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Article

A Discussion on the Existence of Best Proximity Points That Belong to the Zero Set

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Abstract: In this paper, we investigate the existence of best proximity points that belong to the zero set for the α_p-admissible weak (F, φ)-proximal contraction in the setting of M-metric spaces. For this purpose, we establish φ-best proximity point results for such mappings in the setting of a complete M-metric space. Some examples are also presented to support the concepts and results proved herein. Our results extend, improve and generalize several comparable results on the topic in the related literature.

Keywords: m-metric space; proximal α_p-admissible; α_p-admissible weak (F, φ)-proximal contraction; $G-$proximal graphic contraction; φ-best proximity point;

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1. Introduction and Preliminaries

Several real-world problems can be reformulated as a fixed point problem. In other words, the solution of the real-world problem reduces to the solution of a fixed point problem. In some cases getting a fixed point for certain mapping is impossible. In this case, instead of exact solution, it is natural to consider the approximate solution. Roughly speaking, if the equation $F(\xi) = 0$ has no exact solution where $F(\xi) = T(\xi) - \xi$, where T is an opeator defined on a certain distance space. In 1969, Ky Fan [1] suggested an answer to the question that what happen if a given mapping does not possess a fixed point. More precisely, he proved that if A is a compact, convex and nonempty subset of a Banach space S and T is continuous mapping from A to S, then there exists a point $\xi^* \in A$ such that

$$d(\xi^*, T\xi^*) = d(T\xi^*, A) = \inf\{d(\xi, T\xi^*),\ \xi \in A\}.$$

This results is known as best approximation theorem. In the above statement, the point $\xi^* \in A$ is called as approximate fixed point of T or an approximate solution of a fixed point equation $T\xi = \xi$. In general, if A, B are nonempty subsets of a Banach space S and $T : A \to B$, then $\xi^* \in A$ is called best proximity point of T if it satisfies

$$d(\xi^*, T\xi^*) = d(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}.$$

Note that $\xi^* \in A$ turns to be a fixed point of T, if the sets A, B have non-empty intersection. Indeed, if $A \cap B \neq \phi$ or $A = B$, then $d(A, B) = 0$ and hence the best proximity point $\xi^* \in A$ becomes the solution of a fixed point equation $T\xi = \xi$. Attendantly, best proximity point results are natural generalizations of metric fixed point results. For further discussion in this direction, we refer to [2–8].

We underline the fact that a best proximity point $\xi^* \in A$, indeed solves the following optimization problem:

$$\min_{\xi \in A} d(\xi, T\xi).$$

On the other hand, fixed point theory has been extended in several directions. For instance, metric space structure has been changed by some new abstract space which is more general than the standard set-up. One of the significant examples of this trend was given by Matthews [9]. He defined the notion of partial metric space and characterized the Banach contraction principle in that space. Roughly speaking, despite the metric space, in partial metric space self-distance may not be zero. This notion especially provides some simplicity in computer science, in particular, domain theory. A number of authors have involved in this trend with interesting results, see e.g., [10–18] and related reference therein. For the sake of completeness, we recall the concept of partial metric space as follows:

Definition 1 ([9])**.** *A distance function $p : S \times S \to [0, \infty)$, on a non-empty set S, is called partial metric if the followings are fulfilled:*

(p1) $p(\xi, \xi) = p(\eta, \eta) = p(\xi, \eta) \Leftrightarrow \xi = \eta,$
(p2) $p(\xi, \xi) \leq p(\xi, \eta),$
(p3) $p(\xi, \eta) = p(\eta, \xi),$
(p4) $p(\xi, \eta) \leq p(\xi, \zeta) + p(\zeta, \eta) - p(\zeta, \zeta)$

for all $\xi, \eta, \zeta \in S$. A corresponding pair (S, p) is called a partial metric space.

It is evident that $p(\xi, \eta) = 0$, yields $\xi = \eta$. The contrary of the statement is false.

Asadi et al. [19] introduced the notion of an M-metric space and obtained fixed point results in the setup of M-metric spaces. It was indicated that M-metric space is a real generalization of a partial metric space and they supported their claim by providing some constructive examples. For more results in this direction see e.g., [20,21].

The following notations are useful in the sequel.

(1) $m_{\xi\eta} = \min \{\rho(\xi, \xi), \rho(\eta, \eta)\},$
(2) $M_{\xi\eta} = \max \{\rho(\xi, \xi), \rho(\eta, \eta)\}.$

Definition 2 ([19])**.** *A distance function $\rho : S \times S \to [0, \infty)$, on a non-empty set S, is called M-metric if the followings are fulfilled:*

(m1) $\rho(\xi, \xi) = \rho(\eta, \eta) = \rho(\xi, \eta) \Leftrightarrow \xi = \eta,$
(m2) $m_{\xi\eta} \leq \rho(\xi, \eta)$
(m3) $\rho(\xi, \eta) = \rho(\eta, \xi),$
(m4) $\rho(\xi, \eta) - m_{\xi\eta} \leq \rho(\xi, \zeta) - m_{\xi\zeta} + \rho(\zeta, \eta) - m_{\zeta\eta}$

for all $\xi, \eta, \zeta \in S$. A corresponding pair (S, ρ) is called an M-metric space.

Lemma 1 ([19])**.** *Each partial metric forms an M-metric. The converse is false.*

Example 1. *Let $S = \{\xi, \eta, \zeta\}$. Define*

$$\begin{aligned} \rho(\xi, \xi) &= 1, \quad \rho(\eta, \eta) = 9, \quad \rho(\zeta, \zeta) = 5, \\ \rho(\xi, \eta) &= \rho(\eta, \xi) = 10, \quad \rho(\xi, \zeta) = \rho(\zeta, \xi) = 7, \\ \rho(\zeta, \eta) &= \rho(\eta, \zeta) = 7. \end{aligned}$$

It is clear that ρ is an M-metric. Notice that ρ does not form a partial metric.

Definition 3 ([19])**.** *Let (S, ρ) be an M-metric space and $\xi \in S$. A sequence $\{\xi_n\}$ in S is called:*

(1) *$M-$convergent to $\xi \in S$ if and only if*

$$\lim_{n\to\infty} (\rho(\xi_n, \xi) - m_{\xi_n,\xi}) = 0,$$

(2) *$M-$Cauchy sequence if and only if*

$$\lim_{n,m\to\infty} (\rho(\xi_n, \xi_m) - m_{\xi_n,\xi_m}) \text{ and } \lim_{n,m\to\infty} (M_{\xi_n,\xi_m} - m_{\xi_n,\xi_m}),$$

exist (and are finite).

