# Cantor Waves for Signorini Hyperelastic Materials with Cylindrical Symmetry

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**Abstract:** In this paper, local fractional cylindrical wave solutions on Signorini hyperelastic materials are studied. In particular, we focus on the so-called Signorini potential. Cantor-type cylindrical coordinates are used to analyze, both from dynamical and geometrical point of view, wave solutions, so that the nonlinear fundamental equations of the fractional model are explicitly given. In the special case of linear approximation we explicitly compute the fractional wave profile.

**Keywords:** elastic cylindrical waves; signorini hyperelastic potential; nonlinearity; Cantor-type cylindrical coordinate method; local fractional derivative

## 1. Introduction

In this paper, the fractional differential equations for cylindrical waves in hyperelastic materials are given. From mechanical point of view, the considered hyperelastic material is described by the equations of elastic deformation when the stress tensor can be derived from a potential [1–4]. Among the several choices of elastic potentials, we focus on the so-called Signorini potential, which is characterized by the well known Lamé constants $(\lambda, \mu)$ and by a third parameter $c$ [1,5–7], thereby opening new perspectives on the definition and analysis of new hyperelastic materials. There exist many different approaches to describing hyperelastic materials. The theory of hyperelastic materials is mainly based on the assumption that the deformations might be represented by some potential energy or elastic potential. These potentials depend on finite strain-tensor or its fundamental invariants, so that there exist in the literature several different definitions of potentials named after the discoverers: Rivlin-Sanders, John, Murnaghan, Signorini, etc. The differences among them are due both to the order of invariants and to the presence of different phenomenological constants. In the simplest case we have the classical Lamé constants. In the following we will consider hyperelastic materials with the Signorini potential (see (8)). The existence of a potential is a powerful method to analyze complex phenomena. In a recent paper [8] (see also references therein) the authors proposed a geometrical potential to explain the cell orientation in nanofibers.

In the following we are interested on the cylindrical wave solutions, and for this reason it is assumed that there is an axial symmetry in the material, and as a consequence, that there exists a privileged coordinate system; i.e., cylindrical coordinates. These coordinates enable one to simplify the structure of the differential equations for hyperelastic materials as a consequence of symmetries in the corresponding equations. The solutions of a such kinds of equations are in short called cylindrical waves. In [9–11] cylindrical waves are explicitly computed and some nonlinear effects are also shown. In particular, in [9], cylindrical waves in Signorini materials are studied up to the third order nonlinearity. In the papers [9–11] it is also shown that cylindrical waves are obtained as solutions of some generalized Weber equation [4], and they can be easily expressed in terms of some special functions; e.g., the Bessel functions [4,12]. Moreover, cylindrical waves can be studied by using some more generalized differential operators, such as the fractional operators.

In recent papers there has been increasing interest for the so-called fractional differential operators (and fractional differential equations). Indeed the idea that the ordinary differential operator can be generalized to a fractional order derivative was already discovered at the birth of differential calculus by Cauchy, but only recently has fractional calculus significantly grown (see, e.g., [13–16] and references therein). However aside from the many perspectives opened by the fractional calculus, the main drawback is the existence of many families of fractional differential operators, because the fractional derivative is not univocally defined. In the following, we focus on the Yang local fractional derivative [13,15], also known as fractal derivative, which has been successfully applied to the solutions of many interesting problems (see, e.g., [13,17,18]).

The local fractional calculus is somehow based on a quite natural definition of the differential operator which naturally inherits almost all properties of the ordinary differential calculus. The local fractional differential operators are computed, however, in a cylindrical coordinate system defined on local trigonometric functions, also called a Cantor system of coordinates [13,18]. In some recent papers it has been also shown that a fractional differential model can be suitably represented by a a fractal variational model so that it is possible to define a suitable variational principle expressing the stationarity of a physical potential.

This paper is organized as follows: Section 1 deals with some preliminary remarks on Cantor cylindrical coordinates; in Sections 2 and 3, local fractional derivative and local fractional covariant derivative are defined for cylindrical coordinates. In Section 4 the hyperelastic Signorini material is defined and the local fractional covariant equations are derived. Cylindrical waves are computed in Section 5, where the solution of the linear case is explicitly computed and discussed.

