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Continuous Homomorphisms Defined on (Dense) Submonoids of Products of Topological Monoids

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Received: 24 December 2019; Accepted: 13 February 2020; Published: 18 February 2020

Abstract: We study the factorization properties of continuous homomorphisms defined on a (dense) submonoid S of a Tychonoff product $D = \prod_{i\in I} D_i$ of topological or even topologized monoids. In a number of different situations, we establish that every continuous homomorphism $f\colon S \to K$ to a topological monoid (or group) K depends on at most finitely many coordinates. For example, this is the case if S is a subgroup of D and K is a first countable left topological group without small subgroups (i.e., K is an NSS group). A stronger conclusion is valid if S is a finitely retractable submonoid of D and K is a regular quasitopological NSS group of a countable pseudocharacter. In this case, every continuous homomorphism f of S to K has a finite type, which means that f admits a continuous factorization through a finite subproduct of D. A similar conclusion is obtained for continuous homomorphisms of submonoids (or subgroups) of products of topological monoids to Lie groups. Furthermore, we formulate a number of open problems intended to delimit the validity of our results.

Keywords: monoid; homomorphism; character; NSS group; Lie group; factorization; quasitopological group

1. Introduction

The present article is a natural continuation of the study in [1,2], where we considered continuous mappings (homomorphisms) defined on subspaces of products of topological spaces (monoids, groups) and established a kind of irreducible factorization of those mappings (homomorphisms) under quite general assumptions. Our purpose here is to focus on the case when $f\colon S \to K$ is a continuous homomorphism of a submonoid (subgroup) S of a product $D = \prod_{i\in I} D_i$ of (semi)topological monoids or groups. Again, we are interested in identifying conditions under which f admits a factorization in the form:

$$f = g \circ p_J \restriction S, \tag{1}$$

where J is a 'small' subset of the index set I, $p_J\colon D \to D_J = \prod_{i\in J} D_i$ is the projection, and $g\colon p_J(S) \to K$ is a continuous homomorphism. If one can find a finite (countable) set J for which the equality (1) holds true, we say that f has a finite (countable) type.

It is worth noting that if $X = \prod_{i\in I} X_i$ is the product of infinitely many Tychonoff spaces, where each factor satisfies $|X_i| \geq 2$, then there exists a continuous real-valued function on X, which depends on infinitely many coordinates. The case of continuous homomorphisms of topological groups is considerably better. It follows from the Pontryagin–van Kampen duality theory that every continuous homomorphism of a product $D = \prod_{i\in I} D_i$ of compact abelian groups D_i to the circle group $\mathbb{T}$ (called character) has a finite type. In [3], S. Kaplan generalized this fact by proving that if $D = \prod_{i\in I} D_i$ is a product of reflexive topological (not necessarily locally compact) abelian groups, then every continuous character of D has a finite type. An analysis of the proof in [3] shows that the requirement of the reflexivity of the factors in Kaplan's theorem can be dropped. In fact, we show in Corollary 11 that every continuous character of an arbitrary subgroup of a Tychonoff product of paratopological groups (equivalently, topological monoids) has a finite type.

The starting point of the present research is the following very general, but relatively simple fact about continuous homomorphisms of left topological groups (see [4] (Lemma 8.5.4)):

Proposition 1. *A continuous homomorphism $f\colon S \to K$ defined on an arbitrary subgroup S of a product $D = \prod_{i\in I} D_i$ of left topological groups has a countable type provided that the left topological group K is a first countable T_1-space.*

The proof of Proposition 1 given in [4] makes use of inversion in groups and cannot be applied to more general objects of topological algebra like topological monoids or semigroups.

Our general problem, far from being completely solved here, is to extend Proposition 1 to (semi)topological monoids or (semi)topological semigroups and find conditions on D, S, and K guaranteeing that every continuous homomorphism $f\colon S \to K$ has a countable or even finite type.

However, considering submonoids of products of topological monoids and continuous homomorphisms of them presents several difficulties. First, the shifts in monoids need not be surjective. Second, in a topological monoid, the left and right shifts can fail to be open mappings. Indeed, one can take the unit interval $\mathbb{I} = [0, 1]$ with the usual interval topology and define continuous multiplication in $\mathbb{I}$ by $xy = \max\{x, y\}$ for all $x, y \in \mathbb{I}$. Then, the interval $\mathbb{I}$ with the given topology and multiplication is a compact topological monoid. However, the shifts in $\mathbb{I}$ are neither open nor surjective (see, e.g., [4] (Example 1.3.7)). Third, a monoid can have no invertible elements, except for the identity. This makes it impossible to extend 'traditional' arguments that work for groups (see, e.g., the proof of Proposition 2) to the more general case of submonoids of products of topological monoids. Nevertheless, some work in this direction has been done in [5], where a factorization theorem was established for Mal'cev subspaces of products of left semitopological Mal'cev spaces. Some arguments presented in the proofs of Propositions 1 and 2 in [6] can be applied to the study of continuous homomorphisms of certain subsemigroups of products of topological semigroups. Purely algebraic aspects of factoring homomorphisms of infinite products of groups can be found in [7]. For example, if f is an arbitrary homomorphism of a product $\prod_{i\in I} D_i$ of abelian groups to a slender group K, then f depends on finitely many coordinates provided the cardinality of the index set I is less than the first uncountable Ulam measurable cardinal (it is consistent with ZFC that such a cardinal does not exist; see [8]). The simplest nontrivial example of a slender group is an infinite cyclic group [9].

To overcome some of the aforementioned difficulties, we consider finitely retractable and ω-retractable submonoids of products of monoids, the notions introduced in [2] (see Definition 1 below). In a sense, these are close to Σ- and σ-products described on page 4. The article [2] contained several results on factorization of continuous homomorphisms of ω-retractable or finitely retractable submonoids S of products of topologized monoids that, in special cases, extend Proposition 1 to monoids. We complement those results here by proving in Corollary 3 that if S is a dense submonoid of a product $D = \prod_{i\in I} D_i$ of semitopological monoids with open shifts and $f\colon S \to K$ is a continuous homomorphism to a Hausdorff first countable paratopological group K, then f depends on countably many coordinates. This conclusion is somewhat weaker than claiming that f has a countable type (see Problem 2).

In several situations, we are able to prove that a continuous homomorphism $f\colon S \to K$ has a finite type. For example, we establish in Theorem 2 that this happens if S is a finitely retractable submonoid of a product $\prod_{i\in I} D_i$ of topologized monoids and K is a regular quasitopological NSS group K satisfying $\psi(K) \le \omega$.

We also show in Corollary 4 that every continuous homomorphism defined on a finitely retractable submonoid S of a product $D = \prod_{i\in I} D_i$ of topologized monoids has a finite type provided the homomorphism takes values in a topological NSS group K (i.e., there exists a neighborhood of the identity in K that does not contain nontrivial subgroups). The same conclusion is valid if instead of the finite retractability of S, we require S to satisfy the inclusions $\sigma D \subset S \subset D$, where σD is the σ-product of the factors D_i (see Corollary 5).

In Section 3, we switch to considering more general submonoids of products. We prove in Theorem 3 that every continuous homomorphism f of S to a topological NSS group K has a finite type provided S is a submonoid of a product D of left topological monoids, D is pseudo-ω_1-compact, and S fills all countable subproducts in D. It is possible to omit the requirement of the pseudo-ω_1-compactness of D if, instead, we strengthen the requirements on the factors D_i of the product D and the codomain K. According to Proposition 6, a continuous homomorphism $f\colon S \to K$ has a finite type if S is a submonoid of a product D of semitopological monoids with open shifts, which fills all countable subproducts of D and K is a first countable topological NSS group.

In the case of continuous homomorphisms to Lie groups, one can advance even further. We prove in Theorem 4 that if S is a submonoid of a product D of topological monoids with open shifts and either S is open and dense in D or fills all finite subproducts of D, then every continuous homomorphism of S to a Lie group has a finite type. Notice that under either assumption, S is a dense submonoid of D. The density of S in D can be dropped if we assume that S is a subgroup of a product of topological monoids. In this case, every continuous homomorphism of S to a Lie group has a finite type as well (see Theorem 5).

Several open problem are presented in Sections 2 and 3. In these problems, we propose to find out which conditions in our results can be either weakened, modified, or dropped.

Notation and Auxiliary Results

Let $\mathbb{C}$ be the field of complex numbers with the usual Euclidean topology. The torus $\mathbb{T}$ is identified with the multiplicative subgroup $\{z \in \mathbb{C} : |z| = 1\}$ of $\mathbb{C}$.

A semigroup is a nonempty set S with a binary associative operation (called multiplication). A semigroup with an identity is called a monoid. Clearly, a monoid has a unique identity.

Assume that G is a semigroup (monoid, group) with a topology. If the left shifts in G are continuous, then G is called a left topological semigroup (monoid, group). If both the left and right shifts in G are continuous, then G is said to be a semitopological semigroup (monoid, group). If multiplication in G is jointly continuous, we say that G is a topological semigroup (monoid). A semitopological group with continuous inversion is called a quasitopological group. Further, if G is a group and multiplication in G is jointly continuous, we say that G is a paratopological group. A paratopological group with continuous inversion is a topological group.

