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Nonlocal Fractional Boundary Value Problems Involving Mixed Right and Left Fractional Derivatives and Integrals

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Abstract: In this paper, we study the existence of solutions for nonlocal single and multi-valued boundary value problems involving right-Caputo and left-Riemann–Liouville fractional derivatives of different orders and right-left Riemann–Liouville fractional integrals. The existence of solutions for the single-valued case relies on Sadovskii's fixed point theorem. The first existence results for the multi-valued case are proved by applying Bohnenblust-Karlin's fixed point theorem, while the second one is based on Martelli's fixed point theorem. We also demonstrate the applications of the obtained results.

Keywords: fractional differential equations; fractional differential inclusions; existence; fixed point theorems

1. Introduction

Fractional calculus has emerged as an interesting and fruitful subject in view of wide applications of its tools in modeling complex dynamical systems. Mathematical models based on fractional-order operators provide insight into the past history of the underlying phenomena. Examples include constitutive equations (fractional law) in the viscoelastic materials [1], Caputo power law in transport processes [2], dynamic memory describing the economic processes, see [3,4].

Widespread applications of fractional differential equations motivated many researchers to develop the theoretical aspects of the topic. During the last few decades, one can witness the remarkable development on initial and boundary value problems of fractional differential equations and inclusions. Much of the literature on such problems include Caputo, Riemann–Liouville, Hadamard type fractional derivatives, and different kinds of classical and non-classical boundary conditions. For some recent works on fractional order boundary value problems, for example, see the articles [5–12] and the references cited therein. Fractional differential equations involving left and right fractional derivatives also received considerable attention, for instance, see [13–16]. These derivatives appear in the study of Euler-Lagrange equations [17], steady heat-transfer in fractal media [18], electromagnetic waves phenomena in a variety of dielectric media with susceptibility [19], etc.

Multivalued (inclusions) problems are found to be of great utility in studying dynamical systems and stochastic processes, for example, see [20,21]. In the text [22], one can find the details on stochastic processes, queueing networks, optimization and their application in finance, control, climate control, etc. Monotone differential inclusions were applied to study the nonlinear dynamics of wheeled vehicles in [23]. In [24], a fractional differential inclusion with oscillatory potential was studied. In [25], the authors investigated the mild solutions to the time fractional Navier-Stokes delay differential inclusions. Other applications include polynomial control systems [20], synchronization of piecewise continuous systems of fractional order [21], oscillation and nonoscillation of impulsive fractional differential inclusions [26], etc. For some recent existence and controllability results on fractional differential inclusions, we refer the reader to articles [27–33] and the references cited therein.

Recently, in [34], the authors studied existence and uniqueness of solutions for a new kind of boundary value problem involving right-Caputo and left-Riemann–Liouville fractional derivatives of different orders and right-left Riemann–Liouville fractional integrals, subject to nonlocal boundary conditions of the form

$$\begin{cases} {}^{C}D^{\alpha}_{1-}\,{}^{RL}D^{\beta}_{0+}y(t) + \lambda I^{p}_{1-}I^{q}_{0+}h(t,y(t)) = f(t,y(t)), \ \ t \in J := [0,1], \\ y(0) = y(\xi) = 0, \quad y(1) = \delta y(\mu), \ \ 0 < \xi < \mu < 1, \end{cases} \tag{1}$$

where ${}^{C}D^{\alpha}_{1-}$ and ${}^{RL}D^{\beta}_{0+}$ denote the right Caputo fractional derivative of order $\alpha \in (1,2]$ and the left Riemann–Liouville fractional derivative of order $\beta \in (0,1]$, I^{p}_{1-} and I^{q}_{0+} denote the right and left Riemann–Liouville fractional integrals of orders $p,q > 0$ respectively, $f,h : [0,1] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\delta, \lambda \in \mathbb{R}$.

Here we emphasize that the importance of nonlocal conditions can be understood in the sense that such conditions are used to model the peculiarities occurring inside the domain of physical and chemical processes as the classical initial and boundary conditions fail to cater this situation. The present problem is motivated by useful applications of nonlocal boundary data in petroleum exploitation, thermodynamics, elasticity, and wave propagation, etc., for instance, see [35,36] and the details therein.

The existence results for the problem (1) were derived by applying a fixed point theorem due to Krasnoselski and Leray–Schauder nonlinear alternative, while the uniqueness result was established via Banach contraction mapping principle.

The objective of the present work is to enrich the results on this new class of problems. We firstly prove another existence result for the problem (1) with the aid of Sadovskii's fixed point theorem. Afterwards, we initiate the study of the multi-valued analogue of the problem (1) by considering the following inclusions problem:

$$\begin{cases} {}^{C}D^{\alpha}_{1-}\,{}^{RL}D^{\beta}_{0+}y(t) \in F(t,y(t)) - \lambda I^{p}_{1-}I^{q}_{0+}H(t,y(t)), \ \ t \in [0,1], \\ y(0) = y(\xi) = 0, \quad y(1) = \delta y(\mu), \ \ 0 < \xi < \mu < 1, \end{cases} \tag{2}$$

where $F, H : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ are compact multivalued maps, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and the other quantities are the same as defined in problem (1). Existence results for the problem (2) are established via fixed point theorems due to Bohnenblust-Karlin [37] and Martelli [38].

The rest of the paper is arranged as follows. In Section 2 we recall some preliminary concepts and a known lemma [34]. In Section 3 we prove an existence result for the problem (1) by applying Sadovskii's fixed point theorem. Section 4 presents the existence results for the problem (2). Applications and examples are discussed in Section 5.

2. Preliminaries

Let us collect some important definitions on fractional calculus.

Definition 1. *[39] The left and right Riemann–Liouville fractional integrals of order $\delta > 0$ for $g \in L_1[a,b]$, existing almost everywhere on $[a,b]$, are respectively defined by*

$$I^{\delta}_{a+}g(t) = \int_a^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} g(s)ds \ \ \text{and} \ \ I^{\delta}_{b-}g(t) = \int_t^b \frac{(s-t)^{\delta-1}}{\Gamma(\delta)} g(s)ds.$$

In addition, according to the classical theorem of Vallee-Poussin and the Young convolution theorem, $I^{\delta}_{a+}g, I^{\delta}_{b-}g \in L_1[a,b], \delta > 0$.