## 2. Cantor Metric Tensor

Let $(\xi, \eta, \zeta)$, be the Lagrangian cylindrical coordinate system defined with respect to a fixed coordinate system as

$$x = \xi \cos \eta \quad , \quad y = \xi \sin \eta \quad , \quad z = \zeta .$$

The Cantor-type cylindrical coordinates $\xi = r^\alpha$, $\eta = \vartheta^\alpha$, $\zeta = z^\alpha$ are defined as [13,18,19]:

$$\begin{cases} x^\alpha = r^\alpha \cos_\alpha \left( \vartheta^\alpha \right), \\ y^\alpha = r^\alpha \sin_\alpha \left( \vartheta^\alpha \right), \\ z^\alpha = z^\alpha, \end{cases} \tag{1}$$

where $\alpha \in (0,1]$, $r^\alpha \in (0,+\infty)$, $z^\alpha \in (-\infty,+\infty)$, $\vartheta \in (0,2\pi]$, $x^{2\alpha} + y^{2\alpha} = r^{2\alpha}$. This definition depends on the additional fractional order parameter $\alpha$, so that when $\alpha = 1$ we get the ordinary cylindrical coordinates. The functions $\sin_\alpha \left( \vartheta^\alpha \right), \cos_\alpha \left( \vartheta^\alpha \right)$ are defined, according to [13,18,19], as power series with Gamma function coefficients

$$\sin_\alpha \left( \vartheta^\alpha \right) = \sum_{n=0}^\infty \frac{(-1)^n \, \vartheta^{(2n+1)\alpha}}{\Gamma \left( 1 + (2n+1)\alpha \right)}$$

$$\cos_\alpha \left( \vartheta^\alpha \right) = \sum_{n=0}^\infty \frac{(-1)^n \, \vartheta^{2n\alpha}}{\Gamma \left( 1 + 2n\alpha \right)} .$$

The Euler's gamma function is defined as

$$\Gamma \left( 1 + \alpha \right) = \int_0^\infty x^{\alpha-1} e^{-x} dx .$$

It goes also, by definition

$$\begin{pmatrix} \alpha \\ i \end{pmatrix} = \frac{\Gamma(1+\alpha)}{\Gamma(1+i)\,\Gamma(1+\alpha-i)} \; .$$

By using the Mittag Leffler function

$$E_\alpha(\vartheta^\alpha) = \sum_{n=0}^{\infty} \frac{\vartheta^{n\alpha}}{\Gamma(1+n\alpha)} \; ,$$

we can also write

$$\sin_\alpha(\vartheta^\alpha) = \frac{E_\alpha(i^\alpha \vartheta^\alpha) - E_\alpha(-i^\alpha \vartheta^\alpha)}{2i^\alpha}$$

$$\cos_\alpha(\vartheta^\alpha) = \frac{E_\alpha(i^\alpha \vartheta^\alpha) + E_\alpha(-i^\alpha \vartheta^\alpha)}{2}$$

The infinitesimal fundamental form is

$$(ds)^2 = g_{ik} d\theta^i d\theta^k \qquad i,k = 1,2,3$$

with cylindrical coordinates

$$\theta^1 = r \quad ,\theta^2 = \vartheta \quad , \quad \theta^3 = z \; .$$

By assuming the Cantor-type cylindrical coordinates (1), and the cylindrical symmetry

$$g_{ik} = 0 \quad i \neq k$$

we have the fundamental Cantor form in cylindrical coordinates

$$(ds)^2 = g_{ik} d^\alpha \theta^i d^\alpha \theta^k = (d^\alpha r)^2 + r^2 (d^\alpha \vartheta)^2 + (d^\alpha z)^2 \tag{2}$$

where the Cantor metric tensor of cylindrical coordinates is defined as

$$\|g_{ik}\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad \|g^{ik}\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{Vmatrix} . \tag{2'}$$

In order to study the differential properties of the manifold, we need to define the partial derivative. We assume, as fractional operator the Yang local fractional derivative model (see, e.g., [13,17,18]). In the next section, the main properties of this operator are briefly summarized.

## 3. Local Fractional Calculus

### 3.1. Yang Local Fractional Derivative

The local fractional derivative of the function $f_\alpha(\theta)$ of order $\alpha$ $(0 < \alpha < 1)$ at the fixed value $\theta = \theta_0$ is defined by [13,18]:

$$D^{(\alpha)} f_\alpha(\theta) = \frac{d^\alpha f_\alpha(\theta)}{d\theta^\alpha}\bigg|_{\theta=\theta_0} = \lim_{\theta \to \theta_0} \frac{\Delta^\alpha (f_\alpha(\theta) - f_\alpha(\theta_0))}{(\theta - \theta_0)^\alpha}, \tag{3}$$

being

$$\Delta^\alpha (f_\alpha(\theta) - f_\alpha(\theta_0)) \cong \Gamma(1+\alpha)\, \Delta(f_\alpha(\theta) - f_\alpha(\theta_0)),$$

and the forward difference $\Delta$ defined as

$$\Delta f(\theta) = f(\theta) - f(\theta_0)$$

so that

$$\Delta[f(\theta) - f(\theta_0)] = f(\theta) - f(\theta_0).$$

It is also

$$d\theta^\alpha \cong \Gamma(1+\alpha)\, d\theta$$

According to the definition (3), it can be easily shown (see, e.g., [13,18]) that

$$D^{(\alpha)}\theta^\alpha = \frac{d^\alpha \theta^\alpha}{d\theta^\alpha} = \Gamma(1+\alpha) \quad , \quad d^\alpha \theta^\alpha = \Gamma(1+\alpha)\, d\theta^\alpha$$