Definition 4 ([19])**.** *An M-metric space is said to be $M-$complete if every $M-$Cauchy sequence $\{\xi_n\}$ in S converges with respect to τ_m (topology induced by m) to a point $\xi \in S$ such that*

$$\lim_{n\to\infty} (\rho(\xi_n, \xi) - m_{\xi_n,\xi}) = 0 \text{ and } \lim_{n\to\infty} (M_{\xi_n,\xi} - m_{\xi_n,\xi}) = 0.$$

Remark 1 ([19])**.** *Let (S, ρ) be an M-metric space and for every $\xi, \eta \in (S, \rho)$, we have*

$(r1)$ $0 \le M_{\xi\eta} + m_{\xi\eta} = \rho(\xi, \xi) + \rho(\eta, \eta)$,
$(r2)$ $0 \le M_{\xi\eta} - m_{\xi\eta} = [\rho(\xi, \xi) - \rho(\eta, \eta)]$,
$(r3)$ $M_{\xi\eta} - m_{\xi\eta} \le (M_{\xi\zeta} - m_{\xi\zeta}) + (M_{\zeta\eta} - m_{\zeta\eta})$.

The set $\{\xi^* \in A : \varphi(\xi^*) = 0\}$ of all zeros of the function $\varphi : A \to [0, \infty)$ is denoted by Z_φ. By using this notion, Jleli et al. [22] introduced the notion of φ-fixed point as follows: If S is a non empty set, $T : S \to S$ and $\varphi : S \to [0, \infty)$ is a given function, then $\xi^* \in S$ is said to be φ- fixed point of T if and only if $T(\xi^*) = \xi^*$ and $\varphi(\xi^*) = 0$. We denote the set of all φ-fixed points of T by $\varphi_F(S)$, that is,

$$\varphi_F(S) = \{\xi^* \in S : T(\xi^*) = \xi^* \text{ and } \varphi(\xi^*) = 0\}.$$

In [22], the authors also considered the concept of control function $F : [0, \infty)^3 \to [0, \infty)$ defined as follows:

$(F1)$ $\max\{s, t\} \le F(s, t, r)$, for all $s, t, r \in [0, \infty)$,
$(F2)$ F is continuous,
$(F3)$ $F(0, 0, 0) = 0$.

The set of such control functions is denoted by $\mathcal{F}$. An immediate examples of the control functions are collected below:

Example 2 ([22])**.** *Let $i = \{1, 2, 3\}$. Define $F_i : [0, \infty)^3 \to [0, \infty)$ as follows:*

$$F_1(a, b, c) = a + b + c, \ F_2(a, b, c) = \max\{a, b\} + c \text{ and } F_3(a, b, c) = a + a^2 + b + c,$$

for all $a, b, c \in [0, \infty)$. Note that $F_1, F_2, F_3 \in \mathcal{F}$.

In [22], the notion of (F, φ)-contraction mapping was defined and the existence of a fixed point for such mappings were considered.

Definition 5 ([22])**.** *Let (S, d) be a complete metric space and $\varphi : S \to [0, \infty)$. A mapping $T : S \to S$ is said to be an (F, φ)-contraction mapping if there exist $F \in \mathcal{F}$ and $k \in [0, 1)$ such that*

$$F(d(T\xi, T\eta), \varphi(T\xi), \varphi(T\eta)) \le kF(d(\xi, \eta), \varphi(\xi), \varphi(\eta)), \text{ for all } \xi, \eta \in S.$$

Later, this result has been followed by several authors, see e.g., [23–26].

Let Ψ denote the set of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$, for all $t > 0$, where ψ^n is an $n-$iterate of ψ. A function ψ is called a $(c)-$comparison function if $\psi \in \Psi$. Note that if $\psi \in \Psi$, then $\psi(0) = 0$ and $\psi(t) < t$, for all $t > 0$ [27].

Remark 2 ([27])**.** *Note that $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$ implies $\lim_{n\to\infty} \psi^n(t) = 0$, for all $t \in (0,\infty)$.*

In what follows we introduce the notion of "φ-best proximity point".

Definition 6. *Let (S,ρ) be an M-metric space, A, B are two subsets of S. An element $\xi^* \in Z_\varphi$ is said to be a φ-best proximity point of the operator $T : A \to B$ if and only if $\rho(\xi^*, T\xi^*) = \rho(A,B)$, where $\rho(A,B) = \inf\{\rho(a,b) : a \in A,\ b \in B\}$ and $\varphi(\xi^*) = 0$.*

We denote the set of all φ-best proximity points of T by $\varphi_T(A)$, that is,

$$\varphi_T(A) = \{\xi^* \in A : \rho(\xi^*, T\xi^*) = \rho(A,B) \text{ and } \varphi(\xi^*) = 0\}.$$

The following definitions are also needed in the sequel. Before we state the definition, we underline the following assumption: Throughout the paper, all sets and subsets are supposed non-empty. We characterize the following sets (that plays a crucial role in best proximity theory) in the setting of M-metric space.

Definition 7. *Let (S,ρ) be an M-metric space, and A, B be two subsets of S. Define*

$$\begin{aligned} A_0 &= \{\xi \in A : \rho(\xi,\eta) = \rho(A,B), \textit{ for some } \eta \in B\} \textit{ and} \\ B_0 &= \{\xi \in B : \rho(\xi,\eta) = \rho(A,B), \textit{ for some } \eta \in A\}. \end{aligned}$$

Definition 8. *Let (S,ρ) be an M-metric space, and let A, B be two subsets of S. If $\alpha : A \times A \to [-\infty,\infty)$, then mapping $T : A \to B$ is said to be proximal α_p-admissible if*

$$\left.\begin{array}{r} \alpha(\xi,\eta) \geq 0 \\ \rho(u,T\xi) = \rho(A,B) \\ \rho(v,T\eta) = \rho(A,B) \end{array}\right\} \Longrightarrow \alpha(u,v) \geq 0,$$

for all $\xi, \eta, u, v \in A$.

Definition 9. *Let (S,ρ) be an M-metric space, and $T : A \to B$. In addition, let A be a subset of S, and $\alpha : A \times A \to [-\infty,\infty)$. Then A is said to be $\alpha-$regular, if $\{\xi_n\}$ is a sequence in A such that $\alpha(\xi_n, \xi_{n+1}) \geq 0$ and $\xi_n \to \xi \in A$ as $n \to \infty$, then $\alpha(\xi_n, \xi) \geq 0$ for all $n \in N$.*

In this paper, we introduce the notion of φ-best proximity point and prove the φ-best proximity point result in the setting of M-metric space. We also present an example to support our result.

