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Generalized Nabla Differentiability and Integrability for Fuzzy Functions on Time Scales

R. Leelavathi ¹, G. Suresh Kumar ¹ , Ravi P. Agarwal ^{2,*}, Chao Wang ³  and M.S.N. Murty ^{4,†} 

¹ Department of Mathematics, Koneru Lakshmaiah Education Foundation, Vaddeswaram 522502, Guntur, Andhra Pradesh, India; leelaammu36@gmail.com (R.L.); drgsk006@kluniversity.in (G.S.K.)

² Department of Mathematics, Texas A&M University-Kingsville, Kingsville, TX 78363-8202, USA

³ Department of Mathematics, Yunnan University, Kunming 650091, China; chaowang@ynu.edu.cn

⁴ Sainivas, D.No. 21-47, Opp. State Bank of India, Bank Street, Nuzvid 521201, Krishna, Andhra Pradesh, India; drmsn2002@gmail.com

* Correspondence: ravi.agarwal@tamuk.edu

† Professor of Mathematics (Retd.).

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Abstract: This paper mainly deals with introducing and studying the properties of generalized nabla differentiability for fuzzy functions on time scales via Hukuhara difference. Further, we obtain embedding results on $\mathbb{E}_n$ for generalized nabla differentiable fuzzy functions. Finally, we prove a fundamental theorem of a nabla integral calculus for fuzzy functions on time scales under generalized nabla differentiability. The obtained results are illustrated with suitable examples.

Keywords: fuzzy functions time scales; Hukuhara difference; generalized nabla Hukuhara derivative; fuzzy nabla integral

1. Introduction

The theory of dynamic equations on time scales is a genuinely new subject and the research related to this area is developing rapidly. Time scale theory has been developed to unify continuous and discrete structures, and it allows solutions for both differential and difference equations at a time and extends those results to dynamic equations. Basic results in time scales and dynamic equations on time scales are found in [1–6]. In [7], the author illustrated an example where delta derivative needs more assumptions than nabla derivative. Some recent studies in economics [8], production, inventory models [9], adaptive control [10], neural networks [11], and neural cellular networks [12] suggest nabla derivative is also preferable and it has fewer restrictions than delta derivative on time scales.

On the other hand, when we expect to investigate a real world phenomenon absolutely, it is important to think about a number of unsure factors too. To specify these vague or imprecise notions, Zadeh [13] established fuzzy set theory. The theory of fuzzy differential equations (FDEs) and its applications was developed and studied by Kaleva [14], Lakshmikantham and Mohapatra [15]. The concept based on Hukuhara differentiability has a shortcoming that the solution to a FDEs exists only for increasing length of support. To overcome this shortcoming, Bede and Gal [16] studied generalized Hukuhara differentiability for fuzzy functions. In light of this preferred advantage, many authors [17–19] tend their enthusiasm to the generalized Hukuhara differentiability for fuzzy set valued functions.

The calculus of fuzzy functions on time scales was studied by Fard and Bidgoli [20]. Vasavi et al. [21–24] introduced Hukuhara delta derivative, second-type Hukuhara delta derivative, and generalized Hukuhara delta derivatives by using Hukuhara difference, and they studied fuzzy dynamic equations on time scales. Wang et al. [25] introduced and studied almost periodic

fuzzy vector-valued functions on time scales. Deng et al. [26] studied fractional nabla-Hukuhara derivative on time scales. Recently, Leelavathi et al. [27] introduced and studied properties of nabla Hukuhara derivative for fuzzy functions on time scales. However, this derivative has the disadvantage that it exists only for the fuzzy functions on time scales which have a diameter with an increasing length. For the fuzzy functions with decreasing length of diameter on time scales, Leelavathi et al. [28] introduced the second-type nabla Hukuhara derivative and studied its properties. Later, they continued to study fuzzy nabla dynamic equations under the first and second-type nabla Hukuhara derivatives in [29] under generalized differentiability by using generalized Hukuhara difference in [30]. Consider a simple fuzzy function $F(s) = s \odot c$, $s \in \mathbb{T} \cap [-2, 2]$, where $c = (1, 2, 3)$ is a triangular fuzzy number. Clearly, $F(s)$ has decreasing length of diameter in $\mathbb{T} \cap [-2, 0]$ and increasing length of diameter in $\mathbb{T} \cap [0, 2]$. Therefore, the fuzzy function $F(s)$ is neither a nabla Hukuhara differentiable (as defined in [27]) nor a second-type nabla Hukuhara differentiable (as defined in [28]) on $\mathbb{T} \cap [-2, 2]$. In this context, it is required to define a nabla Hukuhara derivative for a fuzzy function which may have both increasing and decreasing length of diameter on a time scale. To address this issue, in the present work, we define a new derivative called generalized nabla derivative for fuzzy functions on time scales via Hukuhara difference and study their properties. In [31], the authors introduced a nabla integral for fuzzy functions on time scales and obtained fundamental properties. In the present work, we continue to study nabla integral for fuzzy functions on time scales and prove a fundamental theorem of nabla integral calculus for generalized nabla differentiable functions.

The rest of this paper is arranged as follows. In Section 2, we present some basic definitions, properties, and results relating to the calculus of fuzzy functions on time scales. In Section 3, we establish the nabla Hukuhara generalized derivative for fuzzy functions on time scales and obtain its fundamental properties. The results are highlighted with suitable examples. In Section 4, we prove an embedding theorem on $\mathbb{E}_n$ and obtain the results connecting to generalized nabla differentiability on time scales. Using these results, we finally prove the fundamental theorem of nabla integral calculus for fuzzy functions on time scales under generalized nabla differentiability and a numerical example is provided to verify the validity of the theorem.

2. Preliminaries

Let $\mathcal{R}_k(\mathbb{R}^n)$ be the family of all nonempty convex compact subsets of $\mathbb{R}^n$. Define the set addition and scalar multiplication in $\mathcal{R}_k(\mathbb{R}^n)$ as usual. Then, by [14], $\mathcal{R}_k(\mathbb{R}^n)$ is a commutative semi-group under addition with cancellation laws. Further, if $\beta, \gamma \in \mathcal{R}$ and $U, V \in \mathcal{R}_k(\mathbb{R}^n)$, then

$$\beta \odot (U \oplus V) = (\beta \odot U) \oplus (\beta \odot V), \quad \beta(\gamma \odot U) = (\beta\gamma) \odot U, \quad 1 \odot U = U, \quad \text{and if}$$

$$\beta, \gamma \geq 0 \text{ then } (\beta \oplus \gamma)U = \beta \odot U \oplus \gamma \odot U.$$

Let P and Q be two bounded nonempty subsets of $\mathbb{R}^n$. By using the Pompeiu–Hausdorff metric, we define the distance between P and Q as follows:

$$d_H(P, Q) = \max\left\{\sup_{p \in P} \inf_{q \in Q} \|p - q\|, \sup_{q \in Q} \inf_{p \in P} \|p - q\|\right\},$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. Then, $(\mathcal{R}_k(\mathbb{R}^n), d_H)$ becomes a separable and complete metric space [14].

Define:

$$\mathbb{E}_n = \{u : \mathbb{R}^n \rightarrow [0, 1] \mid u \text{ satisfies (a)–(d) below}\}, \text{ where}$$

- If there exists a $t \in \mathbb{R}^n$ such that $u(t) = 1$, then u is said to be normal.
- u is fuzzy convex.
- u is upper semi-continuous.
- The closure of $\{t \in \mathbb{R}^n \mid u(t) > 0\} = [u]^0$ is compact.

For $0 \leq \lambda \leq 1$, denote $[u]^\lambda = \{t \in \mathbb{R}^n : u(t) \geq \lambda\}$; then, from the above conditions, we have that the λ -level set $[u]^\lambda \in \mathfrak{R}_k(\mathbb{R}^n)$. By Zadeh's extension principle, a mapping $h : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ can be extended to $g : \mathbb{E}_n \times \mathbb{E}_n \rightarrow \mathbb{E}_n$ by

$$g(s, t)(z) = \sup_{z=g(x,y)} \min\{s(x), t(y)\}.$$

We have $[h(p, q)]^\lambda = h([p]^\lambda, [q]^\lambda)$, for all $p, q \in \mathbb{E}_n$ and h is continuous. The scalar multiplication $\odot$ and addition $\oplus$ of $p, q \in \mathbb{E}_n$ is defined as $[p \oplus q]^\lambda = [p]^\lambda + [q]^\lambda$, $[c \odot p]^\lambda = c[p]^\lambda$, where $p, q \in \mathbb{E}_n$, $c \in \mathbb{R}$, $0 \leq \lambda \leq 1$.

Define $D_H : \mathbb{E}_n \times \mathbb{E}_n \rightarrow [0, \infty)$ by the equation

$$D_H(s, t) = \sup_{0 \leq \lambda \leq 1} d_H([s]^\lambda, [t]^\lambda),$$

where d_H is the Pompeiu–Hausdorff metric defined in $\mathfrak{R}_k(\mathbb{R}^n)$. Then, $(\mathbb{E}_n, D_H)$ is a complete metric space [14]. The following theorem extends the properties of addition and scalar multiplication of fuzzy number valued functions ($\mathfrak{R}_F = \mathbb{E}_1$) to $\mathbb{E}_n$ [14].

The properties of addition and scalar multiplication of fuzzy number valued functions ($\mathfrak{R}_F = \mathbb{E}_1$) are easily extended to $\mathbb{E}_n$.

Theorem 1 ([32]).

- (a) If we denote $\hat{0} = \chi_{\{0\}}$, then $\hat{0} \in \mathbb{E}_n$ is the zero element with respect to $\oplus$, i.e., $p \oplus \hat{0} = \hat{0} \oplus p = p$, $\forall p \in \mathbb{E}_n$.
- (b) For any $p \in \mathbb{E}_n$ has no inverse with respect to $\oplus$.
- (c) For any $\gamma, \beta \in \mathbb{R}$ with $\gamma, \beta \geq 0$ or $\gamma, \beta \leq 0$ and $p \in \mathbb{E}_n$, $(\gamma + \beta) \odot p = (\gamma \odot p) \oplus (\beta \odot p)$.
- (d) For any $\gamma \in \mathbb{R}$ and $p, q \in \mathbb{E}_n$, we have $\gamma \odot (p \oplus q) = (\gamma \odot p) \oplus (\gamma \odot q)$.
- (e) For any $\gamma, \beta \in \mathbb{R}$ and $p \in \mathbb{E}_n$, we have $\gamma \odot (\beta \odot p) = (\gamma\beta) \odot p$.

Definition 1 ([14]). Let $K, L \in \mathbb{E}_n$. If there exists $M \in \mathbb{E}_n$ such that $K = L \oplus M$, then we say that M is the Hukuhara difference of K and L and is denoted by $K \ominus_h L$.

For any $K, L, M, N \in \mathbb{E}_n$ and $\beta \in \mathbb{R}$, the following hold:

- (a) $D_H(K, L) = 0 \Leftrightarrow K = L$;
- (b) $D_H(\beta \odot K, \beta \odot L) = |\beta| D_H(K, L)$;
- (c) $D_H(K \oplus M, L \oplus M) = D_H(K, L)$;
- (d) $D_H(K \ominus_h M, L \ominus_h M) = D_H(K, L)$;
- (e) $D_H(K \oplus L, M \oplus N) \leq D_H(K, M) + D_H(L, N)$; and
- (f) $D_H(K \ominus_h L, M \ominus_h N) \leq D_H(K, M) + D_H(L, N)$.

provided the Hukuhara differences exists.

A triangular fuzzy number is denoted by three points as $t = (t_1, t_2, t_3)$. This representation is denoted as membership function

$$\mu_t(x) = \begin{cases} 0, & x < t_1 \\ \frac{x - t_1}{t_2 - t_1}, & t_1 \leq x \leq t_2 \\ \frac{t_3 - x}{t_3 - t_2}, & t_2 \leq x \leq t_3 \\ 0, & x > t_3 \end{cases}$$

In addition, λ -level sets of triangular fuzzy number t is an interval defined by λ -cut operation, $t_\lambda = [(t_2 - t_1)\lambda + t_1, t_3 - (t_3 - t_2)\lambda]$, for all $\lambda \in [0, 1]$. Clearly, the triangular fuzzy number is in $\mathbb{E}_1$.

