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Lyapunov Type Theorems for Exponential Stability of Linear Skew-Product Three-Parameter Semiflows with Discrete Time

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Abstract: For linear skew-product three-parameter semiflows with discrete time acting on an arbitrary Hilbert space, we obtain a complete characterization of exponential stability in terms of the existence of appropriate Lyapunov functions. As a nontrivial application of our work, we prove that the notion of an exponential stability persists under sufficiently small linear perturbations.

Keywords: exponential stability; linear skew-product semiflows; Lyapunov functions

1. Introduction

The main objective of this paper is to obtain a complete characterization of exponential stability for linear skew-product semiflows with discrete time acting on an arbitrary Hilbert space in terms of the existence of appropriate Lyapunov functions. We then use this characterization to prove that the notion of an exponential stability persists under sufficiently small linear perturbations.

We stress that the use of Lyapunov functions in the study of the stability of trajectories in the theories of differential equations and dynamical systems has a long history that goes back to the landmark work of Lyapunov [1]. For some early contributions to the theory, we refer to books by LaSalle and Lefschetz [2], Hahn [3] and Bhatia and Szegö [4]. For the first contributions dealing with infinite-dimensional dynamics, we refer to the work of Daleckij and Krein [5].

In the context of nonautonomous dynamics, the relationship between exponential dichotomies and the existence of appropriate Lyapunov functions was first considered by Maizel [6]. His results were further developed by Coppel [7,8] as well as Muldowney [9]. We note that these results considered only the case of continuous time. To the best of our knowledge, the first contributions in the case of discrete time are due to Papaschinopoulos [10]. In the recent years, there has been a renewed interest in this topic. More precisely, various characterizations of nonuniform exponential behaviour in terms of Lyapunov functions were obtained (see [11–13]). In addition, the authors have obtained first results in the context of infinite-dimensional dynamics [14] (see also [15]) which lead to further developments [16–18]. Finally, for some related results in the context of ergodic theory, we refer to [19] and references therein.

The purpose of this paper is to show that techniques we developed in our previous work [14] can be used to obtain Lyapunov-type characterization of exponential stability for a very general type of nonautonomous dynamics. More precisely, we consider the so-called linear skew-product three-parameter semiflows. This notion was introduced by Megan and Stoica [20] and includes various previously studied notions as a particular case (see Examples 1 and 2).

Finally, we would like to mention that Lyapunov type characterizations of exponential stability are certainly not the only tool used to study stability of nonautonomous dynamics. Indeed, there is a vast literature devoted to the so-called Perron type characterizations of exponential stability (see [21–26] and references therein) as well as to Datko-Pazy-Rolewicz techniques (see [27–33]). For some other approaches to the study of exponential stability for nonautonomous systems, we refer to [34,35].

The paper is organized as follows. In Section 2 we introduce all relevant notions and recall auxiliary results which will be used in the paper. In Section 3 we state and prove the main results of our paper. Finally, in Section 4 we apply the main result to the study of the robustness property of exponential stability for linear skew-product three-parameter semiflows.

2. Preliminaries

Let (Θ, d) be a metric space and let X be a Hilbert space over $\mathbb{C}$. By $B(X)$ we will denote the space of all bounded operators on X.

Definition 1. *A map $\sigma\colon \Theta \times \mathbb{Z} \times \mathbb{Z} \to \Theta$ is said to be a continuous three-parameter flow (with discrete time) if:*

1. *$\sigma(\theta, n, n) = \theta$ for each $\theta \in \Theta$ and $n \in \mathbb{Z}$;*
2. *$\sigma(\sigma(\theta, m, n), k, m) = \sigma(\theta, k, n)$ for every $\theta \in \Theta$ and $n, m, k \in \mathbb{Z}$;*
3. *$\sigma(\cdot, m, n)$ is a continuous map for each $(m, n) \in \mathbb{Z} \times \mathbb{Z}$.*

Set $\Delta = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m \geq n\}$.

Definition 2. *Let σ be a continuous three-parameter flow. A map $\Phi\colon \Theta \times \Delta \to B(X)$ is said to be a linear skew-product three-parameter semiflow (with discrete time) over σ if:*

1. *$\Phi(\theta, n, n) = \mathrm{Id}$ for $\theta \in \Theta$ and $n \in \mathbb{Z}$;*
2. *$\Phi(\sigma(\theta, m, n), k, m)\Phi(\theta, m, n) = \Phi(\theta, k, n)$ for $\theta \in \Theta$ and $(m, n), (k, m) \in \Delta$;*
3. *$\theta \mapsto \Phi(\theta, m, n)x$ is continuous for each $x \in X$ and $(m, n) \in \Delta$.*

Let us give some examples.

Example 1. *Assume that Θ is a singleton, i.e., that $\Theta = \{p\}$ and let $\sigma(p, m, n) = p$ for $m, n \in \mathbb{Z}$. Furthermore, let $(A_n)_{n \in \mathbb{Z}}$ be a sequence in $B(X)$. For $(m, n) \in \Delta$, set*

$$\Phi(p, m, n) = \begin{cases} A_{m-1} \cdots A_n & \text{for } m > n; \\ \mathrm{Id} & \text{for } m = n. \end{cases}$$

Then, one can easily verify that Φ is a linear skew-product three-parameter semiflow over σ.