2. Main Results

We start the section by introducing the notion of α_p-admissible weak (F,φ)-proximal contraction mappings as follows.

Definition 10. *Let A, B be two subsets of M-metric space (S, ρ) and $F \in \mathcal{F}$. An α_p-admissible mapping $T : A \to B$ is called an α_p-admissible weak (F, φ)-proximal contraction, if there exists a lower semi-continuous function $\varphi : A \to [0, \infty)$ such that*

$$\left.\begin{array}{r}\alpha(\xi,\eta) \geq 0 \\ \rho(u, T\xi) = \rho(A, B) \\ \rho(v, T\eta) = \rho(A, B)\end{array}\right\}$$
$$\Longrightarrow \quad \alpha(\xi,\eta) + F(\rho(u,v), \varphi(u), \varphi(v)) \leq \psi(F(\rho(\xi,\eta), \varphi(\xi), \varphi(\eta))),$$

for all $\xi, \eta, u, v \in A$ and $\psi \in \Psi$.

By taking $\alpha(\xi, \eta) = 0$, we shall get the following definition:

Definition 11. *Let A, B be two subsets of M-metric space (S, ρ) and $F \in \mathcal{F}$. A mapping $T : A \to B$ is said to be a weak (F, φ)-proximal contraction, if there exist two functions $\varphi : A \to [0, \infty)$ and $\psi \in \Psi$ such that*

$$\left.\begin{array}{r}\rho(u, T\xi) = \rho(A, B) \\ \rho(v, T\eta) = \rho(A, B)\end{array}\right\}$$
$$\Longrightarrow \quad F(\rho(u,v), \varphi(u), \varphi(v)) \leq \psi(F(\rho(\xi,\eta), \varphi(\xi), \varphi(\eta))),$$

for all $\xi, \eta, u, v \in A$ and $\psi \in \Psi$.

The main result of the article is below.

Theorem 1. *Let A, B be two subsets of an M-complete M-metric space (S, ρ) and $F \in \mathcal{F}$. Suppose that a mapping $T : A \to B$ is an α_p-admissible weak (F, φ)-proximal contraction. If $T(A_0) \subseteq B_0$ and A_0 is $\alpha-$regular closed set in S, then there exists a φ-best proximity point of T provided that there exist $\xi_0, \xi_1 \in A_0$ such that*

$$\rho(\xi_1, T\xi_0) = \rho(A, B) \text{ and } \alpha(\xi_0, \xi_1) \geq 0.$$

Moreover, if $\alpha(\xi, \eta) \geq 0$ for all $\xi, \eta \in \varphi_T(A)$, then ξ^* is the unique φ-best proximity point of T.

Proof. Let ξ_0, $\xi_1 \in A_0$ be such that $\rho(\xi_1, T\xi_0) = \rho(A, B)$ and $\alpha(\xi_0, \xi_1) \geq 0$. As $T\xi_0 \in T(A_0) \subseteq B_0$, there exists ξ_2 in A_0 such that $\rho(\xi_2, T\xi_1) = \rho(A, B)$. Since T is proximal α_p-admissible, we have $\alpha(\xi_1, \xi_2) \geq 0$. Similarly, by $T(A_0) \subseteq B_0$, we obtain a point $\xi_3 \in A_0$ such that $\rho(\xi_3, T\xi_2) = \rho(A, B)$ which further implies that $\alpha(\xi_2, \xi_3) \geq 0$. Continuing this way, we can obtain a sequence $\{\xi_n\}$ in A_0 such that

$$\begin{aligned} \rho(\xi_n, T\xi_{n-1}) &= \rho(A, B), \\ \rho(\xi_{n+1}, T\xi_n) &= \rho(A, B),\ \alpha(\xi_n, \xi_{n+1}) \geq 0, \text{ for all } n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{1}$$

Since T is α_p-admissible weak (F, φ)-proximal contraction, we have

$$\alpha(\xi_{n-1}, \xi_n) + F(\rho(\xi_n, \xi_{n+1}), \varphi(\xi_n), \varphi(\xi_{n+1})) \leq \psi(F(\rho(\xi_{n-1}, \xi_n), \varphi(\xi_{n-1}), \varphi(\xi_n))).$$

Since $\alpha(\xi, \eta) \geq 0$ for all $\xi, \eta \in A$, we obtain that

$$F(\rho(\xi_n, \xi_{n+1}), \varphi(\xi_n), \varphi(\xi_{n+1})) \leq \psi(F(\rho(\xi_{n-1}, \xi_n), \varphi(\xi_{n-1}), \varphi(\xi_n))).$$

By induction, we get

$$F(\rho(\xi_n, \xi_{n+1}), \varphi(\xi_n), \varphi(\xi_{n+1})) \leq \psi^n(F(\rho(\xi_0, \xi_1), \varphi(\xi_0), \varphi(\xi_1))).$$

It follows from the condition **(F1)** that

$$\max\{\rho(\xi_n,\xi_{n+1}),\varphi(\xi_n)\}\leq\psi^n(F(\rho(\xi_0,\xi_1),\varphi(\xi_0),\varphi(\xi_1))). \tag{2}$$

By (2), we obtain that

$$\rho(\xi_n,\xi_{n+1})\leq\psi^n(F(\rho(\xi_0,\xi_1),\varphi(\xi_0),\varphi(\xi_1))). \tag{3}$$

On the other hand, we get

$$\lim_{n\to\infty}\rho(\xi_n,\xi_{n+1})=0. \tag{4}$$

Using (4) and the condition **(m2)**, we have

$$\begin{aligned}\lim_{n\to\infty}\rho(\xi_n,\xi_n) &= \lim_{n\to\infty}\min\{\rho(\xi_n,\xi_n),\rho(\xi_{n+1},\xi_{n+1})\}\\ &= \lim_{n\to\infty}m_{\xi_n,\xi_{n+1}}\\ &\leq \lim_{n\to\infty}\rho(\xi_n,\xi_{n+1})=0.\end{aligned}$$

