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# A Note on Fernández–Coniglio’s Hierarchy of Paraconsistent Systems

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**Abstract:** A logic is called explosive if its consequence relation validates the so-called principle of ex contradictione sequitur quodlibet. A logic is called paraconsistent so long as it is not explosive. Sette’s calculus  $P^1$  is widely recognized as one of the most important paraconsistent calculi. It is not surprising then that the calculus was a starting point for many research studies on paraconsistency. Fernández–Coniglio’s hierarchy of paraconsistent systems is a good example of such an approach. The hierarchy is presented in Newton da Costa’s style. Therefore, the law of non-contradiction plays the main role in its negative axioms. The principle of ex contradictione sequitur quodlibet has been marginalized: it does not play any leading role in the hierarchy. The objective of this paper is to present an alternative axiomatization for the hierarchy. The main idea behind it is to focus explicitly on the (in)validity of the principle of ex contradictione sequitur quodlibet. This makes the hierarchy less complex and more transparent, especially from the viewpoint of paraconsistency.

**Keywords:** paraconsistent logic; paraconsistency; Sette’s calculus; the law of explosion; the principle of ex contradictione sequitur quodlibet

## 1. Introduction

Let  $var$  denote a (non-empty) denumerable set of all propositional variables. The set of formulas  $\mathcal{F}$  is inductively defined in the following way:

$$\varphi ::= p \mid \sim\alpha \mid \alpha \rightarrow \alpha$$

where  $p \in var$ ,  $\alpha \in \mathcal{F}$  and the symbols  $\sim$ ,  $\rightarrow$  denote negation and implication, respectively. A logic is a pair  $\langle \mathcal{L}, \vdash \rangle$  consisting of a sentential language  $\mathcal{L}$  and a consequence relation  $\vdash$  defined on the (non-empty) set of formulas  $\mathcal{F}$ . A logic is called explosive if its consequence relation validates the principle of ex contradictione sequitur quodlibet, i.e.,  $\{\alpha, \sim\alpha\} \vdash \beta$ , for any formulas  $\alpha, \beta$ . “Paraconsistent logic is defined negatively: any logic is paraconsistent as long as it is not explosive” (cit.per [1]), or, to be more precise,

**Definition 1.** A logic  $\langle \mathcal{L}, \vdash \rangle$  is said to be paraconsistent if  $\{\alpha, \sim\alpha\} \not\vdash \beta$ , for some formulas  $\alpha, \beta$ .

Already at first glance, it is striking that the definition is very broad as it includes some logics that have potentially nothing in common with paraconsistency (cf. [2], p. 19). Nonetheless, the definition reveals a tendency to view paraconsistent logic through the lens of negation understood as a connective symbol rather than a truth-function (cf. [3]. For a more extensive discussion on the paraconsistency, see, e.g., [4–6].)

In the early 1970s of the Twentieth Century, Sette published a paper devoted to one of the most remarkable paraconsistent calculi. The calculus, denoted as  $P^1$ , has some unusual properties: it behaves

in a paraconsistent way only at the level of propositional variables, that is a pair of the formulas  $\alpha$  and  $\sim \alpha$  yields any  $\beta$  if, and only if the formula  $\alpha$  is not a propositional variable.

The calculus  $P^1$  is axiomatized by the following axiom schemas:

- (A1)  $\alpha \rightarrow (\beta \rightarrow \alpha)$
- (A2)  $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$
- (A3)  $(\sim \alpha \rightarrow \sim \beta) \rightarrow ((\sim \alpha \rightarrow \sim \sim \beta) \rightarrow \alpha)$
- (A4)  $\sim(\alpha \rightarrow \sim \sim \alpha) \rightarrow \alpha$
- (A5)  $(\alpha \rightarrow \beta) \rightarrow \sim \sim(\alpha \rightarrow \beta)$

and the rule of detachment (MP)  $\alpha \rightarrow \beta, \alpha / \beta$ .

The connectives of  $\sim$  and  $\rightarrow$  are taken here as primitives. As for the other connectives such as the conjunction, disjunction, and equivalence, they are introduced via the definitions ([7], pp. 178–179):

$$\begin{aligned}\alpha \wedge \beta &=_{df} (((\alpha \rightarrow \alpha) \rightarrow \alpha) \rightarrow \sim((\beta \rightarrow \beta) \rightarrow \beta)) \rightarrow \sim(\alpha \rightarrow \sim \beta) \\ \alpha \vee \beta &=_{df} (\alpha \rightarrow \sim \sim \alpha) \rightarrow (\sim \alpha \rightarrow \beta) \\ \alpha \leftrightarrow \beta &=_{df} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha).\end{aligned}$$

The definitions are complex and often too awkward to handle. More user-friendly definitions are given in [8] (pp. 8–9 of the preprint) and [9] (p. 59):

$$\begin{aligned}\alpha \wedge \beta &=_{df} \sim(\alpha \rightarrow \sim(\sim \beta \rightarrow \beta)) \\ \alpha \vee \beta &=_{df} \sim(\sim \alpha \rightarrow \alpha) \rightarrow \beta \\ \alpha \leftrightarrow \beta &=_{df} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha).\end{aligned}$$

It is noteworthy that the disjunction connective can be also defined as in the three-valued Lukasiewicz logic, namely,  $\alpha \vee \beta =_{df} (\alpha \rightarrow \beta) \rightarrow \beta$  (cf. [10], Section 2).

Many important theorems, which hold in the classical propositional calculus, can be proven for Sette's system, too. Below we recall some of them needed for our further discussion.