Example 2. *Let Θ be an arbitrary Banach space and $\rho\colon \Theta \to \Theta$ a homeomorphism. We define $\sigma\colon \Theta \times \mathbb{Z} \times \mathbb{Z} \to \Theta$ by*

$$\sigma(\theta, m, n) = \rho^{m-n}(\theta), \quad \text{for } \theta \in \Theta \text{ and } m, n \in \mathbb{Z}.$$

One can easily verify that σ is a continuous three-parameter flow. Let $\mathcal{A}\colon \Theta \times \mathbb{N}_0 \to B(X)$ be a linear cocycle over ρ, i.e., $\mathcal{A}$ satisfies the following conditions:

- *$\mathcal{A}(\theta, 0) = \mathrm{Id}$ for $\theta \in \Theta$;*
- *$\mathcal{A}(\theta, m + n) = \mathcal{A}(\rho^m(\theta), n)\mathcal{A}(\theta, m)$ for $\theta \in \Theta$ and $n, m \in \mathbb{N}_0$;*
- *$\theta \mapsto \mathcal{A}(\theta, 1)x$ is continuous for each $x \in X$.*

For $\theta \in \Theta$ and $(m, n) \in \Delta$, set

$$\Phi(\theta, m, n) = \mathcal{A}(\theta, m - n).$$

Then, it is easy to show that Φ is a linear skew-product three-parameter semiflow over σ.

Example 3. *Let σ be a continuous three-parameter flow on a metric space Θ. Furthermore, take a map $A\colon \Theta \to B(X)$ such that $\theta \mapsto A(\theta)x$ is continuous for each $x \in X$. For $(\theta, n) \in \Theta \times \mathbb{Z}$ and $x \in X$, let us consider a Cauchy problem*

$$y_{m+1} = A(\sigma(\theta, m, n))y_m \quad m \geq n, \quad y_n = x.$$

Let $\Phi(\theta, m, n)x$ denote the value of the solution of this problem at time m. Then, Φ is a linear skew-product three-parameter semiflow over σ. We observe that

$$\Phi(\theta, m, n) = A(\sigma(\theta, m-1, n)) \cdots A(\sigma(\theta, n+1, n))A(\theta),$$

for $\theta \in \Theta$ and $(m, n) \in \Delta$.

We now introduce the notion of exponential stability.

Definition 3. *For a linear skew-product three-parameter semiflow Φ we say that it is exponentially stable if there exist $D, \lambda > 0$ such that*

$$\|\Phi(\theta, m, n)\| \leq De^{-\lambda(m-n)}, \quad \text{for } \theta \in \Theta \text{ and } (m, n) \in \Delta. \tag{1}$$

We also introduce some additional notation that will be used throughout this paper. More precisely, for a linear skew-product three-parameter semiflow Φ over σ, we introduce a map $\bar{\Phi}\colon \Theta \times \mathbb{Z} \to B(X)$ by

$$\bar{\Phi}(\theta, n) = \Phi(\theta, n+1, n), \quad \text{for } (\theta, n) \in \Theta \times \mathbb{Z}.$$

Furthermore, we define $\bar{\sigma}\colon \Theta \times \mathbb{Z} \to \Theta \times \mathbb{Z}$ by

$$\bar{\sigma}(\theta, n) = (\sigma(\theta, n+1, n), n+1) \quad \text{for } (\theta, n) \in \Theta \times \mathbb{Z}.$$

Clearly, $\bar{\sigma}$ is invertible and in fact,

$$\bar{\sigma}^m(\theta, n) = (\sigma(\theta, n+m, n), n+m), \quad \text{for } (\theta, n) \in \Theta \times \mathbb{Z} \text{ and } m \in \mathbb{Z}.$$

Some Auxiliary Results

We also recall some useful results established by Daleckij and Krein [5].

Lemma 1. *Assume that $\mathcal{H}$ is a Hilbert space and that T is a bounded operator on $\mathcal{H}$. Furthermore, suppose that the spectrum of T does not cover the whole unit circle S^1. Then every self-adjoint operator bounded operator W on $\mathcal{H}$ with the property that there exists $\delta > 0$ such that*

$$T^*WT - W \leq -\delta \mathrm{Id} \tag{2}$$

is invertible.

We will also use the following result (also taken from [5]).

Lemma 2. *Assume that $\mathcal{H}$ is a Hilbert space and that T is a bounded operator on $\mathcal{H}$. Furthermore, assume that there exists an invertible, self-adjoint and bounded linear operator W on $\mathcal{H}$ such that (2) holds for some $\delta > 0$. Then, the spectrum of T does not intersect S^1 and there exist $\delta' > 0$ satisfying*

$$TW^{-1}T^* - W^{-1} \leq -\delta' \mathrm{Id}.$$

Moreover, if $W \geq 0$ (that is, $\langle Wx, x\rangle \geq 0$ for $x \in \mathcal{H}$) then the spectrum of T is contained in $\{z \in \mathbb{C} : |z| < 1\}$.

3. Main Results

The following is our first main result.