Definition 2. *[39] For $g \in AC^n[a,b]$, the left Riemann–Liouville and the right Caputo fractional derivatives of order $\delta \in (n-1, n], n \in \mathbb{N}$, existing almost everywhere on $[a,b]$, are respectively defined by*

$$^{RL}D^{\delta}_{a+}g(t) = \frac{d^n}{dt^n}\int_a^t \frac{(t-s)^{n-\delta-1}}{\Gamma(n-\delta)}g(s)ds \quad \text{and} \quad {}^{C}D^{\delta}_{b-}g(t) = (-1)^n\int_t^b \frac{(s-t)^{n-\delta-1}}{\Gamma(n-\delta)}g^{(n)}(s)ds.$$

The following known lemma [34] plays a key role in proving the main results.

Lemma 1. *Let $H, F \in C[0,1] \cap L(0,1)$ and $y \in C([0,1], \mathbb{R})$. Then the linear problem*

$$\begin{cases} {}^{C}D^{\alpha}_{1-}\,{}^{RL}D^{\beta}_{0+}y(t) + \lambda I^{p}_{1-}I^{q}_{0+}H(t) = F(t), \quad t \in J := [0,1], \\ y(0) = y(\xi) = 0, \quad y(1) = \delta y(\mu), \end{cases} \tag{3}$$

is equivalent to the fractional integral equation:

$$\begin{aligned} y(t) = & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\Big[I^{\alpha}_{1-}F(s) - \lambda I^{\alpha+p}_{1-}I^{q}_{0+}H(s)\Big]ds \\ & + a_1(t)\Big\{\delta\int_0^{\mu}\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Big[I^{\alpha}_{1-}F(s) - \lambda I^{\alpha+p}_{1-}I^{q}_{0+}H(s)\Big]ds \\ & - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\Big[I^{\alpha}_{1-}F(s) - \lambda I^{\alpha+p}_{1-}I^{q}_{0+}H(s)\Big]ds\Big\} \\ & + a_2(t)\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\Big[I^{\alpha}_{1-}F(s) - \lambda I^{\alpha+p}_{1-}I^{q}_{0+}H(s)\Big]ds, \end{aligned} \tag{4}$$

where

$$a_1(t) = \frac{1}{\Lambda}\Big[\xi^{\beta+1}t^{\beta} - \xi^{\beta}t^{\beta+1}\Big], \; a_2(t) = \frac{1}{\Lambda}\Big[t^{\beta}(1-\delta\mu^{\beta+1}) - t^{\beta+1}(1-\delta\mu^{\beta})\Big], \tag{5}$$

and it is assumed that

$$\Lambda = \xi^{\beta+1}(1-\delta\mu^{\beta}) - \xi^{\beta}(1-\delta\mu^{\beta+1}) \neq 0. \tag{6}$$

3. Existence Result for the Single-Valued Problem (1) via Sadovskii's Fixed Point Theorem

Our existence result for the problem (1) is based on Sadovskii's fixed point theorem. Before proceeding further, let us recall some related auxiliary material. In the sequel, we use the norm $\|.\| = \sup_{t\in[0,1]}|.|$.

Definition 3. *Let M be a bounded set in metric space (X,d). The Kuratowski measure of noncompactness, $\alpha(M)$, is defined as*
$\inf\{\epsilon :$ *M covered by a finitely many sets such that the diameter of each set* $\leq \epsilon\}$.

Definition 4. *[40] Let $\Phi : \mathcal{D}(\Phi) \subseteq X \to X$ be a bounded and continuous operator on Banach space X. Then Φ is called a condensing map if $\alpha(\Phi(B)) < \alpha(B)$ for all bounded sets $B \subset \mathcal{D}(\Phi)$, where α denotes the Kuratowski measure of noncompactness.*

Lemma 2. *[41, Example 11.7] The map $K + C$ is a k-set contraction with $0 \leq k < 1$, and thus also condensing, if*

(i) *$K, C : \mathcal{D} \subseteq X \to X$ are operators on the Banach space X;*
(ii) *K is k-contractive, that is, for all $x, y \in \mathcal{D}$ and a fixed $k \in [0,1)$,*

$$\|Kx - Ky\| \leq k\|x-y\|;$$

(iii) *C is compact.*

Lemma 3. *[42] Let B be a convex, bounded and closed subset of a Banach space X and $\Phi : B \to B$ be a condensing map. Then Φ has a fixed point.*

In the sequel, we set

$$\Lambda_1 = \frac{\Delta_1}{\Gamma(\alpha+1)}, \ \Lambda_2 = \frac{|\lambda|\Delta_1}{\Gamma(\alpha+p+1)\Gamma(q+1)}, \ \Lambda_3 = \frac{\Delta_2}{\Gamma(\alpha)}, \ \Lambda_4 = \frac{|\lambda|\Delta_2}{\Gamma(\alpha+p)\Gamma(q)}, \tag{7}$$

where

$$\Delta_1 = \frac{1}{\Gamma(\beta+1)}\Big[1+\bar{a}_1(|\delta|\mu^\beta+1)+\bar{a}_2\xi^\beta\Big], \ \Delta_2 = \frac{1}{\Gamma(\beta+1)}\Big[1+\bar{a}_1(|\delta|+1)+\bar{a}_2\Big],$$
$$\bar{a}_1 = \max_{t\in[0,1]}|a_1(t)|, \ \bar{a}_2 = \max_{t\in[0,1]}|a_2(t)|.$$

Theorem 1. *Assume that:*

($\mathbf{B_1}$) *There exist $L>0$ such that $|f(t,x)-f(t,y)| \le L|x-y|$, $\forall t \in [0,1]$, $x,y \in \mathbb{R}$;*
($\mathbf{B_2}$) *$|f(t,y)| \le \sigma(t)$ and $|h(t,y)| \le \rho(t)$, where $\sigma,\rho \in C([0,1],\mathbb{R}^+)$.*

Then the problem (1) has at least one solution on $[0,1]$ if

$$Q := L\Lambda_1 < 1.$$

where Λ_1 is given by (7).