**Theorem 1.** *The deduction theorem holds for  $P^1$ .*

**Proof.** It is enough to observe that  $P^1$  includes (A1), (A2), and the sole rule of inference in  $P^1$  is (MP).  $\square$

**Theorem 2.** *For every  $\Gamma, \Delta \subseteq \mathcal{F}$  and  $\alpha, \beta, \gamma \in \mathcal{F}$ , we have:*

1. *if  $\alpha \in \Gamma$ , then  $\Gamma \vdash_{P^1} \alpha$ ,*
2. *if  $\Gamma \subseteq \Delta$  and  $\Gamma \vdash_{P^1} \alpha$ , then  $\Delta \vdash_{P^1} \alpha$ ,*
3. *if  $\Delta \vdash_{P^1} \alpha$  and, for every  $\beta \in \Delta$  it is true that  $\Gamma \vdash_{P^1} \beta$ , then  $\Gamma \vdash_{P^1} \alpha$ ,*
4. *if  $\Gamma \cup \{\alpha\} \vdash_{P^1} \gamma$  and  $\Delta \vdash_{P^1} \alpha$ , then  $\Gamma \cup \Delta \vdash_{P^1} \gamma$*   
*(in particular, if  $\Gamma \cup \{\alpha\} \vdash_{P^1} \gamma$  and  $\emptyset \vdash_{P^1} \alpha$ , then  $\Gamma \vdash_{P^1} \gamma$ ),*
5.  *$\Gamma \vdash_{P^1} \alpha$  iff for some finite  $\Delta \subseteq \Gamma$ ,  $\Delta \vdash_{P^1} \alpha$ .*

**Proof.** The proof proceeds analogously to that of the classical propositional calculus. We refer the reader to [11,12] for details.  $\square$

**Theorem 3.** *Some (weaker) variants of the indirect deduction theorem hold for  $P^1$ , viz.:*

1. *if  $\Gamma, \alpha \vdash_{P^1} \{\sim \beta, \sim \sim \beta\}$ , then  $\Gamma \vdash_{P^1} \sim \alpha$ ,*
2. *if  $\Gamma, \sim \alpha \vdash_{P^1} \{\sim \beta, \sim \sim \beta\}$ , then  $\Gamma \vdash_{P^1} \alpha$ ,*

3. if  $\Gamma, \alpha \rightarrow \beta \vdash_{P^1} \{\gamma \rightarrow \delta, \sim(\gamma \rightarrow \delta)\}$ , then  $\Gamma \vdash_{P^1} \sim(\alpha \rightarrow \beta)$ ,
4. if  $\Gamma, \sim(\alpha \rightarrow \beta) \vdash_{P^1} \{\gamma \rightarrow \delta, \sim(\gamma \rightarrow \delta)\}$ , then  $\Gamma \vdash_{P^1} \alpha \rightarrow \beta$ ,

for every  $\Gamma \subseteq \mathcal{F}$  and  $\alpha, \beta, \gamma, \delta \in \mathcal{F}$ . Note that the notation  $\Gamma \vdash \{\phi, \psi\}$  is an abbreviation of ' $\Gamma \vdash \{\phi\}$  and  $\Gamma \vdash \{\psi\}$ '.

Sette's calculus is sound and complete with respect to the matrix  $\mathcal{M}_{P^1} = \langle \{T_0, T_1, F\}, \{T_0, T_1\}, \sim, \rightarrow \rangle$ , where  $\{T_0, T_1, F\}$  and  $\{T_0, T_1\}$  are the sets of logical and designated values, respectively. The connectives of  $\rightarrow$  and  $\sim$  are defined by the truth tables:

$\rightarrow$	$T_0$	$T_1$	$F$	$\sim$	
$T_0$	$T_0$	$T_0$	$F$	$T_0$	$F$
$T_1$	$T_0$	$T_0$	$F$	$T_1$	$T_0$
$F$	$T_0$	$T_0$	$T_0$	$F$	$T_0$

A  $P^1$ -valuation is any function  $v$  from the set of formulas to the set of logical values ( $v : \mathcal{F} \rightarrow \{T_0, T_1, F\}$ , in symbols) compatible with the above truth-tables (see [7], pp. 176–178). A  $P^1$ -tautology is a formula that under every valuation  $v$  takes on the designated values  $\{T_0, T_1\}$ .

The logical meaning of the  $P^1$ -valuation is clear, but it was never stated in [7] how to interpret philosophically the three-valued semantics. This gave an impulse for further research, and several new semantics for the calculus were proposed (see, e.g., [8,13–16]). Notice that the principle of ex contradictione sequitur quodlibet does not play any significant part in  $P^1$ . Metaphorically speaking, paraconsistency is hidden somewhere between the lines of Sette's paper. Only at one point in his whole paper does Sette refer to paraconsistency: "(...) N.C.A da Costa presents a hierarchy  $C^n$  ( $1 < n < \omega$ ) of propositional calculi which can be used as subjacent propositional logics for inconsistent (but not absolutely inconsistent) formal systems. The purpose of this note is to present a new propositional calculus  $P^1$  which can be used as subjacent logic for inconsistent (but not absolutely inconsistent) formal systems (...)" ([7], p. 173.). In [17], we proposed an alternative axiomatization for  $P^1$ . The idea behind it was to focus explicitly on the (in)validity of ex contradictione sequitur quodlibet, or equivalently, the so-called law of explosion (DS)  $\alpha \rightarrow (\sim\alpha \rightarrow \beta)$ . This concept is directly reflected below in the axiomatization.

**Remark 1.** The calculus  $P^1$  can be axiomatized by the set of formulas:

- (A1)  $\alpha \rightarrow (\beta \rightarrow \alpha)$
- (A2)  $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$
- (PL)  $((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha$
- (DS $\sim$ )  $\sim\alpha \rightarrow (\sim\sim\alpha \rightarrow \beta)$
- (DS $\rightarrow$ )  $(\alpha \rightarrow \beta) \rightarrow (\sim(\alpha \rightarrow \beta) \rightarrow \gamma)$
- (CM)  $(\sim\alpha \rightarrow \alpha) \rightarrow \alpha$

with (MP) as the only primitive rule (see [17], for details).