Theorem 1. *Assume that $\Phi\colon \Theta \times \Delta \to B(X)$ is an exponentially stable linear skew-product three-parameter semiflow over a continuous three-parameter flow σ. Then, there exists a family $S_{(\theta,n)}$, $(\theta,n) \in \Theta \times \mathbb{Z}$ of bounded, self-adjoint and invertible operators on X and $K, \delta > 0$ such that for $(\theta,n) \in \Theta \times \mathbb{Z}$:*

1. $S_{(\theta,n)} \geq 0$;
2. $$\|S_{(\theta,n)}\| \leq K \quad \text{and} \quad \|S^{-1}_{(\theta,n)}\| \leq K; \tag{3}$$
3. $$\bar{\Phi}(\theta,n)^* S_{\bar{\sigma}(\theta,n)} \bar{\Phi}(\theta,n) - S_{(\theta,n)} \leq -\delta \mathrm{Id}; \tag{4}$$
4. $$\bar{\Phi}(\theta,n) S^{-1}_{(\theta,n)} \bar{\Phi}(\theta,n)^* - S^{-1}_{\bar{\sigma}(\theta,n)} \leq -\delta \mathrm{Id}; \tag{5}$$

Proof. For $(\theta,n) \in \Theta \times \mathbb{Z}$, set

$$S_{(\theta,n)} := \sum_{k=n}^{+\infty} \Phi(\theta,k,n)^* \Phi(\theta,k,n).$$

It follows from (1) that

$$\begin{aligned}
\langle S_{(\theta,n)} x, x\rangle &= \sum_{k=n}^{+\infty} \|\Phi(\theta,k,n)x\|^2 \\
&\leq \sum_{k=n}^{+\infty} D^2 e^{-2\lambda(k-n)} \|x\|^2 \\
&= K\|x\|^2,
\end{aligned}$$

where $K = \frac{D^2}{1-e^{-2\lambda}} > 0$. Obviously, $S_{(\theta,n)}$ is self-adjoint, $S_{(\theta,n)} \geq 0$ and therefore

$$\|S_{(\theta,n)}\| = \sup_{\|x\|=1} \langle S_{(\theta,n)} x, x\rangle \leq K, \quad \text{for } (\theta,n) \in \Theta \times \mathbb{Z}.$$

Hence, the first inequality (3) holds. Furthermore, we have that

$$\begin{aligned}
&\bar{\Phi}(\theta,n)^* S_{\bar{\sigma}(\theta,n)} \bar{\Phi}(\theta,n) - S_{(\theta,n)} \\
&= \bar{\Phi}(\theta,n)^* \sum_{k=n+1}^{+\infty} \Phi(\sigma(\theta,n+1,n),k,n+1)^* \Phi(\sigma(\theta,n+1,n),k,n+1) \bar{\Phi}(\theta,n) \\
&\quad - \sum_{k=n}^{+\infty} \Phi(\theta,k,n)^* \Phi(\theta,k,n) \\
&= \sum_{k=n+1}^{+\infty} \Phi(\theta,k,n)^* \Phi(\theta,k,n) - \sum_{k=n}^{+\infty} \Phi(\theta,k,n)^* \Phi(\theta,k,n) \\
&= -\Phi(\theta,n,n) \\
&= -\mathrm{Id},
\end{aligned}$$

which implies that (4) holds with $\delta = 1$.

Set now

$$l^2 := \left\{ \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \subset X : \sum_{n=-\infty}^{+\infty} \|x_n\|^2 < +\infty \right\}.$$

Clearly, l^2 is a Hilbert space with respect to the scalar product

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{n=-\infty}^{+\infty} \langle x_n, y_n \rangle,$$

for $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ and $\mathbf{y} = (y_n)_{n\in\mathbb{Z}}$ in l^2. For $(\theta, n) \in \Theta \times \mathbb{Z}$, we define $\mathbb{A}_{(\theta,n)} \colon l^2 \to l^2$ by

$$\begin{aligned}(\mathbb{A}_{(\theta,n)}\mathbf{x})_m &= \bar{\Phi}(\bar{\sigma}^{m-1}(\theta, n))x_{m-1} \\ &= \bar{\Phi}(\sigma(\theta, n+m-1, n), n+m-1)x_{m-1} \\ &= \Phi(\sigma(\theta, n+m-1, n), n+m, n+m-1)x_{m-1},\end{aligned}$$

for $m \in \mathbb{Z}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in l^2$. It follows from (1) that

$$\begin{aligned}\sum_{m=-\infty}^{+\infty} \|(\mathbb{A}_{(\theta,n)}\mathbf{x})_m\|^2 &= \sum_{m=-\infty}^{+\infty} \|\bar{\Phi}(\sigma(\theta, n+m-1, n), n+m-1)x_{m-1}\|^2 \\ &\leq D^2 \sum_{m=-\infty}^{+\infty} \|x_{m-1}\|^2,\end{aligned}$$

for every $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in l^2$. Hence, $\mathbb{A}_{(\theta,n)}$ is well-defined and bounded linear operator for each $(\theta, n) \in \Theta \times \mathbb{Z}$.

We need the following auxiliary results.

Lemma 3. *We have that*

$$(\mathbb{A}^*_{(\theta,n)}\mathbf{x})_m = \bar{\Phi}(\bar{\sigma}^m(\theta, n))^* x_{m+1}, \quad \text{for } (\theta, n) \in \Theta \times \mathbb{Z} \text{ and } m \in \mathbb{Z}.$$

Proof of the Lemma. Take $(\theta, n) \in \Theta \times \mathbb{Z}$, we define $\mathbb{B}_{(\theta,n)} \colon l^2 \to l^2$ by

$$(\mathbb{B}_{(\theta,n)}\mathbf{x})_m = \bar{\Phi}(\bar{\sigma}^m(\theta, n))^* x_{m+1}, \quad \text{for } (\theta, n) \in \Theta \times \mathbb{Z} \text{ and } m \in \mathbb{Z}.$$