$$D^{(\alpha)}\theta^{k\alpha} = \frac{d^\alpha \theta^{k\alpha}}{d\theta^\alpha} = \frac{\Gamma(1+\alpha)}{\Gamma(1+(k-1)\alpha)}\theta^{(k-1)\alpha}$$

$$D^{(\alpha)}\sin_\alpha \theta^\alpha = \frac{d^\alpha}{d\theta^\alpha}\sin_\alpha \theta^\alpha = \cos_\alpha \theta^\alpha$$

$$D^{(\alpha)}\cos_\alpha \theta^\alpha = \frac{d^\alpha}{d\theta^\alpha}\cos_\alpha \theta^\alpha = \sin_\alpha \theta^\alpha$$

In some recent papers [20,21] the authors proposed a variational principle to derive some anisotropic waves, and by using the fractal derivative (3), it is possible to study an interesting alternative approach to the Yang fractional derivative

### 3.2. Local Fractional Covariant Derivatives in Cantor Cylindrical Coordinates

In order to define the local fractional equations in Cantor cylindrical coordinates we need to compute first the local fractional covariant derivative.

The (ordinary) covariant derivatives of a vector $\{v_i\}$ is

$$\nabla_i v^k = \frac{\partial v^k}{\partial \theta^i} + v^j \Gamma^k_{ji}, \quad \nabla_i v_j = \frac{\partial v_j}{\partial \theta^i} - v_k \Gamma^k_{ji},$$

which can be easily computed by means of the Christoffel's symbols

$$\Gamma^m_{ki} = \frac{1}{2}g^{mn}\left(\frac{\partial g_{kn}}{\partial \theta^i} + \frac{\partial g_{in}}{\partial \theta^k} - \frac{\partial g_{ki}}{\partial \theta^n}\right)$$

and the metric tensor values (2).

Thanks to (1) the only non-vanishing components of these symbols are

$$\Gamma^1_{22} = -r, \quad \Gamma^2_{12} = \Gamma^2_{21} = (1/r).$$

The covariant fractional derivatives of a vector $\{v_i\}$ are defined as

$$\nabla_i^\alpha v^k = \frac{\partial^\alpha v^k}{\partial \theta^i_\alpha} + v^j \overset{\alpha}{\Gamma}{}^k_{ji}, \quad \nabla_i^\alpha v_j = \frac{\partial^\alpha v_j}{\partial \theta^i_\alpha} - v_k \overset{\alpha}{\Gamma}{}_{ji}{}^k$$

The fractional covariant derivative can be easily computed by means of the fractional Christoffel's symbols

$$\overset{\alpha}{\Gamma}{}^m_{ki} = \frac{1}{2}g^{mn}\left(\frac{\partial^\alpha g_{kn}}{\partial \theta^i_\alpha} + \frac{\partial^\alpha g_{in}}{\partial \theta^k_\alpha} - \frac{\partial^\alpha g_{ki}}{\partial \theta^n_\alpha}\right),$$

the metric values (2) and the definition (3). Thanks to (1), the only non-vanishing components of these symbols are

$$\overset{\alpha}{\Gamma}{}^{1}_{22} = -\Gamma\left(1+\alpha\right)r, \quad \overset{\alpha}{\Gamma}{}^{2}_{12} = \overset{\alpha}{\Gamma}{}^{2}_{21} = \Gamma\left(1+\alpha\right)\left(1/r\right).$$

## 4. Local Fractional Covariant Equations in Cylindrical Coordinates for Signorini Hyperelastic Materials

In this section we explicitly compute the fundamental covariant equations for a hyperelastic material, by characterizing first the so-called Signorini materials [2,5–7,9–11].

### 4.1. Signorini Hyperelastic Materials

In order to write the fundamental equations for an elastic material, we need first to compute the components of the Cauchy-Green strain tensor, which is defined for a fractional local derivative by

$$\varepsilon^{\alpha}_{ik} = \frac{1}{2}\left(\nabla^{\alpha}_{i}u_{k} + \nabla^{\alpha}_{k}u_{i} + \nabla^{\alpha}_{i}u_{j}\nabla^{\alpha}_{k}u^{j}\right) \tag{4}$$

$\vec{u} = \{u_{i}\}$ being the displacement vector (in each point of the continuum). When the displacement is characterized in terms of cylindrical coordinates

$$\vec{u}\left(\theta^{1},\theta^{2},\theta^{3}\right) = \vec{u}\left(r,\vartheta,z\right) = \{u_{1} = u_{r}\left(r\right), u_{2} = r\cdot u_{\vartheta} = u_{3} = u_{z} = 0\} \tag{5}$$

we obtain cylindrical waves [1–7,9–11]. In this case, by taking into account (1)-(5) the only non-zero components of the strain tensor are

$$\varepsilon^{\alpha}_{11} = \varepsilon^{\alpha}_{rr} = \Gamma\left(1+\alpha\right)u_{r,r} + \tfrac{1}{2}\left(\Gamma\left(1+\alpha\right)u_{r,r}\right)^{2},$$
$$\varepsilon^{\alpha}_{22} = r^{2}\varepsilon^{\alpha}_{\vartheta\vartheta} = \Gamma\left(1+\alpha\right)ru_{r} + \tfrac{1}{2}\left(\Gamma\left(1+\alpha\right)u_{r}\right)^{2} \tag{6}$$

where the comma stands for fractional partial derivative.