Let $T = (t_1, t_2, t_3)$, $S = (s_1, s_2, s_3)$ be two triangular fuzzy numbers in $\mathbb{E}_1$. The addition and scalar multiplication are defined as:

$$S \oplus T = (t_1 + s_1, t_2 + s_2, t_3 + s_3),$$

$$k \odot T = \begin{cases} (kt_1, kt_2, kt_3) & \text{if } k > 0, \\ (kt_3, kt_2, kt_1) & \text{if } k < 0, \\ \hat{0} & \text{if } k = 0 \end{cases}$$

Remark 1. From Theorem 1(c), we can deduce that, for any $\beta, \gamma \in \mathbb{R}$ and $s \in \mathbb{E}_n$.

- (a) If $\beta > \gamma \geq 0$, then $(\beta \odot s) \ominus_h (\gamma \odot s)$ exists and $(\beta \odot s) \ominus_h (\gamma \odot s) = (\beta - \gamma) \odot s$.
- (b) If $\beta < \gamma \leq 0$, then $(\beta \odot s) \ominus_h (\gamma \odot s)$ exists and $(\beta \odot s) \ominus_h (\gamma \odot s) = (\beta - \gamma) \odot s$.

Proof.

- (a) Since $\beta - \gamma > 0$ and $\gamma > 0$, from Theorem 1(c), we get $(\beta - \gamma) \odot s \oplus \gamma \odot s = (\beta - \gamma + \gamma) \odot s = \beta \odot s$. Therefore, $(\beta - \gamma) \odot s \oplus \gamma \odot s = \beta \odot s$. Hence, $(\beta \odot s) \ominus_h (\gamma \odot s) = (\beta - \gamma) \odot s$.
- (b) Since $\beta - \gamma < 0$ and $\gamma < 0$, from Theorem 1(c), it is easily proven that $(\beta \odot s) \ominus_h (\gamma \odot s) = (\beta - \gamma) \odot s$.

□

Now, we discuss the differentiability and integrability of fuzzy functions on $I = [a, b] \subset \mathbb{R}$ (where I is a compact interval).

Definition 2 ([14]). A mapping $\Phi : I \rightarrow \mathbb{E}_n$ is said to be strongly measurable if, for each $\lambda \in [0, 1]$, the fuzzy function $\Phi_\lambda : I \rightarrow \mathbb{R}_k(\mathbb{R}^n)$ defined by $\Phi_\lambda(s) = [\Phi(s)]^\lambda$ is measurable.

Remark 2 ([14]). A mapping $\Phi : I \rightarrow \mathbb{E}_n$ is said to be integrably bounded if there exists an integrable function h such that $\|x\| \leq h(s)$, for all $x \in \Phi_0(s)$.

Definition 3 ([14]). Let $\Phi : I \rightarrow \mathbb{E}_n$. The integral of Φ over I is denoted by $\int_I \Phi(s)ds$ or $\int_x^y \Phi(s)ds$,

$$\left[\int_I \Phi(s)ds \right]^\lambda = \int_I \Phi_\lambda(s)ds$$

$$= \left\{ \int_I g(s)ds \mid \phi : I \rightarrow \mathbb{R}^n \right\},$$

where g is a level wise selection of measurable functions of Φ_λ for $0 < \lambda \leq 1$.

A mapping $\Phi : I \rightarrow \mathbb{E}_n$ is said to be integrable over I if Φ is integrably bounded and strongly measurable function and also $\int_I \Phi(s)ds \in \mathbb{E}_n$.

Theorem 2 ([14]). Let $\Phi, \Psi : I \rightarrow \mathbb{E}_n$ be integrable. Then,

- (a) $\int \Phi \oplus \Psi = \int \Phi \oplus \int \Psi$;
- (b) $\int \alpha \odot \Phi = \alpha \odot \int \Phi$, where $\alpha \in \mathbb{R}$;
- (c) $\int_x^y \Phi = \int_x^z \Phi \oplus \int_z^y \Phi$, where $z \in \mathbb{R}$;
- (d) $D_H(\Phi, \Psi)$ is integrable; and
- (e) $D_H(\int \Phi, \int \Psi) \leq \int D_H(\Phi, \Psi)$.

Definition 4 ([18]). A fuzzy function $\Phi : I \rightarrow \mathbb{E}_n$ is said to be differentiable from left at s_0 if for $\delta > 0$, there exists $P \in \mathbb{E}_n$, such that the following holds:

(a) for $0 < h < \delta$, $\Phi(s_0) \ominus_h \Phi(s_0 - h)$ exist and $\lim_{h \rightarrow 0^+} \frac{1}{h} \odot (\Phi(s_0) \ominus_h \Phi(s_0 - h)) = P$;

or

(b) for $0 < h < \delta$, $\Phi(s_0 - h) \ominus_h \Phi(s_0)$ exist and $\lim_{h \rightarrow 0^+} \frac{-1}{h} \odot (\Phi(s_0 - h) \ominus_h \Phi(s_0)) = P$.

Here, P is the derivative of Φ from left at s_0 and is denoted as $\Phi'_-(s_0)$.

Definition 5 ([18]). A fuzzy function $\Phi : I \rightarrow \mathbb{E}_n$ is said to be differentiable from right at s_0 if, for $\delta > 0$, there exists $P \in \mathbb{E}_n$, such that the following holds:

(a) for $0 < h < \delta$, $\Phi(s_0 + h) \ominus_h \Phi(s_0)$ exist and $\lim_{h \rightarrow 0^+} \frac{1}{h} \odot (\Phi(s_0 + h) \ominus_h \Phi(s_0)) = P$;

or

(b) for $0 < h < \delta$, $\Phi(s_0) \ominus_h \Phi(s_0 + h)$ exist and $\lim_{h \rightarrow 0^+} \frac{-1}{h} \odot (\Phi(s_0) \ominus_h \Phi(s_0 + h)) = P$.

Here, P is the derivative of Φ from right at s_0 and is denoted as $\Phi'_+(s_0)$. The limits are taken over $(\mathbb{E}_n, D_H)$.

Definition 6 ([18]). If Φ is both left-differentiable and right-differentiable at s_0 , then Φ is said to be differentiable at s_0 and $\Phi'_-(s_0) = \Phi'_+(s_0) = P$. Here, P is called the derivative of Φ at s_0 and we consider one-sided derivative at the end points of I .

Remark 3 ([18]). If Φ is differentiable at s_0 , then there exists a $\delta > 0$, such that:

(a) For $0 < h < \delta$, $\Phi(s_0 - h) \ominus_h \Phi(s_0)$ or $\Phi(s_0) \ominus_h \Phi(s_0 - h)$ exists.

(b) For $0 < h < \delta$, $\Phi(s_0 + h) \ominus_h \Phi(s_0)$ or $\Phi(s_0) \ominus_h \Phi(s_0 + h)$ exists.

3. Generalized Nabla Hukuhara Differentiability on Time Scales

This section is concerned with defining and studying the properties of ∇^s derivative for fuzzy functions on time scales. In addition, we illustrate the results with suitable examples.

Definition 7 ([21]). For any given $\epsilon > 0$, there exists a $\delta > 0$, such that the fuzzy function $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ has a unique $\mathbb{T}$ -limit $P \in \mathbb{E}_n$ at $s \in \mathbb{T}^{[a,b]}$ if $D_H(\Phi(s) \ominus_h P, \hat{0}) \leq \epsilon$, for all $s \in N_{\mathbb{T}^{[a,b]}}(s, \delta)$ and it is denoted by $\mathbb{T} - \lim_{s \rightarrow s_0} \Phi(s)$.

Here, $\mathbb{T}$ -limit denotes the limit on time scale in the metric space $(\mathbb{E}_n, D_H)$.

Remark 4. From the above definition, we have

$$\mathbb{T} - \lim_{s \rightarrow s_0} \Phi(s) = P \in \mathbb{E}_n \iff \mathbb{T} - \lim_{s \rightarrow s_0} (\Phi(s) \ominus_h P) = \hat{0},$$

where the zero element in $\mathbb{E}_n$ is given by $\hat{0}$.

Definition 8. A fuzzy mapping $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ is continuous at $s_0 \in \mathbb{T}$, if $\mathbb{T} - \lim_{s \rightarrow s_0} \Phi(s) \in \mathbb{E}_n$ exists and $\mathbb{T} - \lim_{s \rightarrow s_0} \Phi(s) = \Phi(s_0)$, i.e.,

$$\mathbb{T} - \lim_{s \rightarrow s_0} (\Phi(s) \ominus_h \Phi(s_0)) = \hat{0}.$$

Remark 5. If $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ is continuous at $s_0 \in \mathbb{T}^{[a,b]}$, then, for every $\epsilon > 0$, there exists a $\delta > 0$, such that

$$D_H(\Phi(s) \ominus_h \Phi(s_0), \hat{0}) \leq \epsilon, \text{ for all } s \in N_{\mathbb{T}^{[a,b]}}.$$

Remark 6. Let $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ and $s_0 \in \mathbb{T}^{[a,b]}$.

- (a) If $\mathbb{T} - \lim_{s \rightarrow s_0^+} \Phi(s) = \Phi(s_0)$, then Φ is said to be right continuous at s_0 .
- (b) If $\mathbb{T} - \lim_{s \rightarrow s_0^-} \Phi(s) = \Phi(s_0)$, then Φ is said to be left continuous at s_0 .
- (c) If $\mathbb{T} - \lim_{s \rightarrow s_0^+} \Phi(s) = \Phi(s_0) = \mathbb{T} - \lim_{s \rightarrow s_0^-} \Phi(s)$, then Φ is continuous at s_0 .

Definition 9. A fuzzy function $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ is said to be ∇^g left-differentiable at $s \in \mathbb{T}_k^{[a,b]}$, if there exists an element $\Phi_-^{\nabla^g}(s) \in \mathbb{E}_n$ with the property that, for any given $\epsilon > 0$, there exists a $N_{\mathbb{T}^{[a,b]}}$ of s for some $\delta > 0$ and $0 \leq h \leq \delta$,

$$D_H[\Phi(\varrho(s)) \ominus_h \Phi(s-h), (h-v(s)) \odot \Phi_-^{\nabla^g}(s)] \leq \epsilon |h-v(s)| \quad (1)$$

or

$$D_H[\Phi(s-h) \ominus_h \Phi(\varrho(s)), -(h-v(s)) \odot \Phi_-^{\nabla^g}(s)] \leq \epsilon |-(h-v(s))|, \quad (2)$$

for all $s-h \in N_{\mathbb{T}^{[a,b]}}$, where $v(s) = s - \varrho(s)$, $\Phi_-^{\nabla^g}(s)$ is the generalized nabla left-derivative of Φ at s .

Definition 10. A fuzzy function $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ is said to be ∇^g right-differentiable at $s \in \mathbb{T}_k^{[a,b]}$, if there exists an element $\Phi_+^{\nabla^g}(s) \in \mathbb{E}_n$ with the property that, for every given $\epsilon > 0$, there exists a neighborhood $N_{\mathbb{T}^{[a,b]}}$ of s for some $\delta > 0$ and $0 \leq h \leq \delta$,

$$D_H[\Phi(s+h) \ominus_h \Phi(\varrho(s)), (h+v(s)) \odot \Phi_+^{\nabla^g}(s)] \leq \epsilon |h+v(s)| \quad (3)$$

or

$$D_H[\Phi(\varrho(s)) \ominus_h \Phi(s+h), -(h+v(s)) \odot \Phi_+^{\nabla^g}(s)] \leq \epsilon |-(h+v(s))|, \quad (4)$$

for all $s+h \in N_{\mathbb{T}^{[a,b]}}$, where $v(s) = s - \varrho(s)$, $\Phi_+^{\nabla^g}(s)$ is the generalized nabla right-derivative of Φ at s .