The continuity of the homomorphisms of various objects of topological algebra can usually be deduced from their continuity at the identity of the domain. Lemma 1 below is well known for (left) topological groups (see [4] (Proposition 1.3.4)); it applies in the proofs of several results in this article.

First, we recall that a left topological monoid G has open left shifts if for every $x \in G$, the left shift λ_x of G defined by $\lambda_x(y) = x \cdot y$ for each $y \in G$ is an open mapping of G to itself. Changing 'left' to 'right' in the above definition, we get right topological monoids with right open shifts. If in a semitopological monoid G, all shifts, right and left, are open, we say that G is a semitopological monoid with open shifts.

Lemma 1. *Let G be a left topological monoid with open left shifts and $f\colon G \to H$ be a homomorphism of G to a left topological semigroup H. If f is continuous at the identity e of G, then it is continuous.*

Proof. Take an arbitrary element $x \in G$ and an open neighborhood V of the element $y = f(x)$ in H. Clearly, we have:
$$y = f(x) = f(x \cdot e) = f(x) \cdot f(e) = y \cdot f(e).$$
By the continuity of the left shift λ_y in H, there exists an open neighborhood W of $f(e)$ in H such that $yW \subset V$. Since f is continuous at e, there exists an open neighborhood U of e in G such that

$f(U) \subset W$. Then, xU is an open neighborhood of x in G and $f(xU) = f(x)f(U) \subset yW \subset V$. Hence, f is continuous. □

The next result complements Lemma 1.

Lemma 2. *Let $f\colon S \to K$, $p\colon S \to T$, and $g\colon T \to K$ be homomorphisms of left topological monoids with open left shifts such that $f = g \circ p$ and $p(S) = T$. Assume that f and p are continuous and that for every neighborhood O of the identity in K, there exists an open neighborhood V of the identity in T such that $f(p^{-1}(V)) \subset O$. Then, g is continuous.*

Proof. By Lemma 1, it suffices to verify that g is continuous at the identity of T. Let O be an arbitrary neighborhood of the identity in K. By our assumptions, there exists an open neighborhood V of the identity e_T in T such that $f(p^{-1}(V)) \subset O$. It follows from the equality $f = g \circ p$ and the surjectivity of p that $g(V) = f(p^{-1}(V)) \subset O$, which implies the continuity of g at e_T. Hence, g is continuous. □

The corollary below is close to [4] (Proposition 1.5.10).

Corollary 1. *Let S, T, K be left topological groups and $f\colon S \to K$ and $p\colon S \to T$ be continuous homomorphisms, where K satisfies the T_1 separation axiom and p is surjective. Assume that for every neighborhood O of the identity in K, there exists a neighborhood V of the identity in T such that $f(p^{-1}(V)) \subset O$. Then, there exists a continuous homomorphism $g\colon T \to K$ satisfying $f = g \circ p$.*

Proof. One of the conditions of the corollary can be reformulated by saying that for every neighborhood O of the identity in K, there exists a neighborhood V of the identity in T such that $p^{-1}(V) \subset f^{-1}(O)$. Since K is a T_1-space, this implies that $\ker p \subset \ker f$. As S, T, K are groups, we can apply the first part of [4] (Proposition 1.5.10) to conclude that there exists an (abstract) homomorphism $g\colon T \to K$ satisfying $f = g \circ p$. Hence, Lemma 2 implies the continuity of g. □

The next algebraic fact is known as the induced homomorphism theorem (see [10] (Theorem 1.48) or [11] (Theorem 1.6)).

Lemma 3. *Let $f\colon D \to H$ and $p\colon D \to F$ be homomorphisms of semigroups such that the equality $p(x) = p(y)$ implies that $f(x) = f(y)$ whenever $x, y \in D$. If p is surjective, then there exists a unique homomorphism $g\colon F \to H$ satisfying $f = g \circ p$.*

A character of an arbitrary monoid G is a (not necessarily continuous) homomorphism of G to the torus $\mathbb{T}$. The continuity of a character, if applied, will always be specified explicitly.

Let $X = \prod_{i \in I} X_i$ be the product of a family $\{X_i : i \in I\}$ of spaces endowed with the Tychonoff product topology and $a \in X$ be an arbitrary point. For every $i \in I$, the projection of X to the factor X_i is denoted by p_i. Furthermore, for every $x \in X$, we put:

$$\mathrm{diff}(x, a) = \{i \in I : p_i(x) \neq p_i(a)\}.$$

Then:

$$\Sigma X(a) = \{x \in X : |\mathrm{diff}(x, a)| \leq \omega\}$$

and:

$$\sigma X(a) = \{x \in X : |\mathrm{diff}(x, a)| < \omega\}$$

are dense subspaces of X, which are called respectively the Σ-product and σ-product of the family $\{X_i : i \in I\}$ with the center at a. If every X_i is a monoid (group), we will always choose a to be the identity e of X. In the latter case, $\Sigma X(e)$ and $\sigma X(e)$ are dense submonoids (subgroups) of the product monoid (group) X, and we shorten $\Sigma X(e)$ and $\sigma X(e)$ to ΣX and σX, respectively.

Assume that Y is a subset of the product $X = \prod_{i \in I} X_i$ of a family $\{X_i : i \in I\}$ of sets and $f\colon Y \to Z$ is an arbitrary mapping. We say that f depends on J, for some $J \subset I$, if the equality $f(x) = f(y)$ holds for all $x, y \in Y$ with $p_J(x) = p_J(y)$, where $p_J\colon X \to \prod_{i \in J} X_i$ is the projection. It is clear that if f depends on J, then there exists a mapping g of $p_J(Y)$ to Z satisfying $f = g \circ p_J \restriction Y$.

The family of all sets $J \subset I$ such that f depends on J is denoted by $\mathcal{J}(f)$ (see [1]). Observe that $\emptyset \in \mathcal{J}(f)$ if and only if f is constant. In general, for a non-constant f, the family $\mathcal{J}(f)$ can fail to be a filter, even if f is a continuous homomorphism of topological groups [1] (Example 2.14). For a detailed study of $\mathcal{J}(f)$, see [1,2].

In the following definition, we introduce the notion of a retractable subspace of a product of monoids, which is widely used in the article.

Definition 1. *Assume that D_i is a monoid with identity e_i, where $i \in I$. For a nonempty subset J of I, we define a retraction r_J of the product $D = \prod_{i \in I} D_i$ by letting:*

$$r_J(x)_i = \begin{cases} x_i & \text{if } i \in J; \\ e_i & \text{if } i \in I \setminus J, \end{cases} \tag{2}$$

for each element $x \in D$. A subset S of D is said to be retractable if $r_J(S) \subset S$, for each $J \subset I$. If κ is an infinite cardinal and the latter inclusion is valid for all subsets J of I with $|J| \leq \kappa$, we say that S is κ-retractable. Similarly, if the inclusion $r_J(S) \subset S$ holds for each finite set $J \subset I$, we call S finitely retractable.

Sometimes, we use "countably retractable" in place of "ω-retractable".

We recall that a subspace Y of a product $X = \prod_{i \in I} X_i$ is mixing if for arbitrary points $x, y \in Y$ and any set $J \subset I$, there exists a point $z \in Y$ such that $p_J(z) = p_J(x)$ and $p_{I \setminus J}(z) = p_{I \setminus J}(y)$ (see [1] (Definition 1.2)).

The next lemma is very close to [1] (Lemma 3).

Lemma 4. *Let $D = \prod_{i \in I} D_i$ be a product of monoids and e_i be the identity of D_i, where $i \in I$. Let also S be a finitely retractable submonoid of D. If $x, y \in S$ and K, L are finite disjoint subsets of the index set I, then there exists an element $s \in S$ such that $p_K(s) = p_K(x)$, $p_L(s) = p_L(y)$ and $s_i = e_i$ for each $i \in I \setminus (K \cup L)$. Furthermore, if S is retractable, then it is mixing.*

Proof. Since S is finitely retractable, $r_K(x)$ and $r_L(y)$ are in S. Then, the element $s = r_K(x) \cdot r_L(y) \in S$ satisfies the equalities in the first claim of the lemma.

Assume that S is retractable. Let J be a subset of I and x, y be arbitrary elements of S. Then, both $x_1 = r_J(x)$ and $y_1 = r_{I \setminus J}(y)$ are elements of S, so $z = x_1 \cdot y_1 \in S$ satisfies $p_J(z) = p_J(x)$ and $p_{I \setminus J}(z) = p_{I \setminus J}(y)$. Thus, S is mixing. $\square$

Projections of submonoids preserve the properties of being finitely retractable or κ-retractable; a straightforward proof of this fact is omitted:

Lemma 5. *Let $D = \prod_{i \in I} D_i$ be a product of monoids and S be a finitely retractable submonoid of D. Then, for every set $J \subset I$, the projection $p_J(S)$ is a finitely retractable submonoid of $D_J = \prod_{i \in J} D_i$. The same conclusion is valid for κ-retractability, for each $\kappa \geq \omega$.*

The density, network weight and pseudocharacter of a space X are denoted by $d(X)$, $nw(X)$, and $\psi(X)$, respectively. Notice that the pseudocharacter of X is defined only if X is a T_1-space. Regular spaces are assumed to satisfy the T_1 separation axiom.