Obviously, $\mathbb{B}_{(\theta,n)}$ is a well-defined and bounded linear operator. For $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ and $\mathbf{y} = (y_n)_{n\in\mathbb{Z}}$ in l^2, we have that

$$\begin{aligned}\langle \mathbb{A}_{(\theta,n)}\mathbf{x}, \mathbf{y} \rangle &= \sum_{m=-\infty}^{+\infty} \langle (\mathbb{A}_{(\theta,n)}\mathbf{x})_m, y_m \rangle \\ &= \sum_{m=-\infty}^{+\infty} \langle \bar{\Phi}(\bar{\sigma}^{m-1}(\theta, n))x_{m-1}, y_m \rangle \\ &= \sum_{m=-\infty}^{+\infty} \langle x_{m-1}, \bar{\Phi}(\bar{\sigma}^{m-1}(\theta, n))^* y_m \rangle \\ &= \sum_{m=-\infty}^{+\infty} \langle x_{m-1}, (\mathbb{B}_{(\theta,n)}\mathbf{y})_{m-1} \rangle \\ &= \langle \mathbf{x}, \mathbb{B}_{(\theta,n)}\mathbf{y} \rangle,\end{aligned}$$

which readily implies the desired conclusion. □

Lemma 4. *There exists $t \in (0, 1)$ such that spectrum of $\mathbb{A}_{(\theta,n)}$ is contained in $\{z \in \mathbb{C} : |z| \leq t\}$, for each $(\theta, n) \in \Theta \times \mathbb{Z}$.*

Proof of the Lemma. Fix $(\theta, n) \in \Theta \times \mathbb{Z}$. Then, for each $k \in \mathbb{N}$ and $\mathbf{x} = (x_m)_{m\in\mathbb{Z}} \in l^2$ we have that

$$(\mathbb{A}^k_{(\theta,n)}\mathbf{x})_m = \Phi(\sigma(\theta, n+m-k, n), n+m, n+m-k)x_{m-k}$$

and consequently (1) implies that

$$\|(\mathbb{A}^k_{(\theta,n)}\mathbf{x})_m\| \le De^{-\lambda k}\|x_{m-k}\|,$$

for each $m \in \mathbb{Z}$. This readily yields that $\|(\mathbb{A}^k_{(\theta,n)}\| \le De^{-\lambda k}$. Since $k \in \mathbb{N}$ was arbitrary we conclude that the statement of the lemma holds with $t = e^{-\lambda} < 1$. $\square$

For $(\theta, n) \in \Theta \times \mathbb{Z}$ we define $\mathbb{W}_{(\theta,n)} : l^2 \to l^2$ by

$$(\mathbb{W}_{(\theta,n)}\mathbf{x})_m = S_{\bar{\sigma}^m(\theta,n)}x_m, \quad \text{for } (\theta, n) \in \Theta \times \mathbb{Z} \text{ and } m \in \mathbb{Z}.$$

It follows easily from the already proved first inequality in (3) that $\mathbb{W}_{(\theta,n)}$ is a well-defined and bounded linear operator on l^2. Moreover, it is easy to show that $\mathbb{W}_{(\theta,n)}$ is self-adjoint.

On the other hand, observe that for $(\theta, n) \in \Theta \times \mathbb{Z}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in l^2$, we have that

$$(\mathbb{A}^*_{(\theta,n)}\mathbb{W}_{(\theta,n)}\mathbb{A}_{(\theta,n)}\mathbf{x})_m = \bar{\Phi}(\bar{\sigma}^m(\theta, n))^* S_{\bar{\sigma}^{m+1}(\theta,n)}\bar{\Phi}(\bar{\sigma}^m(\theta, n))x_m,$$

for each $m \in \mathbb{Z}$. Hence, the already proved inequality (4) (we recall that it holds with $\delta = 1$) implies that

$$\mathbb{A}^*_{(\theta,n)}\mathbb{W}_{(\theta,n)}\mathbb{A}_{(\theta,n)} - \mathbb{W}_{(\theta,n)} \le -\mathrm{Id} \quad \text{on } l^2, \tag{6}$$

for each $(\theta, n) \in \Theta \times \mathbb{Z}$. Hence, Lemmas 1 and 4 imply that $\mathbb{W}_{(\theta,n)}$ is invertible for every $(\theta, n) \in \Theta \times \mathbb{Z}$.

Lemma 5. *We have that*

$$\sup_{(\theta,n)\in\Theta\times\mathbb{Z}} \|\mathbb{W}^{-1}_{(\theta,n)}\| < +\infty.$$

Proof of the Lemma. For $(\theta, n) \in \Theta \times \mathbb{Z}$, set

$$\mathbb{H}_{(\theta,n)} := -\mathbb{A}^*_{(\theta,n)}\mathbb{W}_{(\theta,n)}\mathbb{A}_{(\theta,n)} + \mathbb{W}_{(\theta,n)}.$$

Then, $\mathbb{H}_{(\theta,n)} \ge \mathrm{Id}$. It is easy to verify that

$$(\mathbb{A}^*_{(\theta,n)} - \mathrm{Id})\mathbb{W}_{(\theta,n)}(\mathbb{A}_{(\theta,n)} + \mathrm{Id}) + (\mathbb{A}^*_{(\theta,n)} + \mathrm{Id})\mathbb{W}_{(\theta,n)}(\mathbb{A}_{(\theta,n)} - \mathrm{Id}) = -2\mathbb{H}_{(\theta,n)}$$