By using the components of the Cauchy-Green tensor, we can easily compute the three fundamental invariants. These invariants are the fundamental tools for the computation of the potential.

By neglecting displacements of order higher than three, we have for the three fundamental invariants

$$\begin{cases} I^{\alpha}_{1}\left(\varepsilon^{\alpha}_{ik}\right) = \varepsilon^{\alpha}_{ik}g^{ik}_{\alpha} = \varepsilon^{\alpha}_{11} + \varepsilon^{\alpha}_{22} \\[2mm] \quad = \Gamma\left(1+\alpha\right)u_{r,r} + \tfrac{1}{2}\left(\Gamma\left(1+\alpha\right)u_{r,r}\right)^{2} + ru_{r} + \tfrac{1}{2}\left(u_{r}\right)^{2} \\[2mm] I^{\alpha}_{2}\left(\varepsilon^{\alpha}_{ik}\right) = \varepsilon^{\alpha}_{im}\varepsilon^{\alpha}_{nk}g^{ik}_{\alpha}g^{nm}_{\alpha} = \left(\varepsilon^{\alpha}_{11}\right)^{2} + \tfrac{1}{r^{4}}\left(\varepsilon^{\alpha}_{22}\right)^{2} \\[2mm] \quad \cong \left(\Gamma\left(1+\alpha\right)u_{r,r}\right)^{2} + \left(\Gamma\left(1+\alpha\right)u_{r,r}\right)^{3} + \tfrac{1}{r^{2}}\left(u_{r}\right)^{2} + \tfrac{1}{r^{3}}\left(u_{r}\right)^{3} \\[2mm] I^{\alpha}_{3}\left(\varepsilon^{\alpha}_{ik}\right) = \varepsilon^{\alpha}_{pm}\varepsilon^{\alpha}_{in}\varepsilon^{\alpha}_{kq}g^{im}_{\alpha}g^{pq}_{\alpha}g^{kn}_{\alpha} = \left(\varepsilon^{\alpha}_{11}\right)^{3} + \tfrac{1}{r^{6}}\left(\varepsilon^{\alpha}_{22}\right)^{3} \\[2mm] \quad \cong \left(\Gamma\left(1+\alpha\right)u_{r,r}\right)^{3} + \tfrac{1}{r^{3}}\left(\Gamma\left(1+\alpha\right)u_{r}\right)^{3}. \end{cases} \tag{7}$$

Signorini potential belongs to the family of the polynomial hyperelastic model, and it is defined as [2,5,9]

$$W = \left(1/\sqrt{I_{A3}}\right)\left[cI_{A2} + (1/2)(\lambda + \mu - (c/2))(I_{A1})^{2}\right.$$
$$\left. + (\lambda + (c/2))(1 - I_{A1})\right] - (\mu + (c/2)) \tag{8}$$

where we have

$$
\begin{cases}
I_{A1} & = \dfrac{I_1 + 2\,(I_1)^2 - 2I_2 + 2\,(I_1)^3 - 6I_1 I_2 + 4I_3}{1 + 2I_1 + 2\,(I_1)^2 - 2I_2 + \frac{4}{3}\,(I_1)^3 - 4I_1 I_2 + \frac{8}{3}I_3} \\[2mm]
I_{A2} & = \dfrac{1}{2}\,(I_1)^2 - \dfrac{\frac{1}{2}\,(I_1)^2 - \frac{1}{2}I_2 + (I_1)^3 - 3I_1 I_2 + 2I_3}{1 + 2I_1 + 2\,(I_1)^2 - 2I_2 + \frac{4}{3}\,(I_1)^3 - 4I_1 I_2 + \frac{8}{3}I_3} \\[2mm]
I_{A3} & = \dfrac{2}{3}I_1 I_2 - \dfrac{1}{4\sqrt{3}}\,(I_1)^3\,\dfrac{(I_1)^3 - I_1 I_2 + 2I_3}{1 + 2I_1 + 2\,(I_1)^2 - 2I_2 + \frac{4}{3}\,(I_1)^3 - 4I_1 I_2 + \frac{8}{3}I_3}
\end{cases}
$$