In [8], an interesting hierarchy of the paraconsistent calculi starting from  $P^1$  was proposed. It is based on a language more expressive than that which was given in Remark 1 and used by Sette. The hierarchy is obtained from the system  $C_\omega$  of Newton da Costa, i.e.,

- (A1)  $\alpha \rightarrow (\beta \rightarrow \alpha)$
- (A2)  $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$
- (A3)  $(\alpha \wedge \beta) \rightarrow \alpha$
- (A4)  $(\alpha \wedge \beta) \rightarrow \beta$
- (A5)  $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$
- (A6)  $\alpha \rightarrow (\alpha \vee \beta)$
- (A7)  $\beta \rightarrow (\alpha \vee \beta)$
- (A8)  $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \vee \beta \rightarrow \gamma))$

$$\begin{aligned}
 (NN) \quad & \sim\sim\alpha \rightarrow \alpha \\
 (ExM) \quad & \alpha \vee \sim\alpha \\
 (MP) \quad & \alpha \rightarrow \beta, \alpha / \beta, \text{ (See [18], p. 501.)}
 \end{aligned}$$

by adding to it

$$\begin{aligned}
 (dC) \quad & \sim(\beta \wedge \sim\beta) \rightarrow ((\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \sim\beta) \rightarrow \sim\alpha)) \\
 (nC^\dagger) \quad & \sim((\alpha \dagger \beta) \wedge \sim(\alpha \dagger \beta)), \text{ where } \dagger \in \{\wedge, \vee, \rightarrow\} \\
 (nC^{\sim n}) \quad & \sim(\sim^n \alpha \wedge \sim^{n+1} \alpha), \text{ where } n \in \mathbb{N} \text{ and } \sim^n \alpha \text{ denotes } \overbrace{\sim\sim \dots \sim}^{n\text{-times}} \alpha,
 \end{aligned}$$

as new axiom schemas. Obviously, if  $n = 0$ , then  $P^0$  is the classical propositional calculus; if  $n = 1$ , then  $P^1$  is Sette's system (see [8], p. 9 of the preprint, for details). For a positive integer  $n$ , let  $P^n$  denote the calculus of the Fernández–Coniglio's hierarchy ( $P^n$ -hierarchy), henceforth.

Fernández and Coniglio proposed both a matrix and the so-called society semantics for the  $P^n$ -calculi. The former may be viewed as a generalization of  $\mathcal{M}_{P^1}$  given by Sette in [7] (see p. 176), and da Costa in [18] (see, p. 499), that is,

$$\mathcal{M}_{P^n} = \langle X, D, \sim, \rightarrow \rangle,$$

where  $X = \{T_0, T_1, T_2, \dots, T_n, F\}$  and  $D = X - \{F\} = \{T_0, T_1, T_2, \dots, T_n\}$ ,  $n \in \mathbb{N}$ , are the sets of logical and designated values, respectively. The connectives of  $\rightarrow$  and  $\sim$  are defined in the following way ( $i, k \in \mathbb{N}, i \leq n$ ):

$\rightarrow$	$T_0$	$T_i$	$F$	$\sim$	
$T_0$	$T_0$	$T_0$	$F$	$T_0$	$F$
$T_k$	$T_0$	$T_0$	$F$	$T_k$	$T_{k-1}$
$F$	$T_0$	$T_0$	$T_0$	$F$	$T_0$

A  $P^n$ -valuation is any function  $v : \mathcal{F} \rightarrow X$  compatible with the above truth-tables. A  $P^n$ -tautology is a formula that under every valuation  $v$  takes on the designated values.

## 2. A New Axiomatization

The hierarchy discussed in this section is based on different criteria than those used to determine the  $P^n$ -hierarchy. Firstly, we assume that the connectives of conjunction, disjunction, and equivalence are treated as useful abbreviations, which formally do not appear in formulas; whereas  $\sim$  and  $\rightarrow$  will be taken as primitives. Secondly, the law of explosion is assumed to play a crucial role in defining the new hierarchy. The hierarchy will be obtained from that of Remark 1 by replacing (DS) with a more general schema, i.e.,

$$(DS^{\sim n}) \sim^n \alpha \rightarrow (\sim^{n+1} \alpha \rightarrow \beta),$$

where  $n \in \mathbb{N}$  and  $\sim^n \alpha$  is an abbreviation for  $\overbrace{\sim\sim \dots \sim}^n \alpha$ ; and adding to it the law of double negation,  $\sim\sim\alpha \rightarrow \alpha$ , as a new axiom schema. It is worth mentioning at this point that (NN) is provable in  $P^1$  (see [7], pp. 174–175, and [17], p. 271), but it is not in any  $P^m$ , where  $m > 1$ . To put it more precisely, for each  $n \in \mathbb{N}$ , let  $S^n$  result from the implicational fragment of propositional intuitionistic logic by adding to it the following axiom schemas:

$$\begin{aligned}
 (PL) \quad & ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha \\
 (DS^{\sim n}) \quad & \sim^n \alpha \rightarrow (\sim^{n+1} \alpha \rightarrow \beta) \\
 (DS^\rightarrow) \quad & (\alpha \rightarrow \beta) \rightarrow (\sim(\alpha \rightarrow \beta) \rightarrow \gamma) \\
 (CM) \quad & (\sim\alpha \rightarrow \alpha) \rightarrow \alpha \\
 (NN) \quad & \sim\sim\alpha \rightarrow \alpha
 \end{aligned}$$

The other sentential connectives can be introduced by the definitions. Observe that if  $n = 0$ , then  $S^0$  is the classical propositional calculus, and the axioms  $(DS^{\rightarrow})$ ,  $(NN)$  become redundant (cf. [19], p. 437); but if  $n = 1$ , then  $S^1$  is equivalent to Sette's calculus, and  $(NN)$  is provable in  $S^1$  (see [17], p. 268).

**Definition 2.** Let  $\alpha \in \mathcal{F}$  and  $\Gamma \subseteq \mathcal{F}$ . A formula  $\alpha$  is provable from  $\Gamma$  within  $S^n$  ( $\Gamma \vdash_{S^n} \alpha$ , in symbols) iff there is a finite sequence of formulas,  $\beta_1, \beta_2, \dots, \beta_m$ , such that  $\beta_m = \alpha$ , and for each  $i \leq m$ , at least one of the following is true:

1.  $\beta_i \in \Gamma$ ,
2.  $\beta_i$  is an axiom of  $S^n$ ,
3.  $\beta_i$  is obtained from some of the previous  $\beta_j$  by application of the rule of detachment.