By multiplying this identity on the right by $(\mathbb{A}_{(\theta,n)} - \mathrm{Id})^{-1}$ and on the left by $(\mathbb{A}^*_{(\theta,n)} - \mathrm{Id})^{-1}$, we obtain that

$$\begin{aligned}
&\mathbb{W}_{(\theta,n)}(\mathbb{A}_{(\theta,n)} + \mathrm{Id})(\mathbb{A}_{(\theta,n)} - \mathrm{Id})^{-1} + (\mathbb{A}^*_{(\theta,n)} - \mathrm{Id})^{-1}(\mathbb{A}^*_{(\theta,n)} + \mathrm{Id})\mathbb{W}_{(\theta,n)} \\
&= -2(\mathbb{A}^*_{(\theta,n)} - \mathrm{Id})^{-1}\mathbb{H}_{(\theta,n)}(\mathbb{A}_{(\theta,n)} - \mathrm{Id})^{-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\langle(\mathbb{A}^*_{(\theta,n)} - \mathrm{Id})^{-1}\mathbb{H}_{(\theta,n)}(\mathbb{A}_{(\theta,n)} - \mathrm{Id})^{-1}\mathbf{x}, \mathbf{x}\rangle \\
&\le \frac{1}{2}\|\mathbb{W}_{(\theta,n)}\mathbf{x}\| \cdot \|\mathbb{A}_{(\theta,n)} + \mathrm{Id}\| \cdot \|(\mathbb{A}_{(\theta,n)} - \mathrm{Id})^{-1}\| \cdot \|\mathbf{x}\|,
\end{aligned}$$

for every $\mathbf{x} \in l^2$. On the other hand,

$$\begin{aligned}
&2\langle(\mathbb{A}^*_{(\theta,n)} - \mathrm{Id})^{-1}\mathbb{H}_{(\theta,n)}(\mathbb{A}_{(\theta,n)} - \mathrm{Id})^{-1}\mathbf{x}, \mathbf{x}\rangle \\
&= 2\langle\mathbb{H}_{(\theta,n)}(\mathbb{A}_{(\theta,n)} - \mathrm{Id})^{-1}\mathbf{x}, (\mathbb{A}_{(\theta,n)} - \mathrm{Id})^{-1}\mathbf{x}\rangle \\
&\ge 2\|(\mathbb{A}_{(\theta,n)} - \mathrm{Id})^{-1}\mathbf{x}\|^2 \\
&\ge 2\frac{\|\mathbf{x}\|^2}{\|\mathrm{Id} - \mathbb{A}_{(\theta,n)}\|^2}.
\end{aligned}$$

Combining the last two estimates, we obtain that

$$2\frac{\|\mathbf{x}\|^2}{\|\mathrm{Id}-\mathbb{A}_{(\theta,n)}\|^2} \le \|\mathbb{W}_{(\theta,n)}\mathbf{x}\| \cdot \|\mathbb{A}_{(\theta,n)}+\mathrm{Id}\| \cdot \|(\mathbb{A}_{(\theta,n)}-\mathrm{Id})^{-1}\| \cdot \|\mathbf{x}\|,$$

and thus

$$\|\mathbf{x}\| \le \frac{1}{2}\|\mathbb{W}_{(\theta,n)}\mathbf{x}\| \cdot \|\mathbb{A}_{(\theta,n)}+\mathrm{Id}\| \cdot \|(\mathbb{A}_{(\theta,n)}-\mathrm{Id})^{-1}\| \cdot \|\mathrm{Id}-\mathbb{A}_{(\theta,n)}\|^2,$$

for $\mathbf{x} \in l^2$. It follows from Lemma 4 that

$$\sup_{(\theta,n)\in\Theta\times\mathbb{Z}} \|(\mathbb{A}_{(\theta,n)}-\mathrm{Id})^{-1}\| < +\infty.$$

Hence, there exist $R > 0$ such that

$$\|\mathbf{x}\| \le R\|\mathbb{W}_{(\theta,n)}\mathbf{x}\| \quad \text{for } \mathbf{x} \in l^2 \text{ and } (\theta,n) \in \Theta \in \mathbb{Z}.$$

Hence,

$$\sup_{(\theta,n)\in\Theta\times\mathbb{Z}} \|\mathbb{W}^{-1}_{(\theta,n)}\| \le R < +\infty,$$

and the proof of the lemma is completed. □

Lemma 6. *For each $(\theta,n) \in \Theta\times\mathbb{Z}$, $S_{(\theta,n)}$ is invertible. Furthermore, the second inequality in (3) holds.*

Proof of the Lemma. Observe that $S_{(\theta,n)} \ge \mathrm{Id}$ and thus $S_{(\theta,n)}$ is injective. Take $v \in X$ and consider $\mathbf{y} = (y_m)_{m\in\mathbb{Z}} \in l^2$ given by $y_0 = v$ and $y_m = 0$ for $m \neq 0$. Since $\mathbb{W}_{(\theta,n)}$ is invertible, there exists $\mathbf{x} = (x_m)_{m\in\mathbb{Z}} \in l^2$ such that $\mathbb{W}_{(\theta,n)}\mathbf{x} = \mathbf{y}$. Hence,

$$v = y_0 = (\mathbb{W}_{(\theta,n)}\mathbf{x})_0 = S_{(\theta,n)}x_0.$$

Hence, $S_{(\theta,n)}$ is also surjective and thus it is invertible. Moreover,

$$\|S^{-1}_{(\theta,n)}v\| = \|x_0\| \le \|\mathbf{x}\| = \|\mathbb{W}^{-1}_{(\theta,n)}\mathbf{y}\| \le \|\mathbb{W}^{-1}_{(\theta,n)}\| \cdot \|\mathbf{y}\| = \|\mathbb{W}^{-1}_{(\theta,n)}\| \cdot \|v\|.$$

