axioms

MDPI

Article

On the Stability of the Generalized Psi Functional Equation

Gwang Hui Kim [1] and Themistocles M. Rassias [2,*]

[1] Department of Mathematics, Kangnam University, Yongin 16979, Gyeonggi, Korea; ghkim@kangnam.ac.kr
[2] Department of Mathematics, National Technical University of Athens, Zografou Campus, 157 80 Athens, Greece
* Correspondence: trassias@math.ntua.gr

Received: 23 April 2020; Accepted: 18 May 2020; Published: 23 May 2020

Abstract: In this paper, we investigate the generalized Hyers–Ulam stability for the generalized psi functional equation $f(x+p) = f(x) + \varphi(x)$ by the direct method in the sense of P. Găvruta and the Hyers–Ulam–Rassias stability.

Keywords: stability; Hyers–Ulam–Rassias stability; psi functional equation; gamma functional equation

MSC: 39B82; 39B52

---

1. Introduction

Functional equations in a single variable were introduced by Kuczma [1] in 1968. Two years later, Brydak [2] investigated the stability of the generalized single variable functional equation

$$f(\varphi(x)) = g(x)f(x) + F(x). \tag{1}$$

Thereafter, this functional Equation (1) was studied in connection with the iterative functional equation with variable coefficients that could be—for example—a polynomial. Equation (1) is also considered in other forms, such as:

Abel's equation

$$f(\varphi(x)) = f(x) + c,$$

Schröder's equation

$$f(\varphi(x)) = cf(x),$$

the Gamma functional equation

$$f(x+1) = xf(x),$$

the Psi functional equation

$$f(x+1) = f(x) + \frac{1}{x},$$

and various iterative functional equations involving a polynomial.

The stability of the functional Equation (1) as well as similar forms of it has been studied by Baker [3], Choczewski et al. [4], Turdza [5], Lee et al. [6], Agarwal et al. [7], Jung et al. [8] and others.

The stability of iterative equations involving polynomials has been investigated by Kuczma et al. [9], Forti [10], Xu [11], Zhang et al. [12], and others.

The stability of the Gamma functional equation

$$f(x+1) = xf(x)$$

has been studied by Jung [13,14], Kim [15], Kim et al. [16], and others.

Equations with functional perturbations are interesting from many points of view [17,18] and enjoy various applications especially in the theory of integral [19] and functional-differential equations [18].

For further works conducted in the very active domain of the stability of functional equations, the interested reader is referred to [12–31].

The psi (digamma) function is defined by

$$\Psi(x) := \frac{d}{dx}\ln\Gamma(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right) dt, \tag{2}$$

where $\Gamma(x)$ stands for the Gamma function.

The Gamma functional equation is the following:

$$\Gamma(x+1) = x\Gamma(x),\ x > 0.$$

The stability for this functional equation is proved in Jung [13] and Kim [15]. Since the Gamma functional equation implies that

$$\frac{\Gamma'(x+1)}{\Gamma(x+1)} = \frac{\Gamma'(x)}{\Gamma(x)} + \frac{1}{x},$$

it follows that the psi function (2) constitutes the solution of the equation:

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{3}$$

which is the so-called psi functional equation.

Due to (3), we can consider the functional equation

$$f(x+p) = f(x) + \varphi(x) \tag{4}$$

in which f, φ are unknown functions, and x, p are positive real numbers.

Let us recall that, in the Peano axioms, $n' = n+1$ is called the successor of n. Therefore, the functional equation

$$f(x+1) = f(x) + 1, \tag{5}$$

with the unit step is implied, which can be called the unit successor functional equation with unit step. More generally, the functional equation

$$f(x+p) = f(x) + \alpha_p \tag{6}$$

can be considered the α-successor functional equation with p-step, where the constant $\alpha_p = \alpha$ depends on a fixed positive real number p.

The aim of the present paper is to investigate the generalized Hyers–Ulam stability for the functional Equation (4), in the sense of P. Găvruta [21] and the Hyers–Ulam–Rassias stability [22].

As a corollary, we obtain stability results of the successor functional Equations (5) and (6) and the psi functional Equation (3).

