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Best Proximity Points for Monotone Relatively Nonexpansive Mappings in Ordered Banach Spaces

Karim Chaira [1,†], Mustapha Kabil [1,†], Abdessamad Kamouss [1,*,†] and Samih Lazaiz [2,†]

1 Laboratory of Mathematics and Applications, Department of Mathematics, Faculty of Sciences and Technologies Mohammedia, University Hassan II Casablanca, Casablanca 20000, Morocco; chaira_karim@yahoo.fr (K.C.); kabilfstm@gmail.com (M.K.)
2 Laboratory of Mathematical Analysis and Applications, Department of Mathematics, Faculty of Sciences Dhar El Mahraz, University Sidi Mohamed Ben Abdellah, Fes 30050, Morocco; samih.lazaiz@gmail.com
* Correspondence: akamouss@gmail.com
† These authors contributed equally to this work.

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Abstract: In this paper, we give sufficient conditions to ensure the existence of the best proximity point of monotone relatively nonexpansive mappings defined on partially ordered Banach spaces. An example is given to illustrate our results.

Keywords: best proximity point; fixed point; monotone mappings; relatively cyclic nonexpansive mappings; partially ordered Banach spaces

1. Introduction

Let X be a Banach space and (A, B) a pair of nonempty subsets of X. A cyclic mapping on $A \cup B$ is a mapping $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. In case $A \cap B = \varnothing$, T does not possess a fixed point, that is, a solution to the equation $Tx = x$. Therefore, one can consider the following minimization problem:

$$(P) : \begin{cases} \text{find } (x, y) \in A \times B \text{ such that} \\ \|x - Tx\| = \|y - Ty\| = dist(A, B). \end{cases}$$

A point $x \in A \cup B$ is a best proximity point of T if x is a solution of the minimization problem (P). The best proximity point notion can be seen as a generalization of fixed point notion since most fixed point theorems can be derived as corollaries of best proximity point theorems.

The first significant result of best proximity points was studied in [1], using the proximal normal structure, the authors proved that every cyclic relatively nonexpansive mapping from $A \cup B$ to itself has a best proximity point provided that A and B are weakly compact and convex. Furthermore, we find in [2] a similar result without invoking Zorn's lemma, i.e., without proximal normal structure. Recently, Chaira and Lazaiz [3] gave an extension of this last result in modular spaces. For a recent account of the theory we refer the reader to [4–6]. We can also find in ([7], pp. 27–31) an application of a best proximity point theorem to a system of differential equations.

On the other hand, the combination of metric fixed point theory and order theory allows Ran and Reurings in [8] to give a Banach Contraction Principle in partially ordered metric spaces. As consequence, they solved a matrix equation. Nieto and Rodríguez-López [9], extended the Ran–Reurings theorem in order to obtain a periodic solution for a first-order ordinary differential equation with periodic boundary conditions.

Recently, many authors studied the existence of fixed points of monotone nonexpansive mappings defined on partially ordered Banach spaces (see for example [10–15]). Recall that a self mapping T on X is said to be monotone nonexpansive if T is monotone and $\|Tx - Ty\| \leq \|x - y\|$, for every comparable elements x and y. We should mention that monotone nonexpansive mappings may not be continuous. The interested reader can consult the book of Carl and Heikkilä [16] for many applications of fixed point results of monotone mappings.

In this work, motivated by the recent study of a fixed point for monotone mappings, we investigate the existence of the best proximity point of monotone relatively nonexpansive mappings in partially ordered Banach spaces.

2. Preliminaries and Basic Results

Let $(X, \|.\|)$ be a Banach space endowed with a partial order $\preceq$. Throughout, we assume that the order intervals are closed and convex. Recall that an order interval is any of the subsets

$$[a, \rightarrow) = \{x \in X; a \preceq x\} \quad , \quad (\leftarrow, a] = \{x \in X; x \preceq a\}$$

for any $a \in X$. As a direct consequence of this, the subset

$$[a, b] = \{x \in X; a \preceq x \preceq b\} = [a, \rightarrow) \cap (\leftarrow, b]$$

is also closed and convex for any $a, b \in X$.

