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Review

A Comprehensive Survey on Parallel Submanifolds in Riemannian and Pseudo-Riemannian Manifolds

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Abstract: A submanifold of a Riemannian manifold is called a parallel submanifold if its second fundamental form is parallel with respect to the van der Waerden–Bortolotti connection. From submanifold point of view, parallel submanifolds are the simplest Riemannian submanifolds next to totally geodesic ones. Parallel submanifolds form an important class of Riemannian submanifolds since extrinsic invariants of a parallel submanifold do not vary from point to point. In this paper, we provide a comprehensive survey on this important class of submanifolds.

Keywords: parallel submanifold; real space form; complex space form; totally real submanifolds; Kaehler submanifolds; light cone; Thurston 3D geometries; Bianchi–Cartan–Vranceasu spaces

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1. Introduction

In Riemannian geometry, parallel transport is a way of transporting geometrical data along smooth curves in a Riemannian manifold. Following an important idea of T. Levi-Civita [1] in 1917, one can transport vectors of a Riemannian manifold along curves so that they stay parallel with respect to the Levi-Civita connection (or Riemannian connection). Afterwards, a general theory of parallel transportation of tensor fields in Riemannian geometry was studied in the 1920s by T. Levi-Civita, J. A. Schouten, J. D. Struik, H. Weyl, E. Cartan, B. L. van der Waerden and E. Bortolotti among others (cf., e.g., Reference [2]).

For an immersed submanifold M of a Riemannian manifold $(N,\tilde{g})$, there exist two important symmetric tensor fields; namely, the first fundamental form which is the induced metric tensor field g of M and the second fundamental form h which is a normal bundle valued $(1,2)$-tensor field.

It is well known that the first fundamental form g is a parallel tensor field with respect to the Levi-Civita connection. The submanifold M is called a *parallel submanifold* if its second fundamental form h is a parallel tensor field with respect to the van der Waerden–Bortolotti connection. Thus, the extrinsic invariants of a parallel submanifold M do not vary from point to point. Obviously, parallel submanifolds are natural extensions of totally geodesic submanifolds for which the second fundamental form vanishes identically.

Parallel surfaces in a Euclidean 3-space $\mathbb{E}^3$ are classified in 1948 by V. F. Kagan in Reference [3]. Kagan's result states that open parts of planes $\mathbb{E}^2$, of spheres S^2 and of round cylinders $S^1 \times \mathbb{E}^1$ are the only parallel surfaces in $\mathbb{E}^3$. For $n > 2$, parallel hypersurfaces in Euclidean spaces are classified by U.

Simon and A. Weinstein in Reference [4]. A general classification theorem of parallel submanifolds in any Euclidean space is archived in 1974 by D. Ferus [5]. Since then, the study of parallel submanifolds became a very interesting and important research subject in differential geometry.

In this paper, we provide a comprehensive survey on this important subject in differential geometry from classical results to the most recent ones.

2. Preliminaries

An immersion from a manifold M into a pseudo-Riemannian manifold $(N, \tilde{g})$ is called a *pseudo-Riemannian submanifold* if the induced metric g on M is a pseudo-Riemannian metric. For a pseudo-Riemannian submanifold M of N, let ∇ and $\tilde{\nabla}$ be the Levi-Civita connection of g and $\tilde{g}$, respectively. Let us denote the Riemann curvature tensors of M and N by R and $\tilde{R}$, respectively and let $\langle\ ,\ \rangle$ denote the associated inner product for both g and $\tilde{g}$. A pseudo-Riemannian manifold is called a *Lorentzian manifold* if its index is one at each point.

A tangent vector v of a pseudo-Riemannian manifold is called *space-like* (respectively, *time-like*) if $v = 0$ or $\langle v, v\rangle > 0$ (respectively, $\langle v, v\rangle < 0$). A vector v is called *light-like* or *null* if $\langle v, v\rangle = 0$ and $v \neq 0$. A pseudo-Riemannian submanifold M is called *spatial* (or *space-like*) if each tangent vector vector of M is space-like.

A submanifold M of a pseudo-Riemannian manifold is called *non-degenerate* if the induced metric on M is non-degenerate. In particular, a *non-degenerate surface* of a pseudo-Riemannian manifold is either spatial or Lorentzian. *Throughout this article, we assume that every parallel surface M is non-degenerate, that is, the induced metric on M is non-degenerate.*

2.1. Basic Definitions, Formulas and Equations

The formulas of Gauss and Weingarten of a pseudo-Riemannian submanifold M of a pseudo-Riemannian manifold $(N, \tilde{g})$ are given respectively by (cf. References [6–8])

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\ \tilde{\nabla}_X \xi &= -A_\xi X + D_X \xi \end{aligned}$$

for vector fields X, Y tangent to M and ξ normal to M, where h, A and D are the second fundamental form, the shape operator and the normal connection of M. The shape operator and the second fundamental form are related by

$$\tilde{g}(h(X, Y), \xi) = g(A_\xi X, Y)$$

for vector fields X, Y tangent to M and ξ normal to M. The equations of Gauss, Codazzi and Ricci of M in N are given respectively by

$$\begin{aligned} &g(R(X, Y)Z, W) = \tilde{g}(\tilde{R}(X, Y)Z, W) + \langle h(X, W), h(Y, Z)\rangle - \langle h(X, Z), h(Y, W)\rangle, \\ &(\tilde{R}(X, Y)Z)^\perp = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z), \\ &\tilde{g}(R^D(X, Y)\xi, \eta) = \tilde{R}(X, Y; \xi, \eta) + \tilde{g}([A_\xi, A_\eta]X, Y), \end{aligned}$$

for vectors X, Y, Z, W tangent to M and vector ξ, η normal to M, where R^D is the normal curvature tensor defined by

$$R^D(X, Y) = [D_X, D_Y] - D_{[X,Y]},$$

and $\bar{\nabla} h$ denotes the covariant derivative of h with respect to the *van der Waerden–Bortolotti connection* $\bar{\nabla} = \nabla \oplus D$ defined by

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The mean curvature vector H of M in N is given by

$$H = \left(\frac{1}{n}\right) \operatorname{Trace} h = \left(\frac{1}{n}\right) \sum_{i=1}^{n} \epsilon_i h(e_i, e_i), \quad n = \dim M,$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal frame of M such that $\langle e_j, e_k \rangle = \epsilon_j \delta_{jk}$.

The *relative null subspace* $\mathcal{N}_p$ of a pseudo-Riemannian submanifold M in N at $p \in M$ is defined by

$$\mathcal{N}_p = \{X \in T_p M : h(X, Y) = 0 \; \forall Y \in T_p M\}.$$

The dimension ν_p of $\mathcal{N}_p$ is called the *relative nullity* at p.

The *first normal space* at a point p of a pseudo-Riemannian submanifold M in $\tilde{M}$ is, by definition, the image space, $\operatorname{Im} h(p)$, of the second fundamental form of M at p, that is,

$$\operatorname{Im} h(p) = \{h(X, Y) : X, Y \in T_p M\}.$$

2.2. Indefinite Real Space Forms

Let $(N, \tilde{g})$ be a pseudo-Riemannian manifold. At a point $p \in N$, a 2-dimensional linear subspace π of the tangent space $T_p N$ is called a *plane section*. For a given basis $\{v, w\}$ of a plane section π, we define a real number by

$$Q(v, w) = \langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle^2 .$$

A plane section π is called *nondegenerate* if $Q(u, v) \neq 0$. For a non-degenerate plane section $\pi \subset T_p N$ at p, the number

$$\tilde{K}(u, v) = \frac{\langle \tilde{R}(u, v)v, u \rangle}{Q(u, v)}$$

is independent of the choice of basis $\{u, v\}$ for π and is called the *sectional curvature* $\tilde{K}(\pi)$ of π.