Throughout this paper, let $\mathbb{R}$ and $\mathbb{R}_+$ be the set of real numbers and the set of all positive real numbers, respectively. Set $\mathbb{R}_* := \mathbb{R}_+ \cup \{0\}$. Let $p, \delta > 0$ be fixed real numbers, and n be a non-negative integer.

2. Stability of the Functional Equation (4)

In this section, we will investigate the Hyers–Ulam–Rassias stability as well as the stability in the sense of P. Găvruta, for the functional Equation (4)

Theorem 1. *Let a mapping $\theta : \mathbb{R}_+ \to \mathbb{R}_*$ satisfy the inequality*

$$\Theta(x) := \sum_{i=0}^{\infty} \theta(x + pi) < \infty. \tag{7}$$

Assume that $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_*$ satisfies the inequality

$$|f(x + p) - f(x) - \varphi(x)| < \theta(x). \tag{8}$$

Then, there exists a unique solution $F : \mathbb{R}_+ \longrightarrow \mathbb{R}_*$ of the Equation (4) with

$$|F(x) - f(x)| \leq \Theta(x). \tag{9}$$

Proof. For any $x > 0$ and for every positive integer n, we define

$$P_n(x) := f(x + pn) - \sum_{i=0}^{n-1} \varphi(x + pi). \tag{10}$$

By (8), we have

$$\begin{aligned} |P_{n+1}(x) - P_n(x)| &= |f(x + pn + p) - f(x + pn) - \varphi(x + pn)| \\ &\leq \theta(x + pn). \end{aligned} \tag{11}$$

Indeed, for $n \geq m$, we have

$$|P_n(x) - P_m(x)| \leq \sum_{i=m}^{n-1} |P_{i+1}(x) - P_i(x)| \leq \sum_{i=m}^{n-1} \theta(x + pi). \tag{12}$$

The right-hand-side of (12) converges to zero as $m \longrightarrow \infty$, by (7). In view of (12), the sequence $\{P_n(x)\}$ is a Cauchy sequence for all $x \in \mathbb{R}_+$.

Hence, we can define a function $F : \mathbb{R}_+ \to \mathbb{R}_*$ by

$$F(x) := \lim_{n \to \infty} P_n(x).$$

By induction on n, we show that

$$|P_n(x) - f(x)| \leq \sum_{i=0}^{n-1} \theta(x + pi), \tag{13}$$

for all n.

For $n = 1$, the inequality (13) follows immediately from (8). Assume that (13) holds true for some n. Then, from (11) and (13), it follows that

$$\begin{aligned} |P_{n+1}(x) - f(x)| &\leq |P_{n+1}(x) - P_n(x)| + |P_n(x) - f(x)| \\ &\leq \sum_{i=0}^{n} \theta(x + pi). \end{aligned}$$

Therefore, (13) holds true for all positive integers n.

Hence, by (13), we have

$$
\begin{aligned}
|F(x)-f(x)| &= \lim_{n\to\infty}|P_n(x)-f(x)| \\
&\le \sum_{i=0}^{\infty}\theta(x+pi)=\theta(x),
\end{aligned}
$$

which completes the proof of (9).

From the definition of P_n, it follows that F satisfies the functional Equation (4)

$$
\begin{aligned}
F(x+p) &= \lim_{n\to\infty}\left(f(x+p(n+1))-\sum_{i=1}^{n}\varphi(x+pi)\right) \\
&= \lim_{n\to\infty}P_{n+1}(x)+\varphi(x)=F(x)+\varphi(x).
\end{aligned}
$$

If $G:\mathbb{R}_+\longrightarrow\mathbb{R}_*$ is another function which satisfies (9) and (4), then it follows from (10) and (9) that for all n, it holds

$$
\begin{aligned}
|F(x)-G(x)| &= \lim_{n\to\infty}|P_n(x)-Q_n(x)| \\
&\le \lim_{n\to\infty}|P_n(x)-f(x+p(n-1))|+|f(x+p(n-1))-Q_n(x)| \\
&= \lim_{n\to\infty}2\theta(x+p(n-1))
\end{aligned}
$$

where

$$
G:=\lim_{n\to\infty}Q_n.
$$

Thus, the uniqueness of the solution of Equation (4) is established, and this completes the proof of Theorem 1. □

For the stability in the sense of Găvruta [21] to be valuable, there must exist a convergent sequence which satisfies the assumption (7) of the Theorem.