We will say that $x, y \in X$ are comparable whenever $x \preceq y$ or $y \preceq x$. The linear structure of X is assumed to be compatible with the order structure in the following sense:

(i) $x \preceq y$ implies $x + z \preceq y + z$ for all $x, y, z \in X$;
(ii) $x \preceq y$ implies $\alpha x \preceq \alpha y$ for all $x, y \in X$ and $\alpha \in \mathbb{R}^+$.

Let us recall the definition of a uniformly convex Banach space.

Definition 1. *Let $(X, \|.\|)$ be a Banach space. We say that X is uniformly convex (in short, UC) if for every $\epsilon > 0$ we have $\delta(\epsilon) > 0$ such that*

$$\delta(\epsilon) = inf\left\{1 - \left\|\frac{x+y}{2}\right\|; \|x\| \leq 1; \|y\| \leq 1; \|x - y\| \geq \epsilon\right\}.$$

The function δ is known as the modulus of uniform convexity of X. Note that any UC Banach space is reflexive.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a partially ordered set $(X, \preceq)$ is said to be

(i) monotone increasing if $x_n \preceq x_{n+1}$, for all $n \in \mathbb{N}$;
(ii) monotone decreasing if $x_{n+1} \preceq x_n$, for all $n \in \mathbb{N}$;
(iii) monotone sequence if it is either monotone increasing or decreasing.

The following technical lemmas will be useful to establish the main results.

Lemma 1. *Let X be a Banach space endowed with a partial order $\preceq$. Assume that $\{x_n\}$ and $\{y_n\}$ are two sequences on X which are weakly convergent to $\bar{x}$ and $\bar{y}$ respectively and $x_n \preceq y_n$ for any $n \in \mathbb{N}$, then*

$$\bar{x} \preceq \bar{y}.$$

Proof. Note that the positive sequence $\{y_n - x_n\}_n$ converges weakly to $\bar{y} - \bar{x}$. Since closed convex subsets are also weakly closed, the positive cone is weakly closed and so we conclude that $\bar{y} - \bar{x}$ is positive. □

Lemma 2. *[17] Let $\{x_n\}$ be a bounded monotone sequence in X, and assume that X is reflexive. Then $\{x_n\}$ is weakly convergent.*

Lemma 3. *[18] Let C be a nonempty closed convex subset of a UC Banach space $(X, \|.\|)$. Let $\tau : C \to [0, \infty)$ be a type function, i.e., there exists a bounded sequence $\{x_n\} \in X$ such that*

$$\tau(x) = \limsup_{n\to\infty} \|x_n - x\|,$$

for every $x \in C$. Then τ has a unique minimum point $z \in C$ such that

$$\tau(z) = \inf\{\tau(x); x \in C\} = \tau_0.$$

Moreover, if $\{z_n\}$ is a minimizing sequence in C, i.e., $\lim_{n\to\infty} \tau(z_n) = \tau_0$, then $\{z_n\}$ converges strongly to z.

The norm $\|.\|$ of X is said to be monotone if

$$u \preceq v \preceq w \quad \text{implies} \quad \max\{\|w - v\|, \|v - u\|\} \leq \|w - u\|,$$

for any $u, v, w \in X$. If the norm is monotone and $\{x_n\}$ is monotone increasing (respectively, decreasing), then the sequence $\{\|x_n - y\|\}$ is decreasing for any y such that $x_n \preceq y$ (respectively, $y \preceq x_n$), for any $n \in \mathbb{N}$. In this case,

$$\liminf_{n\to\infty} \|x_n - y\| = \lim_{n\to\infty} \|x_n - y\| = \inf_{n\in\mathbb{N}} \|x_n - y\|.$$

Recall that a mapping $T : X \to X$ is said to be

(i) monotone increasing if $x \preceq y$ implies $T(x) \preceq T(y)$, for all $x, y \in X$;
(ii) monotone decreasing if $x \preceq y$ implies $T(y) \preceq T(x)$, for all $x, y \in X$.