A pseudo-Riemannian manifold is said to have *constant curvature* if its sectional curvature function is constant. It is well known that if a pseudo-Riemannian manifold N is of constant curvature c, then its curvature tensor $\tilde{R}$ satisfies

$$\tilde{R}(X, Y)Z = c\{\langle Y, Z \rangle X - \langle X, Z \rangle Y\}.$$

Example 1. *(see, e.g., Reference [6]) Let $\mathbb{E}^n_t$ denote the pseudo-Euclidean n-space equipped with the canonical pseudo-Euclidean metric of index t given by*

$$g_0 = -\sum_{i=1}^{t} du_i^2 + \sum_{j=t+1}^{n} du_j^2,$$

where $(u_1, \ldots, u_n)$ is a rectangular coordinate system of $\mathbb{E}^n_t$. For a non-zero real number c, we put

$$S^k_s(\mathbf{x}_0, c) = \left\{ \mathbf{x} \in \mathbb{E}^{k+1}_s : \langle \mathbf{x} - \mathbf{x}_0, \mathbf{x} - \mathbf{x}_0 \rangle = c^{-1} > 0 \right\}, \quad s > 0,$$
$$H^k_s(\mathbf{x}_0, c) = \left\{ \mathbf{x} \in \mathbb{E}^{k+1}_{s+1} : \langle \mathbf{x} - \mathbf{x}_0, \mathbf{x} - \mathbf{x}_0 \rangle = c^{-1} < 0 \right\}, \quad s > 0,$$
$$H^k(c) = \left\{ \mathbf{x} \in \mathbb{E}^{k+1}_1 : \langle \mathbf{x}, \mathbf{x} \rangle = c^{-1} < 0 \text{ and } x_1 > 0 \right\},$$

where $\langle\ ,\ \rangle$ is the associated scalar product. $S^k_s(\mathbf{x}_0, c)$ and $H^k_s(\mathbf{x}_0, c)$ are pseudo-Riemannian manifolds of constant curvature c with index s, known as a pseudo-sphere and a pseudo-hyperbolic space, respectively. The point $\mathbf{x}_0$ is called the center of $S^m_s(\mathbf{x}_0, c)$ and $H^m_s(\mathbf{x}_0, c)$. If $\mathbf{x}_0$ is the origin o of the pseudo-Euclidean spaces, we denote $S^k_s(o, c)$ and $H^k_s(o, c)$ by $S^k_s(c)$ and $H^k_s(c)$, respectively. The pseudo-Riemannian manifolds $\mathbb{E}^k_s$, $S^k_s(c)$, $H^k_s(c)$ are the standard models of the indefinite real space forms. In particular, $\mathbb{E}^k_1$, $S^k_1(c)$, $H^k_1(c)$ are the standard models of Lorentzian space forms.

The Riemannian manifolds $\mathbb{E}^k$, $S^k(c)$ and $H^k(c)$ (with $s = 0$) are of constant curvature, called real space forms. The Euclidean k-space $\mathbb{E}^k$, the k-sphere $S^k(c)$ and the hyperbolic k-space $H^k(c)$ are complete simply-connected Riemannian manifolds of constant curvature $0, c > 0$ and $c < 0$, respectively. A complete simply-connected pseudo-Riemannian k-manifold, $k \geq 3$, of constant curvature c and with index s is isometric to $\mathbb{E}^k_s$, or $S^k_s(c)$ or $H^k_s(c)$ according to $c = 0$, or $c > 0$ or $c < 0$, respectively.

In the following, we denote a k-dimensional indefinite space form of constant curvature c and index s by $R^k_s(c)$. Also we denote an indefinite space form $R^k_0(c)$ (with index $s = 0$) simply by $R^k(c)$.

For a pseudo-Riemannian submanifold M of a pseudo-Riemannian manifold $R^k_s(c)$ of constant curvature c with index s, the equations of Gauss, Codazzi and Ricci reduce to (see, e.g., Reference [6])

$$\begin{aligned}
\langle R(X,Y)Z,W\rangle &= c\left(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\right) \\
&\quad + \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle, \\
(\bar\nabla_X h)(Y,Z) &= (\bar\nabla_Y h)(X,Z), \\
\left\langle R^D(X,Y)\xi, \eta\right\rangle &= \langle [A_\xi, A_\eta]X, Y\rangle
\end{aligned}$$

for vectors X, Y, Z, W tangent to M and ξ, η normal to M.

2.3. Gauss Image

The classical Gauss map of a surface in $\mathbb{E}^3$ was introduced by C. F. Gauss in his fundamental paper on the theory of surfaces [9]. He used it to define the Gauss curvature. Since then the Gauss maps became one of the important tools in differential geometry. The classical Gauss map can be extended to arbitrary Euclidean submanifolds as follows:

Let $G(n, m-n)$ denote the Grassmann manifold consisting of linear n-subspaces of $\mathbb{E}^m$. Then the Grassmann manifold $G(n, m-n)$ admits a canonical Riemannian metric via Plücker embedding which makes $G(n, m-n)$ into a symmetric space. For an n-dimensional submanifold M of $\mathbb{E}^m$, the Gauss map Γ of M in $\mathbb{E}^m$ is defined to be the mapping

$$\Gamma : M \to G(n, m-n)$$

which carries a point $p \in M$ into the linear n-subspace of $\mathbb{E}^m$ obtained via the parallel displacement of the tangent space T_pM of M at p. The image $\Gamma(M)$ of M in $G(n, m-n)$ via Γ is called the *Gauss image* of M (cf. References [10,11]). In the following, we shall assume that the Gauss maps are regular maps.

The following result of B.-Y. Chen and S. Yamaguchi in Reference [10] provides a simple characterization of Euclidean submanifolds having totally geodesic Gauss image.

Theorem 1. *A submanifold M of a Euclidean space has totally geodesic Gauss image if and only if its second fundamental form h satisfies*

$$(\bar\nabla_X h)(Y,Z) = h(\nabla^G_X Y, Z) - h(\nabla_X Y, Z)$$

for any vector fields X, Y, Z tangent to M, where ∇ is the Levi-Civita connection of M and ∇^G is the Levi-Civita connection of the Gauss image with the induced metric via Γ.

3. Some General Properties of Parallel Submanifolds

In this section, we present some basic properties of parallel submanifolds.

Definition 1. *A pseudo-Riemannian submanifold M of a pseudo-Riemannian submanifold $(N, \tilde{g})$ is called a curvature-invariant submanifold if each tangent space of M is invariant under the curvature transformation, that is, $\tilde{R}(X,Y)(T_pM) \subset T_p(M)$ for any vector fields X, Y tangent to M.*

The following is an immediate consequence of the equation of Codazzi.

Lemma 1. *Any parallel pseudo-Riemannian submanifold M of a pseudo-Riemannian manifold $(N, \tilde{g})$ is curvature-invariant.*

The following properties of parallel submanifolds are also well-known.

Lemma 2. *Every parallel submanifold M of a Riemannian manifold $(N, \tilde{g})$ has constant relative nullity, that is, the dimension of the relative null subspace is constant.*

Lemma 3. *Every parallel submanifold M of a Riemannian symmetric space $(N, \tilde{g})$ is locally symmetric, that is, the Riemannian curvature tensor R of M satisfies $\nabla R = 0$.*

Further, every parallel submanifold in $\mathbb{E}^m$ is of finite type in the sense of Chen (cf. e.g., References [12–14]). Also, if a rank one compact symmetric space N is regarded as a submanifold of a Euclidean space $\mathbb{E}^m$ via its first standard embedding, then any parallel submanifold of N via its first standard embedding is of finite type in $\mathbb{E}^m$ (see, e.g., References [13,14]).

4. Parallel Submanifolds of Euclidean Spaces

In this section, we present basic properties, characterizations and classification of parallel submanifolds of Euclidean spaces.

4.1. Gauss Map and Parallel Submanifolds

As before, let $G(n, m-n)$ denote the Grassmann manifold of n-planes through the origin in $\mathbb{E}^m$ endowed with its natural Riemannian symmetric space metric and let $G(\mathbb{E}^m)$ denote the group of Euclidean motions on $\mathbb{E}^m$.

The following result was obtained by J. Vilms in Reference [15].