Since $\lim_{n\to\infty}\rho(\xi_n,\xi_n)=0$, we have

$$\lim_{n,m\to\infty}m_{\xi_n,\xi_m}=0. \tag{5}$$

We shall indicate that $\{\xi_n\}$ is an M-Cauchy sequence. Consider $m,n\in\mathbb{N}$ such that $m>n$. On using (3) and the condition **(m4)**, we have

$$\begin{aligned}\rho(\xi_n,\xi_m)-m_{\xi_n,\xi_m} &\leq \rho(\xi_n,\xi_{n+1})-m_{\xi_n,\xi_{n+1}}+\rho(\xi_{n+1},\xi_m)-m_{\xi_{n+1},\xi_m}\\ &\leq \rho(\xi_n,\xi_{n+1})-m_{\xi_n,\xi_{n+1}}+\rho(\xi_{n+1},\xi_{n+2})-m_{\xi_{n+1},\xi_{n+2}}+\rho(\xi_{n+2},\xi_m)-m_{\xi_{n+2},\xi_m}\\ &\leq \rho(\xi_n,\xi_{n+1})-m_{\xi_n,\xi_{n+1}}+\rho(\xi_{n+1},\xi_{n+2})-m_{\xi_{n+1},\xi_{n+2}}\\ &\qquad +\ldots+\rho(\xi_{m-1},\xi_m)-m_{\xi_{m-1},\xi_m}\\ &\leq \rho(\xi_n,\xi_{n+1})+\rho(\xi_{n+1},\xi_{n+2})+\ldots+\rho(\xi_{m-1},\xi_m)\\ &\leq \psi^n(F(\rho(\xi_0,\xi_1),\varphi(\xi_0),\varphi(\xi_1)))+\psi^{n+1}(F(\rho(\xi_0,\xi_1),\varphi(\xi_0),\varphi(\xi_1)))+\\ &\quad \ldots+\psi^{m-1}(F(\rho(\xi_0,\xi_1),\varphi(\xi_0),\varphi(\xi_1)))\\ &\leq \sum_{i=1}^{m-1}\psi^i(F(\rho(\xi_0,\xi_1),\varphi(\xi_0),\varphi(\xi_1)))-\sum_{j=1}^{n-1}\psi^j(F(\rho(\xi_0,\xi_1),\varphi(\xi_0),\varphi(\xi_1))).\end{aligned} \tag{6}$$

It follows from Remark 2 and (6) that $\rho(\xi_n,\xi_m)-m_{\xi_n,\xi_m}\to 0$ as $n\to\infty$. On the other hand, by (5), we obtain that

$$\lim_{n,m\to\infty}(M_{\xi_n,\xi_m}-m_{\xi_n,\xi_m})=0.$$

Thus $\{\xi_n\}$ is an M-Cauchy sequence in $A_0\subseteq A\subset S$. By the completeness of S and closeness of A_0, there exists $\xi^*\in A_0$ such that

$$\lim_{n\to\infty}\rho(\xi_n,\xi^*)-m_{\xi_n,\xi^*}=0 \text{ and } \lim_{n\to\infty}M_{\xi_n,\xi^*}-m_{\xi_n,\xi^*}=0.$$

Since $\lim_{n\to\infty}\rho(\xi_n,\xi_n)=0$, we have

$$\lim_{n\to\infty}\rho(\xi_n,\xi^*)=0 \text{ and } \lim_{n\to\infty}M_{\xi_n,\xi^*}=0. \tag{7}$$

Thus by Remark 1, we get that

$$\lim_{n\to\infty}\rho(\xi^*,\xi^*)=\lim_{n\to\infty}[M_{\xi_n,\xi^*}+m_{\xi_n,\xi^*}-\rho(\xi_n,\xi_n)]=0.$$

This implies that
$$\rho(\xi^*, \xi^*) = 0.$$
Now we need to show that $\varphi(\xi^*) = 0$. Using (2), we have
$$\varphi(\xi_n) \leq \psi^n(F(\rho(\xi_0, \xi_1), \varphi(\xi_0), \varphi(\xi_1))).$$
Letting $n \to \infty$ on the inequality above, we obtain
$$\lim_{n\to\infty} \varphi(\xi_n) = 0. \tag{8}$$
Since φ is lower semi continuous, it follows from (7) and (8) that
$$0 \leq \varphi(\xi^*) \leq \lim_{n\to\infty} \inf \varphi(\xi_n) = 0.$$
Hence $\varphi(\xi^*) = 0$. Since A_0 is $\alpha-$regular, $\alpha(\xi_n, \xi^*) \geq 0$. As $\xi^* \in A_0$, $T(A_0) \subseteq B_0$, $T\xi^* \in B_0$, we may choose a point $z \in A_0$ such that $z \neq \xi^*$ and
$$\rho(z, T\xi^*) = \rho(A, B). \tag{9}$$
We shall prove that $z = \xi^*$. On the contrary suppose that $z \neq \xi^*$. Since T is α_p-admissible weak (F, φ)-proximal contraction, by using (1) and (9) we have
$$\begin{aligned}
\rho(\xi_{n+1}, z) &\leq \max\{\rho(\xi_{n+1}, z), \varphi(\xi_{n+1})\} \\
&\leq F(\rho(\xi_{n+1}, z), \varphi(\xi_{n+1}), \varphi(z)) \\
&\leq \alpha(\xi_n, \xi^*) + F(\rho(\xi_{n+1}, z), \varphi(\xi_{n+1}), \varphi(z)) \\
&\leq \psi(F(\rho(\xi_n, \xi^*), \varphi(\xi_n), \varphi(\xi^*))) \\
&< F(\rho(\xi_n, \xi^*), \varphi(\xi_n), \varphi(\xi^*)) \\
&= F(\rho(\xi_n, \xi^*), \varphi(\xi_n), 0).
\end{aligned}$$
Letting $n \to \infty$ on the inequality above, we have
$$\begin{aligned}
\lim_{n\to\infty} \rho(\xi_{n+1}, z) &= \lim_{n\to\infty} F(\rho(\xi_n, \xi^*), \varphi(\xi_n), 0) \\
&= F(0, 0, 0) = 0,
\end{aligned}$$
which implies that
$$\lim_{n\to\infty} \rho(\xi_{n+1}, z) = 0.$$
By using the condition **(m4)**, we have
$$\begin{aligned}
\rho(\xi^*, z) - m_{\xi^*, z} &\leq \rho(\xi^*, \xi_{n+1}) - m_{\xi^*, \xi_{n+1}} + \rho(\xi_{n+1}, z) - m_{\xi_{n+1}, z} \\
\rho(\xi^*, z) - m_{\xi^*, z} &\leq \rho(\xi^*, \xi_{n+1}) + \rho(\xi_{n+1}, z) \\
\lim_{n\to\infty} \rho(\xi^*, z) - m_{\xi^*, z} &\leq \lim_{n\to\infty} \rho(\xi^*, \xi_{n+1}) + \lim_{n\to\infty} \rho(\xi_{n+1}, z) \\
\lim_{n\to\infty} \rho(\xi^*, z) - m_{\xi^*, z} &\leq 0.
\end{aligned}$$
Since $\rho(\xi^*, \xi^*) = 0$, $\xi^* = z$. This is a contradiction. Attendantly, we have
$$\rho(\xi^*, T\xi^*) = \rho(A, B).$$