Proof. Let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \le r\}$ be a closed bounded and convex subset of $C([0,1],\mathbb{R})$, where r is a fixed constant. In view of Lemma 1, we introduce an operator $\mathcal{G} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ associated with the problem (1) as follows:

$$\begin{aligned}
\mathcal{G}y(t) \ = \ & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}f(s,y(s)) - \lambda I_{1-}^{\alpha+p}I_{0+}^{q}h(s,y(s))\Big]ds \\
& +a_1(t)\Bigg[\delta\int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}f(s,y(s)) - \lambda I_{1-}^{\alpha+p}I_{0+}^{q}h(s,y(s))\Big]ds \\
& -\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}f(s,y(s)) - \lambda I_{1-}^{\alpha+p}I_{0+}^{q}h(s,y(s))\Big]ds\Bigg] \\
& +a_2(t)\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}f(s,y(s)) - \lambda I_{1-}^{\alpha+p}I_{0+}^{q}h(s,y(s))\Big]ds.
\end{aligned}$$

Let us split the operator $\mathcal{G} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ on B_r as $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$, where

$$\begin{aligned}
\mathcal{G}_1y(t) =& \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}f(s,y(s))ds + a_1(t)\Bigg[\delta\int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}f(s,y(s))ds \\
& -\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}f(s,y(s))ds\Bigg] + a_2(t)\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}f(s,y(s))ds, \\
\mathcal{G}_2y(t) =& -\lambda\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha+p}I_{0+}^{q}h(s,y(s))ds - \lambda a_1(t)\Bigg[\delta\int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha+p}I_{0+}^{q}h(s,y(s))ds \\
& -\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha+p}I_{0+}^{q}h(s,y(s))ds\Bigg] - \lambda a_2(t)\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha+p}I_{0+}^{q}h(s,y(s))ds.
\end{aligned}$$

We shall show that the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy all the conditions of Lemma 3. The proof will be given in several steps.

Step 1. $\mathcal{G}B_r \subset B_r$.

Let us select $r \geq \|\sigma\|\Lambda_1 + \|\rho\|\Lambda_2$, where Λ_1, Λ_2 are given by (7). For any $y \in B_r$, we have

$$
\begin{aligned}
& \|\mathcal{G}y\| \\
\leq\ & \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha}|f(s,y(s))| + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| \Big] ds \\
& + |a_1(t)| \Bigg\{ |\delta| \int_0^{\mu} \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha}|f(s,y(s))| + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| \Big] ds \\
& + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha}|f(s,y(s))| + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| \Big] ds \Bigg\} \\
& + |a_2(t)| \int_0^{\xi} \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha}|f(s,y(s))| + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| \Big] ds \Bigg\} \\
\leq\ & \|\sigma\| \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}(1) ds + |a_1(t)| \Bigg[|\delta| \int_0^{\mu} \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}(1) ds \\
& + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}(1) ds \Bigg] + |a_2(t)| \int_0^{\xi} \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}(1) ds \Bigg\} \\
& + \|\rho\| |\lambda| \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q}(1) ds + |a_1(t)| \Bigg[|\delta| \int_0^{\mu} \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q}(1) ds \\
& + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q}(1) ds \Bigg] + |a_2(t)| \int_0^{\xi} \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q}(1) ds \Bigg\} \\
\leq\ & \left\{ \frac{\|\sigma\|}{\Gamma(\alpha+1)} + \frac{\|\rho\| |\lambda|}{\Gamma(\alpha+p+1)\Gamma(q+1)} \right\} \Delta_1 \\
=\ & \|\sigma\|\Lambda_1 + \|\rho\|\Lambda_2 < r,
\end{aligned}
$$

which implies that $\mathcal{G}B_r \subset B_r$.

Step 2. $\mathcal{G}_2$ is compact.

Observe that the operator $\mathcal{G}_2$ is uniformly bounded in view of Step 1. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $y \in B_r$. Then we have

$$
\begin{aligned}
|\mathcal{G}_2 y(t_2) - \mathcal{G}_2 y(t_1)| \leq\ & |\lambda| \left| \int_0^{t_1} \frac{(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| ds \right| \\
& + |\lambda| \left| \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| ds \right| \\
& + |\lambda| |a_1(t_2) - a_1(t_1)| \Bigg\{ |\delta| \int_0^{\mu} \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| ds \\
& + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| ds \Bigg\} \\
& + |\lambda| |a_2(t_2) - a_2(t_1)| \Bigg\{ \int_0^{\xi} \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| ds \Bigg\} \\
\leq\ & \frac{|\lambda| \|\rho\|}{\Gamma(\beta+1)\Gamma(\alpha+p+1)\Gamma(q+1)} \Big\{ 2(t_2-t_1)^{\beta} + |t_2{}^{\beta} - t_1{}^{\beta}|
\end{aligned}
$$

$$+\frac{(|\delta|\mu^{\beta}+1)}{|\Lambda|}\Big(\xi^{\beta+1}|t_2^{\beta}-t_1^{\beta}|+\xi^{\beta}|t_2^{\beta+1}-t_1^{\beta+1}|\Big)$$
$$+\frac{\xi^{\beta}}{|\Lambda|}\Big(|1-\delta\mu^{\beta+1}||t_2^{\beta}-t_1^{\beta}|+|1-\delta\mu^{\beta}||t_2^{\beta+1}-t_1^{\beta+1}|\Big)\Big\},$$

which tends to zero independent of y as $t_2 \to t_1$. This shows that $\mathcal{G}_2$ is equicontinuous. It is clear from the foregoing arguments that the operator $\mathcal{G}_2$ is relatively compact on B_r. Hence, by the Arzelá-Ascoli theorem, $\mathcal{G}_2$ is compact on B_r.

Step 3. $\mathcal{G}_1$ is Q-contractive.

Using $(\mathbf{B_1})$ and $(\mathbf{B_2})$, it is easy to show that

$$\begin{aligned}
\|\mathcal{G}_1 y-\mathcal{G}_1 x\| \le\; & \sup_{t\in[0,1]}\Big\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}|f(s,y(s))-f(s,x(s))|ds \\
& +|a_1(t)|\Big[\delta\int_0^{\mu}\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}|f(s,y(s))-f(s,x(s))|ds \\
& +\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}|f(s,y(s))-f(s,x(s))|ds\Big] \\
& +|a_2(t)|\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}|f(s,y(s))-f(s,x(s))|ds\Big\} \\
\le\; & \frac{L}{\Gamma(\beta+1)\Gamma(\alpha+1)}\Big[1+\bar{a}_1(|\delta|\mu^{\beta}+1)+\bar{a}_2\xi^{\beta}\Big]\|y-x\| \\
=\; & L\Lambda_1\|y-x\|,
\end{aligned}$$

which is Q-contractive, since $Q := L\Lambda_1 < 1$.