Therefore, $\|S^{-1}_{(\theta,n)}\| \le \|\mathbb{W}^{-1}_{(\theta,n)}\|$ for all $(\theta,n) \in \Theta\times\mathbb{Z}$. Now the second inequality in (3) follows directly from the previous lemma. □

It remains to establish (5). Using the same notation as in the proof of Lemma 5 we have

$$\begin{aligned}&-2\mathbb{W}^{-1}_{(\theta,n)}(\mathbb{A}^*_{(\theta,n)}-\mathrm{Id})^{-1}\mathbb{H}_{(\theta,n)}(\mathbb{A}_{(\theta,n)}-\mathrm{Id})^{-1}\mathbb{W}^{-1}_{(\theta,n)}\\ &= (\mathbb{A}_{(\theta,n)}+\mathrm{Id})(\mathbb{A}_{(\theta,n)}-\mathrm{Id})^{-1}\mathbb{W}^{-1}_{(\theta,n)}\\ &\quad + \mathbb{W}^{-1}_{(\theta,n)}(\mathbb{A}^*_{(\theta,n)}-\mathrm{Id})^{-1}(\mathbb{A}^*_{(\theta,n)}+\mathrm{Id}).\end{aligned}$$

Moreover, multiplying this equality on the left by $\mathbb{A}_{(\theta,n)}-\mathrm{Id}$ and on the right by $\mathbb{A}^*_{(\theta,n)}-\mathrm{Id}$ yields that

$$\begin{aligned}&-2(\mathbb{A}_{(\theta,n)}-\mathrm{Id})\mathbb{W}^{-1}_{(\theta,n)}(\mathbb{A}^*_{(\theta,n)}-\mathrm{Id})^{-1}\mathbb{H}_{(\theta,n)}(\mathbb{A}_{(\theta,n)}-\mathrm{Id})^{-1}\mathbb{W}^{-1}_{(\theta,n)}(\mathbb{A}^*_{(\theta,n)}-\mathrm{Id})\\ &= (\mathbb{A}_{(\theta,n)}+\mathrm{Id})\mathbb{W}^{-1}_{(\theta,n)}(\mathbb{A}^*_{(\theta,n)}-\mathrm{Id}) + (\mathbb{A}_{(\theta,n)}-\mathrm{Id})\mathbb{W}^{-1}_{(\theta,n)}(\mathbb{A}^*_{(\theta,n)}+\mathrm{Id})\\ &= 2\mathbb{A}_{(\theta,n)}\mathbb{W}^{-1}_{(\theta,n)}\mathbb{A}^*_{(\theta,n)} - 2\mathbb{W}^{-1}_{(\theta,n)}.\end{aligned}$$

Hence,

$$
\begin{aligned}
&-(\mathbb{A}_{(\theta,n)}-\mathrm{Id})\mathbb{W}_{(\theta,n)}^{-1}(\mathbb{A}_{(\theta,n)}^*-\mathrm{Id})^{-1}\mathbb{H}_{(\theta,n)}(\mathbb{A}_{(\theta,n)}-\mathrm{Id})^{-1}\mathbb{W}_{(\theta,n)}^{-1}(\mathbb{A}_{(\theta,n)}^*-\mathrm{Id})\\
&=\mathbb{A}_{(\theta,n)}\mathbb{W}_{(\theta,n)}^{-1}\mathbb{A}_{(\theta,n)}^*-\mathbb{W}_{(\theta,n)}^{-1}.
\end{aligned}
$$

Observe that for each $\mathbf{x}\in l^2$, we have that

$$
\begin{aligned}
&\langle(\mathbb{A}_{(\theta,n)}-\mathrm{Id})\mathbb{W}_{(\theta,n)}^{-1}(\mathbb{A}_{(\theta,n)}^*-\mathrm{Id})^{-1}\mathbb{H}_{(\theta,n)}(\mathbb{A}_{(\theta,n)}-\mathrm{Id})^{-1}\mathbb{W}_{(\theta,n)}^{-1}(\mathbb{A}_{(\theta,n)}^*-\mathrm{Id})\mathbf{x},\mathbf{x}\rangle\\
&\geq\|(\mathbb{A}_{(\theta,n)}-\mathrm{Id})^{-1}\mathbb{W}_{(\theta,n)}^{-1}(\mathbb{A}_{(\theta,n)}^*-\mathrm{Id})\mathbf{x}\|^2\\
&\geq\frac{\|\mathbf{x}\|^2}{\|\mathbb{A}_{(\theta,n)}-\mathrm{Id}\|\cdot\|\mathbb{W}_{(\theta,n)}\|\cdot\|(\mathbb{A}_{(\theta,n)}^*-\mathrm{Id})^{-1}\|}.
\end{aligned}
$$

Since there exists $L>0$ such that

$$\|\mathbb{A}_{(\theta,n)}-\mathrm{Id}\|\cdot\|\mathbb{W}_{(\theta,n)}\|\cdot\|(\mathbb{A}_{(\theta,n)}^*-\mathrm{Id})^{-1}\|\leq L,$$

for $(\theta,n)\in\Theta\times\mathbb{Z}$, we conclude that

$$\langle\mathbb{A}_{(\theta,n)}\mathbb{W}_{(\theta,n)}^{-1}\mathbb{A}_{(\theta,n)}^*\mathbf{x},\mathbf{x}\rangle-\langle\mathbb{W}_{(\theta,n)}^{-1}\mathbf{x},\mathbf{x}\rangle\leq-\frac{1}{L}\langle\mathbf{x},\mathbf{x}\rangle, \tag{7}$$

for every $\mathbf{x}\in l^2$. By applying (7) for $\mathbf{x}=(x_m)_{m\in\mathbb{Z}}\in l^2$ given by $x_m=0$ for $m\neq1$ and $x_1=v$, where $v\in X$ is arbitrary, we conclude that (5) holds with $\delta=\frac{1}{L}>0$. □

We now establish the converse of Theorem 1.