Uniqueness: Let $\alpha(\xi,\eta) \geq 0$, for all $\xi,\eta \in \varphi_T(A)$. Suppose that ξ^* and w are two φ-best proximity points of T with $\xi^* \neq w$. Hence

$$\rho(w,Tw) = \rho(A,B),$$

and

$$\varphi(\xi^*) = \varphi(w) = 0.$$

Since T is α_p-admissible weak (F,φ)-proximal contraction, we have

$$\begin{aligned} F(\rho(\xi^*,w),0,0) &\leq \alpha(\xi^*,w) + F(\rho(\xi^*,w),\varphi(\xi^*),\varphi(w)) \\ &\leq \psi(F(\rho(\xi^*,w),\varphi(\xi^*),\varphi(w))) \\ &< F(\rho(\xi^*,w),0,0), \end{aligned}$$

a contradiction. Consequently, we find that ξ^* is a unique φ-best proximity point of T. □

Corollary 1. *Let A,B be two subsets of an M-complete M-metric space (S,ρ) and $F \in \mathcal{F}$. Suppose that a mapping $T : A \to B$ is a weak (F,φ)-proximal contraction. If $T(A_0) \subseteq B_0$ and A_0 is closed set in S, then there exist a unique φ-best proximity point of T provided that there exist $\xi_0,\xi_1 \in A_0$ such that*

$$\rho(\xi_1,T\xi_0) = \rho(A,B).$$

Proof. It is derived from Theorem 1 by choosing $\alpha(\xi,\eta) = 0$. □

Since an M-metric space is a partial metric space, from the Theorem 1 we deduce immediately the following result. Note that in the following result we consider the notions in Definition 10 and Definition 11 in the setting of partial metric spaces.

Corollary 2. *Let A,B be two subsets of a complete partial metric space (S,p) and $F \in \mathcal{F}$. Suppose that a mapping $T : A \to B$ is an α_p-admissible weak (F,φ)-proximal contraction. If $T(A_0) \subseteq B_0$ and A_0 is $\alpha-$regular closed set in S, then there exists a φ-best proximity point of T provided that there exist $\xi_0,\xi_1 \in A_0$ such that*

$$p(\xi_1,T\xi_0) = p(A,B) \text{ and } \alpha(\xi_0,\xi_1) \geq 0,$$

$p(A,B) = \inf\{p(a,b) : a \in A,\ b \in B\}$. Moreover, if $\alpha(\xi,\eta) \geq 0$ for all $\xi,\eta \in \varphi_T(A)$, then ξ^* is the unique φ-best proximity point of T.

Proof. Since an M-metric space is a generalization of partial metric space, from Theorem 1 we deduce the result. □

Corollary 3. *Let A,B be two subsets of a complete partial metric space (S,p) and $F \in \mathcal{F}$. Suppose that a mapping $T : A \to B$ is a weak (F,φ)-proximal contraction. If $T(A_0) \subseteq B_0$ and A_0 is closed set in S, then there exist a unique φ-best proximity point of T provided that there exist $\xi_0,\xi_1 \in A_0$ such that*

$$p(\xi_1,T\xi_0) = p(A,B).$$

Proof. It is deduced from Corollary 2 by choosing $\alpha(\xi,\eta) = 0$. □

To support Corollary 1, we provide the following example.

Example 3. *Let $S = [0,1]$ and $\rho : S \times S \to [0,\infty)$ be defined by*

$$\rho(\xi,\eta) = |\xi - \eta|,$$

otherwise. Then (S, ρ) is an M-metric space. Suppose that $A = \{0, 0.4, 0.6, 0.9\}$ and $B = \{0.1, 0.3, 0.7, 1\}$. Note that $\rho(A, B) = 0.1$, $A = A_0$ and $B = B_0$. Define a mapping $T : A \to B$ as:

$$T(0) = 0.1,\ T(0.4) = 0.1,\ T(0.6) = 0.1,\ T(0.9) = 0.3.$$

Note that $T(A_0) \subseteq B_0$. Define functions $\psi : [0, \infty) \to [0, \infty)$, $F : [0, \infty)^3 \to [0, \infty)$ and $\varphi : A \to [0, \infty)$ by

$$\begin{aligned} \psi(t) &= \frac{2t}{3}, \\ F(a, b, c) &= \max\{a, b\} + c, \textit{ for all } a, b, c \in [0, \infty) \\ \textit{and } \varphi(\xi) &= \xi, \textit{ for all } \xi \in A. \end{aligned}$$

If we take $\xi = 0.6$, $\eta = 0.9$, $u = 0$ and $v = 0.4$, then we have

$$\rho(u, T\xi) = \rho(v, T\eta) = 0.1 = \rho(A, B),$$

which implies that

$$F(\rho(u, v), \varphi(u), \varphi(v)) = 0.8 \leq 1 = \psi(F(\rho(\xi, \eta), \varphi(\xi), \varphi(\eta))).$$

Hence T forms a weak (F, φ)-proximal contraction. Thus, all the conditions of Corollary 1 are satisfied. Moreover, $\xi^* = 0$ is a unique φ-best proximity point.

To support Corollary 3, we provide the following example.

Example 4. *Let $S = [0, 1] \cup [2, 3]$. Define the mapping $p : S \times S \to [0, \infty)$ by*

$$p(\xi, \eta) = \begin{cases} \max\{\xi, \eta\}, & \{\xi, \eta\} \cap [2, 3] \neq \phi, \\ |\xi - \eta|, & \{\xi, \eta\} \subseteq [0, 1]. \end{cases}$$

Then (S, p) is a partial metric space. Suppose that $A = \{0, 0.4, 0.6, 0.9\}$ and $B = \{0.1, 0.3, 0.7, 1\}$. Note that $p(A, B) = 0.1$, $A = A_0$ and $B = B_0$. Define a mapping $T : A \to B$ as:

$$T(0) = 0.1,\ T(0.4) = 0.1,\ T(0.6) = 0.1,\ T(0.9) = 0.3.$$

Note that $T(A_0) \subseteq B_0$. Define mappings $\psi : [0, \infty) \to [0, \infty)$, $F : [0, \infty)^3 \to [0, \infty)$ and $\varphi : A \to [0, \infty)$ by

$$\begin{aligned} \psi(t) &= \frac{t}{2}, \\ F(a, b, c) &= a + b + c, \textit{ for all } a, b, c \in [0, \infty) \\ \textit{and } \varphi(\xi) &= \xi, \textit{ for all } \xi \in A. \end{aligned}$$

If we take $\xi = 0.6$, $\eta = 0.9$, $u = 0$ and $v = 0.4$, then we have

$$p(u, T\xi) = p(v, T\eta) = 0.1 = p(A, B),$$

which implies that

$$F(p(u, v), \varphi(u), \varphi(v)) = 0.8 \leq 0.9 = \psi(F(p(\xi, \eta), \varphi(\xi), \varphi(\eta))).$$

Hence, T forms a weak (F, φ)-proximal contraction. Thus all the conditions of Corollary 3 are satisfied. Moreover $\xi^* = 0$ is a unique φ-best proximity point.