Definition 11. A fuzzy function $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ is said to be ∇^g differentiable at $s \in \mathbb{T}_k^{[a,b]}$, if Φ is both right- and left-differentiable at $s \in \mathbb{T}_k^{[a,b]}$ and

$$\Phi_+^{\nabla^g}(s) = \Phi_-^{\nabla^g}(s) = \Phi^{\nabla^g}(s).$$

Here, $\Phi_+^{\nabla^g}(s)$ or $\Phi_-^{\nabla^g}(s)$ is called ∇^g -derivative of Φ at $s \in \mathbb{T}_k^{[a,b]}$ and it is denoted by $\Phi^{\nabla^g}(s)$. Moreover, if ∇^g derivative exists at each $s \in \mathbb{T}_k^{[a,b]}$, then Φ is ∇^g differentiable on $\mathbb{T}_k^{[a,b]}$.

Theorem 3. Let $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ be a fuzzy function and $s \in \mathbb{T}_k^{[a,b]}$, then:

- (a) If $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ is ∇^g differentiable at s , then Φ is continuous at $s \in \mathbb{T}_k^{[a,b]}$.
- (b) If s is left dense and $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ is ∇^g differentiable at s iff the limits

$$\lim_{h \rightarrow 0^+} \frac{1}{h} \odot (\Phi(s) \ominus_h \Phi(s-h)) \text{ or } \lim_{h \rightarrow 0^+} \frac{-1}{h} \odot (\Phi(s-h) \ominus_h \Phi(s))$$

and

$$\lim_{h \rightarrow 0^+} \frac{1}{h} \odot (\Phi(s+h) \ominus_h \Phi(s)) \text{ or } \lim_{h \rightarrow 0^+} \frac{-1}{h} \odot (\Phi(s) \ominus_h \Phi(s+h))$$

exist as a finite number and holds any one of the following:

$$(i) \lim_{h \rightarrow 0^+} \frac{1}{h} \odot (\Phi(s) \ominus_h \Phi(s-h)) = \Phi^{\nabla^g}(s) = \lim_{h \rightarrow 0^+} \frac{1}{h} \odot (\Phi(s+h) \ominus_h \Phi(s));$$

$$(ii) \lim_{h \rightarrow 0^+} \frac{1}{h} \odot (\Phi(s) \ominus_h \Phi(s-h)) = \Phi^{\nabla^g}(s) = \lim_{h \rightarrow 0^+} \frac{-1}{h} \odot (\Phi(s) \ominus_h \Phi(s+h));$$

$$(iii) \lim_{h \rightarrow 0^+} \frac{-1}{h} \odot (\Phi(s-h) \ominus_h \Phi(s)) = \Phi^{\nabla^g}(s) = \lim_{h \rightarrow 0^+} \frac{1}{h} \odot (\Phi(s+h) \ominus_h \Phi(s));$$

$$(iv) \lim_{h \rightarrow 0^+} \frac{-1}{h} \odot (\Phi(s-h) \ominus_h \Phi(s)) = \Phi^{\nabla^g}(s) = \lim_{h \rightarrow 0^+} \frac{-1}{h} \odot (\Phi(s) \ominus_h \Phi(s+h)).$$

Proof. (a) Suppose that Φ is ∇^g differentiable at s . Let $\epsilon \in (0, 1)$. Choose $\epsilon^1 = \epsilon[1 + K + 2\nu(s)]^{-1}$, where $K = D_H[\Phi^{\nabla^g}(s), \hat{0}]$. Clearly, $\epsilon^1 \in (0, 1)$. Since Φ is ∇^g left-differentiable, there exists $N_{\mathbb{T}[a,b]}$ a neighborhood of s such that, for all $h \geq 0$ with $s-h \in N_{\mathbb{T}[a,b]}$,

$$D_H[\Phi(\varrho(s)) \ominus_h \Phi(s-h), (h-\nu(s)) \odot \Phi_-^{\nabla^g}(s)] \leq \epsilon|h-\nu(s)|,$$

or

$$D_H[\Phi(s-h) \ominus_h \Phi(\varrho(s)), -(h-\nu(s)) \odot \Phi_-^{\nabla^g}(s)] \leq \epsilon|-(h-\nu(s))|.$$

For $0 \leq h < \epsilon^1$ and for all $h \geq 0$, to each $s-h \in N_{\mathbb{T}[a,b]} \cap (s-h, s+h)$, we have,

$$\begin{aligned} D_H[\Phi(s), \Phi(s-h)] &= D_H[\Phi(s) \ominus_h \Phi(s-h), \hat{0}] \\ &= D_H[\Phi(s) \ominus_h \Phi(\varrho(s)) \oplus \Phi(\varrho(s)) \ominus_h \Phi(s-h), \\ &\quad (h-\nu(s)) \odot \Phi_-^{\nabla^g}(s) \oplus \nu(s) \odot \Phi_-^{\nabla^g}(s) \\ &\quad \oplus (-h) \odot \Phi_-^{\nabla^g}(s)] \\ &\leq D_H[\Phi(\varrho(s)) \ominus_h \Phi(s-h), (h-\nu(s)) \odot \Phi_-^{\nabla^g}(s)] \\ &\quad + D_H[\Phi(s) \ominus_h \Phi(\varrho(s)), \nu(s) \odot \Phi_-^{\nabla^g}(s)] \\ &\quad + hD_H[\Phi_-^{\nabla^g}(s), \hat{0}] \\ &\leq \epsilon^1|h-\nu(s)| + \epsilon^1\nu(s) + hK \\ &= \epsilon^1h + hK + 2\epsilon^1\nu(s) \\ &< \epsilon^1(1 + K + 2\nu(s)) = \epsilon. \end{aligned}$$

Similarly, we can prove Φ is continuous at s , if ∇^g is right-differentiable at s .

(b) Suppose that Φ is ∇^g differentiable at s and s is left dense. To each $\epsilon \geq 0$, there exists a neighborhood $N_{\mathbb{T}[a,b]}$ of s such that

$$D_H[\Phi(\varrho(s)) \ominus_h \Phi(s-h), (h-\nu(s)) \odot \Phi_-^{\nabla^g}(s)] \leq \epsilon|h-\nu(s)|$$

or

$$D_H[\Phi(s-h) \ominus_h \Phi(\varrho(s)), (\nu(s)-h) \odot \Phi_-^{\nabla^g}(s)] \leq \epsilon|-(h-\nu(s))|,$$

and

$$D_H[\Phi(s+h) \ominus_h \Phi(\varrho(s)), (h+\nu(s)) \odot \Phi_+^{\nabla^g}(s)] \leq \epsilon|h+\nu(s)|$$

or

$$D_H[\Phi(\varrho(s)) \ominus_h \Phi(s+h), -(h+\nu(s)) \odot \Phi_+^{\nabla^g}(s)] \leq \epsilon|-(h+\nu(s))|,$$

for all $s - \hbar, s + \hbar \in N_{\mathbb{T}[a,b]}, 0 \leq \hbar \leq \delta$. Since s is left dense, $\varrho(s) = s, \nu(s) = 0$, we have

$$D_H \left[\frac{1}{\hbar} [\Phi(s) \ominus_h \Phi(s - \hbar)], \Phi_-^{\nabla^g}(s) \right] \leq \epsilon$$

or

$$D_H \left[\frac{-1}{\hbar} [\Phi(s - \hbar) \ominus_h \Phi(s)], \Phi_-^{\nabla^g}(s) \right] \leq \epsilon$$

and

$$D_H \left[\frac{1}{\hbar} [\Phi(s + \hbar) \ominus_h \Phi(s)], \Phi_+^{\nabla^g}(s) \right] \leq \epsilon$$

or

$$D_H \left[\frac{-1}{\hbar} [\Phi(s) \ominus_h \Phi(s + \hbar)], \Phi_+^{\nabla^g}(s) \right] \leq \epsilon,$$

for $s - \hbar, s + \hbar \in N_{\mathbb{T}[a,b]}, 0 \leq \hbar \leq \delta$. Since ϵ is arbitrary, we get any one of (i)–(iv). $\square$

The converse proposition of Theorem 3(a) may not be true. That is a fuzzy function which is continuous may not be differentiable.

Example 1. Let $\Phi : \mathbb{T}^{[0,4\pi]} \rightarrow \mathbb{E}_1$ be a fuzzy function defined as follows:

$$\Phi(s) = \begin{cases} \sin(s) \odot c, & \text{if } m\pi \leq s \leq (4m+1)\frac{\pi}{4} \\ \cos(s) \odot c, & \text{if } (4m+1)\frac{\pi}{4} \leq s \leq (4m+2)\frac{\pi}{4}, \end{cases}$$

where $m = 0, 1, 2, 3$, $\mathbb{T} = P_{\frac{\pi}{2}, \frac{\pi}{2}} = \bigcup_{k=0}^{\infty} \left[k\pi, k\pi + \frac{\pi}{2} \right]$ and $c = (2, 4, 6)$ is a triangular fuzzy number. Since

$$\mathbb{T} - \lim_{s \rightarrow \frac{\pi}{4}^-} \Phi(s) = \sin\left(\frac{\pi}{4}\right) \odot c = \frac{1}{\sqrt{2}} \odot c$$

and

$$\mathbb{T} - \lim_{s \rightarrow \frac{\pi}{4}^+} \Phi(s) = \cos\left(\frac{\pi}{4}\right) \odot c = \frac{1}{\sqrt{2}} \odot c.$$

In addition, $\mathbb{T} - \lim_{s \rightarrow \frac{\pi}{4}} \Phi(s) = \Phi\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \odot c$. Then, from Remark 6(c), Φ is continuous at $s = \frac{\pi}{4}$ (See Figure 1). Since $s = \frac{\pi}{4}$ is dense, $\sin \frac{\pi}{4} > \sin(\frac{\pi}{4} - h) > 0$, for h sufficiently small, and, from Remark 1(a), we have

$$\begin{aligned} \Phi_-^{\nabla^g}(s) &= \lim_{h \rightarrow 0} \frac{1}{h} \odot \left(\Phi\left(\frac{\pi}{4}\right) \ominus_h \Phi\left(\frac{\pi}{4} - h\right) \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \odot \left(\left(\sin \frac{\pi}{4} \odot c \right) \ominus_h \left(\sin\left(\frac{\pi}{4} - h\right) \odot c \right) \right) \\ &= \lim_{h \rightarrow 0} \frac{\left(\sin\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4} - h\right) \right)}{h} \odot c \\ &= \frac{1}{\sqrt{2}} \odot c. \end{aligned}$$

In a similarly way,

$$\begin{aligned}\Phi_+^{\nabla^g}(s) &= \lim_{h \rightarrow 0} \frac{1}{-h} \odot \left(\Phi\left(\frac{\pi}{4} \odot c\right) \ominus_h \Phi\left(\frac{\pi}{4} + h\right) \odot c \right) \\ &= \lim_{h \rightarrow 0} \frac{\left(\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4} + h\right)\right)}{-h} \odot c \\ &= \frac{-1}{\sqrt{2}} \odot c.\end{aligned}$$

Therefore, $\Phi_-^{\nabla^g}(s) \neq \Phi_+^{\nabla^g}(s)$. Hence, Φ is not ∇^g differentiable at $s = \frac{\pi}{4}$.