**Definition 3.** A formula  $\alpha$  is a thesis of  $S^n$  iff  $\emptyset \vdash_{S^n} \alpha$ .

In what follows, we will need two lemmas to prove the key theorem:

**Lemma 1.** Let  $n \in \mathbb{N}$ . Then:

1. The deduction theorem holds for  $S^n$ .
2. Some variants of the indirect deduction theorem hold for  $S^n$ , viz.:
  - a. if  $\Gamma, \alpha \vdash_{S^n} \{\sim^n \beta, \sim^{n+1} \beta\}$ , then  $\Gamma \vdash_{S^n} \sim \alpha$
  - b. if  $\Gamma, \sim \alpha \vdash_{S^n} \{\sim^n \beta, \sim^{n+1} \beta\}$ , then  $\Gamma \vdash_{S^n} \alpha$
  - c. if  $\Gamma, \alpha \rightarrow \beta \vdash_{S^n} \{\gamma \rightarrow \delta, \sim(\gamma \rightarrow \delta)\}$ , then  $\Gamma \vdash_{S^n} \sim(\alpha \rightarrow \beta)$
  - d. if  $\Gamma, \sim(\alpha \rightarrow \beta) \vdash_{S^n} \{\gamma \rightarrow \delta, \sim(\gamma \rightarrow \delta)\}$ , then  $\Gamma \vdash_{S^n} \alpha \rightarrow \beta$

for every  $\Gamma \subseteq \mathcal{F}$  and  $\alpha, \beta, \gamma, \delta \in \mathcal{F}$ .

**Proof.** 1. The proof is exactly the same as in Theorem 1.

2.a. Assume that  $\Gamma, \alpha \vdash_{S^n} \{\sim^n \beta, \sim^{n+1} \beta\}$ . Then, by the deduction theorem, we have  $\Gamma \vdash_{S^n} \{\alpha \rightarrow \sim^n \beta, \alpha \rightarrow \sim^{n+1} \beta\}$ . Since  $\emptyset \vdash_{S^n} (\alpha \rightarrow \sim^n \beta) \rightarrow ((\alpha \rightarrow \sim^{n+1} \beta) \rightarrow \sim \alpha)$  (to prove this claim, apply the deduction theorem,  $(DS^{\sim^n})$ ,  $(HS)$ ,  $(C)$ ,  $(CM2)$ , and  $(MP)$ ), then  $\{\alpha \rightarrow \sim^n \beta, \alpha \rightarrow \sim^{n+1} \beta\} \vdash_{S^n} \sim \alpha$  by the deduction theorem. The relation  $\vdash_{S^n}$  is transitive, so  $\Gamma \vdash_{S^n} \sim \alpha$ .

2.b. Suppose that  $\Gamma, \sim \alpha \vdash_{S^n} \{\sim^n \beta, \sim^{n+1} \beta\}$ , then  $\Gamma \vdash_{S^n} \sim \sim \alpha$  (by 2.a). Since  $\emptyset \vdash_{S^n} \sim \sim \alpha \rightarrow \alpha$ , thus  $\{\sim \sim \alpha\} \vdash_{S^n} \alpha$ , and consequently,  $\Gamma \vdash_{S^n} \alpha$ .

2.c., 2.d. The proofs are similar to those of 2.a and 2.b.  $\square$

**Lemma 2.** The (schemas of the) formulas:

- (IL)  $\alpha \rightarrow \alpha$
- (LoC)  $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow (\alpha \rightarrow \gamma))$
- (HS)  $(\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma))$
- (C)  $(\alpha \rightarrow (\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow \beta)$
- (LoE)  $((\alpha \rightarrow \beta) \rightarrow \gamma) \rightarrow (\alpha \rightarrow (\beta \rightarrow \gamma))$
- (CM2)  $(\alpha \rightarrow \sim \alpha) \rightarrow \sim \alpha$
- (DD $^{\rightarrow}$ )  $(\sim \phi \rightarrow \psi) \rightarrow ((\sim \phi \rightarrow \sim \psi) \rightarrow \phi)$ , where  $\phi := \alpha \rightarrow \beta$ ,  $\psi := \gamma \rightarrow \delta$

are provable in  $S^n$ ,  $n \in \mathbb{N}$ .

**Proof.** (IL), (LoC), (HS), and (C) immediately follow from the deduction theorem and (MP); (LoE) follows from the deduction theorem, (A1) and (MP); (CM2) from the deduction theorem, (NN) (HS), (CM), and (MP); and finally, (DD $^{\rightarrow}$ ) can be easily obtained by the indirect deduction theorem and (MP).  $\square$

The lemmas will be particularly useful for proving the main result of this section.

**Theorem 4.**  $S^n = P^n$ , where  $n \in \mathbb{N}$ .

**Proof.** The proof is divided into two steps. The first is to demonstrate that each axiom schema of  $S^n$  is a  $P^n$ -tautology, and the rule (MP) preserves validity. This can be easily done with the help of the semantics for  $P^n$ . To illustrate the point, we show that  $(PL)$ ,  $(DS^\rightarrow)$ , and  $(DS^{\sim n})$  are valid in  $\mathcal{M}_{P^n}$ .

$(PL)$ . Suppose that  $((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha$  is not a  $P^n$ -tautology. Thus, there is a  $P^n$ -valuation  $v$  such that  $v(((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha) = F$ . There are two main cases to consider. Either  $v((\alpha \rightarrow \beta) \rightarrow \alpha) = T_0$  and  $v(\alpha) = F$ , or  $v((\alpha \rightarrow \beta) \rightarrow \alpha) = T_n$  and  $v(\alpha) = F$ , where  $n \geq 1$ . Case 1. If  $v((\alpha \rightarrow \beta) \rightarrow \alpha) = T_0$  and  $v(\alpha) = F$ , then, no matter which value is assigned to  $\beta$ ,  $v(\alpha \rightarrow \beta) = T_0$ , and consequently,  $v((\alpha \rightarrow \beta) \rightarrow \alpha) = F$ . But this gives a contradiction since  $v((\alpha \rightarrow \beta) \rightarrow \alpha) = T_0$ . Case 2. It follows from the truth-table for implication that  $v((\alpha \rightarrow \beta) \rightarrow \alpha) = T_0$  or  $v((\alpha \rightarrow \beta) \rightarrow \alpha) = F$ , for any  $\alpha, \beta \in \mathcal{F}$  and every  $P^n$ -valuation  $v$ . So it is not possible that  $v((\alpha \rightarrow \beta) \rightarrow \alpha) = T_n$ , where  $n \neq 0$ . Consequently, there is no  $P^n$ -valuation  $v$  such that  $v(((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha) = F$ , which means that  $(PL)$  is a  $P^n$ -tautology.