We conclude this section by extending the concept of relatively cyclic nonexpansive mapping to monotone relatively cyclic nonexpansive mapping as follows:

Definition 2. *Let $(X, \|.\|, \preceq)$ be a Banach space endowed with a partially order and (A, B) a pair of nonempty subset of X. The mapping $T : A \cup B \to A \cup B$ is said to be monotone increasing (respectively decreasing) relatively cyclic nonexpansive if*

1. *$T(A) \subseteq B$ and $T(B) \subseteq A$,*
2. *T is monotone increasing (respectively decreasing),*
3. *$\|Tx - Ty\| \leq \|x - y\|$, whenever $x \in A$ and $y \in B$ are comparables.*

3. Main Result

Throughout we assumed that $(X, \|.\|, \preceq)$ is a Banach space endowed with a partial order for which order intervals are convex and closed and the linear structure of X is assumed to be compatible with the order structure.

The following result gives sufficient conditions to obtain a fixed point theorem for a monotone increasing relatively cyclic nonexpansive mapping.

Theorem 1. *Let (A, B) be a nonempty bounded closed convex pair in a partially ordered Banach space $(X, \|.\|, \preceq)$. Assume that $(X, \|.\|)$ is UC. Let $T : A \cup B \to A \cup B$ be a monotone increasing relatively cyclic nonexpansive mapping such that $x_0 \preceq Tx_0$ for some $x_0 \in A$, then $A \cap B \neq \emptyset$ and there exists $a^* \in A \cap B$ such that $Ta^* = a^*$.*

Proof. We assume that $x_0 \preceq Tx_0$ and we define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. By using the monotonicity of T we get

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

Since A and B are bounded and closed, the sequence $\{x_n\}$ is bounded increasing in the reflexive space X. By Lemma 2,

$$x_{2n} \xrightarrow{w} \bar{x}_1 \in A \quad \text{and} \quad x_{2n+1} \xrightarrow{w} \bar{x}_2 \in B.$$

By uniqueness of the weak limit, $\bar{x} = \bar{x}_1 = \bar{x}_2$. We claim that $A \cap B \neq \emptyset$.

Let $K = \{x \in A \cap B, \quad x_n \preceq x \quad \text{for all } n \in \mathbb{N}\}$. It is clear that K is nonempty, closed and convex set. Since $\{x_n\}$ is a bounded sequence in X, we can define the type function as follows

$$\tau(x) = \limsup_{n \to \infty} \|x_n - x\|,$$

for any $x \in K$. From Lemma 3, it follows that there exists a unique $a^* \in K$ such that

$$\tau(a^*) = \inf_{x \in K} \tau(x).$$

We have

$$\tau(Ta^*) = \limsup_{n \to \infty} \|x_n - Ta^*\| = \limsup_{n \to \infty} \|Tx_{n-1} - Ta^*\|.$$

Since $x_{n-1} \preceq a^*$ and T is monotone relatively cyclic nonexpansive mapping,

$$\tau(Ta^*) \leq \limsup_{n \to \infty} \|x_{n-1} - a^*\| = \tau(a^*).$$

Hence, $\tau(Ta^*) = \tau(a^*)$. Thus $Ta^* = a^*$, which completes the proof. □

If $B = A$, we get the next result for a monotone nonexpansive mapping.

Corollary 1. *Let A be a nonempty bounded closed convex set in a partially ordered Banach space $(X, \|.\|, \preceq)$. Let $T : A \to A$ be a monotone increasing nonexpansive mapping. Assume that $(X, \|.\|)$ is UC and there exists $x_0 \in A$ such that $x_0 \preceq Tx_0$, then there exists $a^* \in A$ such that $Ta^* = a^*$.*

Now let $(\mathcal{A}_0^{\preceq}, \mathcal{B}_0^{\preceq})$ denotes the pair obtained from (A, B) upon setting

$$\begin{aligned} \mathcal{A}_0^{\preceq} &= \{x \in A; \|x - y\| = dist(A, B) \text{ for some } y \in B \cap [x, \to)\} \\ \mathcal{B}_0^{\preceq} &= \{y \in B; \|y - x\| = dist(A, B) \text{ for some } x \in A \cap (\leftarrow, y]\}. \end{aligned}$$

Lemma 4. *Let (A, B) be a nonempty bounded closed convex pair in a partially ordered reflexive Banach space $(X, \|.\|, \preceq)$. Then,*

(i) $\mathcal{A}_0^{\preceq} \neq \emptyset$ *if and only if* $\mathcal{B}_0^{\preceq} \neq \emptyset$;
(ii) $dist\left(\mathcal{A}_0^{\preceq}, \mathcal{B}_0^{\preceq}\right) = dist(A, B)$;
(iii) $\left(\mathcal{A}_0^{\preceq}, \mathcal{B}_0^{\preceq}\right)$ *is a closed pair;*
(iv) $\left(\mathcal{A}_0^{\preceq}, \mathcal{B}_0^{\preceq}\right)$ *is a convex pair.*

Proof. Using the definitions of $\mathcal{A}_0^{\preceq}$ and $\mathcal{B}_0^{\preceq}$, we can easily derive (i) and (ii).