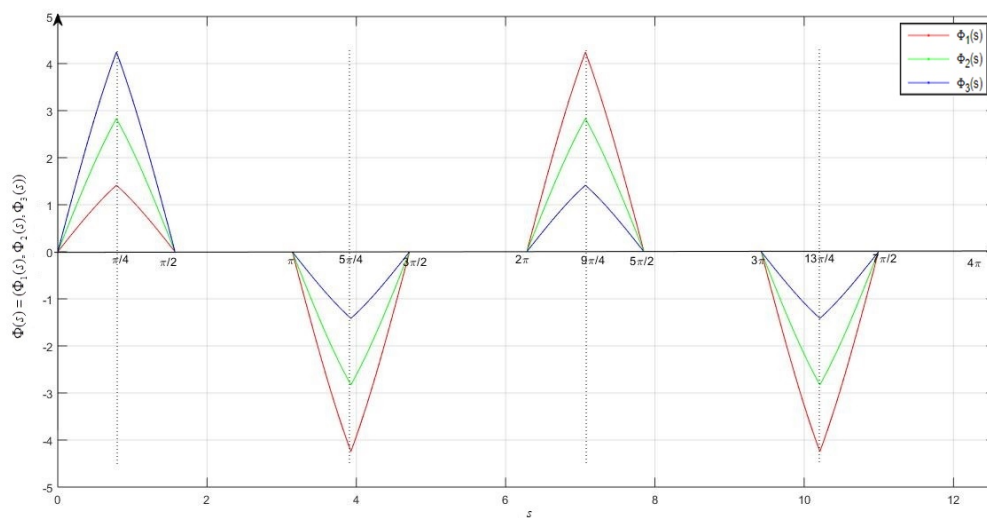


Figure 1. Graphical Representation of $\Phi(s)$ in Example 1.

Definition 11 can equivalently be written as follows:

Remark 7. If $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ is ∇^g differentiable at $s \in \mathbb{T}_k^{[a,b]}$ if and only if there exists an element $\Phi^{\nabla^g}(s) \in \mathbb{E}_n$, such that any one of the following holds:

(GH1) for $0 < h < \delta$, provided the Hukuhara difference $\Phi(q(s)) \ominus_h \Phi(s - h)$, $\Phi(s + h) \ominus_h \Phi(q(s))$ and the limits exist

$$\begin{aligned}\mathbb{T} - \lim_{h \rightarrow 0} \frac{1}{h - v(s)} \odot (\Phi(q(s)) \ominus_h \Phi(s - h)) \\ = \mathbb{T} - \lim_{h \rightarrow 0} \frac{1}{h + v(s)} \odot (\Phi(s + h) \ominus_h \Phi(q(s))) \\ = \Phi^{\nabla^g}(s)\end{aligned}$$

or

(GH2) for $0 < h < \delta$, provided the Hukuhara difference $\Phi(s - h) \ominus_h \Phi(q(s))$, $\Phi(q(s)) \ominus_h \Phi(s + h)$ and the limits exist

$$\begin{aligned}\mathbb{T} - \lim_{h \rightarrow 0} \frac{-1}{h - v(s)} \odot (\Phi(s - h) \ominus_h \Phi(q(s))) \\ = \mathbb{T} - \lim_{h \rightarrow 0} \frac{-1}{h + v(s)} \odot (\Phi(q(s)) \ominus_h \Phi(s + h)) \\ = \Phi^{\nabla^g}(s)\end{aligned}$$

or

(GH3) for $0 < \hbar < \delta$, provided the Hukuhara difference $\Phi(\varrho(s)) \ominus_h \Phi(s - \hbar)$, $\Phi(\varrho(s)) \ominus_h \Phi(s + \hbar)$ and the limits exist

$$\begin{aligned} & \mathbb{T} - \lim_{\hbar \rightarrow 0} \frac{1}{\hbar - \nu(s)} \odot (\Phi(\varrho(s)) \ominus_h \Phi(s - \hbar)) \\ &= \mathbb{T} - \lim_{\hbar \rightarrow 0} \frac{-1}{\hbar + \nu(s)} \odot (\Phi(\varrho(s)) \ominus_h \Phi(s + \hbar)) \\ &= \Phi^{\nabla^g}(s) \end{aligned}$$

or

(GH4) for $0 < \hbar < \delta$, provided the Hukuhara difference $\Phi(s - \hbar) \ominus_h \Phi(\varrho(s))$, $\Phi(s + \hbar) \ominus_h \Phi(\varrho(s))$ and the limits exist

$$\begin{aligned} & \mathbb{T} - \lim_{\hbar \rightarrow 0} \frac{-1}{\hbar - \nu(s)} \odot (\Phi(s - \hbar) \ominus_h \Phi(\varrho(s))) \\ &= \mathbb{T} - \lim_{\hbar \rightarrow 0} \frac{1}{\hbar + \nu(s)} \odot (\Phi(s + \hbar) \ominus_h \Phi(\varrho(s))) \\ &= \Phi^{\nabla^g}(s). \end{aligned}$$

Thus, $\Phi^{\nabla^g} : \mathbb{T}_k^{[a,b]} \rightarrow \mathbb{E}_n$ is called the ∇^g derivative of Φ on $\mathbb{T}_k^{[a,b]}$.

Remark 8. Let $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ be ∇^g differentiable.

- (a) If Φ is (GH1)-nabla differentiable at $s \in \mathbb{T}_k^{[a,b]}$, then there exists a $\delta > 0$, such that, for $0 \leq \lambda \leq 1$, we have

$$\begin{aligned} \text{diam}[\Phi(s - \hbar)]^\lambda &\leq \text{diam}[\Phi(\varrho(s))]^\lambda \\ &\leq \text{diam}[\Phi(s + \hbar)]^\lambda, \text{ for } 0 < \hbar < \delta. \end{aligned}$$

Thus, if Φ is (GH1)-nabla differentiable on $\mathbb{T}^{[a,b]}$, then $\text{diam}[\Phi(s)]^\lambda$ is non-decreasing on $\mathbb{T}^{[a,b]}$.

- (b) If Φ is (GH2)-nabla differentiable at $s \in \mathbb{T}_k^{[a,b]}$, then there exists a $\delta > 0$, such that, for $0 \leq \lambda \leq 1$, we have

$$\begin{aligned} \text{diam}[\Phi(s - \hbar)]^\lambda &\geq \text{diam}[\Phi(\varrho(s))]^\lambda \\ &\geq \text{diam}[\Phi(s + \hbar)]^\lambda, \text{ for } 0 < \hbar < \delta. \end{aligned}$$

Thus, if Φ is (GH2)-nabla differentiable on $\mathbb{T}^{[a,b]}$, then $\text{diam}[\Phi(s)]^\lambda$ is non-increasing on $\mathbb{T}^{[a,b]}$.

- (c) If Φ is (GH3)-nabla differentiable at $s \in \mathbb{T}_k^{[a,b]}$, then there exists a $\delta > 0$, such that, for $0 \leq \lambda \leq 1$, we have

$$\begin{aligned} \text{diam}[\Phi(s - \hbar)]^\lambda &\leq \text{diam}[\Phi(\varrho(s))]^\lambda \text{ and} \\ \text{diam}[\Phi(s + \hbar)]^\lambda &\leq \text{diam}[\Phi(\varrho(s))]^\lambda, \text{ for } 0 < \hbar < \delta. \end{aligned}$$

Therefore, $\text{diam}[\Phi(s)]^\lambda$ is non-decreasing in the left neighborhood and non-increasing in the right neighborhood of s . Thus, monotonicity of $\text{diam}[\Phi(s)]^\lambda$ fails at s .

- (d) If Φ is (GH4)-nabla differentiable at $s \in \mathbb{T}_k^{[a,b]}$, then there exists a $\delta > 0$ such that, for $0 \leq \lambda \leq 1$,

$$\begin{aligned} \text{diam}[\Phi(\varrho(s))]^\lambda &\leq \text{diam}[\Phi(s - \hbar)]^\lambda \text{ and} \\ \text{diam}[\Phi(\varrho(s))]^\lambda &\leq \text{diam}[\Phi(s + \hbar)]^\lambda, \text{ for } 0 < \hbar < \delta. \end{aligned}$$

Therefore, $\text{diam}[\Phi(s)]^\lambda$ is non-increasing in the left neighborhood and non-decreasing in the right neighborhood of s . Thus, monotonicity of $\text{diam}[\Phi(s)]^\lambda$ fails at s .

Example 2. Let $\Phi : \mathbb{T}^{[0,3\pi]} \rightarrow \mathbb{E}_1$ be a fuzzy function defined as $\Phi(s) = \sin(s) \odot c$, where $c = (2, 4, 6)$ is a triangular fuzzy number. Let $\mathbb{T} = P_{\pi, \pi} = \bigcup_{k=0}^{\infty} [2k\pi, (2k+1)\pi]$.

In Figure 2, it is easily seen that $\Phi(s)$ is (GH1)-nabla differentiable on $\mathbb{T}^{[0, \frac{\pi}{2}] \cup (2\pi, \frac{5\pi}{2}]}$, $\Phi(s)$ is (GH2)-nabla differentiable on $\mathbb{T}^{(\frac{\pi}{2}, \pi] \cup (\frac{5\pi}{2}, 3\pi]}$. Now, we check the ∇^g differentiability at $s = \frac{\pi}{2}$. Since $s = \frac{\pi}{2}$ is dense, $v(s) = 0$. In addition, $\sin(\frac{\pi}{2}) > \sin(\frac{\pi}{2} + h) > 0$, and, from Remark 1(a), we have $(\sin(\frac{\pi}{2}) \odot c) \ominus_h (\sin(\frac{\pi}{2} + h) \odot c) = (\sin(\frac{\pi}{2}) - \sin(\frac{\pi}{2} + h)) \odot c$. Consider

$$\begin{aligned} \Phi_+^{\nabla^g}\left(\frac{\pi}{2}\right) &= \lim_{h \rightarrow 0^+} \frac{-1}{h} \odot \left(\Phi\left(\frac{\pi}{2}\right) \ominus_h \Phi\left(\frac{\pi}{2} + h\right)\right) \\ &= \lim_{h \rightarrow 0} \frac{-1}{h} \odot \left(\left(\sin \frac{\pi}{2} \odot c\right) \ominus_h \left(\sin\left(\frac{\pi}{2} + h\right) \odot c\right)\right) \\ &= \lim_{h \rightarrow 0} \frac{(\sin(\frac{\pi}{2}) - \sin(\frac{\pi}{2} + h))}{-h} \odot c \\ &= 0 \odot c = \hat{0}. \end{aligned}$$

In a similar way, we get $\Phi_-^{\nabla^g}(\frac{\pi}{2}) = \hat{0}$. Hence, Φ is (GH3)-nabla differentiable at $s = \frac{\pi}{2}$. Similarly, we can show that Φ is also (GH3)-nabla differentiable at $s = \frac{5\pi}{2}$.

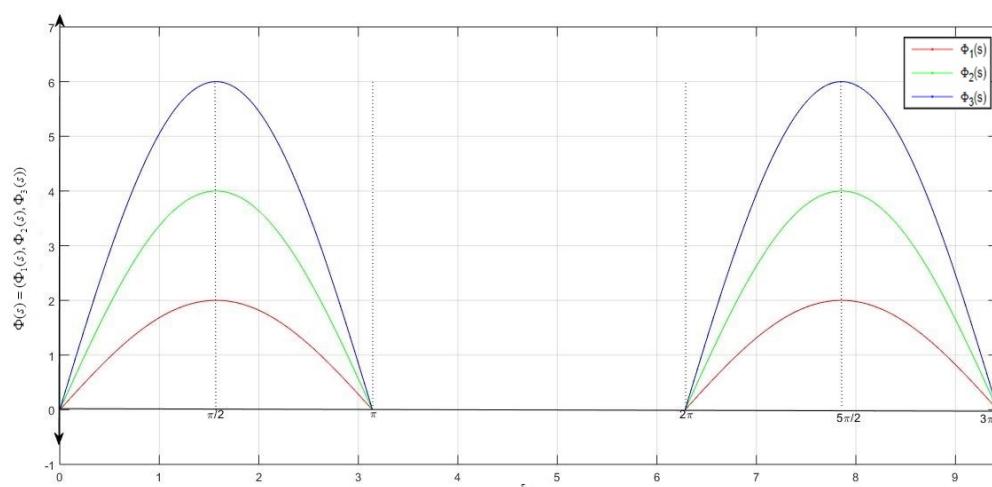


Figure 2. Graphical Representation of $\Phi(s)$ in Example 2.