2. Finitely Retractable and ω-Retractable Submonoids of Products

Let us recall that a topologized group K (i.e., a group with an arbitrary topology) is said to be an NSS (No Small Subgroups) group if there exists a neighborhood U of the identity e_K in K that does not contain nontrivial subgroups. This is equivalent to saying that for every element $x \in K$ distinct from e_K, there exists an integer n such that $x^n \notin U$. Clearly, every subgroup H of an NSS group G is also an NSS group provided H inherits its topology from G.

Proposition 2. *Let $D = \prod_{i \in I} D_i$ be a product of left topological monoids, S an arbitrary subgroup of D, and $f\colon S \to K$ a continuous homomorphism to a first countable left topological group K satisfying the T_1 separation axiom. If K is an NSS group, then f depends on a finite set $E \subset I$.*

Proof. Replacing each D_i with the projection $p_i(S)$ we can assume that the factors D_i are left topological groups. According to Proposition 1, one can find a countable set $J \subset I$ and a continuous homomorphism $g\colon p_J(S) \to K$ satisfying $f = g \circ p_J \restriction S$, where $p_J\colon D \to \prod_{i \in J} D_i$ is the projection. Therefore, we can assume that the index set I is countable, say $I = \omega$ and $D = \prod_{n \in \omega} D_n$. Suppose for a contradiction that the conclusion of the proposition fails to be true. Then, for every $k \in \omega$, one can find elements $x_k, y_k \in S$ such that $p_i(x_k) = p_i(y_k)$ for $i = 0, \ldots, k$ and $f(x_k) \neq f(y_k)$. The element $z_k = x_k^{-1} \cdot y_k \in S$ satisfies $p_i(z_k) = e_{D_i}$ for each $i \leq k$, and $f(z_k) = f(x_k)^{-1} \cdot f(y_k) \neq e_K$.

Let U be a neighborhood of the identity in K that does not contain nontrivial subgroups. For every $k \in \omega$, choose an integer n_k such that $t_k = f(z_k)^{n_k} \notin U$. Notice that $p_i(z_k)^{n_k} = e_{D_i}$ for each $i = 0, \ldots, k$. Hence, the sequence $\{z_k^{n_k} : k \in \omega\} \subset S$ converges to the identity element of D. By the continuity of f, the sequence $\{t_k : k \in \omega\}$ converges to the identity of K. However, the latter contradicts the fact that $t_k \notin U$, for each $k \in \omega$. This contradiction implies the required conclusion. $\square$

One can try to strengthen the conclusion of Proposition 2 as follows:

Question 1. *Is it true, under the conditions of Proposition 2, that f has a finite type? In other words, can one guarantee the continuity of the homomorphism $g\colon p_E(S) \to K$ satisfying $f = g \circ p_E \restriction S$, for a finite set $E \subset I$?*

We answer Question 1 in the negative, even if S is a dense subgroup of D. To present a counterexample, we need the following result (see [12] (Lemma 4.7)):

Proposition 3. *Let G be an uncountable separable topological abelian group such that the torsion subgroup of G is countable. Then, there exists a discontinuous homomorphism $h\colon G \to \mathbb{T}$ such that the graph $Gr(h) = \{(x, h(x)) : x \in G\}$ is a dense subgroup of $G \times \mathbb{T}$.*

In fact, Proposition 3 is slightly more general than Lemma 4.7 in [12] since G is assumed to be torsion-free there. However, almost the same argument works under the conditions of Proposition 3 as well.

Example 1. *There exist a dense subgroup S of the compact topological group $\mathbb{T}^\omega$ and a continuous homomorphism $f\colon S \to K$ to a second countable topological NSS group K such that f fails to have a finite type.*

Proof. Our construction of S, K, and f is quite simple. Let G_0 be the usual compact torus group $\mathbb{T}$. Notice that the torsion subgroup of G_0 is countable. Assume that for some $n \in \omega$, we have defined a dense subgroup G_n of $\mathbb{T}^n$ algebraically isomorphic to $\mathbb{T}$. The topological group G_n is second countable and, hence, separable. By Proposition 3, there exists a discontinuous homomorphism $h_n\colon G_n \to \mathbb{T}$ such that $G_{n+1} = Gr(h_n) = \{(x, h_n(x)) : x \in G_n\}$ is a dense subgroup of $G_n \times \mathbb{T}$. Denote by p_n^{n+1} the restriction to G_{n+1} of the projection $G_n \times \mathbb{T} \to G_n$. Clearly, p_n^{n+1} is a continuous isomorphism of G_{n+1} onto G_n. It follows, by induction, that each G_n is a dense subgroup of $\mathbb{T}^n$, which admits a continuous isomorphism (but not a homeomorphism) onto $\mathbb{T}$.

Denote by S the limit of the inverse sequence $\{G_n,\ p_n^{n+1} : n \in \omega\}$. Alternatively, one can describe the group S as follows. For every $n \in \omega$, let $\pi_n \colon \mathbb{T}^\omega \to \mathbb{T}^n$ be the projection. Then, S is the subgroup of $\mathbb{T}^\omega$, which consists of all $x \in \mathbb{T}^n$ satisfying $\pi_n(x) \in G_n$ for each $n \in \omega$. Since each G_n is dense in $\mathbb{T}^n$, the latter description of S implies that it is dense in $\mathbb{T}^\omega$.

Let $K = S$ and f be the identity isomorphism of S onto K. It is clear that S admits a continuous isomorphism onto the NSS group $\mathbb{T}$, so S and K are second countable NSS groups. It remains to verify that f does not have a finite type. If f has a finite type, one can find $n \in \omega$ and a continuous homomorphism $g \colon G_n \to K$ such that $f = g \circ \pi_n \restriction S$. Since f is a topological isomorphism, so is $\pi_n \restriction S$. It follows from $\pi_n \restriction S = p_n^{n+1} \circ \pi_{n+1} \restriction S$ that p_n^{n+1} is a topological isomorphism of G_{n+1} onto G_n. Let $q_n \colon G_n \to G_{n+1}$ be the mapping inverse to p_n^{n+1}. Then, q_n is also a topological isomorphism. As G_{n+1} is the graph of the homomorphism $h_n \colon G_n \to \mathbb{T}$, we have the equality $h_n = p_2 \circ q_n$, where p_2 is the projection of $G_n \times \mathbb{T}$ to the second factor. Hence, h_n is continuous, which is a contradiction. □

Under additional restrictions on S or K, we answer Question 1 affirmatively in Corollary 2 and Theorems 2 and 5. In the second of these results, we actually consider a more general case of a submonoid S of a product of topologized monoids. We assume, however, that S is finitely retractable and K is a quasitopological group satisfying $\psi(K) \le \omega$. In Theorem 5, we turn back to considering subgroups S of products, but assume that the range K of the homomorphism f is a Lie group.

Corollary 2. *Let $D = \prod_{i \in I} D_i$ be a product of left topological groups, S a subgroup of D, and $f \colon S \to K$ a continuous homomorphism to a first countable left topological NSS group K satisfying the T_1 separation axiom. If S satisfies $p_F(S) = \prod_{i \in F} D_i$ for each finite set $F \subset I$, then f has a finite type.*

Proof. It follows from our assumptions about S that for every finite $F \subset I$, the restriction to S of the projection $p_F \colon D \to \prod_{i \in F} D_i = D_F$ is an open homomorphism of S onto D_F. By Proposition 2, f depends on a finite set $E \subset I$. Hence, there exists a homomorphism $g \colon D_E \to K$ satisfying $f = g \circ p_E \restriction S$. Since $p_E \restriction S$ is an open continuous homomorphism, we conclude that g is continuous. Thus, f has a finite type. □

We will show in Proposition 8 that the above corollary remains valid for an arbitrary topological NSS group K, without the assumption that K is first countable.

Problem 1. *Can one weaken in Corollary 2 the first countability of K to $\psi(K) \le \omega$ (assuming that K is regular)?*

Let us study the dependence of f on a subset J of the index set I in more detail.

Lemma 6. *Let S be a finitely retractable subspace of a product $D = \prod_{i \in I} D_i$ of topologized monoids and $f \colon S \to K$ be a continuous mapping to a Hausdorff space K. Then, the following hold:*

(a) $T = S \cap \sigma D$ is dense in S.
(b) If $f^* = f \restriction T$ depends on a set $J \subset I$, then so does f. Hence, $\mathcal{J}(f) = \mathcal{J}(f^*)$.