(iii) Let $\{x_n\} \subset \mathcal{A}_0^{\preceq}$ be a sequence which converges to some $\bar{x}$ in A. Then there exists a sequence $\{y_n\} \subset B$ such that

$$\|x_n - y_n\| = \text{dist}(A, B) \text{ and } x_n \preceq y_n.$$

Since B is closed and bounded in a reflexive Banach space, there exists a subsequence $\{y_{\varphi(n)}\}$ of $\{y_n\}$ such that $y_{\varphi(n)} \xrightarrow{w} \bar{y} \in B$. From Lemma 1, it follows that $\bar{x} \preceq \bar{y}$. On the other hand,

$$\|\bar{x} - \bar{y}\| \leq \liminf_{n\to\infty} \|x_{\varphi(n)} - y_{\varphi(n)}\| = dist(A, B).$$

Therefore, we have $\bar{x} \in \mathcal{A}_0^{\preceq}$, and hence, $\mathcal{A}_0^{\preceq}$ is closed. By the same arguments we get that $\mathcal{B}_0^{\preceq}$ is also closed.

(iv) Let x and x' in $\mathcal{A}_0^{\preceq}$. Then there exist y and y' in B such that

$$\begin{cases} \|x - y\| &= \operatorname{dist}(A, B) \quad \text{and} \quad x \preceq y, \\ \|x' - y'\| &= \operatorname{dist}(A, B) \quad \text{and} \quad x' \preceq y'. \end{cases}$$

By using the fact that the linear structure of X is compatible with the order structure, we get for any $t \in [0, 1]$

$$\begin{aligned} \|tx + (1-t)x' - ty - (1-t)y'\| &= \|t(x-y) + (1-t)(x'-y')\| \\ &\leq t\|x - y\| + (1-t)\|x' - y'\| \\ &= dist(A, B). \end{aligned}$$

This implies that $tx + (1-t)x' \in \mathcal{A}_0^{\preceq}$. It follows that $\mathcal{A}_0^{\preceq}$ is convex, as claimed. Similarly we prove that $\mathcal{B}_0^{\preceq}$ is also convex.

□

Remark 1. *Note that if T is a monotone decreasing relatively cyclic nonexpansive mapping, we have $T\left(\mathcal{A}_0^{\preceq}\right) \subset \mathcal{B}_0^{\preceq}$ and $T\left(\mathcal{B}_0^{\preceq}\right) \subset \mathcal{A}_0^{\preceq}$. Indeed, let $x \in \mathcal{A}_0^{\preceq}$ then there exists $y \in B$ such that*

$$\|x - y\| = dist(A, B) \quad \textit{and} \;\; x \preceq y.$$

Thus,

$$\|Tx - Ty\| \leq \|x - y\| = dist(A, B) \quad \textit{and} \quad Ty \preceq Tx.$$

This implies $Tx \in \mathcal{B}_0^{\preceq}$. Consequently $T\left(\mathcal{A}_0^{\preceq}\right) \subset \mathcal{B}_0^{\preceq}$.

For the sake of simplicity, we use the following notation

$$\mathcal{A}_T = \left\{(x_0, x_0') \in A \times A;\; x_0 \preceq Tx_0';\; \|x_0 - Tx_0'\| = dist\,(A, B)\right\}.$$

The next lemma gives sufficient conditions such that $\mathcal{A}_T$ is nonempty.

Lemma 5. *Let (A, B) be a nonempty bounded closed convex pair in a partially ordered Banach space $(X, \|.\|, \preceq)$ such that $\mathcal{A}_0^{\preceq}$ is nonempty. Let $T : A \cup B \to A \cup B$ be a monotone relatively cyclic nonexpansive mapping. Then $\mathcal{A}_T$ is nonempty.*

Proof. Suppose that T is a monotone decreasing relatively cyclic nonexpansive mapping. Since $\mathcal{A}_0^{\preceq} \neq \emptyset$, we can find a x_0' in $\mathcal{A}_0^{\preceq}$ such that there exists an $y \in B \cap \left[x_0', \to\right)$ satisfying $\|x_0' - y\| = dist\,(A, B)$.