Theorem 4. If $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ is continuous at s and s is left scattered, then:

(a) Φ is ∇^g differentiable at s as in (GH1) or (GH2) with

$$\begin{aligned} \Phi^{\nabla^g}(s) &= \frac{1}{v(s)} \odot (\Phi(s) \ominus_h \Phi(\varrho(s))) \\ &= \frac{-1}{v(s)} \odot (\Phi(\varrho(s)) \ominus_h \Phi(s)) \end{aligned}$$

and $\Phi^{\nabla^g}(s) = \hat{0}$ (or) $\Phi^{\nabla^g}(s) \in \mathbb{R}^n$;

or

(b) Φ is ∇^s differentiable at s as in (GH3) with $\Phi^{\nabla^s}(s) = \frac{-1}{\nu(s)} \odot (\Phi(\varrho(s)) \ominus_h \Phi(s))$;

or

(c) Φ is ∇^s differentiable at s as in (GH4) with $\Phi^{\nabla^s}(s) = \frac{1}{\nu(s)} \odot (\Phi(s) \ominus_h \Phi(\varrho(s)))$.

Proof. (a) Suppose $s \in \mathbb{T}_k^{[a,b]}$ and Φ is continuous at left scattered point s . Then, from (GH1) or (GH2), we have

$$\mathbb{T} - \lim_{h \rightarrow 0} \frac{1}{h - \nu(s)} \odot (\Phi(\varrho(s)) \ominus_h \Phi(s - h)) = \frac{-1}{\nu(s)} \odot (\Phi(\varrho(s)) \ominus_h \Phi(s)),$$

$$\mathbb{T} - \lim_{h \rightarrow 0} \frac{1}{h + \nu(s)} \odot (\Phi(s + h) \ominus_h \Phi(\varrho(s))) = \frac{1}{\nu(s)} \odot (\Phi(s) \ominus_h \Phi(\varrho(s))).$$

Since the Hukuhara differences $(\Phi(\varrho(s)) \ominus_h \Phi(s))$, $(\Phi(s) \ominus_h \Phi(\varrho(s)))$ exists, then

$$\Phi(\varrho(s)) = \Phi(s) \oplus u(s) \quad \text{and} \quad \Phi(s) = \Phi(\varrho(s)) \oplus v(s),$$

where $u(s), v(s)$ are in $\mathbb{E}_n$. By adding the above equations, we get $u(s) \oplus v(s) = \hat{0}$. Then, $u(s) = \hat{0} = v(s)$ or $u(s), v(s)$ are in $\mathbb{R}^n$ and hence the result is obvious.

(b) Suppose $s \in \mathbb{T}_k^{[a,b]}$ and Φ is continuous at left scattered point s . Then, from (GH3), we have

$$\mathbb{T} - \lim_{h \rightarrow 0} \frac{1}{h - \nu(s)} \odot (\Phi(\varrho(s)) \ominus_h \Phi(s - h)) = \frac{-1}{\nu(s)} \odot (\Phi(\varrho(s)) \ominus_h \Phi(s))$$

$$\mathbb{T} - \lim_{h \rightarrow 0} \frac{-1}{h + \nu(s)} \odot (\Phi(\varrho(s)) \ominus_h \Phi(s + h)) = \frac{-1}{\nu(s)} \odot (\Phi(\varrho(s)) \ominus_h \Phi(s))$$

$$\text{Hence, } \Phi^{\nabla^s}(s) = \frac{-1}{\nu(s)} \odot (\Phi(\varrho(s)) \ominus_h \Phi(s)).$$

(c) Suppose $s \in \mathbb{T}_k^{[a,b]}$ and Φ is continuous at left scattered point s . Then, from (GH4), we have

$$\mathbb{T} - \lim_{h \rightarrow 0} \frac{-1}{h - \nu(s)} \odot (\Phi(s - h) \ominus_h \Phi(\varrho(s))) = \frac{1}{\nu(s)} \odot (\Phi(s) \ominus_h \Phi(\varrho(s))),$$

$$\mathbb{T} - \lim_{h \rightarrow 0} \frac{1}{h + \nu(s)} \odot (\Phi(s + h) \ominus_h \Phi(\varrho(s))) = \frac{1}{\nu(s)} \odot (\Phi(s) \ominus_h \Phi(\varrho(s))).$$

$$\text{Hence, } \Phi^{\nabla^s}(s) = \frac{1}{\nu(s)} \odot (\Phi(s) \ominus_h \Phi(\varrho(s))). \quad \square$$

Remark 9. A fuzzy function $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_1$ is defined as $\Phi(s) = (\phi_1(s), \phi_2(s), \phi_3(s))$, where $\phi_k : \mathbb{T}^{[a,b]} \rightarrow \mathbb{R}$, $k = 1, 2, 3$ are nabla differentiable such that $\phi_1(s) < \phi_2(s) < \phi_3(s)$, for all $s \in \mathbb{T}^{[a,b]}$.

(a) If Φ is ∇^s differentiable as in (GH1) at ld-point s or ∇^s differentiable as (GH4) at left scattered point s , then $\Phi^{\nabla^s}(s) = (\phi_1^{\nabla}, \phi_2^{\nabla}, \phi_3^{\nabla})$, for $s \in \mathbb{T}_k^{[a,b]}$.

(b) If Φ is ∇^s differentiable as (GH2) at ld-point s or ∇^s differentiable as (GH3) at left scattered point s , then $\Phi^{\nabla^s}(s) = (\phi_3^{\nabla}, \phi_2^{\nabla}, \phi_1^{\nabla})$, for $s \in \mathbb{T}_k^{[a,b]}$.

Theorem 5. Let $\Phi, \Psi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ be ∇^s differentiable at $s \in \mathbb{T}_k^{[a,b]}$.

(1) If Φ and Ψ are both ∇^s differentiable of same kind, then:

(a) $(\Phi \oplus \Psi) : \mathbb{T}_k^{[a,b]} \rightarrow \mathbb{E}_n$ is also ∇^s differentiable of same kind at s with

$$(\Phi \oplus \Psi)^{\nabla^s}(s) = \Phi^{\nabla^s}(s) \oplus \Psi^{\nabla^s}(s).$$

(b) $(\Phi \ominus_h \Psi) : \mathbb{T}_k^{[a,b]} \rightarrow \mathbb{E}_n$ also ∇^s differentiable of same kind at s , provided $(\Phi \ominus_h \Psi)$ exists and

$$(\Phi \ominus_h \Psi)^{\nabla^s}(s) = \Phi^{\nabla^s}(s) \ominus_h \Psi^{\nabla^s}(s).$$

(2) If Φ and Ψ are different kinds of ∇^s differentiable at s , and $(\Phi \ominus_h \Psi)$ exists for $s \in \mathbb{T}_k^{[a,b]}$, then $(\Phi \ominus_h \Psi)$ is ∇^s differentiable at s with $(\Phi \ominus_h \Psi)^{\nabla^s}(s) = \Phi^{\nabla^s}(s) \oplus (-1) \odot \Psi^{\nabla^s}(s)$.

Proof. If s is ld-point, then $\varrho(s) = s, \nu(s) = 0$. The proof of this theorem is similar to the proof of Lemma 4 and Theorem 4 in [17].

1(a). Suppose that Φ and Ψ are both (GH3)-nabla differentiable at left scattered point $s \in \mathbb{T}_k^{[a,b]}$. Then, $\Phi(\varrho(s)) \ominus_h \Phi(s)$ exists with $\Phi(\varrho(s)) = \Phi(s) \oplus u(s)$ and $\Psi(\varrho(s)) \ominus_h \Psi(s)$ exists with $\Psi(\varrho(s)) = \Psi(s) \oplus v(s)$. Now,

$$(\Phi(\varrho(s)) \ominus_h \Phi(s)) \oplus (\Psi(\varrho(s)) \ominus_h \Psi(s)) = u(s) \oplus v(s).$$

Multiplying the above equation with $\frac{-1}{\nu(s)}$, we get

$$\begin{aligned} & \frac{-1}{\nu(s)} \odot ((\Phi(\varrho(s)) \ominus_h \Phi(s)) \oplus (\Psi(\varrho(s)) \ominus_h \Psi(s))) \\ &= \frac{-1}{\nu(s)} \odot (u(s) \oplus v(s)), \end{aligned}$$

and it follows that

$$\frac{(\Phi \oplus \Psi)(\varrho(s)) \ominus_h (\Phi \oplus \Psi)(s)}{-\nu(s)} = \frac{u(s)}{-\nu(s)} \oplus \frac{v(s)}{-\nu(s)}.$$

Hence, $(\Phi \oplus \Psi)$ is ∇^s differentiable as in (GH3) with

$$(\Phi \oplus \Psi)^{\nabla^s}(s) = \Phi^{\nabla^s}(s) \oplus \Psi^{\nabla^s}(s).$$

The case when Φ and Ψ are ∇^s differentiable as in (GH4) is similar to the previous one.

1(b). Suppose Φ and Ψ are both (GH3)-nabla differentiable at left scattered points $s \in \mathbb{T}_k^{[a,b]}$, similar to 1(a), we have $\Phi(\varrho(s)) = \Phi(s) \oplus u(s)$ and $\Psi(\varrho(s)) = \Psi(s) \oplus v(s)$. Consider

$$\begin{aligned} (\Phi \ominus_h \Psi)(\varrho(s)) &= \Phi(\varrho(s)) \ominus_h \Psi(\varrho(s)) \\ &= (\Phi(s) \oplus u(s)) \ominus_h (\Psi(s) \oplus v(s)) \\ &= (\Phi(s) \ominus_h \Psi(s)) \oplus (u(s) \ominus_h v(s)). \end{aligned}$$

It implies that

$$(\Phi \ominus_h \Psi)(\varrho(s)) \ominus_h (\Phi \ominus_h \Psi)(s) = u(s) \ominus_h v(s).$$

Multiplying the above equation with $\frac{-1}{\nu(s)}$, we get the desired result. In a similar way, we can easily prove the other case.

(2). Suppose that Φ is ∇^s differentiable as in (GH3) and Ψ is ∇^s differentiable as in (GH4) at left scattered points $s \in \mathbb{T}_k^{[a,b]}$, then the Hukuhara difference $\Phi(\varrho(s)) \ominus_h \Phi(s)$ exists with $\Phi(\varrho(s)) =$

$\Phi(s) \oplus u(s)$ and $\Psi(s) \ominus_h \Psi(\varrho(s))$ exists with $\Psi(s) = \Psi(\varrho(s)) \oplus v(s)$. Now, by adding these equations, we get

$$\Phi(\varrho(s)) \oplus \Psi(s) = \Phi(s) \oplus u(s) \oplus \Psi(\varrho(s)) \oplus v(s).$$

Since the Hukuhara difference of $\Phi(\varrho(s)) \ominus_h \Psi(\varrho(s))$ and $\Phi(s) \ominus_h \Psi(s)$ exist, we have

$$(\Phi(\varrho(s)) \ominus_h \Psi(\varrho(s))) \ominus_h (\Phi(s) \ominus_h \Psi(s)) = u(s) \oplus v(s). \quad (5)$$

Now, by multiplying (5) with $\frac{1}{-v(s)}$, we get $\Phi \ominus \Psi$ is (GH3)-nabla differentiable.

In a similar way, if Φ is ∇^g differentiable as in (GH4) and Ψ is ∇^g differentiable as in (GH3) at left scattered points $s \in \mathbb{T}_k^{[a,b]}$, then we can easily prove that

$$(\Phi(s) \ominus_h \Psi(s)) \ominus_h (\Phi(\varrho(s)) \ominus_h \Psi(\varrho(s))) = \tilde{u}(s) + \tilde{v}(s). \quad (6)$$

Now, by multiplying (6) with $\frac{1}{v(s)}$, we get $\Phi \ominus \Psi$ is (GH4)-nabla differentiable. Therefore,

$$(\Phi \ominus_h \Psi)^{\nabla^g}(s) = \Phi^{\nabla^g}(s) \oplus (-1) \odot \Psi^{\nabla^g}(s).$$

□

The following example illustrates the feasibility of Theorem 5.