$(DS^\rightarrow)$ . Assume that  $(\alpha \rightarrow \beta) \rightarrow (\sim(\alpha \rightarrow \beta) \rightarrow \gamma)$  is not a  $P^n$ -tautology. So there is a  $P^n$ -valuation  $v$  such that  $v((\alpha \rightarrow \beta) \rightarrow (\sim(\alpha \rightarrow \beta) \rightarrow \gamma)) = F$ . Hence, either  $v(\alpha \rightarrow \beta) = T_0$  and  $v(\sim(\alpha \rightarrow \beta) \rightarrow \gamma) = F$ , or  $v(\alpha \rightarrow \beta) = T_n$  and  $v(\sim(\alpha \rightarrow \beta) \rightarrow \gamma) = F$ . The latter is impossible due to the truth table for implication. Therefore, if  $v(\alpha \rightarrow \beta) = T_0$ , then  $v(\sim(\alpha \rightarrow \beta)) = F$ , and consequently,  $v(\sim(\alpha \rightarrow \beta) \rightarrow \gamma) = T_0$ . But this results in a contradiction because  $v(\sim(\alpha \rightarrow \beta) \rightarrow \gamma) = F$ . As a consequence, there is no  $P^n$ -valuation  $v$  such that  $v((\alpha \rightarrow \beta) \rightarrow (\sim(\alpha \rightarrow \beta) \rightarrow \gamma)) = F$ . The formula  $(DS^\rightarrow)$  is a  $P^n$ -tautology.

$(DS^{\sim n})$ . Suppose that  $\sim^n \alpha \rightarrow (\sim^{n+1} \alpha \rightarrow \beta)$  is not a  $P^n$ -tautology, where  $n \geq 1$ . Then, there is a  $P^n$ -valuation  $v$  such that  $v(\sim^n \alpha \rightarrow (\sim^{n+1} \alpha \rightarrow \beta)) = F$ . As a result, either  $v(\sim^n \alpha) = T_0$  and  $v(\sim^{n+1} \alpha \rightarrow \beta) = F$ , or  $v(\sim^n \alpha) = T_{n-1}$  and  $v(\sim^{n+1} \alpha \rightarrow \beta) = F$ . Let  $v(\sim^n \alpha) = T_0$  and  $v(\sim^{n+1} \alpha \rightarrow \beta) = F$ . Hence,  $v(\sim^{n+1} \alpha) = F$  by the truth tables for negation. Since  $v(\sim^{n+1} \alpha) = F$ , then, no matter which value is assigned to  $\beta$ ,  $v(\sim^{n+1} \alpha \rightarrow \beta) = T_0$ . But this entails a contradiction since  $v(\sim^{n+1} \alpha \rightarrow \beta) = F$ . Now, let  $v(\sim^n \alpha) = T_{n-1}$  and  $v(\sim^{n+1} \alpha \rightarrow \beta) = F$ . Consequently, either  $v(\sim^{n+1} \alpha) = T_0$  and  $v(\beta) = F$ , or  $v(\sim^{n+1} \alpha) = T_n$  and  $v(\beta) = F$ . If  $v(\sim^{n+1} \alpha) = T_0$ , then, according to the truth table for negation,  $v(\sim^n \alpha) = F$ . But  $v(\sim^n \alpha) = T_{n-1}$ . On the other hand, if  $v(\sim^{n+1} \alpha) = T_n$ , then  $v(\sim^n \alpha) = T_{n+1}$ . But  $v(\sim^n \alpha) = T_{n-1}$ . Therefore, there is no  $P^n$ -valuation  $v$  such that  $v(\sim^n \alpha \rightarrow (\sim^{n+1} \alpha \rightarrow \beta)) = F$ . The formula  $(DS^{\sim n})$  is a  $P^n$ -tautology.

For the second part of the proof, we have to demonstrate that each axiom schema of  $P^n$  is provable in  $S^n$  and (MP) is its admissible rule, where  $n \in \mathbb{N}$ . To begin with, notice that  $(A1)$ ,  $(A2)$ , and  $(NN)$  are the axiom schemas of  $S^n$ , and (MP) is its sole rule of inference.

$(A3)$ . We show that  $(\alpha \wedge \beta) \rightarrow \alpha$  is a thesis of  $S^n$ , or, to be more precise, that  $\sim(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta)) \rightarrow \alpha$  is provable in  $S^n$ . To see that this claim is true, consider the following sequence of formulas:

1.  $\sim(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta))$  by the deduction theorem,
  2.  $(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta)) \rightarrow (\sim(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta)) \rightarrow \alpha)$  by  $(DS^\rightarrow)$ ,
  3.  $\sim(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta)) \rightarrow ((\alpha \rightarrow \sim(\sim\beta \rightarrow \beta)) \rightarrow \alpha)$  by  $(LoC)$ , 2, (MP),
  4.  $(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta)) \rightarrow \alpha$  by 1, 3, (MP),
  5.  $\alpha$  by  $(PL)$ , 4, (MP),
  6.  $\sim(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta)) \rightarrow \alpha$  by the deduction theorem,
- and finally,
7.  $(\alpha \wedge \beta) \rightarrow \alpha$  by the definition of  $\wedge$ .

$(A4)$ . We prove that  $(\alpha \wedge \beta) \rightarrow \beta$ , i.e.,  $\sim(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta)) \rightarrow \beta$ , is a thesis of  $S^n$ . To see this, consider the sequence of formulas:

1.  $\sim(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta))$  by the deduction theorem,
  - 2.–5. Proceed as in the preceding case,
  6.  $(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta)) \rightarrow (\sim(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta)) \rightarrow (\sim\beta \rightarrow \beta))$  by  $(DS^{\rightarrow})$ ,
  7.  $(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta)) \rightarrow (\sim\beta \rightarrow \beta)$  by  $(LoC)$ , 6, 1, (MP),
  8.  $\alpha \rightarrow (\sim(\sim\beta \rightarrow \beta) \rightarrow (\sim\beta \rightarrow \beta))$  by  $(LoE)$ , 7, (MP),
  9.  $\sim(\sim\beta \rightarrow \beta) \rightarrow (\sim\beta \rightarrow \beta)$  by 5, 8, (MP),
  10.  $\sim\beta \rightarrow \beta$  by  $(CM)$ , 9, (MP),
  11.  $\beta$  by  $(CM)$ , 10, (MP),
  12.  $\sim(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta)) \rightarrow \beta$  by the deduction theorem,
- and consequently,
13.  $(\alpha \wedge \beta) \rightarrow \beta$  by the definition of  $\wedge$ .