Step 4. $\mathcal{G}$ is condensing. Since $\mathcal{G}_1$ is continuous, Q-contraction and $\mathcal{G}_2$ is compact, therefore, by Lemma 2, $\mathcal{G} : B_r \to B_r$ with $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$ is a condensing map on B_r.

From the above four steps, we conclude by Lemma 3 that the map $\mathcal{G}$ has a fixed point which, in turn, implies that the problem (1) has a solution on $[0,1]$. □

4. Existence Results for the Multi-Vaued Problem (2)

For a normed space $(X, \|\cdot\|)$, we have $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$, $\mathcal{P}_{b,cl,c}(\mathbb{R}) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded, closed and convex}\}$. We also define the *sets of selections* of the multi-valued maps F and H as

$$S_{F,y} := \{f \in L^1([0,1],\mathbb{R}) : f(t) \in F(t,y)\},$$
$$\widehat{S}_{H,y} := \{h \in L^1([0,1],\mathbb{R}) : h(t) \in H(t,y)\}.$$

By Lemma 1, we define a solution of the boundary value problem (2) as follows (see also [43,44]).

Definition 5. *A function $y \in C([0,1],\mathbb{R})$ is a solution of the boundary value problem (2) if $y(0) = y(\xi) = 0, y(1) = \delta y(\mu)$, and there exist functions $f \in S_{F,y}, h \in \widehat{S}_{H,y}$ a.e. on $[0,1]$ and*

$$\begin{aligned}
y(t) =\; & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}f(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h(s)\Big]ds \\
& +a_1(t)\Big\{\delta\int_0^{\mu}\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}f(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h(s)\Big]ds
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha} f(s) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q} h(s) \Big] ds \Big\} \\
&+ a_2(t) \int_0^{\xi} \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha} f(s) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q} h(s) \Big] ds.
\end{aligned}
$$

Now we provide the lemmas which will be used in the main existence results in this section.

Lemma 4. (Bohnenblust-Karlin) *([37]) Let D be a nonempty, bounded, closed, and convex subset of X. Let $\Phi : D \to \mathcal{P}(\mathbb{R})$ be upper semi-continuous with closed, convex values such that $\Phi(D) \subset D$ and $\overline{\Phi(D)}$ is compact. Then Φ has a fixed point.*

Lemma 5. *([45]) Let X be a separable Banach space. Let $F : J \times X \to \mathcal{P}_{cp,c}(X)$ be measurable with respect to t for each $y \in X$ and upper semi-continuous with respect to y for almost all $t \in J$ and $S_{F,y} \neq \emptyset$, for any $y \in C(J,X)$, and let Θ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$\Theta \circ S_F : C(J,X) \to \mathcal{P}_{cp,c}(C(J,X)), \;\; y \mapsto (\Theta \circ S_F)(y) = \Theta(S_{F,y})$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

In the first result, we study the existence of the solution for the multi-valued problem (2) by applying Bohnenblust–Karlin fixed point theorem.

Theorem 2. *Suppose that:*

($\mathbf{M_1}$) *$F, H : [0,1] \times \mathbb{R} \to \mathcal{P}_{b,c,cp}(\mathbb{R})$; $(t,y) \to f(t,y)$ and $(t,y) \to h(t,y)$ be measurable with respect to t for each $y \in \mathbb{R}$, upper semi-continuous with respect to y for almost everywhere $t \in [0,1]$, and for each fixed $y \in \mathbb{R}$, the sets $S_{F,y}$ and $\widehat{S}_{H,y}$ are nonempty for almost everywhere $t \in [0,1]$.*

($\mathbf{M_2}$) *For each $\rho > 0$, there exist functions $\phi_\rho, \psi_\rho \in L^1([0,1], \mathbb{R}_+)$ such that*

$$\|F(t,y)\| = \sup\{|f| : f(t) \in F(t,y)\} \leq \phi_\rho(t),$$

$$\|H(t,y)\| = \sup\{|h| : h(t) \in H(t,y)\} \leq \psi_\rho(t),$$

for each $(t,y) \in [0,1] \times \mathbb{R}$ with $\|y\| \leq \rho$, and

$$\liminf_{\rho \to +\infty} \frac{1}{\rho} \int_0^1 \phi_\rho(t) dt = \zeta_1 < \infty, \;\; \liminf_{\rho \to +\infty} \frac{1}{\rho} \int_0^1 \psi_\rho(t) dt = \zeta_2 < \infty. \tag{8}$$

Then the boundary value problem (2) has at least one solution on [0,1] provided that

$$\zeta_1 \Lambda_3 + \zeta_2 \Lambda_4 < 1, \tag{9}$$

where ζ_1, ζ_2 are defined by (8), and Λ_3, Λ_4 are given by (7).

Proof. To transform the problem (2) into a fixed point problem, we define a multi-valued map $\mathcal{U} : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ as

$$
\begin{aligned}
\mathcal{U}(y) = \Big\{ g \in C([0,1],\mathbb{R}) : \;\; g(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha} f(s) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q} h(s) \Big] ds \\
&+ a_1(t) \Bigg[\delta \int_0^{\mu} \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha} f(s) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q} h(s) \Big] ds \\
&- \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha} f(s) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q} h(s) \Big] ds \Bigg]
\end{aligned}
$$

$$+a_2(t)\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}f(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h(s)\right]ds\Big\},$$

for $f\in S_{F,y}, h\in\widehat{S}_{H,y}$.

Now we prove that the operator $\mathcal{U}$ satisfies the hypothesis of Lemma 4 and thus it will have a fixed point which corresponds to a solution of problem (2). Here we show that $\mathcal{U}$ is a compact and upper semi-continuous multi-valued map with convex closed values. This will be established in a sequence of steps.