Proof. (a) is almost immediate. Indeed, take an arbitrary element $x \in S$. For every finite set $F \subset I$, the element $t = r_F(x)$ is in $S \cap \sigma D = T$ (the retraction r_F appears in Definition 1) and satisfies $p_F(t) = p_F(x)$. This implies the density of T in S.

(b) If f depends on a set $J \subset I$, then $f^* = f \restriction T$ also depends on J, so $\mathcal{J}(f) \subset \mathcal{J}(f^*)$ (see [1] (Lemma 1.1)). Clearly, the first part of (b) is equivalent to the inclusion $\mathcal{J}(f^*) \subset \mathcal{J}(f)$. Hence, to complete the proof, it suffices to verify the latter inclusion.

Suppose for a contradiction that there exist $J \in \mathcal{J}(f^*)$ and elements $x, y \in S$ such that $p_J(x) = p_J(y)$ and $f(x) \ne f(y)$. Choose disjoint neighborhoods O_x and O_y of $f(x)$ and $f(y)$, respectively, in K. By the continuity of f, there exist canonical open neighborhoods U_x and U_y of x and y, respectively, in D such that $f(S \cap U_x) \subset O_x$ and $f(S \cap U_y) \subset O_y$. Take a finite set $C \subset I$ such that $U_x = p_C^{-1} p_C(U_x)$ and $U_y = p_C^{-1} p_C(U_y)$. Since S is finitely retractable, we have that $r_C(x) \in S \cap U_x$ and $r_C(y) \in S \cap U_y$. Hence, our choice of the sets U_x and U_y implies that $f(r_C(x)) \in f(S \cap U_x) \subset O_x$ and $f(r_C(y)) \in f(S \cap U_y) \subset O_y$. It follows from the equality $p_J(x) = p_J(y)$ and the definition of the retraction r_C (see (2)) that $p_J(r_C(x)) = p_J(r_C(y))$. Since $r_C(x) \in S \cap \sigma D = T$, $r_C(y) \in S \cap \sigma D = T$ and f^* depends on J, we have the equality $f(r_C(x)) = f(r_C(y))$. Therefore, $f(r_C(x)) \in O_x \cap O_y \neq \emptyset$, which contradicts the choice of O_x and O_y. We have thus proven that f depends on J and $J \in \mathcal{J}(f)$. □

Notice that the mapping f in Lemma 6 is not assumed to be a homomorphism. Furthermore, every nonempty topological space X can be given the structure of a topologized monoid by retaining the topology of X, choosing an element $e \in X$, and defining an associative multiplication in X by $x \cdot y = y$ for all $x, y \in X$ with $y \neq e$ and $x \cdot e = e \cdot x = x$ for each $x \in X$. The assumption in Lemma 6 that the factors D_i are monoids is used only for the possibility to apply the retractions r_C of D, with $C \subset I$, which in turn requires only a choice of a 'fixed' point $e_D \in D$, the identity of D in our case. Therefore, Lemma 6 can be easily reformulated in purely topological terms.

A different version of Lemma 6 is presented below. In it, we assume that the restriction of f to a dense subspace T of S satisfies a condition considered in Lemma 2 and Corollary 1.

Proposition 4. *Let S be a dense submonoid of a product $D = \prod_{i \in I} D_i$ of semitopological monoids with open shifts and $f\colon S \to K$ be a continuous homomorphism to a Hausdorff paratopological group K. Let also T be a dense subspace of S and J be a subset of I such that for every neighborhood O of the identity in K, there exists an open neighborhood U of the identity in D such that $U = p_J^{-1} p_J(U)$ and $f(T \cap U) \subset O$. Then, f depends on J.*

Proof. Suppose that there exist $x, y \in S$ such that $p_J(x) = p_J(y)$ and $f(x) \neq f(y)$. Let $a = f(x)$ and $b = f(y)$ and choose disjoint open neighborhoods O_a and O_b of a and b, respectively, in K. Furthermore, by the continuity of multiplication in K, there exists an open neighborhood O of the identity e_K in K such that $OaO \subset O_a$ and $ObO \subset O_b$.

Since f is continuous, we can find canonical open neighborhoods U_x and U_y of x and y, respectively, in D such that:

$$f(S \cap U_x) \subset aO, \ \ f(S \cap U_y) \subset Ob.$$

By our assumptions, there exists a canonical open neighborhood U_e of the identity e in D such that $U_e = p_J^{-1} p_J(U_e)$ and $f(T \cap U_e) \subset O$. Take a finite set $C \subset I$ such that $U_x = p_C^{-1} p_C(U_x)$, $U_y = p_C^{-1} p_C(U_y)$ and $U_e = p_C^{-1} p_C(U_e)$. Notice that $U_e = p_F^{-1} p_F(U_e)$, where $F = J \cap C$.

Our aim is to define elements $x', y' \in T \cap U_e$ and $z \in S$ satisfying $zx' \in U_x$ and $y'z \in U_y$. Let $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$. For $i \in I$, let $U_{e,i} = p_i(U_e)$, $U_{x,i} = p_i(U_x)$ and $U_{y,i} = p_i(U_y)$, where p_i is the projection of D to the factor D_i. For every $i \in C$, choose an open neighborhood W_i of the identity e_i in D_i such that $x_i W_i \cup W_i x_i \subset U_{x,i}$ and $y_i W_i \cup W_i y_i \subset U_{y,i}$. Since S is dense in D and the set $x_i W_i$ is open in D_i for each $i \in C$, we can find an element $z \in S$ such that:

$$p_C(z) \in \prod_{i \in F} x_i W_i \times \prod_{i \in C \setminus F} W_i.$$

Let $z = (z_i)_{i \in I}$. Then, $z_i \in x_i W_i$ if $i \in F$ and $z_i \in W_i$ if $i \in C \setminus F$. For every $i \in F$, the set $x_i W_i$ is open in D_i. Since $z_i e = z_i \in x_i W_i$, we can find an open neighborhood V_i of e_i in D_i such that $V_i \subset U_{e,i}$ and $z_i V_i \subset x_i W_i$. Further, for $i \in C \setminus F$, it follows from $z_i \in W_i$ that $z_i e_i x_i = z_i x_i \in W_i x_i$, so there exists an open neighborhood V_i of e_i in D_i such that $z_i V_i x_i \subset W_i x_i$. Again, making use of the density of T in S and D, we can find an element $x' \in T$ such that:

$$p_C(x') \in \prod_{i \in F} V_i \times \prod_{i \in C \setminus F} V_i x_i.$$

Let $x' = (x'_i)_{i\in I}$. Then, $x'_i \in V_i$ if $i \in F$ and $x'_i \in V_i x_i$ if $i \in C \setminus F$. Since $U_e = p_F^{-1} p_F(U_e)$ and $x'_i \in V_i \subset U_{e,i}$ for each $i \in F$, we see that $x' \in U_e$. It follows from our choice of z and x' that $z_i x'_i \in z_i V_i \subset x_i W_i \subset U_{x,i}$ if $i \in F$ and $z_i x'_i \in z_i V_i x_i \subset W_i x_i \subset U_{x,i}$ if $i \in C \setminus F$. Therefore, $zx' \in U_x$. A similar argument applies to finding an element $y' \in T \cap U_e$ such that $y'z \in U_y$.

Notice that our choice of the elements $x', y' \in T \cap U_e$ implies that $f(x') \in f(T \cap U_e) \subset O$ and $f(y') \in f(T \cap U_e) \subset O$. It follows from $zx' \in S \cap U_x$ and $y'z \in S \cap U_y$ that:

$$f(z)f(x') = f(zx') \in f(S \cap U_x) \subset aO$$

and:

$$f(y')f(z) = f(y'z) \in f(S \cap U_y) \subset Ob.$$

Therefore, we have that $f(z)f(x') \in f(z)O \cap aO \neq \emptyset$ and $f(y')f(z) \in Of(z) \cap Ob \neq \emptyset$. This implies that $f(z) \in aOO^{-1} \cap O^{-1}Ob \neq \emptyset$, and therefore, $OaO \cap ObO \neq \emptyset$. Since $OaO \subset O_a$ and $ObO \subset O_b$, we infer that $O_a \cap O_b \neq \emptyset$, which contradicts our choice of the sets O_a and O_b. Hence, f depends on J. □

Theorem 1. *Let S be a dense submonoid of a product $\prod_{i\in I} D_i$ of semitopological monoids with open shifts and $f\colon S \to K$ be a continuous homomorphism to a Hausdorff paratopological group K. Let also J be a subset of I such that for every neighborhood O of the identity in K, there exists an open neighborhood U of the identity in D such that $U = p_J^{-1} p_J(U)$ and $f(S \cap U) \subset O$. Then, f depends on J.*

Proof. The required conclusion follows directly from Proposition 4 if we take $T = S$. □

Corollary 3. *Let S be a dense submonoid of a product $\prod_{i\in I} D_i$ of semitopological monoids with open shifts and $f\colon S \to K$ be a continuous homomorphism to a Hausdorff first countable paratopological group K. Then, f depends on a countable set $J \subset I$.*