We can show that the infinite series of the undefined function θ of the condition (7) converges, by the improper integral test, the p-series test, or the ratio test for the infinite series.

By replacing the function θ in the stability inequality (8) by an arbitrary exponential function, the assumption (7) of Theorem 1 can be omitted.

Corollary 1. *Assume that $f:\mathbb{R}_+\longrightarrow\mathbb{R}_*$ satisfies the inequality*

$$
|f(x+p)-f(x)-\varphi(x)|<\theta_a(x):=\begin{cases}a^x, & 0<a<1\\ a^{-x}, & 1<a.\end{cases}
$$

Then, there exists a unique solution $F:\mathbb{R}_+\longrightarrow\mathbb{R}_*$ of the Equation (4) with

$$
|F(x)-f(x)|\le\Theta_a(x):=\begin{cases}\sum_{i=0}^{\infty}a^{x+pi}, & 0<a<1\\ \sum_{i=0}^{\infty}a^{-(x+pi)}, & 1<a.\end{cases}
$$

Proof. The limit of the ratio test implies that

$$
L:=\lim_{x\to\infty}\frac{\theta_a(x+p(i+1))}{\theta_a(x+pi)}<1,
$$

respectively. □

The Hyers–Ulam–Rassias stability follows.

Corollary 2. *Assume that* $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_*$ *satisfies the inequality*

$$|f(x+p) - f(x) - \varphi(x)| < \frac{\delta}{x^r} \ ,$$

for fixed $r > 1$.

Then, there exists a unique solution $F : \mathbb{R}_+ \longrightarrow \mathbb{R}_*$ *of the Equation* (4) *with*

$$|F(x) - f(x)| \leq \sum_{i=0}^{\infty} \frac{\delta}{(x+i)^r} \ , \tag{14}$$

Proof. Set $\theta(x) = \frac{\delta}{x^r}$ in Theorem 1. Since the convergence condition of Ψ is satisfied by the p-series test in the case when $r > 1$, Corollary 2 follows. □

The result (14) of Corollary 2 is the following:

$$\sum_{i=0}^{\infty} \frac{\delta}{(x+i)^r} \leq \begin{cases} (i)\delta\left(\frac{1}{r-1} - \frac{1}{x^r}\right), & 0 \leq x < 1 \\ (ii)\frac{\delta}{r-1}, & 1 \leq x < 2 \\ (iii)\delta\left(\frac{1}{r-1} - \sum_{n=1}^{\lceil x \rceil - 1} \frac{1}{n^r}\right), & 2 \leq \ x, \end{cases}$$

where $\lceil \cdot \rceil$ stands for the Gaussian notation.

The results below concern the Hyers–Ulam–Rassias stability of the successor functional Equations (5) and (6), and the psi functional Equation (3).

Corollary 3. *Assume that* $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_*$ *satisfies the inequality*

$$|f(x+p) - f(x) - \alpha_p| < \frac{\delta}{x^r} \ ,$$

for fixed $r > 1$*, and constant* α_p*, which depends on* p*.*

Then, there exists a unique solution $F : \mathbb{R}_+ \longrightarrow \mathbb{R}_*$ *of the equation*

$$f(x+p) = f(x) + \alpha_p$$

with

$$|F(x) - f(x)| \leq \sum_{i=0}^{\infty} \frac{\delta}{(x+i)^r} \ ,$$

Proof. Let $\varphi(x) := \varphi(p) = \alpha_p$ that is a constant. Namely, we define

$$P_n(x) := f(x+pn) - n\alpha_p.$$

The following process is similar to that of Theorem 1. □

The next result constitutes the Hyers–Ulam–Rassias stability for the psi functional Equation (3).