(A5). We show that  $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$ , i.e.,  $\alpha \rightarrow (\beta \rightarrow \sim(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta)))$ , is provable in  $S^n$ . Consider the sequence of formulas:

1.  $\alpha$ ,
  2.  $\beta$ ,
  3.  $\sim\sim(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta))$  by the indirect deduction theorem,
  4.  $\alpha \rightarrow \sim(\sim\beta \rightarrow \beta)$  by  $(NN)$ , 3, (MP),
  5.  $\sim(\sim\beta \rightarrow \beta)$  by 1, 4, (MP),
  6.  $\sim\beta \rightarrow \beta$  by  $(A1)$ , 2, (MP),
- a contradiction (5, 6). This entails that:
7.  $\sim(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta))$ ,
  8.  $\alpha \rightarrow (\beta \rightarrow \sim(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta)))$  by the deduction theorem 1, 2, 7, (MP),
- and finally,
9.  $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$  by the definition of  $\wedge$ .

(A6). We demonstrate that  $\alpha \rightarrow (\alpha \vee \beta)$ , i.e.,  $\alpha \rightarrow (\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta)$ , is a thesis of  $S^n$ . To see that this claim holds, consider the sequence of formulas:

1.  $\alpha$ ,
  2.  $\sim(\sim\alpha \rightarrow \alpha)$  by the deduction theorem,
  3.  $\sim\alpha \rightarrow \alpha$  by  $(A1)$ , 1, (MP),
  4.  $(\sim\alpha \rightarrow \alpha) \rightarrow (\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta)$  by  $(DS^{\rightarrow})$ ,
  5.  $\beta$  by 4, 3, 2, (MP),
  6.  $\alpha \rightarrow (\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta)$  by the deduction theorem,
- and consequently,
7.  $\alpha \rightarrow (\alpha \vee \beta)$  by the definition of  $\vee$ .

(A7). We show that  $\beta \rightarrow (\alpha \vee \beta)$ , i.e.,  $\beta \rightarrow (\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta)$ , is provable in  $S^n$ . To see this, consider the sequence of formulas:

1.  $\beta$ ,
  2.  $\sim(\sim\alpha \rightarrow \alpha)$  by the deduction theorem,
  3.  $\beta$  by 1,
  4.  $\beta \rightarrow (\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta)$  by the deduction theorem,
- and finally,
5.  $\beta \rightarrow (\alpha \vee \beta)$  by the definition of  $\vee$ .

(A8). We prove that  $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \vee \beta \rightarrow \gamma))$ , i.e.,  $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta) \rightarrow \gamma))$ , is a thesis of  $S^n$ . To see that this claim is true, consider the following sequence of formulas:

1.  $\alpha \rightarrow \gamma$ ,
2.  $\beta \rightarrow \gamma$ ,
3.  $\sim((\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta) \rightarrow \gamma)$  by the indirect deduction theorem,

Let  $\phi := (\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta) \rightarrow \gamma$ . Then,

4.  $\phi \rightarrow (\sim\phi \rightarrow (\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta))$  by  $(DS^{\rightarrow})$ ,
5.  $\phi \rightarrow (\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta)$  by  $(LoC)$ , 4, 3, (MP),
6.  $(\phi \rightarrow (\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta)) \rightarrow (\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta)$  by  $(PL)$ ,
7.  $\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta$  by 5, 6, (MP),
8.  $\phi \rightarrow (\sim\phi \rightarrow \sim(\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta))$  by  $(DS^{\rightarrow})$ ,
9.  $\phi \rightarrow \sim(\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta)$  by  $(LoC)$ , 8, 3, (MP).

If  $\phi := (\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta) \rightarrow \gamma$ , then,

10.  $((\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta) \rightarrow \gamma) \rightarrow \sim(\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta)$ ,
11.  $(\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta) \rightarrow (\gamma \rightarrow \sim(\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta))$  by  $(LoE)$ , 10, (MP),
12.  $\gamma \rightarrow \sim(\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta)$  by 11, 7, (MP),
13.  $\beta \rightarrow \sim(\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta)$  by  $(HS)$ , 2, 12, (MP),
14.  $\sim(\sim\alpha \rightarrow \alpha) \rightarrow \sim(\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta)$  by  $(HS)$ , 7, 13, (MP),
15.  $\beta \rightarrow (\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta)$  by  $(A1)$ ,
16.  $\sim(\sim\alpha \rightarrow \alpha) \rightarrow (\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta)$  by  $(HS)$ , 7, 15, (MP),

Let  $\chi := \sim\alpha \rightarrow \alpha$  and  $\psi := \sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta$ , then,

17.  $(\sim\chi \rightarrow \psi) \rightarrow ((\sim\chi \rightarrow \sim\psi) \rightarrow \chi)$  by  $(DD^{\rightarrow})$ ,
18.  $(\sim\chi \rightarrow \sim\psi) \rightarrow \chi$  by 17, 16, (MP),
19.  $\chi$  by 18, 14, (MP).

If  $\chi := \sim\alpha \rightarrow \alpha$ , then,

20.  $\sim\alpha \rightarrow \alpha$ ,
21.  $\alpha$  by  $(CM)$ , 20, (MP),
22.  $\gamma$  by 21, 1, (MP),
23.  $\gamma \rightarrow ((\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta) \rightarrow \gamma)$  by  $(A1)$ ,
24.  $(\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta) \rightarrow \gamma$  by 23, 22, (MP),

a contradiction (3, 24). This yields that:

25.  $(\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta) \rightarrow \gamma$ ,
26.  $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta) \rightarrow \gamma))$  by the deduction theorem, and consequently,
27.  $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \vee \beta \rightarrow \gamma))$  by the definition of  $\vee$ .

$(ExM)$ . We show that  $\alpha \vee \sim\alpha$ , i.e.,  $\sim(\sim\alpha \rightarrow \alpha) \rightarrow \sim\alpha$ , is provable  $S^n$ .