Since $x_0' \preceq y$ and T is monotone decreasing relatively cyclic nonexpansive mapping, $Ty \preceq Tx_0'$ and $\|Tx_0' - Ty\| \leq \|x_0' - y\| = dist\,(A, B)$, give that $Tx_0' \in \mathcal{B}_0^{\preceq}$.

Next, for Tx_0' there exists an element $x_0 \in \mathcal{A}_0^{\preceq}$ such that

$$x_0 \preceq Tx_0' \quad \text{and} \quad \|x_0 - Tx_0'\| = dist\,(A, B)\,.$$

Now, suppose that T is a monotone increasing relatively cyclic nonexpansive mapping. Since $\mathcal{A}_0^{\preceq} \neq \emptyset$, we can find a x in $\mathcal{A}_0^{\preceq}$ such that there exists an $y \in \mathcal{B}_0^{\preceq}$ satisfying $x \preceq y$ and $\|x-y\| = dist\,(A,B)$.

Since T is monotone increasing, $Tx \preceq Ty$ and

$$\|T^2x - T^2y\| \leq \|Tx - Ty\| \leq \|x-y\| = dist\,(A,B)\,.$$

Take $x_0 = T^2x \in A$ and $x_0^{'} = Ty \in A$. We have clearly,

$$x_0 \preceq Tx_0^{'} \quad \text{and} \quad \|x_0 - Tx_0^{'}\| = dist\,(A,B)\,.$$

Thus $\mathcal{A}_T \neq \emptyset$. □

In the following, we give a best proximity result for monotone increasing relatively cyclic nonexpansive mapping.

Theorem 2. *Let $(X, \|.\|, \preceq)$ be a partially ordered Banach space. Assume that $(X, \|.\|)$ is UC. Let (A,B) be a nonempty bounded closed convex pair in X. Let $T : A \cup B \to A \cup B$ be a monotone increasing relatively cyclic nonexpansive mapping. Assume that T is weakly sequentially continuous, the norm $\|.\|$ of X is monotone and there exists $\left(x_0, x_0^{'}\right) \in \mathcal{A}_T$ such that $x_0 \preceq x_0^{'} \preceq T^2x_0$ then there exist $\bar{x} \in A$ and $\bar{y} \in B$ such that $\|\bar{x} - T\bar{x}\| = \|\bar{y} - T\bar{y}\| = dist\,(A,B)$.*

Proof. Suppose that there exists $\left(x_0, x_0^{'}\right) \in A \times A$ such that

$$\left\|x_0 - Tx_0^{'}\right\| = dist\,(A,B) \quad \text{and} \quad x_0 \preceq x_0^{'} \preceq T^2x_0.$$

Let $\{x_n\}$ and $\{y_n\}$ be two sequences defined as follows :

$$\begin{cases} x_n = T^{2n}x_0 \\ y_n = T^{2n+1}x_0^{'} \end{cases} \quad \text{for all } n \in \mathbb{N}.$$

Note that, since $x_0 \preceq Tx_0^{'}$ we get $T^{2n}x_0 \preceq T^{2n+1}x_0^{'}$ for all $n \geq 0$, that is, $x_n \preceq y_n$ for all $n \geq 0$.

Since T is monotone increasing relatively cyclic nonexpansive mapping, we get

$$\|x_n - y_n\| = \left\|T^{2n}x_0 - T^{2n+1}x_0^{'}\right\| \leq \left\|x_0 - Tx_0^{'}\right\| = dist\,(A,B)\,,$$

that is, $\|x_n - y_n\| = dist\,(A,B)$, for all $n \in \mathbb{N}$.

Since $x_0 \preceq T^2x_0$, $x_1 = T^2x_0 \preceq T^4x_0 = x_2$ and by induction on n, we can get

$$x_n \preceq x_{n+1} \quad \text{for all} \quad n \in \mathbb{N}.$$

In the same manner, we get

$$y_n \preceq y_{n+1} \quad \text{for all} \quad n \in \mathbb{N}.$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded increasing sequences in reflexive space, we get from Lemma 2, $x_n \overset{w}{\rightharpoonup} \bar{x}$ and $y_n \overset{w}{\rightharpoonup} \bar{y}$.