Corollary 4. *Assume that* $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_*$ *satisfies the inequality*

$$|f(x+1) - f(x) - \frac{1}{x}| < \frac{\delta}{x^r} \ ,$$

for a fixed real number $r > 1$.

Then, there exists a unique solution $F : \mathbb{R}_+ \longrightarrow \mathbb{R}_*$ *of Equation* (3) *with*

$$|F(x) - f(x)| \leq \sum_{i=0}^{\infty} \frac{\delta}{(x+i)^r} \ ,$$

Proof. Set
$$p = 1,\ \varphi(x) = \frac{1}{x}\ ,\ \text{and}\ \theta(x) = \frac{\delta}{x^r}$$
in Theorem 1. By applying the p-series test, the result follows. □

Corollary 5. *Assume that $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_*$ satisfies the inequality*
$$|f(x+1) - f(x) - 1| < \frac{\delta}{x^r}\ ,$$
for fixed $r > 1$.

Then, there exists a unique solution $F : \mathbb{R}_+ \longrightarrow \mathbb{R}_*$ of the equation
$$F(x+1) = F(x) + 1$$
with
$$|F(x) - f(x)| \leq \sum_{i=0}^{\infty} \frac{\delta}{(x+i)^r}.$$

Proof. Setting
$$p = 1,\ \varphi(1) = \frac{1}{1} = 1,\ \text{and}\ \theta(x) = \frac{\delta}{x^r}$$
in Theorem 1, and applying the p-series test, the result follows. □

Remark 1. *By setting*
$$x + p = \phi(x),$$
this result can be immediately extended to the more general form
$$f(\phi(x)) = f(x) + \varphi(x).$$

3. Conclusions

In this paper, we proved the generalized Hyers–Ulam stability for the generalized psi functional equation
$$f(x+p) = f(x) + \varphi(x)$$
by the direct method in the sense of P. Găvruta and the Hyers–Ulam–Rassias stability. As corollaries, we obtain the generalized Hyers–Ulam stability of the unit successor functional Equation (5) with unit step and the α_p-successor functional Equation (6) with p-step.

Author Contributions: The authors contributed equally for the preparation of this paper. All authors have read and agree to the published version of the manuscript.

Funding: The first author of this work was supported by Kangnam University Research Grant in 2018.

Acknowledgments: We would like to express our thanks to the referees for valuable comments which helped improve the presentation of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kuczma, M. *The Functional Equations in a Single Variable*; Monografie Matematyczne t. 46; PWN-Polish Scientific Publishers: Warsaw, Poland, 1968.
2. Brydak, D. On the stability of the functional equation. *Proc. Am. Math. Soc.* **1970**, *26*, 455–460.
3. Baker, J.A. The stability of certain functional equations. *Proc. Am. Math. Soc.* **1991**, *112*, 729–732. [CrossRef]
4. Choczewski, B.; Turdza, E.; Wegrzyk, R. On the stability of a linear functional equation. *Rocznik Nauk.-Dydakt. Prace Mat. IX* **1979**, *69*, 15–21.