Theorem 2. *Assume that $\Phi\colon\Theta\times\Delta\to B(X)$ is an linear skew-product three-parameter semiflow over a continuous three-parameter flow σ such that*

$$\sup_{(\theta,n)\in\Theta\times\mathbb{Z}}\|\Phi(\theta,n+1,n)\|<+\infty. \tag{8}$$

Furthermore, suppose that there exists a family $S_{(\theta,n)}$, $(\theta,n)\in\Theta\times\mathbb{Z}$ of bounded, self-adjoint and invertible operators on X and $K,\delta>0$ such that $S_{(\theta,n)}\geq0$ and that (3)–(5) hold for each $(\theta,n)\in\Theta\times\mathbb{Z}$. Then, Φ is exponentially stable.

Proof. For $(\theta,n)\in\Theta\times\mathbb{Z}$, let $\mathbb{A}_{(\theta,n)}$ and $\mathbb{W}_{(\theta,n)}$ are as in the proof of Theorem 1. Please note that (8) implies that $\mathbb{A}_{(\theta,n)}$ is a well-defined and bounded linear operator. Furthermore, observe that (4) and (5) imply that

$$\mathbb{A}_{(\theta,n)}^*\mathbb{W}_{(\theta,n)}\mathbb{A}_{(\theta,n)}-\mathbb{W}_{(\theta,n)}\leq-\delta\mathrm{Id}\quad\text{on } l^2,$$

and

$$\mathbb{A}_{(\theta,n)}\mathbb{W}_{(\theta,n)}^{-1}\mathbb{A}_{(\theta,n)}^*-\mathbb{W}_{(\theta,n)}^{-1}\leq-\delta\mathrm{Id}\quad\text{on } l^2.$$

Since $S_{(\theta,n)}\geq0$ on X for $(\theta,n)\in\Theta\times\mathbb{Z}$, we have that $\mathbb{W}_{(\theta,n)}\geq0$ and l^2 for each $(\theta,n)\in\Theta\times\mathbb{Z}$. Consequently, Lemma 2 implies that the spectrum of $\mathbb{A}_{(\theta,n)}$ is contained in $\{z\in\mathbb{C}:|z|<1\}$, for every $(\theta,n)\in\Theta\times\mathbb{Z}$.

Lemma 7. *We have that*

$$\sup_{(\theta,n)\in\Theta\times\mathbb{Z}}\|(\mathrm{Id}-\mathbb{A}_{(\theta,n)})^{-1}\|<+\infty.$$

Proof of the Lemma. By repeating the arguments in the first part of the proof of Lemma 5 that

$$\delta\|(\mathbb{A}_{(\theta,n)} - \mathrm{Id})^{-1}\mathbf{x}\|^2 \leq \|\mathbb{W}_{(\theta,n)}\| \cdot \|\mathbb{A}_{(\theta,n)} + \mathrm{Id}\| \cdot \|(\mathrm{Id} - \mathbb{A}_{(\theta,n)})^{-1}\mathbf{x}\| \cdot \|\mathbf{x}\|,$$

for $(\theta, n) \in \Theta \times \mathbb{Z}$ and $\mathbf{x} \in l^2$. On the other hand, (3) and (8) imply that

$$\sup_{(\theta,n)\in\Theta\times\mathbb{Z}} (\|\mathbb{W}_{(\theta,n)}\| \cdot \|\mathbb{A}_{(\theta,n)} + \mathrm{Id}\|) < +\infty.$$

The conclusion of the lemma now readily follows. □

Take now $(\theta, n) \in \Theta \times \mathbb{Z}$, $v \in X$ and consider a sequence $\mathbf{y} = (y_m)_{m\in\mathbb{Z}}$ by

$$y_m = \begin{cases} v & \text{if } m = 0, \\ 0 & \text{if } m \neq 0. \end{cases}$$

Set $\mathbf{x} = (\mathrm{Id} - \mathbb{A}_{(\theta,n)})^{-1}\mathbf{y} \in l^2$. It is easy to verify that

$$x_m = \begin{cases} 0 & \text{if } m < 0, \\ \Phi(\theta, n+m, n)v & \text{if } m \geq 0. \end{cases}$$

Then, Lemma 7 implies that there exist $C > 0$ such that

$$\left(\sum_{k\geq n} \|\Phi(\theta,k,n)v\|^2\right)^{1/2} \leq C\|v\|, \quad \text{for } (\theta, n) \in \Theta \times \mathbb{Z} \text{ and } v \in X. \tag{9}$$

In particular, (9) implies that

$$\|\Phi(\theta,k,n)v\| \leq C\|v\|, \quad \text{for } (\theta, n) \in \Theta \times \mathbb{Z},\, k \geq n \text{ and } v \in X. \tag{10}$$