Theorem 2. *Assume that M is an n-dimensional parallel submanifold of $\mathbb{E}^m$. If M is complete, then we have:*

(i) *If the relative nullity $\nu = 0$, then M is a complete totally geodesic submanifold of $G(n, m-n)$.*
(ii) *If $\nu \geq 1$, then there exists a $(G(\mathbb{E}^m), \mathbb{E}^m)$-fibration $\pi : M \to B$, where B is a complete totally geodesic submanifold of $G(n, m-n)$ and the fibres are the leaves of the relative nullity foliation. The metric of M is composed from those on base and fibre and the fibration admits an integrable connection with totally geodesic horizontal leaves (i.e., it is a totally geodesic Riemannian submersion).*
(iii) *The original Riemannian connection of M or its projection onto B, respectively, coincides with the connection induced from $G(n, m-n)$.*
(iv) *M has nonnegative curvature and is locally symmetric.*

As an application of Theorem 1, Chen and S. Yamaguchi [10] classified surfaces with totally geodesic Gauss image as follows.

Theorem 3. *Let M be a surface of $\mathbb{E}^m$. If M has totally geodesic Gauss image in $G(2, m-2)$, then M is one of the following surfaces:*

(a) *A surface in an affine 3-space $\mathbb{E}^3$ of $\mathbb{E}^m$.*
(b) *A surface of $\mathbb{E}^m$ with parallel second fundamental form, that is, M is a parallel surface.*
(c) *A surface in an affine 4-space $\mathbb{E}^4$ of $\mathbb{E}^m$ which is locally the Riemannian product of two plane curves of non-zero curvature.*
(d) *A complex curve lying fully in $\mathbb{C}^2$, where $\mathbb{C}^2$ denotes an affine $\mathbb{E}^4$ endowed with some orthogonal almost complex structure.*

Another application of Theorem 1 is the following result of Chen and Yamaguchi obtained in Reference [11].

Theorem 4. *A submanifold M of $\mathbb{E}^m$ is locally the product of some hypersurfaces if and only if M has totally geodesic Gauss image and has flat normal connection.*

Yu A. Nikolaevskij [16] extended Theorem 3 in 1993 to the following.

Theorem 5. *Let M be an n-dimensional submanifold of $\mathbb{E}^m$. Then M has totally geodesic Gauss image in $G(n, m-n)$ if and only if M is the product of submanifolds such that each of the factors is either*

(a) *a real hypersurface or*
(b) *a parallel submanifold or*
(c) *a complex hypersurface.*

4.2. Normal Sections and Parallel Submanifolds

Let M be an n-dimensional submanifold in a Euclidean m-space $\mathbb{E}^m$. For a given point $p \in M$ and a given unit vector t at p tangent to M, the vector t and the normal space $T_p^\perp M$ of M determine an $(m-n+1)$-dimensional subspace $E(p,t)$ in $\mathbb{E}^m$. The intersection of M and $E(p,t)$ gives a curve γ_t (in a neighborhood of p) which is called the *normal section* of M at p in the direction t (cf. References [8,17,18]). In general, the normal section γ_t is a space curve in $E(p,t)$.

For normal sections, Chen proved the following result in References [8,17].

Theorem 6. *Let M be an n-dimensional $(n>2)$ submanifold of a Euclidean m-space $\mathbb{E}^m$. Then M has planar normal sections if and only if the second fundamental form h and its covariant derivative $\bar{\nabla}h$ satisfy*

$$h(t,t) \wedge (\bar{\nabla}_t h)(t,t) = 0$$

for any unit vector t tangent to M.

An immediate consequence of this theorem is the following.

Theorem 7. *Every parallel submanifold M of $\mathbb{E}^m$ with $n = \dim M > 2$ has planar normal sections.*

By a *vertex* of a planar curve $\gamma(s)$ we mean a point x on the curve such that the curvature function $\kappa(s)$ of γ satisfies $\frac{d\kappa^2}{ds} = 0$ at x.

Another application of Theorem 6 is the following simple geometric characterization of parallel submanifolds obtained by Chen in References [8,17].

Theorem 8. *An n-dimensional $(n>2)$ submanifold M of a Euclidean space is a parallel submanifold if and only if, for each $p \in M$, each normal section of M at any point p is a planar curve with p as one of its vertices.*

For further applications of normal sections, see, for example, References [18–28].

4.3. Symmetric Submanifolds and Parallel Submanifolds

The notion of *extrinsic symmetric submanifolds* was defined by D. Ferus in Reference [29]. More precisely, an isometric immersion $\psi : M \to \mathbb{E}^m$ is called *extrinsic symmetric* if for each $p \in M$ there exists an isometry ϕ of M into itself such that $\phi(p) = p$ and $\psi \circ \phi = \sigma_p \circ \psi$, where σ_p denotes the reflection at the normal space $T_p^\perp M$ at p, that is, the motion of $\mathbb{E}^m$ which fixes the space through $\psi(p)$ normal to $\psi_*(T_pM)$ and reflects $\psi(p) + \psi_*(T_pM)$ at $\psi(p)$. The immersed submanifold $\psi : M \to \mathbb{E}^m$ is said to be *extrinsic locally symmetric* if each point $p \in M$ has a neighborhood U and an isometry ϕ of U into itself such that $\phi(p) = p$ and $\psi \circ \phi = \sigma_p \circ \psi$ on U. In other words, a submanifold M of $\mathbb{E}^m$ is extrinsic locally symmetric if each point $p \in M$ has a neighborhood which is invariant under the reflection of $\mathbb{E}^m$ with respect to the normal space at p.

D. Ferus [29] proved the following result.

Theorem 9. *Extrinsic locally symmetric submanifolds of Euclidean spaces have parallel second fundamental form and vice versa.*

Symmetric submanifolds were classified completely by D. Ferus in Reference [5] as being a very special class of orbits of isotropy representations of semisimple symmetric spaces. For some symmetric spaces N, a distinguished class of isotropy orbits (the symmetric R-spaces) are symmetric spaces. They are symmetric submanifolds in the corresponding tangent space T_oN of N. If N is non-compact, the projection of these symmetric submanifolds from T_oN into N via the exponential map at o provides examples of symmetric submanifolds in N.

In Reference [30], J. Berndt et al. extended these symmetric submanifolds to larger one-parameter families of symmetric submanifolds and proved that if N is irreducible and of rank greater than or equal to 2, then every symmetric submanifold of N arises in this way. This result yields the full classification of symmetric submanifolds in Riemannian symmetric spaces. For symmetric submanifolds in non-flat Riemannian manifolds of constant curvature, see References [31–33].

4.4. Extrinsic K-Symmetric Submanifolds as ∇^c-Parallel Submanifolds

A canonical connection on a Riemannian manifold (M, g) is defined as any metric connection ∇^c on M such that the difference tensor $\hat{D}$ between ∇^c and the Levi-Civita connection ∇ of (M, g) is ∇^c-parallel. An embedded submanifold M of $\mathbb{E}^m$ is said to be *extrinsic homogeneous* with constant principal curvatures if, for any given $p, q \in M$ and a given piecewise differentiable curve γ from p to q, there is an isometry ϕ of $\mathbb{E}^m$ satisfying (1) $\phi(M) = M$, (2) $\phi(p) = q$ and (3) $\phi_{*p} : T_p^\perp M \to T_q^\perp M$ coincides with the $\hat{D}$-parallel transport along γ.

C. Olmos and C. Sánchez extended Ferus' result in Reference [34] to the following.

Theorem 10. *Let M be a compact Riemannian submanifold fully in $\mathbb{E}^m$ and let h be its second fundamental form. Then the following three statements are equivalent:*

(1) M admits a canonical connection ∇^c such that $\nabla^c h = 0$,
(2) M is an extrinsic homogeneous submanifold with constant principal curvatures,
(3) M is an orbit of an s-representation, that is, of an isotropy representation of a semisimple Riemannian symmetric space.

Furthermore, C. Sánchez defined in Reference [35] the notion of *extrinsic k-symmetric submanifolds* of $\mathbb{E}^m$ and classified such submanifolds for odd k. Moreover, he proved in Reference [36] that the extrinsic k-symmetric submanifolds are essentially characterized by the property of having parallel second fundamental form with respect to the canonical connection of k-symmetric space. In particular, the above result implies that every extrinsic k-symmetric submanifold of a Euclidean space is an orbit of an s-representation.