Proof. Let $\{O_n : n \in \omega\}$ be a local base at the identity of K. For every $n \in \omega$, take a canonical open neighborhood U_n of the identity in D such that $f(S \cap U_n) \subset O_n$. Then, $U_n = p_{C_n}^{-1} p_{C_n}(U_n)$, for a finite subset C_n of I. Let $J = \bigcup_{n\in\omega} C_n$. Then, $|J| \le \omega$ and $U_n = p_J^{-1} p_J(U_n)$, for each $n \in \omega$. Hence, the required conclusion follows from Theorem 1. □

Under different assumptions on S and K, Corollaries 1 and 2 in [2] state that every continuous homomorphism $f\colon S \to K$ has a countable type. This tempts us to raise the next problem:

Problem 2. *Can one strengthen the conclusion of Corollary 3 by proving that f has a countable type (provided that the space K is regular)?*

The following result is close to [1] (Theorem 2.3). As a matter of fact, it is a version of the latter theorem in the case when the factors of the product space D are topologized monoids and S is a finitely retractable submonoid of D. It can be helpful to mention that a finitely retractable submonoid of $D = \prod_{i\in I} D_i$ is not necessarily finitely mixing (cf. Lemma 4), unless the index set I is finite. Hence, Proposition 5 below does not follow from Theorem 2.3 in [1], but we do use the latter theorem in the proof of the proposition applying it to a special retractable (hence, mixing) submonoid of S.

In what follows, we denote by $[I]^{<\omega}$ the family of all finite subsets of a given set I.

Proposition 5. *Let $D = \prod_{i\in I} D_i$ be a product of topologized monoids, S be a finitely retractable submonoid of D, and $f\colon S \to Z$ be a continuous mapping to a regular space Z. Then, $J = \bigcap \mathcal{J}(f)$ is the smallest element of $\mathcal{J}(f)$, and there exists a continuous mapping g of $p_J(S)$ to Z satisfying $f = g \circ p_J \restriction S$, where p_J is the projection of D to $D_J = \prod_{i\in J} D_i$.*

Proof. Let $T = S \cap \sigma D$. If F is a finite subset of I, then:

$$r_F(T) \subset r_F(S) \cap r_F(\sigma D) \subset S \cap \sigma D = T,$$

so T is finitely retractable. Since $T \subset \sigma D$, we conclude that T is retractable. Hence, T is mixing according to Lemma 4. Let $f_T = f \restriction T$. It follows from [1] (Theorem 2.3) that $J^* = \bigcap \mathcal{J}(f_T)$ is the smallest element of $\mathcal{J}(f_T)$, so f_T depends on J^*. By Lemma 6 (b), f also depends on J^* and $J = \bigcap \mathcal{J}(f) = J^*$. Hence, there exists a mapping $g\colon p_J(S) \to Z$ satisfying $f = g \circ p_J \restriction S$. Let us verify the continuity of g.

Suppose for a contradiction that there exist a point $y \in S$ and a net $\{y^c : c \in C\}$ in S such that $\{p_J(y^c) : c \in C\}$ converges to $p_J(y)$, but $\{f(y^c) : c \in C\}$ does not converge to $f(y)$ and is outside $\overline{W}$, for a neighborhood W of $f(y)$ in Z (the regularity of Z is used here). For $K \in [I]^{<\omega}$, define a point $u^c_K \in D$ equal to the identity e of D on $I \setminus K$, to y on $K \setminus J$ and to y^c on $J \cap K$. Since S is finitely retractable, Lemma 4 implies that $u^c_K \in S$. Then, the net $\{u^c_K : (c,K) \in C \times [I]^{<\omega}\}$ converges to y in S. Indeed, for any canonical neighborhood U of y in D, there exists $c_0 \in C$ such that $p_J(y^c) \in p_J(U)$ for each $c > c_0$. Take a finite set $C \subset I$ such that $U = p_C^{-1} p_C(U)$. If K is a finite subset of I with $K \supset C$ and $c > c_0$, then the point u^c_K belongs to U. Consequently, $\{f(u^c_K) : (c,K) \in C \times [I]^{<\omega}\}$ converges to $f(y)$ and $f(u^c_K) \in W$ for all (c,K) following some (c_1, K_1).

For $c \in C$ and $K \in [I]^{<\omega}$, let y^c_K be an element of S that is equal to e on $I \setminus K$ and to y^c on K. Since $p_J(u^c_K) = p_J(y^c_K)$, we have $f(u^c_K) = f(y^c_K)$. For a fixed $c \in C$, the net $\{y^c_K : K \in [I]^{<\omega}\}$ converges to y^c, whence $\lim_K f(y^c_K) = f(y^c)$. The latter implies that $f(y^c_K) \notin W$ for some $c > c_1$ and $K \supset K_1$, while $f(y^c_K) = f(u^c_K) \in W$. This contradiction completes the proof. □

In Theorem 2 below, we present an affirmative answer to Question 1 in the case when S is a finitely retractable submonoid of a product of topologized monoids and K is a regular quasitopological group. We recall that a quasitopological group is a semitopological group with continuous inversion (see [4] (Section 1.2)). Every quasitopological group has a local base at the identity consisting of symmetric sets.

Theorem 2. *Let S be a finitely retractable submonoid of a product $D = \prod_{i \in I} D_i$ of topologized monoids and $f\colon S \to K$ be a continuous homomorphism to a regular quasitopological NSS group K satisfying $\psi(K) \le \omega$. Then, f has a finite type.*

Proof. Let $J = \bigcap \mathcal{J}(f)$. By Proposition 5, there exists a continuous mapping $g\colon p_J(S) \to Z$ satisfying $f = g \circ p_J \restriction S$. Since p_J and f are homomorphisms, so is g. Let us show that the set J is countable.

As in the proof of Proposition 5, we put $T = S \cap \sigma D$. Then, T is a retractable submonoid of both S and D. Let $f_T = f \restriction T$. It follows from [1] (Corollary 1) (with T in place of S) that $J^* = \bigcap \mathcal{J}(f_T)$ is the smallest element of $\mathcal{J}(f_T)$, $|J^*| \le \omega$ and f_T depends on J^*. Hence, we can apply Lemma 6(b) to conclude that $J = J^*$. Thus, J is countable.

By Lemma 5, the image $p_J(S)$ is a finitely retractable submonoid of $D_J = \prod_{i \in J} D_i$. Therefore, replacing S with $p_J(S)$ and f with g, respectively, we can assume that the index set I is countable.

Claim 1. There exists a finite set $E \subset I$ such that $S \cap p_E^{-1} p_E(e) \subset f^{-1} f(e)$, where e is the identity element of D.

Suppose the claim is false. Since K is a quasitopological NSS group, there exists a symmetric neighborhood O of the identity e_K in K that does not contain nontrivial subgroups. Take a sequence $\{E_n : n \in \omega\}$ of finite subsets of I such that $E_n \subset E_{n+1}$ for each $n \in \omega$ and $I = \bigcup_{n \in \omega} E_n$. By our assumption, for every $n \in \omega$, there exists $x_n \in S$ such that $p_{E_n}(x_n) = p_{E_n}(e)$ and $y_n = f(x_n) \ne e_K$. It follows from our choice of O that there exists an integer k_n such that $y_n^{k_n} \notin O$. Note that $k_n \ne 0$. Since the set O is symmetric, we can assume that $k_n > 0$. Consider the sequence $\eta = \{x_n^{k_n} : n \in \omega\}$ of elements of S. It is clear that $p_{E_n}(x_n^{k_n}) = p_{E_n}(x_n)^{k_n} = p_{E_n}(e)$, for each $n \in \omega$. Hence, η converges to e in S. This contradicts the continuity of f since $f(x_n^{k_n}) = y_n^{k_n} \notin O$, for each $n \in \omega$. Claim 1 is proven.

From now on, we fix a set $E \subset I$ as in Claim 1.

Claim 2. The homomorphism f depends on E.

According to Lemma 6, it suffices to verify that $f \restriction T$ depends on E. Take arbitrary elements $x, y \in T$ with $p_E(x) = p_E(y)$ and suppose for a contradiction that $f(x) \neq f(y)$. It follows from $x, y \in T \subset \sigma D$ that the set $C_0 = \mathrm{diff}(x, e) \cup \mathrm{diff}(y, e)$ is finite. Since K is Hausdorff, we can find disjoint open neighborhoods O_x and O_y of $f(x)$ and $f(y)$, respectively, in K. By the continuity of f, there exist canonical open sets U_x and U_y in D containing x and y, respectively, such that $f(S \cap U_x) \subset O_x$ and $f(S \cap U_y) \subset O_y$. Take a finite set $C \subset I$ such that $U_x = p_C^{-1} p_C(U_x)$ and $U_y = p_C^{-1} p_C(y)$. We can assume without loss of generality that $E \cup C_0 \subset C$. Put $F = C \setminus E$. Then, the elements $x' = r_F(x)$, $y' = r_F(y)$, $x'' = r_E(x)$ and $y'' = r_E(y)$ belong to $S \cap \sigma D = T$ (the retractions r_E and r_F are defined in (2)). It also follows from our definitions that $x = x' \cdot x''$, $y = y' \cdot y''$, $x'' = y''$, and $p_E(x') = p_E(y') = p_E(e)$. Our choice of the set E (see Claim 1) implies that $f(x') = f(y') = e_K$. Therefore, we have:

$$f(x) = f(x' \cdot x'') = f(x') \cdot f(x'') = f(y') \cdot f(y'') = f(y' \cdot y'') = f(y).$$

This contradicts our assumption that $f(x) \neq f(y)$ and proves Claim 2.