The corresponding fractional potential can be obtained by replacing the fundamental invariants with the corresponding fractional invariant. Thus from the previous equation, by taking into account (7), we have the approximation

$$
\begin{aligned}
I^{\alpha}_{A1} & \cong \left(\varepsilon^{\alpha}_{11} + \varepsilon^{\alpha}_{22}\right) + 2\varepsilon^{\alpha}_{11}\varepsilon^{\alpha}_{22} \\[2mm]
I^{\alpha}_{A2} & \cong -\frac{1}{2}\left((\varepsilon^{\alpha}_{11})^2 + \frac{1}{r^4}\,(\varepsilon^{\alpha}_{22})^2\right) \\[2mm]
I^{\alpha}_{A3} & \cong \frac{2}{3}\left(\varepsilon^{\alpha}_{11} + \varepsilon^{\alpha}_{22}\right)\left((\varepsilon^{\alpha}_{11})^2 + \frac{1}{r^4}\,(\varepsilon^{\alpha}_{22})^2\right)
\end{aligned}
\tag{9}
$$

and, according to (6)

$$
\begin{aligned}
I^{\alpha}_{A1} & \cong \left(\Gamma\,(1+\alpha)\,u_{r,r} + \frac{1}{2}\,(\Gamma\,(1+\alpha)\,u_{r,r})^2 + ru_r + \frac{1}{2}\,(u_r)^2\right) + \\
& \quad 2\left[\Gamma\,(1+\alpha)\,u_{r,r} + \frac{1}{2}\,(\Gamma\,(1+\alpha)\,u_{r,r})^2\right]\left[ru_r + \frac{1}{2}\,(u_r)^2\right] \\[2mm]
I^{\alpha}_{A2} & \cong -\frac{1}{2}\left(\left[\Gamma\,(1+\alpha)\,u_{r,r} + \frac{1}{2}\,(\Gamma\,(1+\alpha)\,u_{r,r})^2\right]^2 + \frac{1}{r^4}\left[ru_r + \frac{1}{2}\,(u_r)^2\right]^2\right) \\[2mm]
I^{\alpha}_{A3} & \cong \frac{2}{3}\left(\Gamma\,(1+\alpha)\,u_{r,r} + \frac{1}{2}\,(\Gamma\,(1+\alpha)\,u_{r,r})^2 + ru_r + \frac{1}{2}\,(u_r)^2\right) \times \\
& \quad \left(\left[\Gamma\,(1+\alpha)\,u_{r,r} + \frac{1}{2}\,(\Gamma\,(1+\alpha)\,u_{r,r})^2\right]^2 + \frac{1}{r^4}\left[ru_r + \frac{1}{2}\,(u_r)^2\right]^2\right)
\end{aligned}
\tag{10}
$$

### 4.2. Fractional Covariant Equations

The fundamental fractional equations of motion for hyperelastic materials are

$$
\nabla^{\alpha}_i T^{ik}_{\alpha} - \rho \nabla^{\alpha}_i \varepsilon^{ik}_{\alpha} = \frac{\partial^2 u^k}{\partial t^2}\,,
\tag{11}
$$

where the Piola-Kirchoff stress tensor is defined as $T^{ik}_{\alpha} = \left(\partial^{\alpha} W / \partial \varepsilon^{\alpha}_{ik}\right)$ and $W$ is the hyperelastic potential.

Taking into account that

$$
\frac{\partial^{\alpha} W}{\partial \varepsilon^{\alpha}_{ik}} = \sum_{h=1}^{3} \frac{\partial^{\alpha} W}{\partial I^{\alpha}_{Ah}} \frac{\partial I^{\alpha}_{Ah}}{\partial \varepsilon^{\alpha}_{ik}}
$$

and, according to (6)–(10), we can easily obtain the fractional Piola-Kirchoff tensor for the Signorini model (see also [2,9] for the ordinary model).

$$
\begin{aligned}
T^{ik}_{\alpha} = & \left[\lambda I^{\alpha}_{A1} + c I^{\alpha}_{A2} + \frac{1}{2}\left(\lambda + \mu - \frac{c}{2}\right)\left(I^{\alpha}_{A1}\right)^2\right] g^{ik} \\
& + 2\left[\mu - \left(\lambda + \mu + \frac{c}{2}\right) I^{\alpha}_{A1}\right] \varepsilon^{ik}_{\alpha} + 2c\left(\varepsilon^{ij}_{\alpha}\varepsilon^{\;k}_{\alpha j}\right)\,.
\end{aligned}
$$