5. Symmetric R-Spaces and Parallel Submanifolds of Real Space Forms

Symmetric spaces are the most beautiful and important Riemannian manifolds. Such spaces arise in a wide variety of situations in both mathematics and physics. This class of spaces contains many prominent examples which are of great importance for various branches of mathematics, like compact Lie groups, Grassmannians and bounded symmetric domains. Symmetric spaces are also important objects of study in representation theory, harmonic analysis as well as in differential geometry.

We refer to References [37–41] for general information on compact symmetric spaces.

5.1. Symmetric R-Spaces and Borel Subgroups

An isometry s of a Riemannian manifold is called an *involutive* if $s^2 = id$. A Riemannian manifold M is called a *symmetric space* if for each $p \in M$ there is an involutive isometry s_p such that p is an isolated fixed point of s_p; the involutive isometry $s_p \neq id$ is called the *symmetry at* p.

Let M be a symmetric space. Denote by $G = G_M$ the closure of the group of isometries on M generated by $\{s_p : p \in M\}$ in the compact-open topology. Then G is a Lie group which acts transitively on the symmetric space. Thus, the typical *isotropy subgroup* K, say at a point $o \in M$, is compact and $M = G/K$. Let $I_0(M)$ denote the connected group of isometries of a compact symmetric Riemannian manifold M.

A symmetric R-space is a special type of compact symmetric space for which several characterizations were known. Originally in 1965, T. Nagano defined in Reference [42] a symmetric R-space as a compact symmetric space M which admits a Lie transformation group P which is non-compact and contains the identity component of the isometric group $I_0(M)$ of M as a subgroup.

In the theory of algebraic groups, a *Borel subgroup* of an algebraic group G is a maximal Zariski closed and connected solvable algebraic subgroup (cf. References [43,44]). Subgroups between a Borel subgroup B and the ambient group G are called *parabolic subgroups*. Working over algebraically closed fields, the Borel subgroups turn out to be the minimal parabolic subgroups in this sense. Thus, B is a Borel subgroup when the homogeneous space G/B is a complete variety which is as large as possible.

In 1965, M. Takeuchi used the terminology *symmetric R-space* for the first time in Reference [45]. He gave a cell decomposition of an R-space in Reference [45], which is a kind of generalization of a symmetric R-space. Here, by an *R-space* we mean $M = G/U$ where G is a connected real semisimple Lie group without center and U is a parabolic subgroup of G. A compact symmetric space M is said to have a *cubic lattice* if a maximal torus of M is isometric to the quotient of $\mathbb{E}^r$ by a lattice of $\mathbb{E}^r$ generated by an orthogonal basis of the same length.

In 1985, O. Loos [46] provided another intrinsic characterization of symmetric R-spaces which states that a compact symmetric space M is a symmetric R-spaces if and only if the unit lattice of the maximal torus of M is a cubic lattice. The proof of Loos is based on the correspondence between the symmetric R-spaces and compact Jordan triple systems.

5.2. Classification of Symmetric R-Spaces

An affine subspace of $\mathbb{E}^m$ or a symmetric R-space $M \subset \mathbb{E}^m$, which is minimally embedded in a hypersphere of $\mathbb{E}^m$ as described in Reference [47] by M. Takeuchi and S. Kobayashi, is a parallel submanifold of $\mathbb{E}^m$. The class of symmetric R-spaces includes (see Reference [47]):

(a) all Hermitian symmetric spaces of compact type,
(b) Grassmann manifolds $O(p+q)/O(p) \times O(q), Sp(p+q)/Sp(p) \times Sp(q)$,
(c) the classical groups $SO(m)$, $U(m)$, $Sp(m)$,
(d) $U(2m)/Sp(m)$, $U(m)/O(m)$,

(e) $(SO(p+1) \times SO(q+1))/S(O(p) \times O(q))$, where $S(O(p)\times O(q))$ is the subgroup of $SO(p+1) \times SO(q+1)$ consisting of matrices of the form

$$\begin{pmatrix} \varepsilon & 0 & & \\ 0 & A & & \\ & & \varepsilon & 0 \\ & & 0 & B \end{pmatrix}, \quad \varepsilon = \pm 1, \quad A \in O(p), \quad B \in O(q),$$

(f) the Cayley projective plane $\mathcal{O}P^2$ and
(g) the three exceptional spaces $E_6/Spin(10) \times T$, $E_7/E_6 \times T$, and E_6/F_4.

5.3. Ferus' Theorem

A classification theorem of parallel submanifolds in Euclidean spaces was obtained in 1974 by D. Ferus [5]. He proved that essentially these submanifolds mentioned above exhaust all parallel submanifolds of $\mathbb{E}^m$ in the following sense.

Theorem 11. *A complete full parallel submanifold of the Euclidean m-space $\mathbb{E}^m$ is congruent to*

(1) $M = \mathbb{E}^{m_0} \times M_1 \times \cdots \times M_s \subset \mathbb{E}^{m_0} \times \mathbb{E}^{m_1} \times \cdots \times \mathbb{E}^{m_s} = \mathbb{E}^m$, $s \geq 0$, *or to*
(2) $M = M_1 \times \cdots \times M_s \subset \mathbb{E}^{m_1} \times \cdots \times \mathbb{E}^{m_s} = \mathbb{E}^m$, $s \geq 1$,

where each $M_i \subset \mathbb{E}^{m_i}$ is an irreducible symmetric R-space. Notice that in case (1) *M is not contained in any hypersphere of $\mathbb{E}^m$ but in case* (2) *M is contained in a hypersphere of $\mathbb{E}^m$.*

5.4. Parallel Submanifolds in Spheres

For the standard inclusion of a unit hypersphere S^{m-1} in a Euclidean m-space $\mathbb{E}^m$, a submanifold $M \subset S^{m-1}$ is a parallel submanifold if and only if $M \subset S^{m-1} \subset \mathbb{E}^m$ is a parallel submanifold of $\mathbb{E}^m$. Hence, Ferus' classification theorem given in Section 5.3 implies that M is a parallel submanifold of S^{m-1} if and only if M is obtained by a submanifold of type (2).

For parallel submanifolds of spaces of constant curvature, see also References [48,49].

5.5. Parallel Submanifolds in Hyperbolic Spaces

Parallel submanifolds of a hyperbolic space were classified by M. Takeuchi [49] in 1981 as follows.

Theorem 12. *Let $H^m(\bar{c})$ be the hyperbolic m-space defined by*

$$H^m(\bar{c}) = \{(x_0, \ldots, x_m) \in \mathbb{E}^{m+1} : -x_0^2 + x_1^2 + \cdots + x_m^2 = \bar{c}^{-1}, x_0 > 0\}, \quad \bar{c} < 0.$$

If M is a parallel submanifold of $H^m(\bar{c})$, then we have:

(1) *If M is not contained in any complete totally geodesic hypersurface of $H^m(\bar{c})$, then M is congruent to the product*

$$H^{m_0}(c_0) \times M_1 \times \cdots \times M_s \subset H^{m_0}(c_0) \times S^{m-m_0-1}(c') \subset H^{m_0}(\bar{c})$$

with $c_0 < 0$, $c' > 0, 1/c_0 + 1/c' = 1/\bar{c}$, $s \geq 0$, where $M_1 \times \cdots \times M_s \subset S^{m-m_0-1}(c')$ is a parallel submanifold as described in Ferus' result.

(2) *If M is contained in a complete totally geodesic hypersurface N of $H^m(\bar{c})$, then N is isometric to an $(m-1)$-sphere or to a Euclidean $(m-1)$-space or to a hyperbolic $(m-1)$-space. Consequently, such parallel submanifolds reduce to the parallel submanifolds described before.*

6. Parallel Kaehler Submanifolds

By a *complex space form* $\tilde{M}^m(4c)$, we mean a complex m-dimensional Kaehler manifold of constant holomorphic sectional curvature $4c$. It is well known that a complete simply-connected complex space form $\tilde{M}^m(4c)$ is holomorphically isometric to a complex projective m-space $CP^m(4c)$, a complex Euclidean m-space $\mathbb{C}^m$ or a complex hyperbolic m-space $CH^m(4c)$ depending on $c > 0$, $c = 0$ or $c < 0$, respectively.