By Lemma 3, there exists a homomorphism h of $p_E(S)$ to K satisfying $f = h \circ p_E \restriction S$, where $p_E \colon D \to \prod_{i \in E} D_i$ is the projection. Clearly, every continuous retraction is a quotient mapping. Since $p_E = p_E \circ r_E$ and the restriction to $r_E(S)$ of the projection p_E is a homeomorphism of $r_E(S)$ onto $p_E(S)$, we conclude that $p_E \restriction S$ is a quotient mapping. Hence, the equality $f = h \circ p_E \restriction S$ implies that h is continuous, as required. □

The following auxiliary fact is well known in the topological algebra folklore. For the sake of completeness, we supply the reader with a short proof of it.

Lemma 7. *Every topological NSS group K is Hausdorff and satisfies $\psi(K) \le \omega$.*

Proof. Let e be the identity of K and N be the closure of the singleton $\{e\}$ in K. If K fails to be Hausdorff, then N is a nontrivial subgroup of K. By our assumptions, there exists an open neighborhood U of e in K that does not contain nontrivial subgroups. Hence, $N \setminus U \neq \emptyset$. Pick an element $x \in N \setminus U$, and choose an open symmetric neighborhood V of e in K such that $V^2 \subset U$. Then, $V \cap xV = \emptyset$—otherwise $x \in VV^{-1} = V^2 \subset U$, which contradicts the choice of x. Since xV is an open neighborhood of x, we conclude that x does not belong to the closure of V. Hence, $x \notin N$. This contradiction proves that K is Hausdorff.

To show that $\psi(K) \le \omega$, we take an open neighborhood U of e in K as above. There exists a sequence $\{U_n : n \in \omega\}$ of open symmetric neighborhoods of e in K such that $U_0 \subset U$ and $U_{n+1}^2 \subset U_n$, for each $n \in \omega$. Then, $H = \bigcap_{n \in \omega} U_n$ is a subgroup of K satisfying $H \subset U_0 \subset U$. Hence, our choice of U implies that $H = \{e\}$, whence the required inequality $\psi(K) \le \omega$ follows. □

Combining Theorem 2 and Lemma 7, we deduce the following:

Corollary 4. *Let S be a finitely retractable submonoid of a product $\prod_{i \in I} D_i$ of topologized monoids and $f \colon S \to K$ be a continuous homomorphism to a topological NSS group K. Then, f has a finite type.*

Corollary 5. *Let S be a submonoid of a product $D = \prod_{i \in I} D_i$ of topologized monoids and $f \colon S \to K$ be a continuous homomorphism to a topological NSS group K. If $\sigma D \subset S$, then f has a finite type.*

Proof. It follows from $\sigma D \subset S$ that $r_C(S) \subset \sigma D \subset S$, for each finite subset C of I. Hence, S is finitely retractable. By Lemma 7, K is Hausdorff (hence, regular) and satisfies $\psi(K) \leq \omega$. It remains to apply Theorem 2. $\square$

Proposition 6. *Let S be a submonoid of a product $D = \prod_{i\in I} D_i$ of semitopological monoids with open shifts such that $p_J(S) = \prod_{i\in J} D_i$, for each countable set $J \subset I$. Then, every continuous homomorphism of S to a first countable topological NSS group has a finite type.*

Proof. It follows from the assumptions of the proposition that S is dense in D. Therefore, Corollary 3 implies that there exists a countable subset C of I such that f depends on C. Therefore, we can find a homomorphism $g\colon p_C(S) \to K$ satisfying $f = g \circ p_C \restriction S$. Since projections of S fill all countable subproducts of D, the restriction of p_C to S is an open mapping (see, e.g., [2] (Lemma 7)). Hence, g is a continuous homomorphism of $p_C(S) = D_C = \prod_{i\in C} D_i$ to K. Applying Corollary 4 to D_C and g in place of S and f, respectively, we conclude that g has a finite type. Hence, f has a finite type as well. $\square$

Problem 3. *In Proposition 6,*

(a) *is it possible to weaken the assumptions about S assuming that the projections of S fill all finite subproducts of D?*
(b) *is it possible to drop the assumption that K is first countable?*

Our next aim is to present an analogue of Corollary 2 in the case when K is a topological NSS group. This requires an auxiliary result on continuous isomorphisms of topological NSS groups onto metrizable left topological groups (see Corollary 6).

Proposition 7. *Every Hausdorff topological group G with $\psi(G) \leq \omega$ admits a continuous isomorphism onto a left topological group H whose topology is generated by a left invariant metric. If G is an NSS group, then one can choose H to be an NSS group as well.*

Proof. Let G be a topological group satisfying $\psi(G) \leq \omega$. Choose a sequence $\{U_n : n \in \omega\}$ of open symmetric neighborhoods of the identity e in G such that $U_{n+1}^2 \subset U_n$ for each $n \in \omega$ and $\{e\} = \bigcap_{n\in\omega} U_n$. If G is an NSS group, we can additionally assume that U_0 does not contain nontrivial subgroups. According to [4] (Lemma 3.3.10), there exists a continuous prenorm N on G such that:

$$V_n = \{x \in G : N(x) < 1/2^n\} \subset U_n \subset \{x \in G : N(x) \leq 2/2^n\},$$

for each $n \in \omega$. It follows from our choice of the sets U_n and the prenorm N that $N(x) = 0$ if and only if $x = e$. Hence, N is a continuous norm on G.

We define a continuous left invariant metric d on G by letting $d(x, y) = N(x^{-1}y)$ for all $x, y \in G$. Let $\mathcal{T}_d$ be the topology of G generated by d. Clearly, $\mathcal{T}_d$ is coarser than the original topology of G. We claim that the space $H = (G, \mathcal{T}_d)$ with the same multiplication is a left topological group.

First, we verify that the family $\mathcal{B} = \{xV_n : x \in G,\ n \in \omega\}$ is a base for $\mathcal{T}_d$. Indeed, take arbitrary elements xV_p and yV_q of $\mathcal{B}$ and a point $z \in xV_p \cap yV_q$. Then, $x^{-1}z \in V_p$ and $y^{-1}z \in V_q$, so $N(x^{-1}z) < 1/2^p$ and $N(y^{-1}z) < 1/2^q$. There exists $k \in \omega$ such that $N(x^{-1}z) + 1/2^k < 1/2^p$ and $N(y^{-1}z) + 1/2^k < 1/2^q$. Since N is a prenorm, our choice of k implies that $x^{-1}zV_k \subset V_p$ and $y^{-1}zV_k \subset V_q$, whence it follows that $zV_k \subset xV_p \cap yV_q$. Since $zV_k \in \mathcal{B}$, this proves that $\mathcal{B}$ is a base for a topology on G.

For an arbitrary element $x \in G$ and an integer $n \geq 0$, we have the equality:

$$xV_n = \{y \in G : d(x, y) < 1/2^n\}.$$

Hence, the elements of $\mathcal{B}$ are open in H and form a base for the metric topology $\mathcal{T}_d$.

Finally, the left translations in G preserve the base $\mathcal{B}$ and, therefore, are open bijections of H onto itself. Since the inverse of the left translation λ_x, with $x \in G$, is the left translation $\lambda_{x^{-1}}$, which is also open, we conclude that λ_x is an autohomeomorphism of H. This shows that H is a left topological group. Notice that if G is an NSS group, then V_0 is an open neighborhood of the identity in H, which does not contain nontrivial subgroups. Thus, the identity mapping of G onto H is the required continuous isomorphism. □

The next fact follows from Lemma 7 and Proposition 7.

Corollary 6. *Every topological NSS group admits a continuous isomorphism onto a metrizable left topological NSS group.*

We can now present the following result complementing Corollary 2:

Proposition 8. *Let $D = \prod_{i \in I} D_i$ be a product of left topological groups, S a subgroup of D, and $f\colon S \to K$ a continuous homomorphism to a topological NSS group K. If $p_F(S) = \prod_{i \in F} D_i$ for each finite set $F \subset I$, then f has a finite type.*

Proof. According to Lemma 7, the group K is Hausdorff and satisfies $\psi(K) \le \omega$. Hence, Corollary 6 implies that there exists a continuous isomorphism $j\colon K \to L$ onto a metrizable left topological NSS group L. Let $f^* = j \circ f$. We can apply Proposition 2 to f^* and find a finite subset E of the index set I such that f^* depends on E. Therefore, there exists a homomorphism $g\colon p_E(S) \to L$ satisfying $f^* = g \circ p_E{\restriction}S$.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "p_E{\restriction}S"'] \arrow[dr, "f^*"] & K \arrow[d, "j"] \\ p_E(S) \arrow[r, "g"] & L \end{tikzcd}$$

Let $g^* = j^{-1} \circ g$. Clearly, the homomorphism g^* satisfies the equality $f = g^* \circ p_E{\restriction}S$. It follows from our assumptions about S that the restriction $p_E{\restriction}S$ is an open continuous homomorphism of S onto $p_E(S) = D_E$. Since f is continuous, the latter equality implies the continuity of g^*. Hence, f has a finite type. □

Proposition 8 has several applications to the study of continuous homomorphic images. We give here only two of them. Let us recall that a Tychonoff space X is called submetrizable if it admits a weaker metrizable topology.