1.  $\sim(\sim\alpha \rightarrow \alpha)$  by the deduction theorem,
2.  $(\sim\alpha \rightarrow \alpha) \rightarrow (\sim(\sim\alpha \rightarrow \alpha) \rightarrow \sim\alpha)$  by  $(DS^{\rightarrow})$ ,
3.  $(\sim\alpha \rightarrow \alpha) \rightarrow \sim\alpha$  by  $(LoC)$ , 2, 1, (MP),
4.  $((\sim\alpha \rightarrow \alpha) \rightarrow \sim\alpha) \rightarrow \sim\alpha$  by  $(PL)$ ,
5.  $\sim\alpha$  by 4, 3, (MP),
6.  $\sim(\sim\alpha \rightarrow \alpha) \rightarrow \sim\alpha$  by the deduction theorem,

and finally,

7.  $\alpha \vee \sim\alpha$  by the definition of  $\vee$ .

$(dC)$ . We prove that  $\sim(\beta \wedge \sim\beta) \rightarrow ((\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \sim\beta) \rightarrow \sim\alpha))$ , i.e.,  $\sim\sim(\beta \rightarrow \sim(\sim\beta \rightarrow \sim\beta)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \sim\beta) \rightarrow \sim\alpha))$ , is a thesis of  $S^n$ .



1.  $\sim\sim(\beta \rightarrow \sim(\sim\sim\beta \rightarrow \sim\beta))$ ,
2.  $\alpha \rightarrow \beta$ ,
3.  $\alpha \rightarrow \sim\beta$  by the deduction theorem,
4.  $(\sim\sim\beta \rightarrow \sim\beta) \rightarrow (\sim(\sim\sim\beta \rightarrow \sim\beta) \rightarrow \sim\alpha)$  by  $(DS^{\rightarrow})$ ,
5.  $\beta \rightarrow \sim(\sim\sim\beta \rightarrow \sim\beta)$  by  $(NN)$ , 1, (MP),
6.  $\beta \rightarrow (\sim(\sim\sim\beta \rightarrow \sim\beta) \rightarrow \sim\alpha)$  by  $(HS)$ , 4, 5, (MP),
7.  $\alpha \rightarrow (\sim(\sim\sim\beta \rightarrow \sim\beta) \rightarrow \sim\alpha)$  by  $(HS)$ , 2, 6, (MP),
8.  $\sim(\sim\sim\beta \rightarrow \sim\beta) \rightarrow (\alpha \rightarrow \sim\alpha)$  by  $(LoC)$ , 7, (MP),
9.  $\beta \rightarrow (\alpha \rightarrow \sim\alpha)$  by  $(HS)$ , 5, 8, (MP),
10.  $\alpha \rightarrow (\alpha \rightarrow \sim\alpha)$  by  $(HS)$ , 2, 9, (MP),
11.  $\alpha \rightarrow \sim\alpha$  by  $(C)$ , 10, (MP),
12.  $\sim\alpha$  by  $(CM2)$ , (11), (MP),
13.  $\sim\sim(\beta \rightarrow \sim(\sim\sim\beta \rightarrow \sim\beta)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \sim\beta) \rightarrow \sim\alpha))$  by the deduction theorem, and consequently,
14.  $\sim(\beta \wedge \sim\beta) \rightarrow ((\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \sim\beta) \rightarrow \sim\alpha))$  by the definition of  $\wedge$ .

$(nC^{\dagger})$ . We demonstrate that  $\sim((\alpha \dagger \beta) \wedge \sim(\alpha \dagger \beta))$ , i.e.,  $\sim\sim((\alpha \dagger \beta) \rightarrow \sim(\sim\sim(\alpha \dagger \beta) \rightarrow \sim(\alpha \dagger \beta)))$  and  $\dagger \in \{\wedge, \vee, \rightarrow\}$ , is provable in  $S^n$ . Let  $\phi := \alpha \rightarrow \beta$ , if  $\dagger$  is  $\rightarrow$ ;  $\phi := \sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta$ , if  $\dagger$  is  $\vee$ ; and  $\phi := \sim(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta))$ , if  $\dagger$  is  $\wedge$ . As a result, we have:

1.  $\sim\sim\sim(\phi \rightarrow \sim(\sim\sim\phi \rightarrow \sim\phi))$  by the indirect deduction theorem,
2.  $\sim(\phi \rightarrow \sim(\sim\sim\phi \rightarrow \sim\phi))$  by  $(NN)$ , 1, (MP),
3.  $(\phi \rightarrow \sim(\sim\sim\phi \rightarrow \sim\phi) \rightarrow (\sim(\phi \rightarrow \sim(\sim\sim\phi \rightarrow \sim\phi)) \rightarrow \phi))$  by  $(DS^{\rightarrow})$ ,
4.  $(\phi \rightarrow \sim(\sim\sim\phi \rightarrow \sim\phi)) \rightarrow \phi$  by  $(LoC)$ , 3, 2, (MP),
5.  $((\phi \rightarrow \sim(\sim\sim\phi \rightarrow \sim\phi)) \rightarrow \phi) \rightarrow \phi$  by  $(PL)$ ,
6.  $\phi$  by 5, 4, (MP),
7.  $(\phi \rightarrow \sim(\sim\sim\phi \rightarrow \sim\phi)) \rightarrow (\sim(\phi \rightarrow \sim(\sim\sim\phi \rightarrow \sim\phi)) \rightarrow \sim\sim(\sim\sim\phi \rightarrow \sim\phi))$  by  $(DS^{\rightarrow})$ ,
8.  $(\phi \rightarrow \sim(\sim\sim\phi \rightarrow \sim\phi)) \rightarrow \sim\sim(\sim\sim\phi \rightarrow \sim\phi)$  by  $(LoC)$ , 7, 2, (MP),
9.  $\phi \rightarrow (\sim(\sim\sim\phi \rightarrow \sim\phi) \rightarrow \sim\sim(\sim\sim\phi \rightarrow \sim\phi))$  by  $(LoE)$ , 8, (MP),
10.  $\sim(\sim\sim\phi \rightarrow \sim\phi) \rightarrow \sim\sim(\sim\sim\phi \rightarrow \sim\phi)$  by 6, 9, (MP),
11.  $(\sim(\sim\sim\phi \rightarrow \sim\phi) \rightarrow \sim\sim(\sim\sim\phi \rightarrow \sim\phi)) \rightarrow \sim\sim(\sim\sim\phi \rightarrow \sim\phi)$  by  $(CM2)$ ,
12.  $\sim\sim(\sim\sim\phi \rightarrow \sim\phi)$  by 10, 11, (MP),
13.  $\sim\sim\phi \rightarrow \sim\phi$  by  $(NN)$ , 12, (MP),
14.  $\sim\phi$  by  $(CM)$ , 13, (MP),

a contradiction (6, 14). This entails that,

15.  $\sim\sim(\phi \rightarrow \sim(\sim\sim\phi \rightarrow \sim\phi))$ ,

and finally,

16.  $\sim(\phi \wedge \sim\phi)$ .