Note that $\bar{x} = \sup\{x_n; n \in \mathbb{N}\}$ and $\bar{y} = \sup\{y_n;\, n \in \mathbb{N}\}$.

Let $K = \{y \in B;\, y_n \preceq y, \quad \text{for any } n \in \mathbb{N}\}$ and define the type function $\tau : K \to [0,\infty)$ generated by the sequence $\{x_n\}$, that is,

$$\tau\,(y) = \limsup_{n\to\infty}\, \|x_n - y\|\,,$$

for $y \in K$. Using the fact that τ is increasing function, we get

$$\tau\left(\bar{y}\right) = \inf_{y \in K} \tau\left(y\right). \tag{1}$$

Indeed, let $z_1, z_2 \in K$ such that $z_1 \preceq z_2$ then for all $n \in \mathbb{N}$ we have

$$x_n \preceq y_n \preceq z_1 \preceq z_2.$$

Using the fact that the norm $\|.\|$ is monotone, we get

$$\|x_n - z_1\| \leq \|x_n - z_2\|,$$

hence,

$$\tau(z_1) \leq \tau(z_2).$$

From Lemma 3, it follows that there exists a unique $b^* \in K$ such that :

$$\tau\left(b^*\right) = \inf_{y \in K} \tau\left(y\right). \tag{2}$$

Since $\bar{y} = \sup\left\{y_n;\, n \in \mathbb{N}\right\}$ and $b^* \in K$, $\bar{y} \preceq b^*$, that is, $\tau(\bar{y}) \preceq \tau(b^*)$.
Thus, $\tau(\bar{y}) = \tau(b^*)$, i.e., $\bar{y} = b^*$.
We have also

$$\begin{aligned}
\tau\left(T^2\bar{y}\right) &= \limsup_{n \to \infty} \left\|x_n - T^2\bar{y}\right\| \\
&= \limsup_{n \to \infty} \left\|T^2 x_{n-1} - T^2\bar{y}\right\| \\
&\leq \limsup_{n \to \infty} \left\|x_{n-1} - \bar{y}\right\| \\
&= \tau\left(\bar{y}\right),
\end{aligned}$$

hence, $T^2\bar{y} = \bar{y}$.
Furthermore, T is weakly sequentially continuous then $Tx_n \overset{w}{\rightharpoonup} T\bar{x}$ and $Ty_n \overset{w}{\rightharpoonup} T\bar{y}$. By the lower semi continuity of the norm, we get

$$\|\bar{x} - \bar{y}\| \leq \liminf_{n \to \infty} \|x_n - y_n\| = dist\left(A, B\right).$$

Let $\{x_n'\}$ be a sequence defined by $x_n' = T^{2n}x_0'$, for all $n \in \mathbb{N}$. We have

$$y_n = T^{2n+1}x_0' = T(T^{2n}x_0') = Tx_n'.$$

Since $x_0' \preceq T^2x_0'$, $T^{2n}x_0' \preceq T^{2n+2}x_0'$, that is, $x_n' \preceq x_{n+1}'$, for all $n \in \mathbb{N}$. Since $\{x_n'\}$ is bounded increasing sequence in reflexive space, we get by using Lemma 2 $x_n' \overset{w}{\rightharpoonup} \bar{x'}$. Since T is weakly sequentially continuous, $y_n = Tx_n' \overset{w}{\rightharpoonup} T\bar{x'}$. By the uniqueness of the limit, $T\bar{x'} = \bar{y}$, that is,

$$\left\|\bar{x} - T\bar{x'}\right\| = dist\left(A, B\right). \tag{3}$$

Note that $x_0 \preceq x_0' \preceq T^2x_0 \preceq T^2x_0'$, that is, $x_0 \preceq x_0' \preceq x_1 \preceq x_1'$. Then, by induction on n, we can get

$$x_n \preceq x_n' \preceq x_{n+1} \preceq x_{n+1}'.$$

Define the sequence $\{z_n\}$ as follows

$$z_n = \begin{cases} x_n & \text{if } n \text{ is even,} \\ x'_{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Since $\{z_n\}$ is bounded increasing sequence in reflexive space, by using Lemma 2, we get $z_n \overset{w}{\rightharpoonup} \bar{z}$. In particular, the subsequences $\{z_{2n}\}$ and $\{z_{2n+1}\}$ also converge to $\bar{z}$, that is, $\bar{z} = \bar{x} = \bar{x}'$. Thus, by using (3) we get $\|\bar{x} - T\bar{x}\| = dist\,(A, B)$. $\square$