3. Application to Fixed Point Theory

Let us take $A = B = S$, and suppose that T is proximal α_p-admissible mapping. Obviously

$$\alpha(\xi, \eta) \geq 0,$$

and

$$\rho(u, T\xi) = 0 \text{ and } \rho(v, T\eta) = 0,$$

implies that

$$\alpha(T\xi, T\eta) = \alpha(u, v) \geq 0.$$

Hence T is α_p-admissible mapping.

Remark 3. *If $\alpha : S \times S \to [-\infty, \infty)$, $\varphi : S \to [0, \infty)$ and a selfmapping T on S is α_p-admissible weak (F, φ)-contraction, then $\alpha(\xi, \eta) \geq 0$ implies that*

$$\alpha(\xi, \eta) + F(\rho(T\xi, T\eta), \varphi(T\xi), \varphi(T\eta)) \leq \psi(F(\rho(\xi, \eta), \varphi(\xi), \varphi(\eta))), \tag{10}$$

where $F \in \mathcal{F}$,and $\psi \in \Psi$, for all $\xi, \eta \in S$. In other words, we consider the notions in Definition 10 and Definition 11 in the setting of standard metric spaces.

Definition 12. *A self mapping $T : S \to S$ satisfying the above implication is called α_p-admissible weak (F, φ)-contraction.*

Corollary 4. *Let (S, d) be a $M-$complete M-metric space, $F \in \mathcal{F}$, and a self-mapping T be an α_p-admissible weak (F, φ)-contraction. If $\{\xi_n\}$ is a sequence in S such that $\alpha(\xi_n, \xi_{n+1}) \geq 0$ and $\lim_{n\to\infty} \xi_n = \xi \in S$, then $\alpha(\xi_n, \xi) \geq 0$, for all $n \in N$. Then there exists a φ-fixed point of T provided that there exists $\xi_0 \in S$ such that $\alpha(\xi_0, T\xi_0) \geq 0$. Moreover, if $\alpha(\xi, \eta) \geq 0$ for all $\xi, \eta \in \varphi_F(S)$, then ξ^* is the unique φ-fixed point of T.*

Proof. Let us take $A = B = S$ in Theorem 1. We shall show that T is α_p-admissible weak (F, φ)-contraction. Suppose that $\xi, \eta, u, v \in S$ satisfies the following

$$\begin{aligned} \alpha(\xi, \eta) &\geq 0, \\ \rho(u, T\xi) &= \rho(A, B), \\ \rho(v, T\eta) &= \rho(A, B). \end{aligned}$$

As $\rho(A, B) = 0$, we have $u = T\xi$ and $v = T\eta$. Since T satisfies the condition (10), so

$$\alpha(\xi, \eta) + F(\rho(T\xi, T\eta), \varphi(T\xi), \varphi(T\eta)) \leq \psi(F(\rho(\xi, \eta), \varphi(\xi), \varphi(\eta))),$$

that is,

$$\alpha(\xi, \eta) + F(\rho(u, v), \varphi(u), \varphi(v)) \leq \psi(F(\rho(\xi, \eta), \varphi(\xi), \varphi(\eta))),$$

which implies that T is an α_p-admissible weak (F, φ)-contraction Let ξ_0 be an arbitrary point in S. Define a sequence $\{\xi_n\}$ in S by

$$\xi_n = T\xi_{n-1}, \text{ for all } n \in \mathbb{N}.$$

As T is α_p-admissible mapping. So, we have

$$\alpha(\xi_0, \xi_1) = \alpha(\xi_0, T\xi_0) \geq 0 \text{ implies that } \alpha(T\xi_0, T\xi_1) = \alpha(\xi_1, \xi_2) \geq 0.$$

By induction, we get that

$$\alpha(\xi_n, \xi_{n+1}) = \alpha(\xi_n, T\xi_n) \geq 0, \text{ for all } n \in \mathbb{N}. \tag{11}$$

Using (11) and the fact that T is $(F, M, \varphi, \alpha_p, \psi)-$contraction, we obtain

$$\begin{aligned} F(\rho(\xi_n, \xi_{n+1}), \varphi(\xi_n), \varphi(\xi_{n+1})) &= F(\rho(T\xi_{n-1}, T\xi_n), \varphi(T\xi_{n-1}), \varphi(T\xi_n)) \\ &\leq \alpha(\xi_{n-1}, \xi_n) + F(\rho(T\xi_{n-1}, T\xi_n), \varphi(T\xi_{n-1}), \varphi(T\xi_n)) \\ &\leq \psi(F(\rho(\xi_{n-1}, \xi_n), \varphi(\xi_n), \varphi(\xi_{n+1}))), \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Using the arguments similar to those given in the proof of Theorem 1, we obtain that $\{\xi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in S. Since (S, ρ) is M-complete M-metric space, there exists $\xi^* \in S$ such that

$$\lim_{n\to\infty} \rho(\xi_n, \xi^*) = 0 \text{ and } \lim_{n\to\infty} M_{\xi_n, \xi^*} = 0. \tag{12}$$

We now show that $\varphi(\xi^*) = 0$. From (2), we conclude that

$$\varphi(\xi_n) \leq \psi^n(F(\rho(\xi_0, \xi_1), \varphi(\xi_0), \varphi(\xi_1))).$$