6.1. The Segre and Veronese Maps

Let $(z_0^i, \ldots, z_{n_i}^i)$ $(1 \leq i \leq s)$ denote the homogeneous coordinates of CP^{n_i}. Define a map:

$$S_{n_1 \cdots n_s} : CP^{n_1} \times \cdots \times CP^{n_s} \to CP^n, \quad n = \prod_{i=1}^{s}(n_i + 1) - 1,$$

which maps a point $((z_0^1, \ldots, z_{n_1}^1), \ldots, (z_0^s, \ldots, z_{n_s}^s))$ of the product Kaehler manifold $CP^{n_1} \times \cdots \times CP^{n_s}$ to the point $(z_{i_1}^1 \cdots z_{i_s}^s)_{0 \leq i_1 \leq n_1, \ldots, 0 \leq i_s \leq n_s}$ in CP^n. Is it well known that the map $S_{n_1 \cdots n_s}$ is a Kaehler embedding, known as the *Segre embedding*.

B.-Y. Chen [50] and Chen and W. E. Kuan [51,52] proved the following simple characterization for Segre embeddings for $n = 2$ and for $n \geq 3$, respectively (see also References [8,53–55]).

Theorem 13. *Let $M_1, \ldots, M_s$ be Kaehler manifolds of complex dimensions $n_1, \ldots, n_s$, respectively. Then every Kaehler immersion*

$$\phi : M_1 \times \cdots \times M_s \to CP^n, \quad n = \prod_{i=1}^{s}(n_i + 1) - 1,$$

of $M_1 \times \cdots \times M_s$ into CP^n is locally the Segre embedding, that is, $M_1, \ldots, M_s$ are open portions of $CP^{n_1}, \ldots, CP^{n_s}$, respectively and moreover, the Kaehler immersion ϕ is congruent to the Segre embedding.

A complex projective n-space $CP^n(c)$ of constant holomorphic sectional curvature c can be holomorphically isometrically embedded into an $\left(\binom{n+\nu}{\nu} - 1\right)$-dimensional complex projective space of constant holomorphic sectional curvature μc as

$$(z_0, \ldots, z_n) \to \left(z_0^\nu, \sqrt{\nu} z_0^{\nu-1} z_1, \ldots, \sqrt{\frac{\nu!}{\alpha_0! \cdots \alpha_n!}} z_0^{\alpha_0} \cdots z_n^{\alpha_n}, \ldots, z_n^\nu\right), \quad \sum_{i=0}^{n} \alpha_i = \nu,$$

which is called the ν-th *Veronese embedding* of $CP^n(c)$. The degree of the ν-th Veronese embedding is ν (cf. e.g., page 83 of Reference [56]).

The Veronese embeddings were characterized by A. Ros [57] in terms of holomorphic sectional curvature H in the following result.

Theorem 14. *If a compact n-dimensional Kaehler submanifold M immersed in $CP^m(c)$ satisfies*

$$\frac{c}{\nu + 1} < H \leq \frac{c}{\nu},$$

then $M = CP^n(\frac{c}{\nu})$ and the immersion is given by the ν-th Veronese embedding.

6.2. Classification of Parallel Kaehler Submanifolds of CP^m and CH^m

In 1972, K. Ogiue classified parallel complex space forms in complex space forms in Reference [58]. More precisely, he proved the following.

Theorem 15. *Let $M^n(c)$ be a complex space form holomorphically isometrically immersed in another complex space form $M^m(\bar{c})$. If the second fundamental form of the immersion is parallel, then either the immersion is totally geodesic or $\bar{c} > 0$ and the immersion is given by the second Veronese embedding.*

All complete parallel Kaehler submanifolds of a complex projective space were classified by H. Nakagawa and R. Tagaki [59] in 1976 (also [60] by M. Takeuchi in 1978).

Theorem 16. *Let M be a complete parallel Kaehler submanifold in $CP^m(c)$. If M is irreducible, then M is congruent to one of the following six kinds of Kaehler submanifolds:*

$$CP^n(c),\ \ CP^n\left(\frac{c}{2}\right),\ Q_n = SO(n+2)/SO(n)\times SO(2),$$
$$SU(r+2)/S(U(r)\times U(2)),\ r\geq 3,\ \ SO(10)/U(5),$$
$$E_6/Spin(10)\times SO(2).$$

If M is reducible, then M is congruent to $CP^{n_1}\times CP^{n_2}$ with $n=n_1+n_2$ and the embedding is given by the Segre embedding.

On the other hand, M. Kon [61] proved in 1974 the following result for parallel Kaehler submanifolds in complex hyperbolic spaces.

Theorem 17. *Every parallel Kaehler submanifold of $CH^m(-4)$ is totally geodesic.*

6.3. Parallel Kaehler Submanifolds of Hermitian Symmetric Spaces

Parallel submanifolds of Hermitian symmetric spaces were studied in 1985 by K. Tsukada [62] as follows.

Theorem 18. *Let $\phi : M \to \tilde{M}$ be a parallel Kaehler immersion of a connected complete Kaehler manifold M into a simply connected Hermitian symmetric space $\tilde{M}$. Then M is the direct product of a complex Euclidean space and semi-simple Hermitian symmetric spaces. Moreover, $\phi = \phi_2 \circ \phi_1$, where ϕ_1 is a direct product of identity maps and (not totally geodesic) parallel Kaehler embeddings into complex projective spaces and ϕ_2 is a totally geodesic Kaehler embedding.*

All non-totally geodesic parallel Kaehler embeddings into complex projective spaces have been classified earlier by H. Nakagawa and R. Takagi [59] in 1976. More precisely, these are the Veronese maps and the Segre maps applied to complex projective spaces and the first standard embeddings applied to rank two compact irreducible Hermitian symmetric spaces.

6.4. Parallel Kaehler Manifolds in Complex Grassmannian Manifolds

Let $G^{\mathbf{C}}(n,p)$ denote the complex Grassmannian manifold of complex p-planes in $\mathbb{C}^n$. We denote by $S \to G^{\mathbf{C}}(n,p)$ the tautological vector bundle over $G^{\mathbf{C}}(n,p)$ (cf. e.g., Reference [63]). Since the taulogical bundle $S \to G^{\mathbf{C}}(n,p)$ is a subbundle of a trivial bundle $G^{\mathbf{C}}(n,p)\times\mathbb{C}^n \to G^{\mathbf{C}}(n,p)$, one has the quotient bundle $Q \to G^{\mathbf{C}}(n,p)$, which is called the *universal quotient bundle*.

The holomorphic tangent bundle $T_{1,0}(G^{\mathbf{C}}(n,p))$ over $G^{\mathbf{C}}(n,p)$ can be identified with the tensor product of holomorphic vector bundles S^* and Q, where $S^* \to G^{\mathbf{C}}(n,p)$ is the dual bundle of $S \to G^{\mathbf{C}}(n,p)$. If $\mathbb{C}^n$ has a Hermitian inner product, S, Q have Hermitian metrics and Hermitian connections and so $G^{\mathbf{C}}(n,p)$ has a Hermitian metric induced by the identification of $T_{1,0}(G^{\mathbf{C}}(n,p))$ and $S^*\otimes Q$ is called the standard metric on $G^{\mathbf{C}}(n,p)$.

In Reference [64], I. Koga and Y. Nagatomo proved the following result for parallel Kaehler manifolds in a complex Grassmannian manifold.