In the strain components, we are going to neglect those terms with order higher than 3, so that the only unvanishing components of $T_\alpha^{ik}$ are $T_\alpha^{11}$, $T_\alpha^{22}$ and $T_\alpha^{33}$, so that by using Equations (6)–(10) we finally get the fractional Kirchoff tensor in terms of displacements

$$
\begin{aligned}
T_\alpha^{11} \;=\; & T_\alpha^{rr} = (\lambda + 2\mu)\,\Gamma\,(1+\alpha)\,u_{r,r} + \lambda\frac{u_r}{r} + \\[4pt]
& + \; \frac{1}{4}\,(-10\lambda - 4\mu + 5c)\,(\Gamma\,(1+\alpha)\,u_{r,r})^2 \\[4pt]
& + \; \frac{1}{2}\,(2\lambda - 2\mu - 5c)\,\Gamma\,(1+\alpha)\,\frac{1}{r}u_r u_{r,r} + \frac{1}{4}\,(6\lambda + 2\mu + c)\,\frac{1}{r^2}\,(u_r)^2 + \\[4pt]
& + \; \frac{1}{2}\,(6\lambda + 13c)\,(\Gamma\,(1+\alpha)\,u_{r,r})^3 + \frac{1}{4}\,(70\lambda - 18\mu + c)\,\Gamma\,(1+\alpha)\,\frac{1}{r}u_r u_{r,r} + \\[4pt]
& + \; \frac{1}{4}\,(-42\lambda + 10\mu + 15c)\,\frac{1}{r^2}\,(u_r)^2 + \frac{1}{2}\,(4\lambda - 2\mu + 3c)\,\frac{1}{r^3}\,(u_r)^3
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
r^2 T_\alpha^{22} \;=\; & r^2 T_\alpha^{33} = T_\alpha^{\vartheta\vartheta} = (\lambda + 2\mu)\,\frac{u_r}{r} + \lambda\Gamma\,(1+\alpha)\,u_{r,r} + \\[4pt]
& + \; \frac{1}{4}\,(-2\lambda + 2\mu + c)\,(\Gamma\,(1+\alpha)\,u_{r,r})^2 + \frac{1}{2}\,(2\lambda - 2\mu - 5c)\,\Gamma\,(1+\alpha)\,\frac{1}{r}u_r u_{r,r} \\[4pt]
& + \; \tfrac{1}{4}\,(-2\lambda - 4\mu + 5c)\,\tfrac{1}{r^2}\,(u_r)^2 + \\[4pt]
& + \; (2\lambda - \mu + 4c)\,(\Gamma\,(1+\alpha)\,u_{r,r})^3 + \frac{1}{4}\,(-42\lambda + 10\mu + 15c)\,\Gamma\,(1+\alpha)\,\frac{1}{r}u_r u_{r,r} + \\[4pt]
& + \; \frac{1}{4}\,(70\lambda - 18\mu + c)\,\frac{1}{r^2}\,(u_r)^2 + (-\lambda + 4c)\,\frac{1}{r^3}\,(u_r)^3 \quad .
\end{aligned}
$$

### 4.3. Fractional Differential Equations for Longitudinal Waves

From Equations (6), (11) and (12) we get the only non trivial equation

$$
\begin{aligned}
(\lambda + 2\mu)\left(\Gamma\,(1+\alpha)^2\,u_{r,rr} + \Gamma\,(1+\alpha)\,\frac{u_{r,r}}{r} + u_r - \frac{u_r}{r^2}\right) - \rho \ddot{u}_r = & \\[4pt]
= S_1\,\Gamma\,(1+\alpha)^3\,u_{r,rr}u_{r,r} + S_2\Gamma\,(1+\alpha)^2\,\frac{1}{r}u_{r,rr}u_r & \\[4pt]
+ S_3\Gamma\,(1+\alpha)\,\frac{1}{r}\,(u_{r,r})^2 + S_4\Gamma\,(1+\alpha)\,\frac{1}{r^2}u_{r,r}u_r + S_5\frac{1}{r^3}\,(u_r)^2 & \\[4pt]
+ S_6\,\Gamma\,(1+\alpha)^4\,u_{r,rr}\,(u_{r,r})^2 + S_7\Gamma\,(1+\alpha)^2\,\frac{1}{r^3}u_{r,rr}\,(u_r)^2 & \\[4pt]
+ S_8\,\Gamma\,(1+\alpha)^3\,\frac{1}{r}u_{r,rr}u_{r,r}u_r + S_9\frac{1}{r}\Gamma\,(1+\alpha)^3\,(u_{r,r})^3 + S_{10}\frac{1}{r^4}\,(u_r)^3 & \\[4pt]
+ S_{11}\Gamma\,(1+\alpha)^2\,\frac{1}{r^2}\,(u_{r,r})^2\,u_r + S_{12}\Gamma\,(1+\alpha)\,\frac{1}{r^3}u_{r,r}\,(u_r)^2 ,
\end{aligned}
\tag{13}
$$

where the coefficients $S_1, S_2, ..., S_{12}$ depend on Signorini parameters $\lambda, \mu, c$

$$
S_1 = \frac{1}{2}\,(-6\lambda + 4\mu + 5c) \quad,\quad S_2 = \frac{1}{2}\,(4\lambda - 2\mu - 5c) \quad,\quad S_3 = \frac{1}{2}\,(2\lambda - \mu - 3c) \quad,
$$

$$
S_4 = \frac{1}{2}\,(2\mu - 5c) \quad,\quad S_5 = \frac{1}{2}\,(5\mu - 3c) \quad,\quad S_6 = \frac{1}{4}\,(9\lambda - 12\mu + 93c) \quad,
$$

$$
S_7 = \frac{1}{2}\,(24\lambda - 4\mu - 7c) \quad,\quad S_8 = 36\lambda - 10\mu - 2c \quad,\quad S_9 = \frac{1}{2}\,(32\lambda - 13\mu - 2c) \quad,
$$

$$
S_{10} = -\frac{1}{4}\,(10\lambda + c) \quad,\quad S_{11} = \frac{1}{4}\,(-74\lambda + 26\mu + 33c) \quad,\quad S_{12} = \frac{1}{4}\,(22\lambda - 18\mu + 7c) .
$$