Corollary 7. *Let $D = \prod_{i \in I} D_i$ be a product of σ-compact left topological groups, S a subgroup of D, and $f\colon S \to F(X)$ a continuous homomorphism to the free topological group $F(X)$ on a submetrizable space X. If $p_C(S) = \prod_{i \in C} D_i$ for each finite set $C \subset I$, then the subgroup $f(S)$ of $F(X)$ is σ-compact and has a countable network. The same conclusion is valid for the free abelian topological group $A(X)$ in place of $F(X)$.*

Proof. Since X is submetrizable, it follows from the main theorem in [13] (see a correction in [14]) that $F(X)$ is an NSS group. Therefore, by Proposition 8, one can find a finite set $E \subset I$ and a continuous homomorphism $g\colon p_E(S) \to F(X)$ satisfying $f = g \circ p_E{\restriction}S$. In particular, the subgroup $f(S)$ of $F(X)$ is a continuous homomorphic image of the group $p_E(S) = D_E = \prod_{i \in E} D_i$. Since the set E is finite and each factor D_i is σ-compact, so are D_E and its continuous image $f(S) = g(D_E)$.

Further, by [13] (Lemma 1), the group $F(X)$ admits a continuous metric and hence is submetrizable. Since every compact subspace of a submetrizable space has a countable base (hence a countable network) and $f(S)$ is σ-compact, we conclude that $f(S)$ has a countable network. Finally, the above argument applies without changes to the group $A(X)$. □

Remark 1. *If in the assumptions of Corollary 7, one replaces the σ-compactness of the factors D_i with the requirement $d(D_i) \leq \omega$ (resp., $nw(D_i) \leq \omega$) for each $i \in I$, then accordingly, the conclusion changes to $d(f(S)) \leq \omega$ (resp., $nw(f(S)) \leq \omega$). To see it, one can apply the argument in the proof of the corollary along with the fact that continuous mappings do not increase either the density or network weight.*

Another curious application of Proposition 8 is given below. We recall that a space X is feebly compact if every locally finite family of open sets in X is finite. In Tychonoff spaces, feeble compactness and pseudocompactness coincide.

Corollary 8. *Let $D = \prod_{i \in I} D_i$ be a product of feebly compact paratopological groups, S a subgroup of D, and $f \colon S \to K$ a continuous homomorphism onto a topological NSS group K. If $p_F(S) = \prod_{i \in F} D_i$ for each finite set $F \subset I$, then K is a compact Lie group.*

Proof. It follows from Proposition 8 that f has a finite type, so we can find a finite set $E \subset I$ and a continuous homomorphism $g \colon p_E(S) \to K$ satisfying $f = g \circ p_E \restriction S$. Hence, the group K is a continuous homomorphic image of the group $p_E(S) = D_E = \prod_{i \in E} D_i$.

By Ravsky's result (see [15] (Theorem 4.1) or [16] (Theorem 2.7.9)), the product group D_E is feebly compact. Hence, the continuous image K of D_E is also feebly compact. By Lemma 7, the topological group K is Hausdorff and has a countable pseudocharacter. Since the space K is Tychonoff and feeble compactness coincides with pseudocompactness in Tychonoff spaces, we conclude that K is pseudocompact. Furthermore, every pseudocompact topological group of countable pseudocharacter is compact and has a countable base [16] (Proposition 2.3.12). Hence, K is compact. Finally, every (locally) compact topological NSS group is a Lie group, by [17]. $\square$

3. Open, Dense, and More General Submonoids of Products

There exist many results on the subject of when a continuous mapping $f \colon S \to K$ defined on a subspace S of a Tychonoff product $D = \prod_{i \in I} D_i$ of spaces has a countable (or even finite) type. The article [18] by M. Hušek presented a comprehensive survey of results and methods on factoring continuous mappings. Corollaries 1–3 in [2], as well as our results in Section 2 contribute to the corresponding study of continuous homomorphisms.

The wealth of results and methods for factoring continuous mappings and homomorphisms enables us to use the following two-step strategy for solving the general problem formulated in the introduction. First, we impose purely topological restrictions on S, D, and K to guarantee that every continuous mapping (or homomorphism) $f \colon S \to K$ has a countable type. Second, making use of the algebraic structures of S and K and the fact that f is a homomorphism, we try to show that, actually, f has a finite type. In fact, we apply this strategy in the proof of Theorem 2. We use a similar approach in the proof of Theorem 3 below.

A space X is said to be pseudo-ω_1-compact if every locally finite family of open sets in X is countable. Several authors use the term Discrete Countable Chain Condition (DCCC) in place of pseudo-ω_1-compactness. Let us also recall that a space Y has a regular G_δ-diagonal if there exists a countable family $\{O_n : n \in \omega\}$ of open neighborhoods of the diagonal $\Delta_Y = \{(y, y) : y \in Y\}$ in $Y \times Y$ such that $\Delta_Y = \bigcap_{n \in \omega} \overline{O_n}$. Notice that a space with a regular G_δ-diagonal is Hausdorff. It follows from Proposition 7 that every topological group of a countable pseudocharacter has a regular G_δ-diagonal (see also [2] (Lemma 9)).

A subset X of a (semi)topological group G is called ω-narrow (in G) if for every neighborhood U of the identity e in G, there exists a countable set $C \subset G$ such that $X \subset CU \cap UC$. Similarly, G is ω-narrow if it is ω-narrow in itself (see [4] (Section 3.4)). These concepts will be used in the proof of the following theorem.

Theorem 3. *Let S be a submonoid of a product $D = \prod_{i \in I} D_i$ of left topological monoids such that $p_J(S) = \prod_{i \in J} D_i$, for each countable subset J of I. If D is pseudo-ω_1-compact, then the following are valid:*

(a) *every continuous homomorphism $f\colon S \to K$ to a topological NSS group K has finite type;*
(b) *in (a), the image $f(S)$ admits a weaker separable metrizable topology.*

Proof. By virtue of Lemma 7, we have that $\psi(K) \le \omega$. Hence, K has a regular G_δ-diagonal. Since S fills all countable subproducts of D, it follows from [19] (Theorem 3.8) with $\kappa = \omega$ and $\alpha = \omega_1$ (see also [2] (Lemma 8)) that S is pseudo-ω_1-compact. Then, we apply [20] (Theorem 5) or [2] (Proposition 3]) to conclude that f depends on a countable set $J \subset I$. Hence, we can find a mapping g of $p_J(S) = \prod_{i\in J} D_i$ to K satisfying $f = g \circ p_J\restriction S$. It follows from our assumptions about S that the restriction of p_J to S is an open mapping (see, e.g., [2] (Lemma 7)), so the equality $f = g \circ p_J\restriction S$ implies that g is continuous.

Since the monoid $p_J(S) = \prod_{i\in J} D_i = D_J$ is obviously (finitely) retractable, we can apply Corollary 4 to the homomorphism g to conclude that g has a finite type. Hence, the homomorphism $f = g \circ p_J\restriction S$ has a finite type, as well.

Note that $Y = f(S) \subset K$ is pseudo-ω_1-compact as a continuous image of S. Hence, Y is an ω-narrow subspace of K [4] (Proposition 5.1.15). Applying Theorem 5.1.19 of [4], we conclude that the subgroup $L = \langle Y\rangle$ of K generated by Y is also ω-narrow. Further, we have that $\psi(L) \le \psi(K) \le \omega$. By virtue of [4] (Corollary 3.4.25), every Hausdorff ω-narrow topological group of a countable pseudocharacter admits a continuous isomorphism onto a separable metrizable topological group. Hence, the subspace Y of L also admits a weaker separable metrizable topology. $\square$

Problem 4. *Does Theorem 3 (a) remain valid without the assumption that D is pseudo-ω_1-compact? What if, additionally, the group K is first countable?*

The following simple lemma is not used in this article. However, it clarifies the permanence properties of the class of topologized monoids (semigroups) with open shifts.