Take now $\theta \in \Theta$, $v \in X$ and $m \geq n$. Then, for each $n \leq k \leq m$ we have that

$$\|\Phi(\theta,m,n)v\|^2 = \|\Phi(\sigma(\theta,k,n),m,k)\Phi(\theta,k,n)v\|^2 \leq C^2\|\Phi(\theta,k,n)v\|^2.$$

Summing over k and using (9), we obtain that

$$(m-n+1)\|\Phi(\theta,m,n)v\|^2 \leq C^2 \sum_{k\geq n} \|\Phi(\theta,k,n)v\|^2 \leq C^4\|v\|^2.$$

Thus,

$$\|\Phi(\theta,m,n)\| \leq \frac{C^2}{\sqrt{m-n+1}}.$$

Consequently, there exist $N_0 \in \mathbb{N}$ such that

$$\|\Phi(\theta,m,n)\| \leq e^{-1}, \quad \text{for } \theta \in \Theta \text{ and } m, n \in \mathbb{Z} \text{ such that } m - n \geq N_0. \tag{11}$$

Now, (10) and (11) easily imply that Φ is exponentially stable. □

4. Applications

In this section, we use Theorems 1 and 2 to prove that the notion of exponential stability persists under sufficiently small linear perturbations.

Theorem 3. *Assume that $\Phi, \Psi \colon \Theta \times \Delta \to B(X)$ are two linear skew-product three-parameter semiflows over a continuous three-parameter flow σ. Furthermore, suppose that:*

1. *Φ is exponentially stable;*
2. *there exists $c > 0$ such that*

$$\sup_{(\theta,n)\in\Theta\times\mathbb{Z}} \|\Phi(\theta, n+1, n) - \Psi(\theta, n+1, n)\| \le c. \tag{12}$$

Then, if c is sufficiently small, Ψ is also exponentially stable.

Proof. We first observe that since Φ is exponentially stable, (12) implies that

$$\sup_{(\theta,n)\in\Theta\times\mathbb{Z}} \|\Psi(\theta, n+1, n)\| < +\infty.$$

Let $S_{(\theta,n)}$, $(\theta, n) \in \Theta \times \mathbb{Z}$, $K, \delta > 0$ be given by Theorem 1. For each $(\theta, n) \in \Theta \times \mathbb{Z}$ and $v \in X$, (4) implies that

$$\begin{aligned}
&\langle \bar{\Psi}(\theta,n)^* S_{\bar{\sigma}(\theta,n)} \bar{\Psi}(\theta,n)v, v\rangle - \langle S_{(\theta,n)}v, v\rangle \\
&= \langle (\bar{\Psi}(\theta,n) - \bar{\Phi}(\theta,n))^* S_{\bar{\sigma}(\theta,n)} (\bar{\Psi}(\theta,n) - \bar{\Phi}(\theta,n))v, v\rangle \\
&\quad + \langle (\bar{\Psi}(\theta,n) - \bar{\Phi}(\theta,n))^* S_{\bar{\sigma}(\theta,n)} \bar{\Phi}(\theta,n)v, v\rangle \\
&\quad + \langle \bar{\Phi}(\theta,n)^* S_{\bar{\sigma}(\theta,n)} (\bar{\Psi}(\theta,n) - \bar{\Phi}(\theta,n))v, v\rangle \\
&\quad + \langle \bar{\Phi}(\theta,n)^* S_{\bar{\sigma}(\theta,n)} \bar{\Phi}(\theta,n)v, v\rangle - \langle S_{(\theta,n)}v, v\rangle.
\end{aligned}$$

It follows from (1), (3), (4) and (12) that

$$\begin{aligned}
&\langle \bar{\Psi}(\theta,n)^* S_{\bar{\sigma}(\theta,n)} \bar{\Psi}(\theta,n)v, v\rangle - \langle S_{(\theta,n)}v, v\rangle \\
&\le -\delta\langle v, v\rangle + c^2 K\langle v, v\rangle + 2DcK\langle v, v\rangle \\
&= -(\delta - c^2 K - 2DcK)\langle v, v\rangle,
\end{aligned}$$

for $v \in X$. We conclude that

$$\bar{\Psi}(\theta,n)^* S_{\bar{\sigma}(\theta,n)} \bar{\Psi}(\theta,n) - S_{(\theta,n)} \le -\tilde{\delta}\mathrm{Id},$$

where $\tilde{\delta} = \delta - c^2K - 2DcK$. Observe that $\tilde{\delta} > 0$ if c is sufficiently small. Similarly, one can prove that there exists $\tilde{\delta}' > 0$ such that

$$\bar{\Psi}(\theta,n) S^{-1}_{(\theta,n)} \bar{\Psi}(\theta,n)^* - S^{-1}_{\bar{\sigma}(\theta,n)} \le -\tilde{\delta}'\mathrm{Id},$$

for every $(\theta, n) \in \Theta \times \mathbb{Z}$. Putting all this together, Theorem 2 implies that Ψ is exponentially stable and the proof of the theorem is completed. □

5. Conclusions

In this paper, we obtained a complete Lyapunov-type characterization of exponential stability for linear skew-product three-parameter semiflows with discrete time. More precisely, we proved that exponential stability can be described in terms of the existence of appropriate quadratic Lyapunov functions. We then applied these results and prove that the notion of exponential stability persists under sufficiently small linear perturbations.

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