Equation (13) is a nonlinear, second order, partial differential equation for the unknown function of two variables $u_r = u_r(r, t)$. The search of solutions can be greatly simplified by assuming some further hypotheses on the solution. In particular, the solution can be searched as a product of separable functions so that the Equation (13) can be split into two nonlinear, second order, ordinary differential equations; one equation only for the time depending function; and another equation for the radial dependent function, as shown in the next section.

## 5. Local Fractional Longitudinal Waves on Cantor Coordinates

In this section the solution of the Equation (13) $u_r = u_r(r, t)$ is searched for as a product of two separable functions, $t$ and $r$ respectively; that is, in the form

$$u_r = e^{i\omega t} u(r) \tag{14}$$

where the time-harmonic waves $e^{i\omega t}$ are separated by the fractional longitudinal waves $u(r)$, so that

$$\ddot{u}_r = -\omega^2 u_r \quad , \quad \omega = \sqrt{\frac{1 - (\lambda + 2\mu)}{\rho}} \tag{15}$$

where the two dots stand for the second time derivative. The radial function $u(r)$ is the solution of the equation

$$
\begin{aligned}
\left( \Gamma(1+\alpha)^2 u_{,rr} + \Gamma(1+\alpha) \frac{u_{,r}}{r} + u - \frac{u}{r^2} \right) &= \\
&= a_1 \Gamma(1+\alpha)^3 u_{,rr} u_{,r} + a_2 \Gamma(1+\alpha)^2 \frac{1}{r} u_{,rr} u + a_3 \Gamma(1+\alpha)^2 \frac{1}{r} (u_{,r})^2 \\
&+ a_4 \Gamma(1+\alpha) \frac{1}{r^2} u_{,r} u + a_5 \frac{1}{r^3} (u)^2 + a_6 \Gamma(1+\alpha)^4 u_{,rr} (u_{,r})^2 \\
&+ a_7 \Gamma(1+\alpha)^2 \frac{1}{r^3} u_{,rr} (u)^2 + a_8 \Gamma(1+\alpha)^3 \frac{1}{r} u_{,rr} u_{,r} u \\
&+ a_9 \frac{1}{r} \Gamma(1+\alpha)^3 (u_{,r})^3 + a_{10} \frac{1}{r^4} (u)^3 \\
&+ a_{11} \Gamma(1+\alpha)^2 \frac{1}{r^2} (u_{,r})^2 u + a_{12} \Gamma(1+\alpha) \frac{1}{r^3} u_{,r} (u)^2 ,
\end{aligned}
\tag{16}
$$

with $a_i = \dfrac{S_i}{\lambda + 2\mu}, i = 1, ..., 12$ and $u_{,r} = \dfrac{du}{dr}(r), , u_{,rr} = \dfrac{d^2u}{dr^2}(r)$.

Equation (16) gives the more general model of local fractional cylindrical wave propagation for Signorini hyperelastic materials. This is a nonlinear equation up to the third order in $u, u_{,r}, u_{,rr}$, while the coefficients depend both on inverse $r$ up to the $4^{th}$ power, and for the physical parameters $\lambda, \mu, c$ the dependence on the local fractional derivatives is expressed by the presence of the Gamma function. In order to analyze the fractal shape of solution, we consider the linear approximation.

*Linear Equation*

If we neglect the nonlinear terms of Equation (16), we obtain the linear equation

$$\left( \Gamma(1+\alpha)^2 u_{,rr} + \Gamma(1+\alpha) \frac{u_{,r}}{r} + u - \frac{u}{r^2} \right) = 0 \tag{17}$$

which is the well-known (homogeneous) Weber equation [12], and it can be solved by using the Bessel functions. In fact, the Bessel function $J^n(x)$ of order $n$ is the solution of the Weber equation

$$x^2 y'' + xy' + (x^2 - n^2)y = 0 \quad , \quad n \in \mathbb{N}.$$

In particular, when $n = 1$, the more general solution of

$$x^2 y'' + xy' + (x^2 - 1)y = 0$$

is

$$y(x) = c_1 J^I(x) + c_2 J^{II}(x).$$

The Taylor series for the first order Bessel function is

$$J_n^I(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)} \left(\frac{1}{2}x\right)^{2k+n}, x \in (-\varepsilon, \varepsilon)$$

By using the Bessel functions, the solution of (17) can be easily obtained.