Step 1: *$\mathcal{U}(y)$ is convex for each $y\in C([0,1],\mathbb{R})$.* For that, let $g_1,g_2\in\mathcal{U}(y)$. Then there exist $f_1,f_2\in S_{F,y}$, $h_1,h_2\in\widehat{S}_{H,y}$ such that, for each $t\in[0,1]$, we get

$$\begin{aligned}
g_i(t) &= \int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}f_i(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h_i(s)\right]ds\\
&\quad+a_1(t)\Big\{\delta\int_0^{\mu}\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}f_i(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h_i(s)\right]ds\\
&\quad-\int_0^{1}\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}f_i(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h_i(s)\right]ds\Big\}\\
&\quad+a_2(t)\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}f_i(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h_i(s)\right]ds,\ i=1,2.
\end{aligned}$$

For each $t\in[0,1]$ and $0\le\nu\le1$, we can find that

$$\begin{aligned}
&\left[\nu g_1+(1-\nu)g_2\right](t)\\
&=\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}\left[\nu f_1(s)+(1-\nu)f_2(s)\right]-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}\left[\nu h_1(s)+(1-\nu)h_2(s)\right]\right]ds\\
&+a_1(t)\Big\{\delta\int_0^{\mu}\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}\left[\nu f_1(s)+(1-\nu)f_2(s)\right]-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}\left[\nu h_1(s)+(1-\nu)h_2(s)\right]\right]ds\\
&-\int_0^{1}\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}\left[\nu f_1(s)+(1-\nu)f_2(s)\right]-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}\left[\nu h_1(s)+(1-\nu)h_2(s)\right]\right]ds\Big\}\\
&+a_2(t)\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}\left[\nu f_1(s)+(1-\nu)f_2(s)\right]-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}\left[\nu h_1(s)+(1-\nu)h_2(s)\right]\right]ds.
\end{aligned}$$

Since $S_{F,y},\widehat{S}_{H,y}$ are convex valued (F,H have convex values), it follows that $\nu g_1+(1-\nu)g_2\in\mathcal{U}(y)$.

Step 2: *$\mathcal{U}(y)$ maps bounded sets (balls) into bounded sets in $C([0,1],\mathbb{R})$.* Let us define $\mathcal{B}_\rho=\{y\in C([0,1],\mathbb{R}):\|y\|\le\rho\}$ as a bounded closed convex set in $C([0,1],\mathbb{R})$ for each positive constant ρ. We shall prove that there exists a positive number $\bar{\rho}$ such that $\mathcal{U}(\mathcal{B}_{\bar{\rho}})\subseteq\mathcal{B}_{\bar{\rho}}$. If it is not true, then we can find a function $y_\rho\in\mathcal{B}_\rho, g_\rho\in\mathcal{U}(y_\rho)$ with $\|\mathcal{U}(y_\rho)\|>\rho$, such that

$$\begin{aligned}
g_\rho(t) &= \int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}f_\rho(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h_\rho(s)\right]ds\\
&\quad+a_1(t)\Big\{\delta\int_0^{\mu}\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}f_\rho(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h_\rho(s)\right]ds\\
&\quad-\int_0^{1}\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}f_\rho(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h_\rho(s)\right]ds\Big\}\\
&\quad+a_2(t)\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}f_\rho(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h_\rho(s)\right]ds,
\end{aligned}$$

for some $f_\rho\in S_{F,y_\rho}, h_\rho\in\widehat{S}_{H,y_\rho}$.

According to condition $(\mathbf{M_2})$, we obtain

$$\begin{aligned}
\rho &< \|\mathcal{U}(y_\rho)\| \\
&\leq \int_0^t \frac{|t-s|^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}\phi_\rho(s) + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q}\psi_\rho(s)\Big]ds \\
&\quad +|a_1(t)|\Big\{|\delta|\int_0^\mu \frac{|\mu-s|^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}\phi_\rho(s) + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q}\psi_\rho(s)\Big]ds \\
&\quad + \int_0^1 \frac{|1-s|^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}\phi_\rho(s) + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q}\psi_\rho(s)\Big]ds\Big\} \\
&\quad +|a_2(t)|\int_0^\xi \frac{|\xi-s|^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}\phi_\rho(s) + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q}\psi_\rho(s)\Big]ds \\
&\leq \frac{1+\bar{a}_1(|\delta|+1)+\bar{a}_2}{\Gamma(\beta+1)\Gamma(\alpha)}\int_0^1 \phi_\rho(t)dt + \frac{|\lambda|(1+\bar{a}_1(|\delta|+1)+\bar{a}_2)}{\Gamma(\beta+1)\Gamma(\alpha+p)\Gamma(q)}\int_0^1 \psi_\rho(t)dt \\
&\leq \Lambda_3\int_0^1 \phi_\rho(t)dt + \Lambda_4\int_0^1 \psi_\rho(t)dt,
\end{aligned} \tag{10}$$

where Λ_3, Λ_4 are given by (7). In (10), we have used the following estimates ($\alpha \in (1,2]$, $\beta \in (0,1]$, $p > 0$, $q > 1$):

$$\begin{aligned}
\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}\phi_\rho(s)ds &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_\rho(u)du\,ds \\
&\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\int_0^1 \phi_\rho(u)du \\
&\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\frac{1}{\Gamma(\alpha)}ds\int_0^1 \phi_\rho(u)du \\
&\leq \frac{1}{\Gamma(\beta+1)\Gamma(\alpha)}\int_0^1 \phi_\rho(u)du \\
\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q}\psi_\rho(s)ds &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha+p-1}}{\Gamma(\alpha+p)}\int_0^u \frac{(u-r)^{q-1}}{\Gamma(q)}\psi_\rho(r)dr\,du\,ds \\
&\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha+p-1}}{\Gamma(\alpha+p)}\frac{u^{q-1}}{\Gamma(q)}du\,ds\int_0^1 \psi_\rho(r)dr \\
&\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{1}{\Gamma(\alpha+p)}\frac{1}{\Gamma(q)}du\,ds\int_0^1 \psi_\rho(r)dr \\
&\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\frac{1-s}{\Gamma(\alpha+p)\Gamma(q)}ds\int_0^1 \psi_\rho(r)dr \\
&\leq \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+p)\Gamma(q)}\int_0^1 \psi_\rho(t)dt.
\end{aligned}$$

Dividing both sides of (10) by ρ and then taking the lower limit as $\rho \to \infty$, we find by (8) that $\zeta_1\Lambda_3 + \zeta_2\Lambda_4 > 1$, which is a contradiction to the assumption (9). Hence there exists a positive number $\bar{\rho}$ such that $\mathcal{U}(\mathcal{B}_{\bar{\rho}}) \subseteq \mathcal{B}_{\bar{\rho}}$.