Lemma 8. *Let $D = \prod_{i=1}^n D_i$ be a product of topologized monoids (semigroups) with open left shifts. Then the left shifts in D are also open. Further, if $f\colon G \to H$ is an open continuous homomorphism of a topologized monoid (semigroup) G with open shifts onto a topologized monoid (semigroup) H, then the left shifts in H are also open.*

Proof. Take an arbitrary element $x = (x_1, \dots, x_n) \in D$. If $U = U_1 \times \cdots \times U_n$ is a basic open set in D, then the set $xU = x_1U_1 \times \cdots \times x_nU_n$ is open in D. Clearly, every open set in D is the union of a family of basic open sets, so we conclude that the left shifts in D are open.

To prove the second part of the lemma, take an element $y \in H$ and an open set $V \subset H$. Choose $x \in G$ with $f(x) = y$. Since the set $U = f^{-1}(V)$ is open in G, so is xU. Hence, $yV = f(xU)$ is open in H. $\square$

It is worth noting that Lemma 8 is not valid for infinite products. In fact, the left shifts in the product $D = \prod_{i\in I} D_i$ of left topological monoids are open if and only if each factor D_i has open left shifts and almost all the factors D_i, except for finitely many of them, are groups. This result easily follows from the purely algebraic fact that if all left shifts in a monoid D are onto mappings, then D is a group (apply Exercise 14 on page 9 of [21]).

In what follows, the codomain K of a continuous homomorphism $f\colon S \to K$ will be an arbitrary locally compact topological NSS group or, equivalently, a Lie group (see [17]). Prior to the proof of Theorem 4, the main result of this section, we present two lemmas that do almost all the job. The first of them is close to [22] (Corollary 3.4) and, in fact, is a special case of [23] (Theorem 0.5), which states that every compact Hausdorff semigroup with separately continuous multiplication and two-sided cancellation is a topological group.

Lemma 9. *Let P be a compact submonoid of a Hausdorff semitopological group G. Then, P with the topology inherited from G is a topological group.*

Lemma 10. *Let S be an arbitrary submonoid of a product $D = \prod_{i\in I} D_i$ of topological monoids and $f\colon S \to K$ be a continuous homomorphism to a Lie group K. Then, there exists a finite set $E \subset I$ such that for every neighborhood O of the identity in K, one can find a canonical open neighborhood V of the identity in D satisfying $V = p_E^{-1} p_E(V)$ and $f(S \cap V) \subset O$.*

Proof. Clearly, the group K is Hausdorff and first countable. Let O^* be an open symmetric neighborhood of the identity e_K in K that does not contain nontrivial subgroups. There exists an open neighborhood O_* of e_K such that the closure of O_* in K is compact and contained in O^*. Take a canonical open neighborhood U of the identity element e in D such that $f(S \cap U) \subset O_*$. Then, $U = p_E^{-1} p_E(U)$, for a finite subset E of I. We claim that the set E is as required.

Choose a local base $\{O_n : n \in \omega\}$ at the identity in K. Let $\mathcal{V}$ be the family of canonical open neighborhoods V of e in D satisfying $V = p_E^{-1} p_E(V)$. Let us put:

$$P = \bigcap\{\overline{f(S \cap V)} : V \in \mathcal{V}\}.$$

It is easy to see that P is a compact submonoid of K. Indeed, take arbitrary elements $x, y \in P$. Given an element $V \in \mathcal{V}$, choose $W \in \mathcal{V}$ with $W^2 \subset V$. Then, $x, y \in P \subset \overline{f(S \cap W)}$, whence it follows that:

$$xy \in \overline{f(S \cap W)} \cdot \overline{f(S \cap W)} \subset \overline{f(S \cap W) \cdot f(S \cap W)} \subset \overline{f(S \cap W^2)} \subset \overline{f(S \cap V)}.$$

Since the above inclusions are valid for every $V \in \mathcal{V}$, we see that $xy \in P$. This proves that $PP \subset P$. It is also clear that $e_K \in P$, so P is a submonoid of K. It also follows from $U \in \mathcal{V}$ and the inclusion $f(S \cap U) \subset O_*$ that $P \subset \overline{f(S \cap U)} \subset \overline{O_*}$, so P is compact. Thus, P is a compact submonoid of K.

Since the Lie group K is locally compact, it follows from Lemma 9 that P is a subgroup of K. Hence, the inclusions $P \subset \overline{O_*} \subset O^*$ and our choice of O^* together imply that $P = \{e_K\}$. Let $m \in \omega$ be an arbitrary integer. If $f(S \cap V) \setminus O_m \neq \emptyset$ for each $V \in \mathcal{V}$, then the family $\{\overline{f(S \cap V)} \setminus O_m : V \in \mathcal{V}\}$ is a base of a filter, which contains the compact set $\overline{f(S \cap U)} \setminus O_m$. Hence, the set:

$$\bigcap\left\{\overline{f(S \cap V)} \setminus O_m : V \in \mathcal{V}\right\} = \bigcap\left\{\overline{f(S \cap V)} : V \in \mathcal{V}\right\} \setminus O_m = P \setminus O_m$$

is nonempty. Clearly, this is impossible since $P = \{e_K\}$ and $e_K \in O_m$. We have thus proven that there exists $V \in \mathcal{V}$ satisfying $f(S \cap V) \subset O_m$. This completes the proof of the lemma. □

Theorem 4. *Let S be a submonoid of a product $D = \prod_{i\in I} D_i$ of topological monoids with open shifts such that either:*

(a) $p_C(S) = \prod_{i\in C} D_i$, *for each finite set $C \subset I$, or*
(b) *S is open and dense in D.*

Then, every continuous homomorphism $f\colon S \to K$ to a Lie group K has a finite type.

Proof. By Lemma 10, there exists a finite set $E \subset I$ such that for every neighborhood O of the identity in K, one can find a canonical open neighborhood V of the identity e in D satisfying $V = p_E^{-1} p_E(V)$ and $f(S \cap V) \subset O$.

In both cases (a) and (b), S is dense in D. Hence, Theorem 1 implies that f depends on E. Therefore, there exists a homomorphism $g\colon p_E(S) \to K$ satisfying $f = g \circ p_E \restriction S$. We claim that the homomorphism g is continuous. Indeed, in Case (a), we have that $p_C(S) = D_C$ for each finite set $C \subset I$, so the restriction $p_E \restriction S$ is an open continuous mapping of S onto D_E. The same conclusion is valid in Case (b) since S is open in D. Hence, the continuity of g follows from the equality $f = g \circ p_E \restriction S$. We have thus proven that f has a finite type. □

Corollary 9. *Let S be a submonoid of a product $D = \prod_{i\in I} D_i$ of topological monoids with open shifts such that $p_C(S) = \prod_{i\in C} D_i$, for each finite set $C \subset I$. Then, every continuous character of S has a finite type.*

Corollary 10. *Let S be a submonoid of a product $D = \prod_{i\in I} D_i$ of topological monoids with open shifts such that $p_C(S) = \prod_{i\in C} D_i$, for each finite set $C \subset I$. Then, every continuous homomorphism of S to a discrete group has a finite type.*

The next problem suggests a way of generalizing Item (b) of Theorem 4 (see, e.g., Theorem 1).

Problem 5. *Let $D = \prod_{i\in I} D_i$ be a product of topological monoids with open shifts and S be an open submonoid of D. Does every continuous homomorphism of S to a Lie group have a finite type?*

In the special case when S is a subgroup of the product $D = \prod_{i\in I} D_i$ of topological monoids D_i, we solve Problem 5 in the affirmative, even without assuming S to be open in D.

Theorem 5. *Let $D = \prod_{i\in I} D_i$ be a product of topological monoids and S be an arbitrary subgroup of D. Then, every continuous homomorphism $f\colon S \to K$ to a Lie group K has a finite type.*

Proof. Replacing the factors D_i with the projections $S_i = p_i(S)$ of S and observing that each S_i is a paratopological group, we can assume that each factor D_i is a paratopological group. Hence, so are D and S.

Let $\{O_n : n \in \omega\}$ be a local base at the identity e_K of K. It follows from Lemma 10 that there exists a finite set $E \subset I$ such that for every $n \in \omega$, one can find an open neighborhood V of the identity e in D satisfying $V = p_E^{-1} p_E(V)$ and $f(S \cap V) \subset O_n$. Since S, K, and $T = p_E(S) \subset D_E$ are left topological groups, we can apply Corollary 1 to find a continuous homomorphism $g\colon T \to K$ satisfying $f = g \circ p_E \restriction S$. Hence, f has a finite type. □

The following two corollaries are immediate from Theorem 5.

Corollary 11. *Let $D = \prod_{i\in I} D_i$ be a product of topological monoids and S be a subgroup of D. Then, every continuous character of S has a finite type.*

Corollary 12. *Let $D = \prod_{i\in I} D_i$ be a product of topological monoids and S be a subgroup of D. Then, every continuous homomorphism of S to a discrete group has a finite type.*

We do not know whether one can drop the requirement on S to be open in Theorem 4 (b):

Problem 6. *Let $D = \prod_{i\in I} D_i$ be a product of topological monoids with open shifts and S be an arbitrary dense submonoid of D. Does every continuous homomorphism of S to a Lie group have a finite type?*

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

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