$$
\begin{aligned}
u^\alpha(r) &= c_1 r^{\frac{\alpha_*-1}{2\alpha_*}} J^I_{\sqrt{\frac{5-2\alpha_*+\alpha_*^2}{4\alpha_*^2}}}\left(\frac{r}{\alpha_*}\right) + c_2 r^{\frac{\alpha_*-1}{2\alpha_*}} J^{II}_{\sqrt{\frac{5-2\alpha_*+\alpha_*^2}{4\alpha_*^2}}}\left(\frac{r}{\alpha_*}\right), \\
\alpha_* &= \Gamma(1+\alpha)
\end{aligned}
\tag{18}
$$

In Figure 1 the local fractional solution (18) of Equation (17), with $\alpha = 0.7$ is compared with the ordinary solution (i.e., $\alpha = 1$), the corresponding longitudinal wave solution on the Cantor set, by taking into account the time-harmonic contribution given in Equation (14) as well, is shown in Figure 2, while in Figure 3 the smooth longitudinal wave ($\alpha = 1$) is given for comparison.
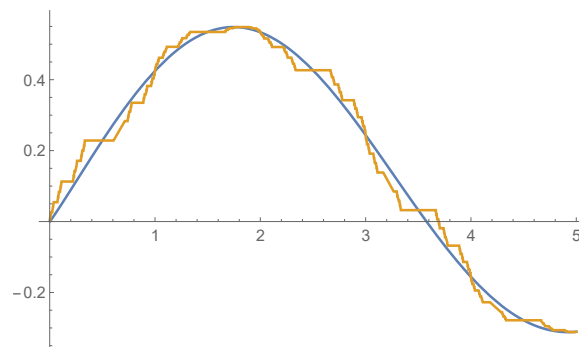


**Figure 1.** The local fractional solution of Equation (17) for the longitudinal displacement on Cantor coordinates $(\alpha = 0.7)$ compared with the smooth solution $(\alpha = 1)$.
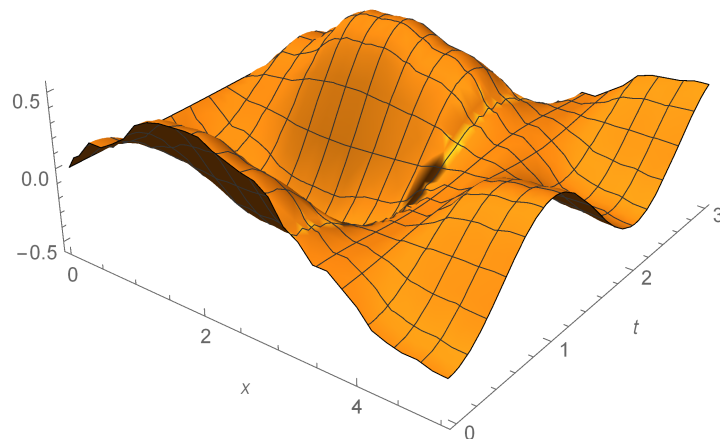


**Figure 2.** Longitudinal fractional wave on Cantor longitudinal coordinates $(\alpha = 0.7)$.
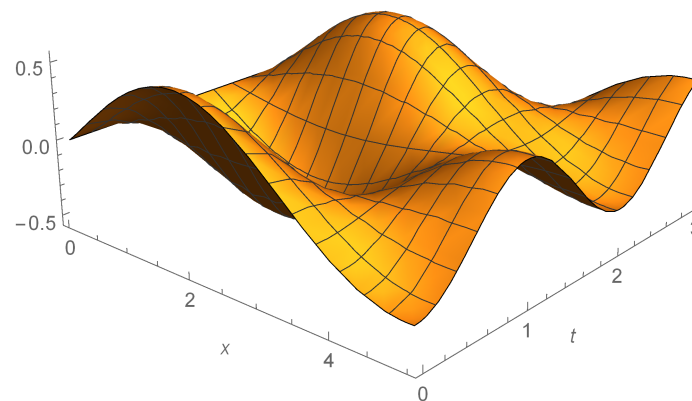
**Figure 3.** Smooth longitudinal wave (with $\alpha = 1$).

## 6. Conclusions

In this work, the local, fractional, longitudinal wave propagation on Cantor cylindrical coordinates has been studied for Signorini hyperelastic materials. By using the local fractional derivative (fractal derivative), the explicit solution of the linear approximation of the complete equations where third order nonlinearities appear has been given. As a further step, the second and third order nonlinearities can be investigated as well. Moreover, it would be interesting to study the same problem, e.g., the Signorini hyperelastic materials and corresponging fractional waves, by using an integral minimization as a consequence of a suitable variational principle. In some recent papers, the author gave also a fractal variational model that could be applied to the Signorini hyperelastic materials as well, thereby opening up some more general perspectives for the analysis of future smart materials.

**Conflicts of Interest:** The author declares no conflict of interest.

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