In the following, we give a best proximity result for monotone decreasing relatively cyclic nonexpansive mapping without assuming the monotonicity of the norm $\|.\|$.

Theorem 3. *Let (A, B) be a nonempty bounded closed convex pair in a partially ordered Banach space $(X, \|.\|, \preceq)$. Let $T : A \cup B \to A \cup B$ be a monotone decreasing relatively cyclic nonexpansive mapping. Assume that $(X, \|.\|)$ is UC, T is weakly sequentially continuous and there exists $\left(x_0, x'_0\right) \in \mathcal{A}_T$ such that $x'_0 \preceq x_0 \preceq T^2 x'_0$, then there exists $(\bar{x}, \bar{y}) \in A \times B$ such that*

$$\|\bar{x} - T\bar{x}\| = \|\bar{y} - T\bar{y}\| = dist\,(A, B)\,.$$

Proof. Let $\left(x_0, x'_0\right) \in \mathcal{A}_T$ such that

$$x'_0 \preceq x_0 \preceq T^2 x'_0.$$

If $A \cap B \neq \emptyset$ then $x_0 = Tx'_0$ by Lemma 5. Since $x'_0 \preceq x_0 \preceq T^2 x'_0$ and T is decreasing, we get $x_0 \preceq Tx_0$ and $Tx_0 \preceq Tx'_0 = x_0$. Thus, $Tx_0 = x_0$.

If $A \cap B = \emptyset$, then we consider the sequences $\{x_n\}$ and $\{z_n\} \subset A$ defined by

$$\left\{ \begin{array}{lcl} z_0 & = & x'_0 \\ x_n & = & T^{2n} x_0 \\ z_n & = & T^{2n} x'_0 \end{array} \right. \quad \text{for all } n \in \mathbb{N}^*.$$

Since $x'_0 \preceq x_0 \preceq T^2 x'_0 = z_1$ and T^2 is a monotone increasing mapping, by induction on n, we get $T^{2n} x'_0 \preceq T^{2n} x_0 \preceq T^{2n+2} x'_0$, which implies

$$z_n \preceq x_n \preceq z_{n+1}, \tag{4}$$

for all $n \geq 0$. Also, since $x_0 \preceq Tx'_0 = Tz_0$ and T^2 is a monotone increasing mapping, by induction on n, we get $T^{2n} x_0 \preceq T(T^{2n} x'_0)$, which implies

$$x_n \preceq Tz_n, \tag{5}$$

for all $n \geq 0$. The sequences $\{x_n\}$ and $\{z_n\}$ are increasing. Indeed, $x_0 \preceq T^2 x'_0 \preceq T^2 x_0$ implies by induction on n that $T^{2n} x_0 \preceq T^{2n+2} x_0$. Thus,

$$x_n \preceq x_{n+1},$$

for all $n \in \mathbb{N}$. Since $\{x_n\}$ and $\{z_n\}$ are bounded increasing sequences in a reflexive space, we get by Lemma 2, $x_n \overset{w}{\rightharpoonup} \bar{x}$ and $z_n \overset{w}{\rightharpoonup} \bar{z}$. Using the fact that T is weakly sequentially continuous we conclude that $Tz_n \overset{w}{\rightharpoonup} T\bar{z}$.