Theorem 19. *Let $G^{\mathbf{C}}(n,p)$ be the complex Grassmannian manifold of complex p-planes in $\mathbb{C}^n$ with the standard metric h_{Gr} induced from a Hermitian inner product on $\mathbb{C}^n$ and ϕ be a holomorphic isometric immersion of a compact Kaehler manifold (M, h_M) with a Hermitian metric h_M into $G^{\mathbf{C}}(n,p)$. We denote by $Q \to G^{\mathbf{C}}(n,p)$ the universal quotient bundle over $G^{\mathbf{C}}(n,p)$ of rank $n-p$. Assume that the pull-back bundle of $Q \to G^{\mathbf{C}}(n,p)$*

is projectively flat. Then ϕ has parallel second fundamental form if and only if the holomorphic sectional curvature of M is greater than or equal to 1.

7. Parallel Totally Real Submanifolds

7.1. Basics on Totally Real Submanifolds

A *totally real submanifold* M of an almost Hermitian manifold $\tilde{M}$ is a submanifold such that the almost Hermitian structure J of $\tilde{M}$ carries each tangent vector of M into the corresponding normal space of M in $\tilde{M}$, that is, $J(T_pM) \subseteq T_p^{\perp}M$ for any point $p \in M$ (cf. Reference [65]). When $\dim_{\mathbf{R}} M = \dim_{\mathbf{C}} \tilde{M}$, the totally real submanifold N in M is also known as a *Lagrangian submanifold*.

The following result of Chen and K. Ogiue in Reference [65] is well-known.

Theorem 20. *A parallel submanifold M of dimension ≥ 2 of a non-flat complex space form is either a Kaehler submanifold or a totally real submanifold.*

H. Naitoh [66] proved in 1981 that the classification of complete totally real parallel submanifolds in complex projective spaces is reduced to that of certain cubic forms of n-variables. Further, H. Naitoh and M. Takeuchi [67] classified in 1982 these submanifolds by the theory of symmetric bounded domains of tube type.

In 1983, H. Naitoh [68,69] proved the following reduction theorem.

Theorem 21. *A parallel totally real submanifold of a complex space form $\tilde{M}^n(c)$ with $c \neq 0$ is either a totally real submanifold which is contained in a totally real totally geodesic submanifold or a totally real submanifold which is contained in a totally geodesic Kaehler submanifold whose dimension is twice of the dimension of the submanifold.*

The classifications of Naitoh and Naitoh-Takeuchi given above rely heavily on the theory of Lie groups and symmetric spaces.

Remark 1. *Theorem 21 implies that the classification of complete parallel submanifolds of complex projective space $CP^m(c)$ is reduced to those of D. Ferus [29] and H. Naitoh and M. Takeuchi [67].*

Remark 2. *For parallel totally real submanifolds in a complex hyperbolic space CH^m, Theorem 21 implies that the classification reduces to those of M. Takeuchi [49].*

7.2. Parallel Lagrangian Submanifolds of CP^n

F. Dillen, H. Li, L. Vrancken and X. Wang gave in Reference [70] explicitly and geometrically classification of parallel Lagrangian submanifolds in $CP^n(4)$ using a different method, which applies the warped products of Lagrangian immersions, called *Calabi products* and the characterization of parallel Lagrangian submanifolds by the Calabi products. For the definition of Calabi products and their characterization, see, for example, References [71,72].

The advantage of this classification given by Dillen et al. is that it allows the study of details for these submanifolds. In particular, for the reduced cases, they obtained the classification theorem as follows:

Theorem 22. *Let M be a parallel Lagrangian submanifold in $CP^n(4)$. Then either M is totally geodesic or*

(1) M is locally the Calabi product of a point with a lower-dimensional parallel Lagrangian submanifold;
(2) M is locally the Calabi product of two lower-dimensional parallel Lagrangian submanifolds; or
(3) M is congruent to one of the following symmetric spaces: (a) $SU(k)/SO(k)$ *with* $n = k(k+1)/2 - 1$ *and* $k \geq 3$, (b) $SU(k)$ *with* $n = k^2 - 1$ *and* $k \geq 3$, $SU(2k)/Sp(k)$ *with* $n = 2k^2 - k - 1$ *and* $k \geq 3$ *or* (c) E_6/F_4 *with* $n = 26$.

7.3. Parallel Surfaces of CP^2 and CH^2

For the explicit classification of parallel surfaces in CP^2 (see Reference [73]).

Theorem 23. *If M is a parallel surface in the complex projective plane $CP^2(4)$, then it is either holomorphic or Lagrangian in $CP^2(4)$.*

(a) If M is holomorphic, then locally either

(a.1) M is a totally geodesic complex projective line $CP^1(4)$ in $CP^2(4)$ or
(a.2) M is the complex quadric Q^1 embedded in $CP^2(4)$ as $\{(z_0, z_1, z_2) \in CP^2(4) \mid z_0^2 + z_1^2 + z_2^2 = 0\}$, where z_0, z_1, z_2 are complex homogeneous coordinates on $CP^2(4)$.

(b) If M is Lagrangian, then locally either

(b.1) M is a totally geodesic real projective plane $RP^2(1)$ in $CP^2(4)$ or
(b.2) M is a flat surface and the immersion is congruent to $\pi \circ L$, where $\pi : S^5(1) \to CP^2(4)$ is the Hopf-fibration and $L : M \to S^5(1) \subseteq \mathbb{C}^3$ is given by

$$L(x,y) = \left(\frac{a\, e^{-ix/a}}{\sqrt{1+a^2}}, \frac{e^{i(ax+by)}}{\sqrt{1+a^2+b^2}} \sin\left(\sqrt{1+a^2+b^2}\, y\right), \right.$$
$$\left. \frac{e^{i(ax+by)}}{\sqrt{1+a^2}} \left(\cos\left(\sqrt{1+a^2+b^2}\, y\right) - \frac{ib}{\sqrt{1+a^2+b^2}} \sin\left(\sqrt{1+a^2+b^2}\, y\right) \right) \right),$$

where a and b are real numbers with $a \neq 0$.

For parallel surfaces in CH^2, we have the following result from Reference [73].

Theorem 24. *If M is a parallel surface in the complex hyperbolic plane $CH^2(-4)$, then it is either holomorphic or Lagrangian in $CH^2(-4)$.*

(a) If M^2 is holomorphic, then it is an open part of a totally geodesic complex submanifold $CH^1(-4)$ in $CH^2(-4)$.
(b) If M is Lagrangian, then locally either

(b.1) M is a totally geodesic real hyperbolic plane $RH^2(-1)$ in $CH^2(-4)$ or
(b.2) M is flat and the immersion is congruent to $\pi \circ L$, where $\pi : H_1^5(-1) \to CH^2(-4)$ is the Hopf fibration and $L : M^2 \to H_1^5(-1) \subseteq \mathbb{C}_1^3$ is one of the following six maps:

(1) $$L = \left(\frac{e^{i(ax+by)}}{\sqrt{1-a^2}} \left(\cosh\left(\sqrt{1-a^2-b^2}\, y\right) - \frac{ib \sinh\left(\sqrt{1-a^2-b^2}\, y\right)}{\sqrt{1-a^2-b^2}} \right), \right.$$
$$\left. \frac{e^{i(ax+by)}}{\sqrt{1-a^2-b^2}} \sinh\left(\sqrt{1-a^2-b^2}\, y\right), \frac{a\, e^{ix/a}}{\sqrt{1-a^2}} \right), \; a, b \in \mathbf{R},\; a \neq 0,\; a^2+b^2 < 1;$$

(2) $$L(x,y) = \left(\left(\frac{i}{b} + y\right) e^{i(\sqrt{1-b^2}x+by)}, y e^{i(\sqrt{1-b^2}x+by)}, \frac{\sqrt{1-b^2}}{b} e^{ix/\sqrt{1-b^2}} \right), \; b \in \mathbf{R},\; 0 < b^2 < 1;$$

