,  $T = T_1 \cup T_2$ . Let  $V_1 : T_1 \rightarrow 2^{\mathcal{A}^P}$  will be such a function that  $p \notin V_1(t_2)$ . Let  $V_2 : T_2 \rightarrow 2^{\mathcal{A}^P}$  will be such a function that:  $V_2(t_1) = V_1(t_1)$ ,  $V_2(t_2) = V_1(t_2) \cup \{p\}$ ,  $V_2(t_3) = V_1(t_3)$ ,  $p \in V_1(t_3)$ . Let  $\mathcal{F} = \{V_1, V_2\}$ . Let  $m_1 = \langle T_1, R_1, V_1 \rangle$ ,  $m_2 = \langle T_2, R_2, V_2 \rangle$ . Let  $\mathfrak{M} = \{m_1, m_2\}$ . By means of structural induction, it can be shown that for any  $\varphi$  sentence without the  $G$  operator we have:

$$\mathfrak{M}, m_1, t_1 \models \varphi \text{ iff } \mathfrak{M}, m_2, t_1 \models \varphi. \quad (51)$$

At the same time,  $\mathfrak{M}, m_2, t_1 \models Gp$  and  $\mathfrak{M}, m_1, t_1 \not\models Gp$ . So the  $G$  operator is not definable in  $\mathbf{IK}_t$ .

Similarly, we can to show that  $P$  and  $H$  are not each other definable in  $\mathbf{IK}_t$ .

□

It is not, however, that the operators  $G, F, H, P$  are completely independent of each other. Certain relationships between the operators  $H$  and  $P$  and  $G$  and  $F$  occur in  $\mathbf{IK}_t$ . We will prove some of them:

**Theorem 18.**

- (a)  $\vdash_{\mathbf{IK}_t} H\neg\varphi \rightarrow \neg P\varphi$ ,
- (b)  $\vdash_{\mathbf{IK}_t} H\varphi \rightarrow \neg P\neg\varphi$ ,
- (c)  $\vdash_{\mathbf{IK}_t} P\neg\varphi \rightarrow \neg H\varphi$ ,
- (d)  $\vdash_{\mathbf{IK}_t} G\neg\varphi \rightarrow \neg F\varphi$ ,
- (e)  $\vdash_{\mathbf{IK}_t} G\varphi \rightarrow \neg F\neg\varphi$ ,
- (f)  $\vdash_{\mathbf{IK}_t} F\neg\varphi \rightarrow \neg G\varphi$ .

**Proof.**

- (a)  $\vdash_{\mathbf{IK}_t} (H\neg\varphi \rightarrow \neg P\varphi)$

1.  $\vdash_{\mathbf{IK}_t} (P\varphi \rightarrow \neg H\neg\varphi) \rightarrow (H\neg\varphi \rightarrow \neg P\varphi)$  axiom 1,
2.  $\vdash_{\mathbf{IK}_t} H\neg\varphi \rightarrow \neg P\varphi$  A5', 1, MP.

- (b)  $\vdash_{\mathbf{IK}_t} H\varphi \rightarrow \neg P\neg\varphi$

1.  $\vdash_{\mathbf{IK}_t} \varphi \rightarrow \neg\neg\varphi$  axiom 1,
2.  $\vdash_{\mathbf{IK}_t} H(\varphi \rightarrow \neg\neg\varphi)$  1, RH,
3.  $\vdash_{\mathbf{IK}_t} H(\varphi \rightarrow \neg\neg\varphi) \rightarrow (H\varphi \rightarrow H\neg\neg\varphi)$  A2',
4.  $\vdash_{\mathbf{IK}_t} H\varphi \rightarrow H\neg\neg\varphi$  2, 3, MP,
5.  $\vdash_{\mathbf{IK}_t} H\neg\neg\varphi \rightarrow \neg P\neg\varphi$  case (a),
6.  $\vdash_{\mathbf{IK}_t} H\varphi \rightarrow \neg P\neg\varphi$  4, 5, SYLL.

- (c)  $\vdash_{\mathbf{IK}_t} (P\neg\varphi \rightarrow \neg H\varphi)$

1.  $\vdash_{\mathbf{IK}_t} (H\varphi \rightarrow \neg P\neg\varphi) \rightarrow (P\neg\varphi \rightarrow \neg H\varphi)$  axiom 1,
2.  $\vdash_{\mathbf{IK}_t} P\neg\varphi \rightarrow \neg H\varphi$  1, case (b), MP.

The proofs of the cases (d), (e) and (f) are similar, so we skip them.  $\square$

## 10. Summary

Temporal logic systems can be built in a variety of ways. They can be based on classical logic, but also, as we presented in this article, based on intuitionistic logic. The discussed systems are minimal systems, which means that no properties have been imposed on the time structure. One can, however, enrich these systems with additional specific axioms, build a temporal logic systems adequate to various time structures, e.g., reflexive, symmetrical, transitive, linear or branched. However, while in tense logic systems based on classical logic, the thesis of logical determinism can be rejected by modifying the structure of time and assuming, as a semantic time, a branching time into the future, in tense logics based on intuitionistic logic, modification of the time structure is not necessary. Formulas expressing the thesis of logical determinism are not theses of the minimal system because of its basic properties, no matter what time structure is adopted as a semantic time.

There is a relationship between the systems being discussed. Each thesis of the  $\mathbf{IK}_t$  system is also the thesis of  $\mathbf{K}_t$ , so:

$$\mathbf{IK}_t \subset \mathbf{K}_t.$$

In addition, as we have shown in this article, intuitionistic temporal logic can be used to represent knowledge that changes over time. Intuitionistic logic and knowledge are closely related. This epistemic approach is the epicenter of Brouwer's intuitionistic explanation of truth as provability by an ideal mathematician, or more generally by an ideal cognitive subject. Kripke's intuitionistic models are good tools for modelling the evolutionary learning process of the cognitive subject.

The intuitionistic temporal logic  $\mathbf{IK}_t$  has many advantages when we understand it as a formal tool for the logical representation of knowledge changing over time. Knowledge is implemented in this system on a semantic level in a natural way. In a natural way, by means of a set of partially ordered states of knowledge, the way of acquiring knowledge is also modeled. However, this system has some imperfections and limitations. The first is the limited applicability of this system. Due to the adopted monotonicity of knowledge, i.e., a fact recognized in a given state of knowledge is known in all states of knowledge with a not lower level of knowledge, this system is a good tool for a modelling of mathematical or logical knowledge that changes over time.

**Conflicts of Interest:** The authors declare no conflict of interest.

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