Example 3. Let $\Omega, \Psi : \mathbb{T}^{[0,3\pi]} \rightarrow \mathbb{E}_1$ be fuzzy functions defined as follows:

$$\Omega(s) = \begin{cases} (\frac{\pi}{2} - s) \odot c, & 0 \leq s \leq \pi \\ (s - \frac{5\pi}{2}) \odot c, & 2\pi \leq s \leq 3\pi \end{cases}$$

and

$$\Psi(s) = \begin{cases} \cos(s) \odot c, & 0 \leq s \leq \pi \\ -\cos(s) \odot c, & 2\pi \leq s \leq 3\pi \end{cases}$$

where $\mathbb{T} = P_{\pi,\pi}$, $c = (2, 4, 6)$ is a triangular fuzzy number.

IN Figures 3 and 4, it is easily seen that Ω and Ψ are (GH2)-nabla differentiable on $\mathbb{T}^{[0, \frac{\pi}{2}) \cup (2\pi, \frac{5\pi}{2})}$, (GH1)-nabla differentiable on $\mathbb{T}^{(\frac{\pi}{2}, \pi] \cup (\frac{5\pi}{2}, 3\pi]}$, and (GH4)-nabla differentiable at $s = \frac{\pi}{2}, \frac{5\pi}{2}$. Thus, $\Omega \oplus \Psi$, $\Omega \ominus_h \Psi$ are ∇^g differentiable at left scattered point $s = 2\pi$. Now, from Remark 1, we have

$$(\Omega \oplus \Psi)(s) = \begin{cases} (\frac{\pi}{2} - s + \cos(s)) \odot c, & s \in [0, \pi] \\ (s - \frac{5\pi}{2} - \cos(s)) \odot c, & s \in [2\pi, 3\pi]. \end{cases}$$

and

$$(\Omega \ominus_h \Psi)(s) = \begin{cases} (\frac{\pi}{2} - s - \cos(s)) \odot c, & s \in [0, \pi] \\ (s - \frac{5\pi}{2} + \cos(s)) \odot c, & s \in [2\pi, 3\pi]. \end{cases}$$

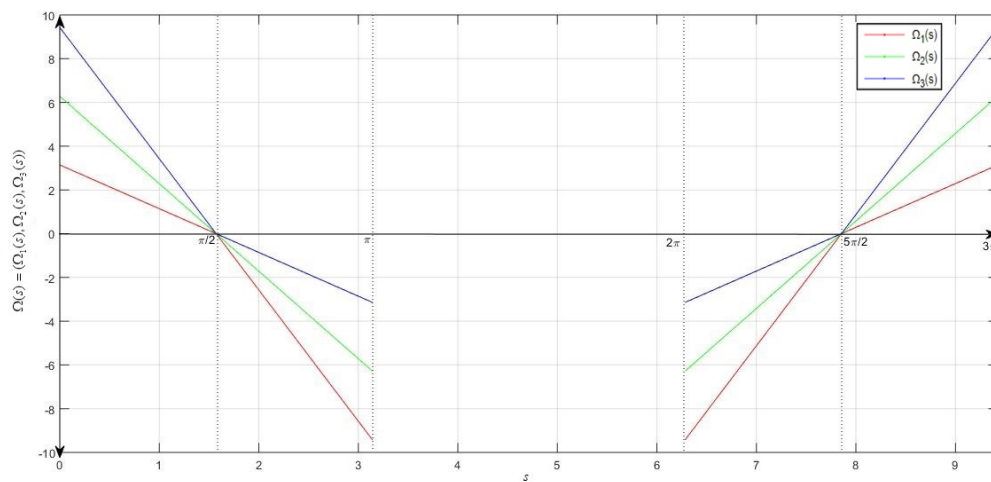


Figure 3. Graphical Representation of $\Omega(s)$ in Example 3.

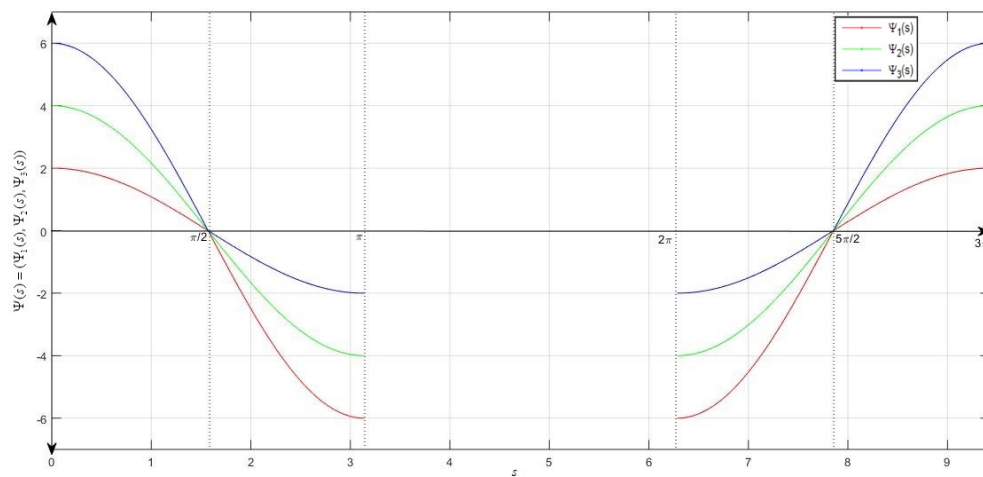


Figure 4. Graphical Representation of $\Psi(s)$ in Example 3.

In Figure 5, $(\Omega \oplus \Psi)$ is (GH2)-nabla differentiable on $\mathbb{T}^{[0, \frac{\pi}{2}] \cup [2\pi, \frac{5\pi}{2}]}$, (GH1)-nabla differentiable on $\mathbb{T}^{(\frac{\pi}{2}, \pi] \cup (\frac{5\pi}{2}, 3\pi]}$. At $s = \frac{\pi}{2}$, Ω and Ψ are (GH4)-nabla differentiable with $\Omega^{\nabla^g}(\frac{\pi}{2}) = (-1) \odot c$, and $\Psi^{\nabla^g}(\frac{\pi}{2}) = (-1) \odot c$. Now,

$$\begin{aligned} (\Omega \oplus \Psi)^{\nabla^g}_+(\frac{\pi}{2}) &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\pi}{2} - \left(\frac{\pi}{2} + h \right) + \cos\left(\frac{\pi}{2} + h\right) \right) \odot c \ominus_h \left(\frac{\pi}{2} - \left(\frac{\pi}{2} \right) + \cos\left(\frac{\pi}{2}\right) \right) \odot c \\ &= \left(\lim_{h \rightarrow 0} \frac{-h + \cos(\frac{\pi}{2} + h)}{h} \right) \odot c \\ &= \left(-1 + (-1) \lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \odot c = -2 \odot c. \end{aligned}$$

Similarly, we can show that $(\Omega \oplus \Psi)^{\nabla^g}_-(\frac{\pi}{2}) = -2 \odot c$. Thus, $(\Omega \oplus \Psi)$ is (GH4)-nabla differentiable at $\frac{\pi}{2}$ and Theorem 5 1(a) is verified.

In Figure 6, it is easily seen that $(\Omega \ominus_h \Psi)$ is (GH2)-nabla differentiable on $\mathbb{T}^{[0, \frac{\pi}{2}) \cup [2\pi, \frac{5\pi}{2})}$ and (GH1)-nabla differentiable on $\mathbb{T}^{(\frac{\pi}{2}, \pi] \cup (\frac{5\pi}{2}, 3\pi]}$. Again, from Remark 1, we have

$$\begin{aligned} (\Omega \ominus_h \Psi)^{\nabla^s}_+(\frac{\pi}{2}) &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\pi}{2} - (\frac{\pi}{2} + h) - \cos(\frac{\pi}{2} + h) \right) \odot c \ominus_h \left(\frac{\pi}{2} - \frac{\pi}{2} - \cos(\frac{\pi}{2}) \right) \odot c \\ &= \left(\lim_{h \rightarrow 0} \frac{-h - \cos(\frac{\pi}{2} + h)}{h} \right) \odot c \\ &= (-1 + \lim_{h \rightarrow 0} \frac{\sin h}{h}) \odot c = 0 \odot c = \hat{0}. \end{aligned}$$

Similarly, we can show that $(\Omega \ominus \Psi)^{\nabla^s}_-(\frac{\pi}{2}) = \hat{0}$. Thus, $(\Omega \ominus \Psi)$ is (GH4)-nabla differentiable at $\frac{\pi}{2}$ and Theorem 5 1(b) is verified.

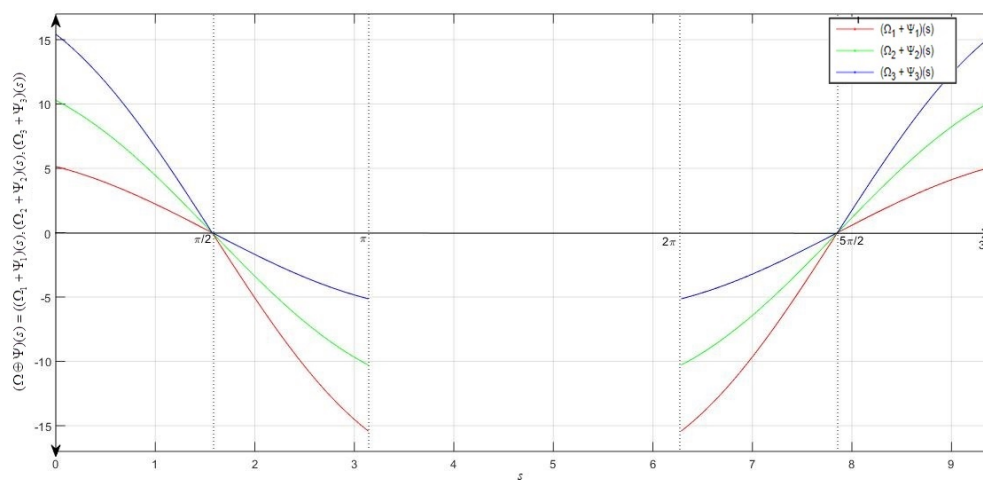


Figure 5. Graphical Representation of $(\Omega \oplus \Psi)(s)$ in Example 3.

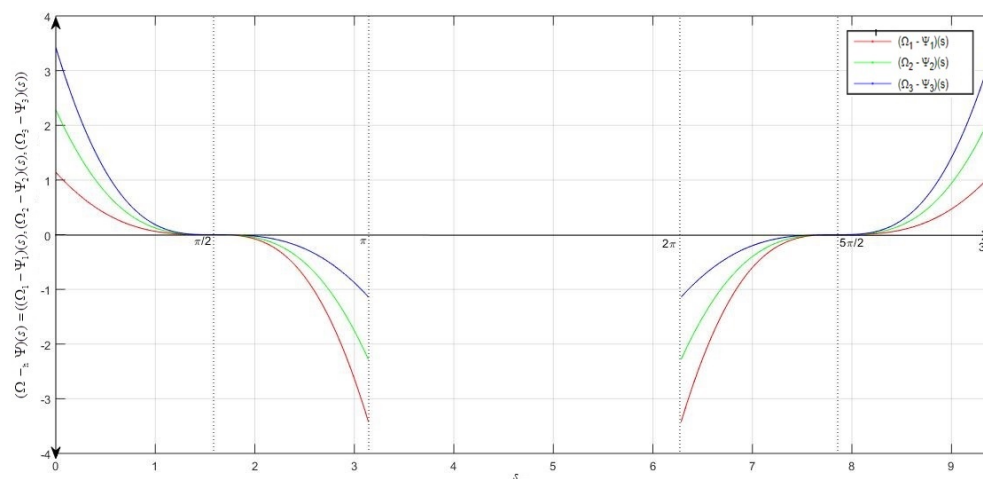


Figure 6. Graphical Representation of $(\Omega \ominus_h \Psi)(s)$ in Example 3.