Again by using the arguments similar to those given in the proof of Theorem 1, we obtain that $\varphi(\xi^*) = 0$. In the view of (11) and (12) we have $\alpha(\xi_n, \xi^*) \geq 0$, for all $n \in \mathbb{N}$. By taking $\xi = \xi_n$ and $\eta = \xi^*$ in the condition (10), we have

$$\begin{aligned} \rho(\xi_{n+1}, T\xi^*) &= \rho(T\xi_n, T\xi^*) \\ &\leq \max\{\rho(T\xi_n, T\xi^*), \varphi(T\xi_n)\} \\ &\leq F(\rho(T\xi_n, T\xi^*), \varphi(T\xi_n), \varphi(T\xi^*)) \\ &\leq \alpha(\xi_n, \xi^*) + F(\rho(T\xi_n, T\xi^*), \varphi(T\xi_n), \varphi(T\xi^*)) \\ &\leq \psi(F(\rho(\xi_n, \xi^*), \varphi(\xi_n), \varphi(\xi^*))) \\ &< F(\rho(\xi_n, \xi^*), \varphi(\xi_n), \varphi(\xi^*)) \\ &= F(\rho(\xi_n, \xi^*), \varphi(\xi_n), 0). \end{aligned}$$

On taking limit as $n \to \infty$ on the both sides of the above inequality, we have

$$\begin{aligned} \lim_{n\to\infty} \rho(\xi_{n+1}, T\xi^*) &= \lim_{n\to\infty} F(\rho(\xi_n, \xi^*), \varphi(\xi_n), 0) \\ &= F(0, 0, 0) = 0, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \rho(\xi_{n+1}, T\xi^*) = 0.$$

By using the condition **(m4)**, we have

$$\begin{aligned} \rho(\xi^*, T\xi^*) - m_{\xi^*, T\xi^*} &\leq \rho(\xi^*, \xi_{n+1}) - m_{\xi^*, \xi_{n+1}} + \rho(\xi_{n+1}, T\xi^*) - m_{\xi_{n+1}, T\xi^*} \\ &\leq \rho(\xi^*, \xi_{n+1}) + \rho(\xi_{n+1}, T\xi^*). \end{aligned}$$

Letting $n \to \infty$ in the inequality above, we deduce that

$$\begin{aligned} \lim_{n\to\infty} \rho(\xi^*, T\xi^*) - m_{\xi^*, T\xi^*} &\leq \lim_{n\to\infty} \rho(\xi^*, \xi_{n+1}) + \lim_{n\to\infty} \rho(\xi_{n+1}, T\xi^*) \\ \lim_{n\to\infty} \rho(\xi^*, T\xi^*) - m_{\xi^*, T\xi^*} &\leq 0. \end{aligned}$$

Since $\rho(\xi^*, \xi^*) = 0$, hence

$$\rho(\xi^*, T\xi^*) = 0,$$

gives that ξ^* is a φ-fixed point of T.

Uniqueness: Let $\alpha(\xi,\eta) \geq 0$ for all $\xi,\eta \in \varphi_F(S)$. Suppose that ξ^* and w are two $\varphi-$fixed point of T with $\xi^* \neq w$. Hence

$$\rho(w, Tw) = 0,$$

and

$$\varphi(\xi^*) = \varphi(w) = 0.$$

Since T is α_p-admissible weak (F,φ)-contraction, we have

$$\begin{aligned} F(\rho(\xi^*, w), 0, 0) &= F(\rho(T\xi^*, Tw), \varphi(T\xi^*), \varphi(Tw)) \\ &\leq \alpha(\xi^*, w) + F(\rho(T\xi^*, Tw), \varphi(T\xi^*), \varphi(Tw)) \\ &\leq \psi(F(\rho(\xi^*, w), \varphi(\xi^*), \varphi(w))) \\ &< F(\rho(\xi^*, w), 0, 0), \end{aligned}$$

a contradiction. Attendantly, we find that ξ^* is a unique φ-fixed point of T. □

4. Application to Graph Theory

Let S be a set and Δ denotes the diagonal of $S \times S$. A graph is a pair (V, E), where the set $V = V(G)$ of its vertices coincides with S and set $E = E(G)$ of its edges which contains all loops, that is, $\Delta \subseteq S \times S$. Furthermore, we assume that the graph G has no parallel edges. In a graph G, by reversing the direction of edges we get the graph G^{-1} whose set of edges and set of vertices are defined as follows:

$$E(G^{-1}) = \{(\xi,\eta) \in S \times S : (\eta,\xi) \in E(G)\} \text{ and } V(G^{-1}) = V(G).$$

We denote the undirected graph by $\widetilde{G}$ obtained from G by ignoring the direction of edges.

Consider the graph $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric, under this convention, we have

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

Definition 13 ([28])**.**
1. *A graph's subgraph is a graph whose vertex set is a subset of $V(G)$ and whose edge set is a subset of $E(G)$.*
2. *Let ξ and η be two vertices of a graph G. A path from ξ to η of length n (where $n \in \mathbb{N} \cup \{0\}$) in a graph G is a sequence $\{\xi_n : n = 0, 1, 2, ..., n\}$ of $n+1$ distinct vertices such that $\xi_0 = \xi$, $\xi_n = \eta$ and $(\xi_i, \xi_{i+1}) \in E(G)$ for $i = 1, 2, \ldots, n$.*
3. *A graph G is called connected graph if there exist a path between any two vertices of graph G and if $\widetilde{G}$ is connected then G is said to be weakly connected graph.*
4. *A path is called elementary if no vertices appear more than once in it.*

Throughout this section, we suppose that (S,ρ) is an M-metric space endowed with a directed graph G and has no parallel edges.

We now introduce a notion of $G-$proximal graphic contraction.

Definition 14. *Let A, B be two subsets of an M-complete M-metric space (S,ρ), $\varphi : S \to [0,\infty)$, $\psi \in \Psi$, $F \in \mathcal{F}$ and G be a graph without parallel edges such that $V(G) = S$. A mapping $T : A \to B$ is said to be a $G-$proximal graphic contraction if for all $\xi, \eta, u, v \in A$, $\xi \neq \eta$, with $(\xi,\eta) \in E(G)$ we have*

$$\left.\begin{array}{l} \rho(u, T\xi) = \rho(A, B) \\ \rho(v, T\eta) = \rho(A, B) \end{array}\right\} \Longrightarrow F(\rho(u, v), \varphi(u), \varphi(v)) \leq \psi(F(\rho(\xi,\eta), \varphi(\xi), \varphi(\eta))),$$

and

$$(u, v) \in E(G).$$

Theorem 2. *Let $\varphi : A \to [0, \infty)$ be a lower semi continuous function and $T : A \to B$ a $G-$proximal graphic contraction. If $T(A_0) \subseteq B_0$, A_0 is closed set in S and there exist a path $(\eta^i)_{i=0}^N \subseteq A_0$ in G between any two elements ξ and η. Then there exist a unique $\varphi-$best proximity point of T provided that there exist $\xi_0, \xi_1 \in A_0$ and an elementary path between them in A_0 and*