However, if  $\phi := \alpha \rightarrow \beta$ , then  $\sim((\alpha \rightarrow \beta) \wedge \sim(\alpha \rightarrow \beta))$ ; if  $\phi := \sim(\sim\alpha \rightarrow \alpha) \rightarrow \beta$ , then  $\sim((\alpha \vee \beta) \wedge \sim(\alpha \vee \beta))$ ; and if  $\phi := \sim(\alpha \rightarrow \sim(\sim\beta \rightarrow \beta))$ , then  $\sim((\alpha \wedge \beta) \wedge \sim(\alpha \wedge \beta))$ . Hence,  $\sim((\alpha \dagger \beta) \wedge \sim(\alpha \dagger \beta))$ , where  $\dagger \in \{\wedge, \vee, \rightarrow\}$ .

$(nC^{\sim n})$ . We show that  $\sim(\sim^n\alpha \wedge \sim^{n+1}\alpha)$ , that is,  $\sim\sim(\sim^n\alpha \rightarrow \sim(\sim^{n+2}\alpha \rightarrow \sim^{n+1}\alpha))$ , where  $n \in \mathbb{N}$ , is provable in  $S^n$ .

1.  $\sim\sim\sim(\sim^n\alpha \rightarrow \sim(\sim^{n+2}\alpha \rightarrow \sim^{n+1}\alpha))$  by the indirect deduction theorem,
2.  $\sim(\sim^n\alpha \rightarrow \sim(\sim^{n+2}\alpha \rightarrow \sim^{n+1}\alpha))$  by  $(NN)$ , 1, (MP),

Let  $\phi := \sim^n\alpha \rightarrow \sim(\sim^{n+2}\alpha \rightarrow \sim^{n+1}\alpha)$ . Then,

3.  $\sim \phi$ ,
  4.  $\phi \rightarrow (\sim \phi \rightarrow \sim^n \alpha)$  by  $(DS^\rightarrow)$ ,
  5.  $\phi \rightarrow \sim^n \alpha$  by  $(LoC)$ , 4, 3, (MP),
  6.  $(\sim^n \alpha \rightarrow \sim (\sim^{n+2} \alpha \rightarrow \sim^{n+1} \alpha)) \rightarrow \sim^n \alpha$  by  $\phi$ ,
  7.  $\sim^n \alpha$  by  $(PL)$ , 6, (MP),
  8.  $\phi \rightarrow (\sim \phi \rightarrow \sim \sim (\sim^{n+2} \alpha \rightarrow \sim^{n+1} \alpha))$  by  $(DS^\rightarrow)$ ,
  9.  $\phi \rightarrow \sim \sim (\sim^{n+2} \alpha \rightarrow \sim^{n+1} \alpha)$  by  $(LoC)$ , 8, 3, (MP),
  10.  $(\sim^n \alpha \rightarrow \sim (\sim^{n+2} \alpha \rightarrow \sim^{n+1} \alpha)) \rightarrow \sim \sim (\sim^{n+2} \alpha \rightarrow \sim^{n+1} \alpha)$  by  $\phi$ ,
  11.  $\sim^n \alpha \rightarrow (\sim (\sim^{n+2} \alpha \rightarrow \sim^{n+1} \alpha) \rightarrow \sim \sim (\sim^{n+2} \alpha \rightarrow \sim^{n+1} \alpha))$  by  $(LoE)$ , 10, (MP),
  12.  $\sim (\sim^{n+2} \alpha \rightarrow \sim^{n+1} \alpha) \rightarrow \sim \sim (\sim^{n+2} \alpha \rightarrow \sim^{n+1} \alpha)$  by 11, 7, (MP),
  13.  $\sim \sim (\sim^{n+2} \alpha \rightarrow \sim^{n+1} \alpha)$  by  $(CM2)$ , 12, (MP),
  14.  $\sim^{n+2} \alpha \rightarrow \sim^{n+1} \alpha$  by  $(NN)$ , 13, (MP),
  15.  $\sim^{n+1} \alpha$  by  $(CM)$ , 14, (MP),
  16.  $\sim^n \alpha \rightarrow (\sim^{n+1} \alpha \rightarrow \phi)$  by  $(DS^{\sim n})$ ,
  17.  $\phi$  by 16, 15, 7, (MP),
- a contradiction (3, 17). This entails that,
18.  $\sim \sim (\sim^n \alpha \rightarrow \sim (\sim^{n+2} \alpha \rightarrow \sim^{n+1} \alpha))$ , and consequently,
  19.  $\sim (\sim^n \alpha \wedge \sim^{n+1} \alpha)$  by the definition of  $\wedge$ .

This finishes the proof of Theorem 4.  $\square$

### 3. Conclusions

In this paper, we proposed a new axiomatization for the  $P^n$ -hierarchy. The main idea behind it was to focus directly on the principle of ex contradictione sequitur quodlibet. This is a remarkable difference between Fernández–Coniglio’s and our proposal, which makes the hierarchy less complex and more transparent from the viewpoint of paraconsistency. Additionally, we followed Sette’s idea and the connectives of negation and implication were taken as primitives. In conclusion let us also mention that the several other hierarchies can be easily generated from  $P^n$ -hierarchy. For instance, by dropping  $(DS^\rightarrow)$ , we get the  $CB^n$ -hierarchy of the paraconsistent calculi (cf. [20]). The interested reader can also find a slightly different hierarchy in [21] (the so-called  $B^n$ -hierarchy).

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