(3) $$L(x,y) = \left(\frac{e^{i(ax+by)}}{\sqrt{1-a^2}} \left(\cos\left(\sqrt{a^2+b^2-1}\, y\right) - \frac{ib \sin\left(\sqrt{a^2+b^2-1}\, y\right)}{\sqrt{a^2+b^2-1}} \right), \right.$$
$$\left. \frac{e^{i(ax+by)}}{\sqrt{a^2+b^2-1}} \sin\left(\sqrt{a^2+b^2-1}\, y\right), \frac{a\, e^{ix/a}}{\sqrt{1-a^2}} \right), \; a, b \in \mathbf{R},\; 0 < a^2 < 1,\; a^2+b^2 > 1;$$

$$(4) \quad L(x,y) = \left(\frac{a\, e^{ix/a}}{\sqrt{a^2-1}}, \frac{e^{i(ax+by)}}{\sqrt{a^2+b^2-1}} \sin\left(\sqrt{a^2+b^2-1}\, y\right), \right.$$

$$\left. \frac{e^{i(ax+by)}}{\sqrt{a^2-1}} \left(\cos\left(\sqrt{a^2+b^2-1}\, y\right) - \frac{ib \sin\left(\sqrt{a^2+b^2-1}\, y\right)}{\sqrt{a^2+b^2-1}} \right) \right), \quad a,b \in \mathbf{R},\ a^2 > 1;$$

$$(5) \quad L(x,y) = \frac{e^{ix}}{8b^2} \left(i + 8b^2(i+x) - 4by, i + 8b^2 x - 4by, 4be^{2iby} \right), \quad \mathbf{R} \ni b \neq 0;$$

$$(6) \quad L(x,y) = e^{ix} \left(1 + \frac{y^2}{2} - ix, y, \frac{y^2}{2} - ix \right).$$

7.4. Parallel Totally Real Submanifolds in Nearly Kaehler S^6

Let $\mathcal{O}$ denote the Cayley numbers. E. Calabi [74] showed in 1958 that any oriented submanifold M^6 of the hyperplane $\operatorname{Im}\mathcal{O}$ of the imaginary octonions carries a $U(3)$-structure, that is, an almost Hermitian structure J.

The almost Hermitian structure J on $S^6(1) \subset \operatorname{Im}\mathcal{O}$ is a nearly Kaehler structure in the sense that the (2,1)-tensor field G on $S^6(1)$, defined by $G(X,Y) = (\tilde{\nabla}_X J)(Y)$, is skew-symmetric, where $\tilde{\nabla}$ is the Riemannian connection on $S^6(1)$. The group of automorphisms of this nearly Kähler structure is the exceptional simple Lie group G_2 which acts transitively on S^6 as a group of isometries.

In 1969, A. Gray proved in Reference [75] the following.

Theorem 25. *(1) Every almost complex submanifold of the nearly Kaehler $S^6(1)$ is a minimal submanifold and (2) the nearly Kaehler $S^6(1)$ has no 4-dimensional almost complex submanifolds.*

N. Ejiri proved in Reference [76] that a 3-dimensional totally real submanifold of the nearly Kaehler $S^6(1)$ is minimal and orientable.

It was proved by B. Opozda in Reference [77] that every 3-dimensional parallel Lagrangian submanifold (respectively, a 2-dimensional totally real and minimal submanifold) of the nearly Kaehler $S^6(1)$ is totally geodesic (see also Reference [78]). Opozda also proved in Reference [77] that a 2-dimensional parallel totally real, minimal surface of the nearly Kaehler $S^6(1)$ is also totally geodesic. The same result holds for Lagrangian submanifolds of the nearly Kaehler $S^3 \times S^3$; namely, a (3-dimensional) parallel Lagrangian submanifold of the nearly Kaehler $S^3 \times S^3$ is totally geodesic (see, e.g., B. Dioos's PhD thesis [79]).

8. Parallel Slant Submanifolds of Complex Space Forms

8.1. Basics on Slant Submanifolds

Besides Kaehler and totally real submanifolds in a Kaehler manifold $\tilde{M}$, there is another important family of submanifolds, called *slant submanifolds* (cf. References [80,81]).

Let N be a submanifold of a Kähler manifold (or an almost Hermitian manifold) (M, J, g). For any vector X tangent to M, we put

$$JX = PX + FX,$$

where PX and FX denote the tangential and the normal components of JX, respectively. Then P is an endomorphism of the tangent bundle TN. For any non-zero vector $X \in T_pN$ at $p \in N$, the angle $\theta(X)$ between JX and the tangent space T_pN is called the *Wirtinger angle* of X.

In 1990, the author of Reference [80] introduced the notion of slant submanifolds as follows.

Definition 2. *A submanifold N of an almost Hermitian manifold (M, J, g) is called a slant submanifold if the Wirtinger angle $\theta(X)$ is independent of the choice of $X \in T_pN$ and of $p \in N$. The Wirtinger angle of a slant submanifold is called the slant angle. A slant submanifold with slant angle θ is simply called θ-slant.*

Complex submanifolds and totally real submanifolds are exactly θ-slant submanifolds with $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant submanifold is called *proper slant* if it is neither complex nor totally real.

The following basic result on slant submanifolds was proved in Reference [82] by Chen and Y. Tazawa.

Theorem 26. *Let M be a slant submanifold in a complex Euclidean m-space $\mathbb{C}^m$. If M is not totally real, then M is non-compact. In particular, there do not exist compact proper slant submanifolds in any complex Euclidean m-space.*

The next result on slant surface was proved in Reference [83] by Chen and Y. Tazawa.

Theorem 27. *Every proper slant surface of CP^2 or of CH^2 is non-minimal.*

8.2. Classification of Parallel Slant Submanifolds

For parallel slant surfaces in $\mathbb{C}^m$, we have the following classification result.

Theorem 28. *Let M be a slant surface of $\mathbb{C}^m$. Then M is a parallel surface if and only if M is one of the following surfaces:*

(a) An open portion of a slant plane in $\mathbb{C}^2 \subset \mathbb{C}^m$;
(b) An open portion of the product surface of two plane circles;
(c) An open portion of a circular cylinder which is contained in a hyperplane of $\mathbb{C}^2 \subset \mathbb{C}^m$.

If case (b) or case (c) occurs, the M is totally real.

Theorem 28 follows from Theorem 1.2 of Reference [81] and that every parallel surface of a Euclidean space lies in affine 4-space of the ambient space.

For higher dimensional parallel slant submanifolds, we have the following result by applying Theorem 19, the list of symmetric R-spaces and Ferus' Theorem.

Theorem 29. *A proper slant submanifold of $\mathbb{C}^m$ is parallel if and only if it is an open part of a slant n-plane of $\mathbb{C}^m$.*

For further results on slant submanifolds, see, for example, References [6,81,84–86].

9. Parallel Submanifolds of Quaternionic Space Forms and Cayley Plane

9.1. Parallel Submanifolds of Quaternionic Space Forms

K. Tsukada [87] classified in 1985 all parallel submanifolds of a quaternionic projective m-space HP^m. Tsukada's results states that such submanifolds are either parallel totally real submanifolds in a totally real totally geodesic submanifold RP^m or parallel totally real submanifolds in a totally complex totally geodesic submanifold CP^m or parallel complex submanifolds in a totally complex totally geodesic submanifold CP^m or parallel totally complex submanifolds in a totally geodesic quaternionic submanifold HP^k whose dimension is twice the dimension of the parallel submanifold. In Reference [87], K. Tsukada also classified parallel submanifolds of the non-compact dual of HP^m.

9.2. Parallel Submanifolds of the Cayley Plane

A result of K. Tsukada [88] in 1985 states that parallel submanifolds of the Cayley plane $\mathcal{O}P^2$ are contained either in a totally geodesic quaternion projective plane HP^2 as parallel submanifolds or in a totally geodesic 8-sphere as parallel submanifolds. Hence, all these immersions are completely known.

The non-compact case is treated in a similar way.

10. Parallel Spatial Submanifolds in Pseudo-Euclidean Spaces

The first classification result of parallel submanifolds in indefinite real space forms was given by M. A. Magid [89] in 1984 in which he classified parallel immersions of $\mathbb{E}^n \to \mathbb{E}^{n+k}_1$, $\mathbb{E}^n_1 \to \mathbb{E}^{n+2}_1$ and $\mathbb{E}^n_1 \to \mathbb{E}^{n+k}_2$. He showed that such immersions are either quadratic in nature, like the flat umbilical immersion with light-like mean curvature vector or the product of the identity map and previously determined low dimensional maps. In this section, we survey known results on parallel pseudo-Riemannian submanifolds in indefinite real space forms.