Step 3: *$\mathcal{U}(y)$ maps bounded sets into equicontinuous sets of* $C([0,1],\mathbb{R})$. For that, let $0 \leq t_1 \leq t_2 \leq 1$, $y \in \mathcal{B}_{\bar{\rho}}$, and $g \in \mathcal{U}(y)$. Then there exist $f \in S_{F,y}, h \in \widehat{S}_{H,y}$ such that, for each $t \in [0,1]$, we find that

$$\begin{aligned}
g(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}f(s) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q}h(s)\Big]ds \\
&\quad +a_1(t)\Big\{\delta\int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}f(s) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q}h(s)\Big]ds
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha} f(s)-\lambda I_{1-}^{\alpha+p} I_{0+}^{q} h(s)\Big] ds\Big\} \\
&+a_2(t) \int_0^{\xi} \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha} f(s)-\lambda I_{1-}^{\alpha+p} I_{0+}^{q} h(s)\Big] ds,
\end{aligned}
$$

and that

$$
\begin{aligned}
& |g(t_2)-g(t_1)| \\
= & \int_0^{t_1} \frac{|(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}|}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}|f(s)|+|\lambda| I_{1-}^{\alpha+p} I_{0+}^{q}|h(s)|\Big] ds \\
& +\int_{t_1}^{t_2} \frac{|t_2-s|^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}|f(s)|+|\lambda| I_{1-}^{\alpha+p} I_{0+}^{q}|h(s)|\Big] ds \\
& +\big|a_1(t_2)-a_1(t_1)\big|\Big\{|\delta| \int_0^{\mu} \frac{|\mu-s|^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}|f(s)|+|\lambda| I_{1-}^{\alpha+p} I_{0+}^{q}|h(s)|\Big] ds \\
& +\int_0^1 \frac{|1-s|^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}|f(s)|+|\lambda| I_{1-}^{\alpha+p} I_{0+}^{q}|h(s)|\Big] ds\Big\} \\
& +\big|a_2(t_2)-a_2(t_1)\big| \int_0^{\xi} \frac{|\xi-s|^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}|f(s)|+|\lambda| I_{1-}^{\alpha+p} I_{0+}^{q}|h(s)|\Big] ds \\
\leq & \Big[|t_2^{\beta}-t_1^{\beta}|+2(t_2-t_1)^{\beta}+\frac{(|\delta|+1)}{|\Lambda|}\Big(\xi^{\beta+1}|t_2^{\beta}-t_1^{\beta}|+\xi^{\beta}|t_1^{\beta+1}-t_2^{\beta+1}|\Big) \\
& +\frac{1}{|\Lambda|}\Big(|1-\delta\mu^{\beta+1}||t_2^{\beta}-t_1^{\beta}|+|1-\delta\mu^{\beta}||t_1^{\beta+1}-t_2^{\beta+1}|\Big)\Big] \\
& \times\Big\{\frac{1}{\Gamma(\beta+1)\Gamma(\alpha)} \int_0^1 \phi_{\rho}(s) ds+\frac{|\lambda|}{\Gamma(\beta+1)\Gamma(\alpha+p)\Gamma(q)} \int_0^1 \psi_{\rho}(s) ds\Big\}.
\end{aligned}
$$

Clearly, the right-hand side of the above inequality tends to zero as $t_2 \to t_1$ independently of $y \in \mathcal{B}_{\bar{\rho}}$. Hence $\mathcal{U}$ is equi-continuous. As $\mathcal{U}$ satisfies the above three steps, it follows by the Ascoli-Arzelá theorem that $\mathcal{U}$ is a compact multi-valued map.

Step 4: *$\mathcal{U}$ has a closed graph.* Let $y_n \to y_*$, $g_n \in \mathcal{U}(y_n)$ and $g_n \to g_*$. Then we need to show that $g_* \in \mathcal{U}(y_*)$. Associated with $g_n \in \mathcal{U}(y_n)$, we can find $f_n \in S_{F,y_n}, h_n \in \widehat{S}_{H,y_n}$ such that, for each $t \in [0,1]$, we have

$$
\begin{aligned}
g_n(t) = & \int_0^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha} f_n(s)-\lambda I_{1-}^{\alpha+p} I_{0+}^{q} h_n(s)\Big] ds \\
& +a_1(t)\Big\{\delta \int_0^{\mu} \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha} f_n(s)-\lambda I_{1-}^{\alpha+p} I_{0+}^{q} h_n(s)\Big] ds \\
& -\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha} f_n(s)-\lambda I_{1-}^{\alpha+p} I_{0+}^{q} h_n(s)\Big] ds\Big\} \\
& +a_2(t) \int_0^{\xi} \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha} f_n(s)-\lambda I_{1-}^{\alpha+p} I_{0+}^{q} h_n(s)\Big] ds.
\end{aligned}
$$

Thus it suffices to show that there exist $f_* \in S_{F,y_*}, h_* \in \widehat{S}_{H,y_*}$ such that for each $t \in [0,1]$,

$$
\begin{aligned}
g_*(t) = & \int_0^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha} f_*(s)-\lambda I_{1-}^{\alpha+p} I_{0+}^{q} h_*(s)\Big] ds \\
& +a_1(t)\Big\{\delta \int_0^{\mu} \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha} f_*(s)-\lambda I_{1-}^{\alpha+p} I_{0+}^{q} h_*(s)\Big] ds \\
& -\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha} f_*(s)-\lambda I_{1-}^{\alpha+p} I_{0+}^{q} h_*(s)\Big] ds\Big\}
\end{aligned}
$$

$$+a_2(t)\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}f_*(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h_*(s)\Big]ds.$$

Let us consider the continuous linear operator $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1])$ so that

$$\begin{aligned}(\Theta(f,h))(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}f(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h(s)\Big]ds\\ &+a_1(t)\bigg\{\delta\int_0^{\mu}\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}f(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h(s)\Big]ds\\ &-\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}f(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h(s)\Big]ds\bigg\}\\ &+a_2(t)\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}f(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h(s)\Big]ds.\end{aligned}$$