$$\rho(\xi_1, T\xi_0) = \rho(A, B).$$

Proof. Let $\xi_0, \xi_1 \in A_0$ such that $\rho(\xi_1, T\xi_0) = \rho(A, B)$. A path $\{s_0^0, s_0^1, s_0^2, \ldots, s_0^N\}$ in G is a sequence containing points of A_0. Consequently, $s_0^0 = \xi_0$, $s_0^N = \xi_1$ and $(s_0^i, s_0^{i+1}) \in E(G)$ for all $0 \leq i \leq N - 1$. Given that $s_0^1 \in A_0$, by $T(A_0) \subseteq B_0$ and the definition of A_0, there exist $s_1^1 \in A_0$ such that $\rho(s_1^1, Ts_0^1) = \rho(A, B)$. Similarly, for each $i = 2, \ldots, N$, there exists $s_1^i \in A_0$ such that $\rho(s_1^i, Ts_0^i) = \rho(A, B)$. As $\{s_0^0, s_0^1, s_0^2, \ldots, s_0^N\}$ is a path in G, $(s_0^0, s_0^1) = (\xi_0, s_0^1) \in E(G)$. From the above argument, we have $\rho(\xi_1, T\xi_0) = \rho(A, B)$ and $\rho(s_1^1, Ts_0^1) = \rho(A, B)$. Since, T is $G-$proximal graphic contraction, it follows that $(\xi_1, s_1^1) \in E(G)$. In similar manner, we have the following:

$$(s_1^{i-1}, s_1^i) \in E(G), \text{ for all } 1 \leq i \leq N.$$

If $\xi_2 = s_1^N$, then $\{s_1^0, s_1^1, s_1^2, \ldots, s_1^N\}$ is a path from $\xi_1 = s_1^0$ to $\xi_2 = s_1^N$. As $s_1^i \in A_0$ and $Ts_1^i \in T(A_0) \subseteq B_0$, or each $i = 1, 2, 3, \ldots, N$, by the definition of B_0, there exists $s_2^i \in A_0$ such that $\rho(s_2^i, Ts_1^i) = \rho(A, B)$. In addition, we have $\rho(\xi_2, T\xi_1) = \rho(A, B)$. As mentioned above, we have

$$(\xi_2, s_2^1) \in E(G) \text{ and } (s_2^{i-1}, s_2^i) \in E(G), \text{ for all } 1 \leq i \leq N.$$

Similarly, by $T(A_0) \subseteq B_0$, there exists a point $\xi_3 \in A_0$ where $\xi_3 = s_2^N$. Then $(s_2^i)_{i=0}^N$ is a path from $s_2^0 = \xi_2$ and $s_2^N = \xi_3$. Continuing in this manner for all $n \in \mathbb{N}$, we obtain a sequence $\{\xi_n\}_{n \in \mathbb{N}}$ where $\xi_{n+1} \in [\xi_n]_G^N$ and $\rho(\xi_{n+1}, T\xi_n) = \rho(A, B)$ by producing a path $\{s_n^0, s_n^1, s_n^2, \ldots, s_n^N\}$ from $\xi_n = s_n^0$ and $\xi_{n+1} = s_n^N$ in such a way that

$$\rho(s_{n+1}^i, Ts_n^i) = \rho(A, B),$$

for all $1 \leq i \leq N$, $n \in \mathbb{N}$. Thus we have

$$\rho(s_n^{i-1}, Ts_{n-1}^{i-1}) = \rho(A, B) = \rho(s_n^i, Ts_{n-1}^i), \text{ for all } 1 \leq i \leq N, \ n \in \mathbb{N}. \tag{13}$$

Now for any positive integer n

$$\begin{aligned}
\rho(\xi_n, \xi_{n+1}) &= \rho(s_n^0, s_n^N) \\
&\leq \rho(s_n^0, s_n^1) - m_{s_n^0, s_n^1} + \rho(s_n^1, s_n^2) - m_{s_n^1, s_n^2} + \ldots + \rho(s_n^{N-1}, s_n^N) - m_{s_n^{N-1}, s_n^N} \\
&\leq \rho(s_n^0, s_n^1) + \rho(s_n^1, s_n^2) + \ldots + \rho(s_n^{N-1}, s_n^N) \\
&= \sum_{i=1}^N \rho(s_n^{i-1}, s_n^i),
\end{aligned} \tag{14}$$

for all $1 \leq i \leq N$ and $n \in \mathbb{N}$. Note that, $(s_{n-1}^{i-1}, s_{n-1}^i) \in E(G)$, and T is $G-$proximal graphic contraction. It follows from (13), that

$$F(\rho(s_n^{i-1}, s_n^i), \varphi(s_n^{i-1}), \varphi(s_n^i)) \leq \psi(F(\rho(s_{n-1}^{i-1}, s_{n-1}^i), \varphi(s_{n-1}^{i-1}), \varphi(s_{n-1}^i))), \text{ for all } 1 \leq i \leq N, \ n \in \mathbb{N}.$$

Again by using the arguments similar to those given in the proof of Theorem 1, we obtain that

$$\rho(s_n^{i-1}, s_n^i) \leq \psi^n(F(\rho(s_0^{i-1}, s_0^i), \varphi(s_0^{i-1}), \varphi(s_0^i))). \tag{15}$$

From (14) and (15), we have

$$\rho(\xi_n, \xi_{n+1}) \leq \psi^n M, \text{ for all } n \in \mathbb{N},$$

where $M = \sum_{i=1}^{N}(F(\rho(s_0^{i-1}, s_1^i), \varphi(s_0^{i-1}), \varphi(s_1^i)))$. Again by using the arguments similar to those given in the proof of Theorem 1, we obtain

$$\varphi(\xi^*) = 0 \text{ and } \rho(\xi^*, T\xi^*) = \rho(A, B).$$

Hence ξ^* is a unique φ-best proximity point of T. $\square$

5. Conclusions

In this paper, we defined φ-best proximity point and α_p-admissible weak (F, φ)-contraction. We proved some φ-best proximity point results in the setting of M-metric spaces. As an application, we derived the φ-fixed point results for some self mappings. We also introduced the notions of $G-$proximal graphic contraction and provided an application to graph theory in the setting of M-complete M-metric space. Some examples are also presented to illustrate the novelty of the result proved herein.

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