Since T is relatively cyclic nonexpansive mapping, we get

$$\begin{aligned} \|x_n - Tz_n\| &= \|T^2x_{n-1} - T^3z_{n-1}\| \\ &\leq \|Tx_{n-1} - T^2z_{n-1}\| \\ &\leq \|x_{n-1} - Tz_{n-1}\|, \end{aligned}$$

for all n in $\mathbb{N}^*$. By induction on n, we prove that

$$\|x_n - Tz_n\| \leq \|x_0 - Tx_0^{'}\| = dist\,(A,B)\,,$$

for all $n \in \mathbb{N}$. By the lower semi continuity of the norm, we get

$$\|\bar{x} - T\bar{z}\| \leq \liminf_{n\to\infty} \|x_n - Tz_n\| = dist\,(A,B)\,. \tag{6}$$

It follows from the Lemma 1 and the inequality (4) that $\bar{z} \preceq \bar{x} \preceq \bar{z}$, and hence, $\bar{z} = \bar{x}$. Finally, by equation (6) it follows that

$$\|\bar{x} - T\bar{x}\| = dist\,(A,B)\,.$$

Let $\bar{y} = T\bar{x}$, then by inequality (5) and Lemma 1 we have $\bar{x} \preceq \bar{y}$ and

$$\|\bar{y} - T\bar{y}\| = \|T\bar{x} - T\bar{y}\| \leq \|\bar{x} - \bar{y}\| = dist\,(A,B)\,.$$

So the proof is complete.
We claim that $T^2\bar{x} = \bar{x}$ and $T^2\bar{y} = \bar{y}$. Indeed, since $x_{n+1} = T^2x_n \overset{w}{\rightharpoonup} \bar{x}$ and $x_{n+1} = T^2x_n \overset{w}{\rightharpoonup} T^2\bar{x}$, the uniqueness of the weak limit implies that $T^2\bar{x} = \bar{x}$. Furthermore, $T\bar{x} = \bar{y}$ then

$$T^2\bar{x} = \bar{x} \Longrightarrow T(T^2\bar{x}) = T\bar{x} \Longrightarrow T^2(T\bar{x}) = \bar{y} \Longrightarrow T^2\bar{y} = \bar{y}.$$

□

The following example illustrates Theorem 3.

Example 1. *Consider $X = \mathbb{R}^2$ with usual norm and the partially order defined by:*

$$(a,b) \preceq (c,d) \quad \textit{iff} \quad (a \leq c \quad \textit{and} \quad b \leq d),$$

for any (a,b), (c,d) in $\mathbb{R}^2$. Suppose that

$$A = \Big\{(x,0) \in \mathbb{R}^2\,;\, x \in [0,2]\Big\} \quad \textit{and}$$

$$B = \Big\{(x,1) \in \mathbb{R}^2\,;\, x \in [2,4]\Big\},$$

we can show that $dist(A,B) = 1$, $\mathcal{A}_0^{\preceq} = \{(2,0)\}$ and $\mathcal{B}_0^{\preceq} = \{(2,1)\}$. Suppose that a mapping $T : A \cup B \to A \cup B$ is defined as follows

$$\begin{cases} T(x,0) = (2,1); & \textit{for all} \quad (x,0) \in A \\ T(x,1) = (4-x,0); & \textit{for all} \quad (x,1) \in B. \end{cases}$$

We have $T(A) \subset B$, $T(B) \subset A$ and T is a decreasing mapping. Also, for any $\Big((x,0),(x',1)\Big) \in A \times B$ we have $(x,0) \preceq (x',1)$ and

$$\begin{cases} \|T(x,0) - T(x',1)\| &= \|(2,1) - (4 - x',0)\| \\ &= \sqrt{(x'-2)^2+1} \\ \|(x,0)-(x',1)\| &= \|(x'-x,1)\| \\ &= \sqrt{(x'-x)^2+1}, \end{cases}$$

thus, $\|T(x,0) - T(x',1)\| \leq \|(x,0)-(x',1)\|$. Then T is a monotone decreasing relatively cyclic nonexpansive mapping.
If we choose $x_0' = (0,0)$ and $x_0 = (2,0)$ in A we get

$$x_0 \preceq Tx_0', \quad \|x_0 - Tx_0'\| = dist\,(A,B) \quad \text{ and } \quad x_0' \preceq x_0 \preceq T^2x_0'.$$

Then there exist $\bar{x} = (2,0) \in A$ and $\bar{y} = (2,1) \in B$ such that $T^2\bar{x} = \bar{x}$, $T^2\bar{y} = \bar{y}$ and

$$\|\bar{x} - T\bar{x}\| = \|\bar{y} - T\bar{y}\| = dist\,(A,B)\,.$$

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