Consider $\Phi(s)$ as in Example 2, Φ is (GH3)-nabla differentiability at $s = \frac{\pi}{2}$ and Ψ is (GH4)-nabla differentiability at $s = \frac{\pi}{2}$. Hence, Φ and Ψ are different kinds of ∇^s differentiable at $s = \frac{\pi}{2}$, and $(\Phi \ominus_h \Psi)$ exists at $s = \frac{\pi}{2}$. Now, from Theorem 5(2), we have

$$\begin{aligned} (\Phi \ominus_h \Psi)^{\nabla^s}(\frac{\pi}{2}) &= \lim_{h \rightarrow 0} \frac{1}{h} \odot (\sin(\frac{\pi}{2}) - \cos(\frac{\pi}{2}) \odot c) \ominus_h ((\sin(\frac{\pi}{2} - h) - \cos(\frac{\pi}{2} - h) \odot c) \\ &= \left(\lim_{h \rightarrow 0} \frac{1 - \cos h}{h} + \lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \odot c = c. \end{aligned}$$

Similarly, we can show that $(\Phi \ominus_h \Psi)^{\nabla^s}(\frac{\pi}{2}) = c$. Hence, Theorem 5(2) is verified.

Now, we check the ∇^s -differentiable at $s = 2\pi$. It is left scattered and $\varrho(2\pi) = \pi$, $\nu(2\pi) = \pi$. Clearly, Ω , Φ , and Ψ are (GH3)- and (GH4)-nabla differentiable at $s = 2\pi$. We get $\Omega^{\nabla^s}(2\pi) = \hat{0}$, $\Phi^{\nabla^s}(2\pi) = \hat{0}$ and $\Psi^{\nabla^s}(2\pi) = \hat{0}$. In addition, the results of Theorem 5 hold at left scattered point $s = 2\pi$.

4. Integration of Fuzzy Functions on Time Scales

In this section, we prove fundamental theorem of nabla integral calculus for fuzzy functions on time scales under generalized fuzzy nabla differentiable functions on time scales.

First, we prove an embedding theorem on $\mathbb{E}_n$ and obtain some results which are useful to prove the main theorem. To prove the these results, we make use of Definitions 1–3 and Theorem 4 in [31].

Let $\mathcal{C}[0, 1]$ be the set of all functions $\mathcal{F} : [0, 1] \rightarrow \mathbb{R}^n$, $\mathcal{F}$ is bounded on $[0, 1]$, left-continuous for each $x \in (0, 1]$, right-continuous on 0, and $\mathcal{F}$ has right limit for each $x \in [0, 1]$. Endowed with the norm $\|\mathcal{F}\|_{\mathcal{C}} = \sup \{|\mathcal{F}(\lambda)|_{\mathbb{R}^n}; x \in [0, 1]\}$, $\mathcal{C}[0, 1]$ is a Banach space. It is known that the following result which embeds $\mathbb{E}_n$ into $X = \mathcal{C}[0, 1] \times \mathcal{C}[0, 1]$ isometrically and isomorphically.

Theorem 6. If we define $i : \mathbb{E}_n \rightarrow X$ by $i(u) = (u_-, u_+)$, where $u_-, u_+ : [0, 1] \rightarrow \mathbb{R}^n$, $u_-(\lambda) = u_-^\lambda$, $u_+(\lambda) = u_+^\lambda$, then $i(\mathbb{E}_n)$ is a closed convex cone with vertex 0 in X (here X is a Banach space with the norm $\|(f, g)\| = \max(\|f\|_{\mathcal{C}}, \|g\|_{\mathcal{C}})$).

Proof. First, we show that $X = \mathcal{C}[0, 1] \times \mathcal{C}[0, 1]$ is a Banach space. Consider a cauchy sequence $l_{n_0} = (f_{n_0}, g_{n_0})$ and for $\epsilon^* > 0$, there exists $N > 0$, $n_0 > N$ such that $n_0, m_0 > N$ implies $\|l_{m_0} - l_{n_0}\| < \epsilon^*$, that is

$$\begin{aligned} \|l_{m_0} - l_{n_0}\| &= \|(f_{m_0}, g_{m_0}) - (f_{n_0}, g_{n_0})\| \\ &= \|(f_{m_0} - f_{n_0}, g_{m_0} - g_{n_0})\| \\ &= \max(\|f\|_{\mathcal{C}}, \|g\|_{\mathcal{C}}). \end{aligned}$$

which yields the result that $f_{n_0}(\lambda) \rightarrow f$ and $g_{n_0}(\lambda) \rightarrow g$ as $n_0 \rightarrow \infty$ where $\|\mathcal{F}\|_{\mathcal{C}} = \sup\{|\mathcal{F}(x)|; x \in [0, 1]\}$, $\mathcal{C}[0, 1]$ is a Banach space. Hence, $X = \mathcal{C}[0, 1] \times \mathcal{C}[0, 1]$ is a Banach space. To obtain i embeds $\mathbb{E}_n$ into $X = \mathcal{C}[0, 1] \times \mathcal{C}[0, 1]$ isometrically and isomorphically, we need to prove the following:

- (a) $i(p \odot u \oplus q \odot v) = pi(u) + qi(v)$, for any $u, v \in \mathbb{E}_n$ and $p, q \geq 0$; and
- (b) $D_H(u, v) = \|i(u) - i(v)\|$.

Let $i(u) = (u_-, u_+)$. The λ -level set of $u \in \mathbb{E}_n$ can be written as

$$[u]^\lambda = \beta u_-^\lambda + (1 - \beta) u_+^\lambda \quad \text{for all } 0 \leq \beta \leq 1.$$

Now,

$$\begin{aligned}[p \odot u \oplus q \odot v]^\lambda &= p[u]^\lambda + q[v]^\lambda \\ &= p[\beta u_-^\lambda + (1 - \beta)u_+^\lambda] \\ &\quad + q[\beta v_-^\lambda + (1 - \beta)v_+^\lambda] \\ &= \beta(u_-^\lambda + v_-^\lambda) + (1 - \beta)(u_+^\lambda + v_+^\lambda).\end{aligned}$$

Therefore,

$$\begin{aligned}i(p \odot u \oplus q \odot v) &= (pu_-^\lambda + qv_-^\lambda, pu_+^\lambda + qv_+^\lambda) \\ &= p(u_-^\lambda, u_+^\lambda) + q(v_-^\lambda, v_+^\lambda) \\ &= pi(u) + qi(v).\end{aligned}$$

Thus, (a) is proved.

Now, consider

$$\begin{aligned}\|i(u) - i(v)\| &= \|(u_-, u_+) - (v_-, v_+)\| \\ &= \|(u_- - v_-), (u_+ - v_+)\| \\ &= \max\{\|u_- - v_-\|_C, \|u_+ - v_+\|_C\} \\ &= \max\{\sup_\lambda \|u_-^\lambda - v_-^\lambda\|_{\mathbb{R}_n}, \sup_\lambda \|u_+^\lambda - v_+^\lambda\|_{\mathbb{R}_n}\} \\ &= \sup_\lambda \{\max\{\|u_-^\lambda - v_-^\lambda\|_{\mathbb{R}_n}, \|u_+^\lambda - v_+^\lambda\|_{\mathbb{R}_n}\}\}, \\ &= \sup_\lambda d_H([u]^\lambda, [v]^\lambda) \\ &= D_H(u, v).\end{aligned}$$

□

We make use the Proposition 3.1 and Remark 3.4 in [18] to prove the following results.

Theorem 7. Suppose $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ is

∇^g left-differentiable at s_0 ; then, $(i \circ \Phi)(s) = i(\Phi(s))$ is nabla-differentiable at $s_0 \in \mathbb{T}^{[a,b]}$. Moreover,

- (a) If there exists a $\delta > 0 \ni (\Phi(s_0 - \hbar) \ominus_h \Phi(\varrho(s_0)))$ exists for $0 < \hbar < \delta$, then $(i \circ \Phi)^\nabla(s_0) = -i^*(\Phi_-^{\nabla^g}(s_0))$.
- (b) If there exists a $\delta > 0 \ni (\Phi(\varrho(s_0)) \ominus_h \Phi(s_0 - \hbar))$ exists for $0 < \hbar < \delta$, then $(i \circ \Phi)^\nabla(s_0) = i^*(\Phi_-^{\nabla^g}(s_0))$.

Proof. Let Φ be ∇^g left-differentiable at $s_0 \in \mathbb{T}^{[a,b]}$.

- (a) If there exists a $\delta > 0$ such that $\Phi(s_0 - \hbar) \ominus_h \Phi(\varrho(s_0))$ exists for $0 < \hbar < \delta$, then

$$\begin{aligned}
& \left\| \frac{-1}{(\hbar - \nu(s_0))} [(i \circ \Phi)(s_0 - \hbar) - (i \circ \Phi)(\varrho(s_0))] - [-i^*(\Phi_-^{\nabla^g}(s_0))] \right\| \\
&= \left\| \frac{-1}{(\hbar - \nu(s_0))} [(i \circ \Phi)(s_0 - \hbar) - (i \circ \Phi)(\varrho(s_0))] + [i^*(\Phi_-^{\nabla^g}(s_0))] \right\| \\
&\leq \left\| \frac{1}{(\hbar - \nu(s_0))} [i(\Phi(\varrho(s_0)) - \Phi(s_0 - \hbar))] \right. \\
&\quad \left. + i^* \left[\frac{1}{(\hbar - \nu(s_0))} \odot [\Phi(\varrho(s_0)) \ominus_h \Phi(s_0 - \hbar)] \right] \right\| \\
&\quad + \left\| -i^* \left[\frac{1}{(\hbar - \nu(s_0))} \odot [\Phi(\varrho(s_0)) \ominus_h \Phi(s_0 - \hbar)] \right] i^*(\Phi_-^{\nabla^g}(s_0)) \right\|.
\end{aligned}$$

From Remark 3.4.1 in [18], we have

$$\begin{aligned}
& \left\| i^* \left[\frac{1}{(\hbar - \nu(s_0))} \odot (\Phi(\varrho(s_0)) \ominus_h \Phi(s_0 - \hbar)) \right] - i^*(\Phi_-^{\nabla^g}(s_0)) \right\| \\
&= D_H \left[\frac{1}{(\hbar - \nu(s_0))} \odot (\Phi(\varrho(s_0)) \ominus_h \Phi(s_0 - \hbar)), \Phi_-^{\nabla^g}(s_0) \right] \rightarrow 0, \text{ as } \hbar \rightarrow 0.
\end{aligned}$$

Consider

$$\begin{aligned}
i^* \left[\frac{1}{(\hbar - \nu(s_0))} \odot (\Phi(\varrho(s_0)) \ominus_h \Phi(s_0 - \hbar)) \right] &= -i \left[\frac{1}{(\hbar - \nu(s_0))} \odot (\Phi(\varrho(s_0)) \ominus_h \Phi(s_0 - \hbar)) \right] \\
&= \frac{-1}{(\hbar - \nu(s_0))} [i(\Phi(\varrho(s_0)) - \Phi(s_0 - \hbar))],
\end{aligned}$$

we have

$$\left\| \frac{(i \circ \Phi)(s_0 - \hbar) - (i \circ \Phi)(\varrho(s_0))}{-(\hbar - \nu(s_0))} - [-i^*(\Phi_-^{\nabla^g}(s_0))] \right\| \rightarrow 0, \text{ as } \hbar \rightarrow 0.$$

Thus, $(i \circ \Phi)^{\nabla}(s_0) = -i^*(\Phi_-^{\nabla^g}(s_0))$.

Similarly, we can prove (b). $\square$

Theorem 8. Suppose $\Phi : \mathbb{T}^{[a,b]} \rightarrow E_n$ is ∇^g right-differentiable s_0 ; then, $(i \circ \Phi)(s) = i(\Phi(s))$ is nabla-differentiable at $s_0 \in \mathbb{T}_k^{[a,b]}$. Moreover,

(a) If there exists a $\delta > 0 \ni (\Phi(s_0 + \hbar) \ominus_h \Phi(\varrho(s_0)))$ exists for $0 < \hbar < \delta$, then

$$(i \circ \Phi)^{\nabla}(s_0) = i(\Phi_+^{\nabla^g}(s_0)).$$

(b) If there exists a $\delta > 0 \ni (\Phi(\varrho(s_0)) \ominus_h \Phi(s_0 + \hbar))$ exists for $0 < \hbar < \delta$, then

$$(i \circ \Phi)^{\nabla}(s_0) = -i^*(\Phi_+^{\nabla^g}(s_0)).$$

Proof. The proof of this theorem is similar to that of Theorem 7. $\square$

Theorem 9. If $\Phi : \mathbb{T}^{[a,b]} \rightarrow E_n$ is ∇^g differentiable at s , then $i \circ \Phi(s)$ is nabla-differentiable and $(i \circ \Phi)^{\nabla}(s) \in i(E_n)$. In this case, either $(i \circ \Phi)^{\nabla}(s) = i(\Phi^{\nabla^g}(s))$ or $(i \circ \Phi)^{\nabla}(s) = -i^*(\Phi^{\nabla^g}(s))$, $s \in \mathbb{T}_k^{[a,b]}$.