Observe that

$$\begin{aligned}&\|g_n(t)-g_*(t)\|\\ =\;&\bigg\|\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}(f_n(s)-f_*(s))-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}(h_n(s)-h_*(s))\Big]ds\\ &+a_1(t)\bigg\{\delta\int_0^{\mu}\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}(f_n(s)-f_*(s))-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}(h_n(s)-h_*(s))\Big]ds\\ &-\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}(f_n(s)-f_*(s))-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}(h_n(s)-h_*(s))\Big]ds\bigg\}\\ &+a_2(t)\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}(f_n(s)-f_*(s))-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}(h_n(s)-h_*(s))\Big]ds\bigg\|\to 0 \text{ as } n\to\infty.\end{aligned}$$

Thus, it follows by Lemma 5 that $\Theta \circ S_B$ is a closed graph operator where $S_B = S_F \cup \widehat{S}_H$. Moreover, we have $g_n(t) \in \Theta(S_{B,y_n})$. Since $y_n \to y_*$, $g_n \to g_*$, therefore, Lemma 5 yields

$$\begin{aligned}g_*(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}f_*(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h_*(s)\Big]ds\\ &+a_1(t)\bigg\{\delta\int_0^{\mu}\frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}f_*(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h_*(s)\Big]ds\\ &-\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}f_*(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h_*(s)\Big]ds\bigg\}\\ &+a_2(t)\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}f_*(s)-\lambda I_{1-}^{\alpha+p}I_{0+}^{q}h_*(s)\Big]ds,\end{aligned}$$

for some $f_* \in S_{F,y_*}, h_* \in \widehat{S}_{H,y_*}$.

Hence, we conclude that $\mathcal{U}$ is a compact and upper semi-continuous multi-valued map with convex closed values. Thus, the hypothesis of Lemma 4 holds true, and therefore its conclusion implies that the operator $\mathcal{U}$ has a fixed point y, which corresponds to a solution of problem (2). This completes the proof. □

Next, we give an existence result based upon the following form of fixed point theorem due to Martelli [38], which is applicable to completely continuous operators.

Lemma 6. *Let X a Banach space, and $T : X \to \mathcal{P}_{b,cl,c}(X)$ be a completely continuous multi-valued map. If the set $\mathcal{E} = \{x \in X : \kappa x \in T(x),\ \kappa > 1\}$ is bounded, then T has a fixed point.*

Theorem 3. *Assume that the following hypotheses hold:*

($\mathbf{M_3}$) *$F, H : [0,1] \times \mathbb{R} \to \mathcal{P}_{b,cl,c}(\mathbb{R})$ are L^1-Carathéodory multi-valued maps; that is, (i) $t \longmapsto F(t,y), t \longmapsto H(t,y)$, are measurable for each $y \in \mathbb{R}$; (ii) $y \longmapsto F(t,y), y \longmapsto H(t,y)$ are upper semicontinuous for almost all $t \in [0,1]$; (iii) for each $r > 0$, there exist $\phi_r, \psi_r \in L^1([0,1], \mathbb{R}^+)$ such that $\|F(t,y)\| = \sup\{|v| : v \in F(t,y)\} \le \phi_r(t)$, $\|H(t,y)\| = \sup\{|v| : v \in F(t,y)\} \le \psi_r(t)$, for all $y \in \mathbb{R}$ with $\|y\| \le r$ and for almost every $t \in [0,1]$.*

($\mathbf{M_4}$) *There exist functions $z, u \in L^1([0,1], \mathbb{R}^+)$ such that*

$$\|F(t,y)\| \le z(t), \ \|H(t,y)\| \le u(t), \text{ for a.e. } t \in [0,1] \text{ and each } y \in \mathbb{R}.$$

Then the problem (2) has at least one solution on $[0,1]$.

Proof. Consider $\mathcal{U}$ defined in the proof of Theorem 2. As in Theorem 2, we can show that $\mathcal{U}$ is convex and completely continuous. It remains to show that the set

$$\mathcal{E} = \{y \in C([0,1], \mathbb{R}) : \kappa y \in \mathcal{U}(y), \ \kappa > 1\}$$

is bounded. Let $y \in \mathcal{E}$, then $\kappa y \in \mathcal{U}(y)$ for some $\kappa > 1$ and there exist functions $f \in S_{F,y}, h \in \widehat{S}_{H,y}$ such that

$$\begin{aligned} y(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha} f(s) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q} h(s) \Big] ds \\ &\quad + a_1(t) \Big\{ \delta \int_0^{\mu} \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha} f(s) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q} h(s) \Big] ds \\ &\quad - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha} f(s) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q} h(s) \Big] ds \Big\} \\ &\quad + a_2(t) \int_0^{\xi} \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha} f(s) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q} h(s) \Big] ds. \end{aligned}$$

For each $t \in [0,1]$, we have

$$\begin{aligned} |y(t)| &\le \int_0^t \frac{|t-s|^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha} z(s) + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q} u(s) \Big] ds \\ &\quad + |a_1(t)| \Big\{ |\delta| \int_0^{\mu} \frac{|\mu-s|^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha} z(s) + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q} u(s) \Big] ds \\ &\quad + \int_0^1 \frac{|1-s|^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha} z(s) + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q} u(s) \Big] ds \Big\} \\ &\quad + |a_2(t)| \int_0^{\xi} \frac{|\xi-s|^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha} z(s) + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q} u(s) \Big] ds \\ &\le \frac{1 + \bar{a}_1(|\delta|+1) + \bar{a}_2}{\Gamma(\beta+1)\Gamma(\alpha)} \|z\|_{L^1} + \frac{|\lambda|(1 + \bar{a}_1(|\delta|+1) + \bar{a}_2)}{\Gamma(\beta+1)\Gamma(\alpha+p)\Gamma(q)} \|u\|_{L^1} \\ &\le \Lambda_3 \|z\|_{L^1} + \Lambda_4 \|u\|_{L^1}, \end{aligned}$$