5. Turdza, E. On the stability of the functional equation $\varphi[f(x)] = g(x)\varphi(x) + F(x)$. *Proc. Am. Math. Soc.* **1971**, *30*, 484–486. [CrossRef]
6. Lee, S.H.; Jun, K.W. The stability of the equation $f(x+p) = kf(x)$. *Bull. Korean Math. Soc.* **1998**, *35*, 653–658.
7. Agarwal, R.P.; Xu, B.; Zhang, W. Stability of functional equations in single variable. *J. Math. Anal. Appl.* **2003**, *288*, 852–869. [CrossRef]
8. Jung, S.-M.; Popa, D.; Rassias, M.T. On the stability of the linear functional equation in a single variable on complete metric groups. *J. Glob. Optim.* **2014**, *59*, 165–171. [CrossRef]
9. Kuczma, M.; Choczewski, B.; Ger, R. Iterative Functional Equations. In *Encyclopedia of Mathematics and Its Applications*; Cambridge Univ. Press: Cambridge, UK, 1990.
10. Forti, G.L. Hyers–Ulam stability of functional equations in several variables. *Aequ. Math.* **1995**, *50*, 146–190. [CrossRef]
11. Xu, B.; Zhang, W. Hyers–Ulam stability for a nonlinear iterative equation. *Colloq. Math.* **2002**, *93*, 1–9. [CrossRef]
12. Zhang, W. Stability of the solution of the iterated equation $\sum_{i=1}^{n} \lambda_i f^i(x) = F(x)$. *Acta Math. Sci.* **1988**, *8*, 421–424. [CrossRef]
13. Jung, S.-M. On the stability of the gamma functional equation. *Results Math.* **1998**, *33*, 306–309. [CrossRef]
14. Jung, S.-M. On the modified Hyers–Ulam–Rassias stability of the functional equation for gamma function. *Mathematica (Cluj)* **1997**, *39*, 233–237.
15. Kim, G.H. On the stability of the generallized gamma functional eqaution. *Int. J. Math. Math. Sci.* **2000**, *23*, 513–520. [CrossRef]
16. Kim, G.H.; Xu, B.; Zhang, W. Notes on stability of the generalized gamma functional equation. *Int. J. Math. Math. Sci.* **2002**, *32*, 57–63. [CrossRef]
17. Sidorov, N.; Trufanov, A. Nonlinear operator equations with a functional perturbation of the argument of neutral type. *Diff. Equat. (Springer)* **2009**, *45*, 1840–1844. [CrossRef]
18. Kolmanovskii, V.; Myshkis, A. *Introduction to the Theory and Applications of Functional-Differential Equations*; Mathematics and its Applications; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1999; Volume 463, p. 648.
19. Sidorov, D. Solvability of systems of Volterra integral equations of the first kind with piecewise continuous kernels. *Russ. Math.* **2013**, *57*, 54–63. [CrossRef]
20. Abdollahpour, M.R.; Rassias, M.T. Hyers–Ulam stability of hypergeometric differential equations. *Aequ. Math.* **2019**, *93*, 691–698. [CrossRef]
21. Găvruta, P. On the stability of some functional equations. In *Stability of Mappings of Hyers–Ulam Type*; Rassias, T.M., Tabor, J., Eds.; Hadronic Press: Palm Harbor, FL, USA, 1994; pp. 93–98.
22. Rassias, T.M. On the stability of the linear mapping in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300. [CrossRef]
23. Abdollahpour, M.R.; Aghayaria, R.; Rassias, M.T. Hyers–Ulam stability of associated Laguerre differential equations in a subclass of analytic functions. *J. Math. Anal. Appl.* **2016**, *437*, 605–612. [CrossRef]
24. Czerwik, S. *Functional Equations and Inequalities*; World Scientific: London, UK, 2002.
25. Hyers, D.H.; Isac, G.; Rassias, T.M. *Stability of Functional Equations in Several Variables*; Birkhäuser: Boston, MA, USA, 1998.
26. Jung, S.-M. *Hyers–Ulam–Rassias Stability of Functional Equations in Nonlinear Analysis*; Springer: Berlin/Heidelberg, Germany, 2011.
27. Kannappan, P. *Functional Equations and Inequalities with Applications*; Springer: Berlin/Heidelberg, Germany, 2009.
28. Lee, Y.-H.; Jung, S.-M.; Rassias, M.T. Uniqueness theorems on functional inequalities concerning cubic-quadratic-additive equation. *J. Math. Inequal.* **2018**, *12*, 43–61. [CrossRef]
29. Mortici, C.; Jung, S.-M.; Rassias, M.T. On the stability of a functional equation associated with the Fibonacci numbers. *Abstr. Appl. Anal.* **2014**, *2014*, 546046. [CrossRef]

30. Park, C.; Rassias, M.T. Additive functional equations and partial multipliers in C*-algebras. *Revista de la Real Academia de Ciencias Exactas Serie A Matemáticas* **2019**, *113*, 2261–2275. [CrossRef]
31. Ulam, S.M. *A Collection of Mathematical Problems*; Problems in Modern Mathematics; Interscience Publ.: New York, NY, USA, 1960.