Proof. Let $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ be ∇^g differentiable at $s \in \mathbb{T}_k^{[a,b]}$ and s is left dense; then, the proof is similar to the proof of Theorem 8 [16]. Now, for s being left scattered, we have

$$\frac{1}{\nu(s)} [i(\Phi(s)) - \Phi(\varrho(s))] = \begin{cases} \frac{1}{\nu(s)} [i \circ \Phi(s) - i \circ \Phi(\varrho(s))] & \text{or} \\ \frac{-1}{\nu(s)} [i \circ \Phi(\varrho(s)) - i \circ \Phi(s)]. \end{cases}$$

Consider

$$\begin{aligned} & \left\| \frac{1}{\nu(s)} [(i \circ \Phi)(s) - (i \circ \Phi)(\varrho(s))] - i(\Phi^{\nabla^g}(s)) \right\| \\ &= \left\| \frac{-1}{\nu(s)} [(i \circ \Phi)(\varrho(s)) - (i \circ \Phi)(s)] - i(\Phi^{\nabla^g}(s)) \right\| \\ &= \left\| i \left(\frac{1}{\nu(s)} \odot [\Phi(s) \ominus_h \Phi(\varrho(s))] \right) - i(\Phi^{\nabla^g}(s)) \right\| \\ &= D_H \left[\frac{1}{\nu(s)} \odot [\Phi(s) \ominus_h \Phi(\varrho(s))], \Phi^{\nabla^g}(s) \right]. \end{aligned}$$

Then, $(i \circ \Phi)^{\nabla}(s) = i(\Phi^{\nabla^g}(s))$.

Again, in the same way,

$$\begin{aligned} & \left\| \frac{1}{\nu(s)} [(i \circ \Phi)(s) - (i \circ \Phi)(\varrho(s))] - [-i^*(\Phi^{\nabla^g}(s))] \right\| \\ &= \left\| \frac{-1}{\nu(s)} [(i \circ \Phi)(\varrho(s)) - (i \circ \Phi)(s)] + [i^*(\Phi^{\nabla^g}(s))] \right\| \\ &\leq \left\| \frac{-1}{\nu(s)} [(i \circ \Phi)(\varrho(s)) - (i \circ \Phi)(s)] \right. \\ &\quad \left. + i^* \left[\frac{1}{\nu(s)} \odot [\Phi(s) \ominus_h \Phi(\varrho(s))] \right] \right\| \\ &\quad + \left\| i^* \left[\frac{1}{\nu(s)} \odot [\Phi(s) \ominus_h \Phi(\varrho(s))] \right] - i^*(\Phi^{\nabla^g}(s)) \right\|. \end{aligned}$$

However,

$$\begin{aligned} & \left\| i^* \left(\frac{1}{\nu(s)} \odot [\Phi(s) \ominus_h \Phi(\varrho(s))] \right) - i^*(\Phi^{\nabla^g}(s)) \right\| \\ &= D_H \left(\frac{1}{\nu(s)} \odot (\Phi(s) \ominus_h \Phi(\varrho(s))), \Phi^{\nabla^g}(s) \right) = 0. \end{aligned}$$

Since $i((-1) \odot \tilde{u}) = i^*(\tilde{u})$, we have

$$\begin{aligned} & \left\| \frac{-1}{\nu(s)} [(i \circ \Phi)(\varrho(s)) - (i \circ \Phi)(s)] - i^* \left[\frac{\Phi(s) \ominus_h \Phi(\varrho(s))}{-\nu(s)} \right] \right\| \\ &= \left\| \frac{-1}{\nu(s)} [(i \circ \Phi)(\varrho(s)) - (i \circ \Phi)(s)] - [-i^*(\Phi^{\nabla^g}(s))] \right\| = 0. \end{aligned}$$

Thus, $\left\| \frac{(i \circ \Phi)(s) - (i \circ \Phi)(\varrho(s))}{\nu(s)} - [-i^*(\Phi^{\nabla^g}(s))] \right\| = 0$. Therefore, $(i \circ \Phi)^{\nabla}(s) = -i^*(\Phi^{\nabla^g}(s))$.

Finally, $(i \circ \Phi)^{\nabla}(s) = i(\Phi^{\nabla^g}(s)) = -i^*(\Phi^{\nabla^g}(s))$. $\square$

From Remark 8, it is clear that, the fuzzy function $\Phi(s)$ is (GH3)- or (GH4)-nabla differentiable at discrete points. For example, if $\Phi(s)$ is ∇^g -differentiable on $\mathbb{T}^{[a,b]}$, $a < c < d < b$ and Φ is only (GH3)-nabla differentiable at $s = c$, (GH4)-nabla differentiable at $s = d$, then Φ is

(GH1)-nabla differentiable on $\mathbb{T}^{[a,c] \cup (d,b]}$ and (GH2)-nabla differentiable on $\mathbb{T}^{(c,d]}$. Therefore, if $\Phi(s)$ is ∇^g -differentiable on $\mathbb{T}^{[a,b]}$, then it is possible to partition the $\mathbb{T}^{[a,b]}$ into sub-intervals such that in each sub-interval $\Phi(s)$ is either (GH1)- or (GH2)-nabla differentiable.

Now, we prove the main theorem of this section fundamental theorem of nabla integral calculus of fuzzy functions on time scales.

Theorem 10. Let $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ and $a = a_0 < a_1 < a_2 < \dots < a_k = b$ be a division of the interval $[a, b]$ such that Φ is (GH1) or (GH2)-nabla differentiable on each of the interval $\mathbb{T}^{[a_{m-1}, a_m]}$, $m = 1, 2, \dots, k$ with same kind of differentiability on each sub-interval. Then,

$$\int_a^b \Phi^{\nabla^g}(\tau) \nabla \tau = \sum_{m \in M} (\Phi(a_m) \ominus_h \Phi(a_{m-1})) \oplus (-1) \odot \sum_{n \in N} (\Phi(a_{n-1}) \ominus_h \Phi(a_n)),$$

where $M = \{m \in \{1, 2, \dots, k\} \text{ such that } \Phi \text{ is (GH1)-nabla differentiable on } \mathbb{T}^{(a_{m-1}, a_m)}\}$ and $N = \{n \in \{1, 2, \dots, k\} \text{ such that } \Phi \text{ is (GH2)-nabla differentiable on } \mathbb{T}^{(a_{n-1}, a_n)}\}$

Proof. Let $\Phi : \mathbb{T}^{[a,b]} \rightarrow \mathbb{E}_n$ is ∇^g differentiable on $\mathbb{T}_k^{[a,b]}$. Suppose Φ is (GH1)-nabla differentiable on (a_{i-1}, a_i) . Then, for $m \in M$, we have

$$\int_{a_{m-1}}^{a_m} \Phi^{\nabla^g}(\tau) \nabla \tau = \Phi(a_m) \ominus_h \Phi(a_{m-1}) \text{ for all } m \in M. \quad (7)$$

Let $n \in N$; using Cauchy formula for functions with values in Banach space, we have

$$(i \circ \Phi)(a_n) = (i \circ \Phi)(a_{n-1}) + \int_{a_{n-1}}^{a_n} (i \circ \Phi)^{\nabla^g}(\tau) \nabla \tau.$$

By Theorem 9, there exists $(i \circ \Phi)^{\nabla}(s)$ and we get $(i \circ \Phi)(a_n) = (i \circ \Phi)(a_{n-1}) + \int_{a_{n-1}}^{a_n} (-i^*(\Phi^{\nabla^g})(\tau) \nabla \tau$.

Since the embedding i commutes with the integral, we obtain

$$(i \circ \Phi)(a_n) = (i \circ \Phi)(a_{n-1}) - i^* \left(\int_{a_{n-1}}^{a_n} \Phi^{\nabla^g}(\tau) \nabla \tau \right).$$

Then, it follows that

$$i^* \left(\int_{a_{n-1}}^{a_n} \Phi^{\nabla^g}(s) \nabla s \right) + (i \circ \Phi)(a_n) = (i \circ \Phi)(a_{n-1}).$$

By the definition of i^* , we obtain

$$i \left((-1) \odot \int_{a_{n-1}}^{a_n} \Phi^{\nabla^g}(\tau) \nabla \tau \right) + i(\Phi(a_n)) = i(\Phi)(a_{n-1}).$$

By the additive property of the embedding i , we have

$$(-1) \odot \int_{a_{n-1}}^{a_n} \Phi^{\nabla^g}(\tau) \nabla \tau = \Phi(a_{n-1}) \ominus_h \Phi(a_n).$$

Finally,

$$\int_{a_{n-1}}^{a_n} \Phi^{\nabla^g}(\tau) \nabla \tau = (-1) \odot \Phi(a_{n-1}) \ominus_h \Phi(a_n), \quad (8)$$

for all $n \in N$. Adding Equations (7) and (8), we get the desired result

$$\int_a^b \Phi^{\nabla^g}(\tau) \nabla \tau = \sum_{m \in M} (\Phi(a_m) \ominus_h \Phi(a_{m-1})) \oplus (-1) \odot \sum_{n \in N} (\Phi(a_{n-1}) \ominus_h \Phi(a_n)).$$

□

Example 4. Consider $\Phi(s)$ as in Example 2. We partition $[0, 3\pi]$ as $a_0 = 0 < a_1 = \frac{\pi}{2} < a_2 = \pi < a_3 = 2\pi < a_4 = \frac{5\pi}{2} < a_5 = 3\pi$ such that $\Phi(s)$ is (GH1)-nabla differentiable on $\mathbb{T}^{[a_{m-1}, a_m]}$, $m \in M = \{1, 3\}$ and (GH2)-nabla differentiable on $\mathbb{T}^{[a_{n-1}, a_n]}$, $n \in N = \{2, 5\}$. Thus, from Theorem 10, we have

$$\begin{aligned} \int_a^b \Phi^{\nabla^g}(\tau) \nabla \tau &= \sum_{m \in M} (\Phi(a_m) \ominus_h \Phi(a_{m-1})) \oplus (-1) \odot \sum_{n \in N} (\Phi(a_{n-1}) \ominus_h \Phi(a_n)) \\ &= (\Phi(\frac{\pi}{2}) \ominus_h \Phi(0)) \oplus (\Phi(\frac{5\pi}{2}) \ominus_h \Phi(2\pi)) \\ &\quad \oplus (-1) \odot (\Phi(\frac{\pi}{2}) \ominus_h \Phi(\pi)) \oplus (-1) \odot (\Phi(\frac{5\pi}{2}) \ominus_h \Phi(3\pi)) \\ &= 2 \odot c \oplus (-2) \odot c \\ &= (4, 8, 12) \oplus (-12, -8, -4) = (-8, 0, 8). \end{aligned}$$

5. Conclusions

This paper is concerned with investigating a new derivative called generalized nabla derivative for fuzzy functions on time scales and studies some basic properties of ∇^g derivative. In addition, we prove a fundamental theorem of nabla integral calculus for fuzzy functions on time scales under generalized differentiability on time scales. The advantage of ∇^g derivative is that it exists even for a fuzzy function having increasing and decreasing length of diameter on a time scale. The results obtained in this paper include results of Leelavathi et al. [27], when the function having only increasing length of diameter, and the results of Leelavathi et al. [28], when the function having only decreasing length of diameter. The obtained results are illustrated with numerical examples. In the future, we propose to study fuzzy nabla dynamic equations on time scales under generalized nabla derivative and their applications.

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