Taking the supremum over $t \in J$, we get

$$\|y\| \le \Lambda_3 \|z\|_{L^1} + \Lambda_4 \|u\|_{L^1} < \infty.$$

Hence the set $\mathcal{E}$ is bounded. As a consequence of Lemma 6 we deduce that $\mathcal{U}$ has at least one fixed point which implies that the problem (2) has a solution on $[0,1]$. □

5. Applications

We consider four different cases for $F(t,y)$ and $H(t,y)$ (in (2)) to demonstrate applications of theorem (2): (**a**) F and H have sub-linear growth in their second variable. (**b**) F and H have linear growth in their second variable. (**c**) F has sub-linear growth in its second variable and H has linear growth. (**d**) F has linear growth in its second variable and H has sub-linear growth.
Case (**a**). For each $(t,y) \in [0,1] \times \mathbb{R}$, there exist functions $\sigma_i(t), \vartheta_i(t) \in L^1([0,1], \mathbb{R}^+), i = 1,2, \gamma \in [0,1)$ such that $\|F(t,y)\| \leq \sigma_1(t)|y|^\gamma + \vartheta_1(t)$ and $\|H(t,y)\| \leq \sigma_2(t)|y|^\gamma + \vartheta_2(t)$ which correspond in this case to $\phi_\rho(t) = \sigma_1(t)\rho^\gamma + \vartheta_1(t)$ and $\psi_\rho(t) = \sigma_2(t)\rho^\gamma + \vartheta_2(t)$ and the condition (9) will take the form $0 \cdot \Lambda_3 + 0 \cdot \Lambda_4 < 1$, that is, $\zeta_1 = \zeta_2 = 0$.
Case (**b**). F and H will satisfy the assumptions $\|F(t,y)\| \leq \sigma_1(t)|y| + \vartheta_1(t)$ and $\|H(t,y)\| \leq \sigma_2(t)|y| + \vartheta_2(t)$, which, in view of ($\mathbf{M_2}$), implies that $\phi_\rho(t) = \sigma_1(t)\rho + \vartheta_1(t)$ and $\psi_\rho(t) = \sigma_2(t)\rho + \vartheta_2(t)$, and the condition (9) becomes $\|\sigma_1\|_{L^1} \cdot \Lambda_3 + \|\sigma_2\|_{L^1} \cdot \Lambda_4 < 1$.
Similarly, one can verify the cases (**c**) and (**d**). Thus, the boundary value problem (2) has at least one solution on $[0,1]$ for all the cases (**a**)–(**d**).

Let us consider the following inclusions problem:

$$\begin{cases} {}^{C}D_{1-}^{5/4}\, {}^{RL}D_{0+}^{3/4} y(t) \in F(t,y(t)) - 2 I_{1-}^{3/2} I_{0+}^{5/2} H(t,y(t)), \ \ t \in [0,1], \\ y(0) = y(1/3) = 0, \quad y(1) = \frac{1}{4} y(2/3), \end{cases} \tag{11}$$

where $\alpha = 5/4,\ \beta = 3/4,\ \lambda = 2,\ p = 3/2,\ q = 5/2,\ \xi = 1/3,\ \mu = 2/3,\ \delta = 1/4$. It is easy to find that

$$\begin{aligned} \bar{a}_1 &= \max_{t \in [0,1]} |a_1(t)| = |a_1(t)|_{t=1} \approx 1.101592729739686, \\ \bar{a}_2 &= \max_{t \in [0,1]} |a_2(t)| = |a_2(t)|_{t=t_{a_2}} \approx 1.055901462873258, \end{aligned}$$

where

$$t_{a_2} = \frac{\beta(1 - \delta\mu^{\beta+1})}{(1 - \delta\mu^\beta)(\beta + 1)} \approx 0.460880265746053 < 1.$$

Using the above given data, we find that $\Lambda_3 \approx 4.120918689155884$, $\Lambda_4 \approx 3.494023466997676$, where Λ_3, Λ_4 are given by (7).
(**a**). We consider $\|F(t,y)\| \leq \sigma_1(t)|y|^{1/3} + \vartheta_1(t)$ and $\|H(t,y)\| \leq \sigma_2(t)|y|^{1/3} + \vartheta_2(t)$ with $\sigma_i(t), \vartheta_i(t) \in L^1([0,1], \mathbb{R}^+), i = 1,2, \gamma \in [0,1)$. In this case, F and H in (11) satisfy all the assumptions of Theorem 2 with $0 \cdot \Lambda_3 + 0 \cdot \Lambda_4 < 1$, which implies that the boundary value problem (11) has at least one solution on $[0,1]$.
(**b**) As a second example, let F and H be such that $\|F(t,y)\| \leq \frac{1}{4(1+t)^2}|y| + 2e^t$ and $\|H(t,y)\| \leq \frac{2}{(4+t)^2}|y| + e^{-t}$. In this case, the condition (9) will take the form $\frac{1}{8} \cdot \Lambda_3 + \frac{1}{10} \cdot \Lambda_4 \approx 0.864517182844253 < 1$. Thus, by the conclusion of Theorem 2, there exists at least one solution for the problem (11) on $[0,1]$.

In a similar manner, one can verify that the problem (2) has at least one solution on $[0,1]$ when we choose the cases: (**c**) $\|F(t,y)\| \leq \sigma_1(t)|y|^{1/3} + \vartheta_1(t)$, $\|H(t,y)\| \leq \frac{2}{(4+t)^2}|y| + e^{-t}$, and (**d**) $\|F(t,y)\| \leq \frac{1}{4(1+t)^2}|y| + 2e^t$, $\|H(t,y)\| \leq \sigma_2(t)|y|^{1/3} + \vartheta_2(t)$.

6. Conclusions

In this paper, we have discussed the existence of solutions for a new class of boundary value problems involving right-Caputo and left-Riemann–Liouville fractional derivatives of different orders and right-left Riemann–Liouville fractional integrals with nonlocal boundary conditions. The existence result for the single-valued case of the given problem is proven via Sadovski's fixed point theorem, while the existence results for the multi-valued case of the problem at hand are derived by means of

Bohnenblust-Karlin and Martelli fixed point theorems. Applications for the obtained results are also presented. By taking $\delta = 0$ in the results of this paper, we obtain the ones for a problem associated with three-point nonlocal boundary conditions: $y(0) = 0, y(\xi) = 0, y(1) = 0$ $(0 < \xi < 1)$ as a special case.

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