First we recall the next lemma which is an easy consequence of Erbacher–Magid's reduction theorem (see Lemma 3.1 of Reference [90]).

Lemma 4. *Let $\psi : M^n_i \to \mathbb{E}^m_s$ be an isometric immersion of a pseudo-Riemannian n-manifold M^n_i into $\mathbb{E}^m_s$. If M is a parallel submanifold, then there exists a complete $(n+k)$-dimensional totally geodesic submanifold E^* such that $\psi(M) \subset E^*$, where k is the dimension of the first normal spaces.*

10.1. Marginally Trapped Surfaces

Now, we recall the notion of marginally trapped surfaces for later use.

The concept of trapped surfaces, introduced R. Penrose in Reference [91] plays very important role in the theory of cosmic black holes. If there is a massive source inside the surface, then close enough to a massive enough source, the outgoing light rays may also be converging; a trapped surface. Everything inside is trapped. Nothing can escape, not even light. It is believed that there will be a marginally trapped surface, separating the trapped surfaces from the untrapped ones, where the outgoing light rays are instantaneously parallel. The surface of a black hole is the marginally trapped surface. As times develops, the marginally trapped surface generates a hypersurface in space-time, a *trapping horizon.*

Spatial surfaces in pseudo-Riemannian manifolds play important roles in mathematics and physics, in particular in general relativity theory. For instance, a *marginally trapped surface* in a space-time is a spatial surface with light-like mean curvature vector field. In this article, we also call a Lorentzian surfaces in a pseudo-Riemannian manifold *marginally trapped* (or *quasi-minimal*) if it has light-like mean curvature vector field (cf., e.g., References [92,93]). A non-degenerate surface in a pseudo-Riemannian manifold is called *trapped* (respectively, *untrapped*) if it has time-like (respectively, space-like) mean curvature vector field.

10.2. Classification of Parallel Spatial Surfaces in $\mathbb{E}^m_s$

In this subsection, we provide the classification of parallel spatial surfaces in indefinite space forms with arbitrary index and arbitrary dimension obtained by Chen in Reference [90] as follows.

Theorem 30. *Let $L : M \to \mathbb{E}^m_s$ be a parallel isometric immersion of a spatial surface into the pseudo-Euclidean m-space $\mathbb{E}^m_s$. Then up to dilations and rigid motions of $\mathbb{E}^m_s$, we have either*

(A) the surface is an open part of one of the following 11 surfaces:

- *(i) a totally geodesic Euclidean 2-plane $\mathbb{E}^2 \subset \mathbb{E}^m_s$ given by $(0, \dots, 0, u, v)$;*
- *(ii) a totally umbilical $S^2(1)$ in a totally geodesic $\mathbb{E}^3$ given by $(0, \dots, 0, \cos u, \sin u \cos v, \sin u \sin v)$;*
- *(iii) a flat cylinder $\mathbb{E}^1 \times S^1$ lying in a totally geodesic $\mathbb{E}^3 \subset \mathbb{E}^m_s$ given by $(0, \dots, 0, u, \cos v, \sin v)$;*

(iv) *a flat torus* $S^1 \times S^1$ *in a totally geodesic* $\mathbb{E}^4$ *given by* $(0, \ldots, 0, a\cos u, a\sin u, b\cos v, b\sin v)$ *with* $a, b > 0$;

(v) *a real projective plane of curvature* $\frac{1}{3}$ *lying in a totally geodesic* $\mathbb{E}^5 \subset \mathbb{E}^m_s$ *given by*

$$\left(0, \ldots, 0, \frac{vw}{\sqrt{3}}, \frac{uw}{\sqrt{3}}, \frac{uv}{\sqrt{3}}, \frac{u^2 - v^2}{2\sqrt{3}}, \frac{1}{6}\left(u^2 + v^2 - 2w^2\right)\right), \ u^2 + v^2 + w^2 = 3;$$

(vi) *a hyperbolic 2-plane* H^2 *in a totally geodesic* $\mathbb{E}^3_1$ *as* $(\cosh u, 0, \ldots, 0, \sinh u \cos v, \sinh u \sin v)$;

(vii) *a flat cylinder* $H^1 \times \mathbb{E}^1$ *lying in a totally geodesic* $\mathbb{E}^3_1 \subset \mathbb{E}^4_1$ *given by* $(\cosh u, 0, \ldots, 0, \sinh u, v)$;

(viii) *a flat surface* $H^1 \times S^1$ *in a totally geodesic* $\mathbb{E}^4_1 \subset \mathbb{E}^m_s$ *given by*

$$\left(a\cosh u, 0, \ldots, 0, a\sinh u, b\cos v, b\sin v\right)$$

with $a, b > 0$;

(ix) *a flat totally umbilical surface of a totally geodesic* $\mathbb{E}^4_1 \subset \mathbb{E}^m_s$ *defined by*

$$\left(u^2 + v^2 + \frac{1}{4}, 0, \ldots, 0, u, v, u^2 + v^2 - \frac{1}{4}\right);$$

(x) *a flat surface* $H^1 \times H^1$ *lying in a totally geodesic* $\mathbb{E}^4_2 \subset \mathbb{E}^m_s$ *given by*

$$\left(a\cosh u, b\cosh v, 0, \ldots, 0, a\sinh u, b\sinh v\right), \ a, b > 0;$$

(xi) *a surface of curvature* $-\frac{1}{3}$ *lying in a totally geodesic* $\mathbb{E}^5_3 \subset \mathbb{E}^m_s$ *given by*

$$\left(\sinh\left(\frac{2s}{\sqrt{3}}\right) - \frac{t^2}{3} - \left(\frac{7}{8} + \frac{t^4}{18}\right)e^{\frac{2s}{\sqrt{3}}}, t + \left(\frac{t^3}{3} - \frac{t}{4}\right)e^{\frac{2s}{\sqrt{3}}}, \frac{1}{2} + \frac{t^2}{2}e^{\frac{2s}{\sqrt{3}}},\right.$$
$$\left. 0, \ldots, 0, t + \left(\frac{t^3}{3} + \frac{t}{4}\right)e^{\frac{2s}{\sqrt{3}}}, \sinh\left(\frac{2s}{\sqrt{3}}\right) - \frac{t^2}{3} - \left(\frac{1}{8} + \frac{t^4}{18}\right)e^{\frac{2s}{\sqrt{3}}}\right), \text{ or}$$

(B) $L = (f_1, \ldots, f_\ell, \phi, f_\ell, \ldots, f_1)$, *where* ϕ *is a surface given by (i), (iii), (iv), (vii), (viii), (ix), or (x) from* (A) *and* $f_1, \ldots, f_\ell$ *are polynomials of degree* ≤ 2 *in* u, v.

10.3. Special Case: Parallel Spatial Surfaces in $\mathbb{E}^3_1$

For parallel surfaces in $\mathbb{E}^3_1$, Theorem 30 implies the following.

Corollary 1. *A parallel spatial surface in* $\mathbb{E}^3_1$ *is congruent to an open part of one of the following eight types of surfaces:*

(1) *the Euclidean plane* $\mathbb{E}^2$ *given by* $(0, u, v)$;
(2) *a hyperbolic plane* H^2 *given by* $a(\cosh u \cosh v, \cosh u \sinh v, \sinh u)$ $a > 0$;
(3) *a cylinder* $H^1 \times \mathbb{E}^1$ *defined by* $(a\cosh u, a\sinh u, v)$, $a > 0$;

Remark 3. *The surfaces (1) is totally geodesic, the surfaces (2) is totally umbilical but not totally geodesic and surfaces (1) and (3) are products of parallel curves in totally geodesic subspaces.*

11. Parallel Spatial Surfaces in S^m_s

11.1. Classification of Parallel Spatial Surfaces in S^m_s

For parallel spatial surfaces in a pseudo-sphere S^m_s, we have the following classification theorem proved in Reference [90].

Theorem 31. *Let* $\psi : M \to S^m_s(1)$ *be a parallel immersion of a spatial surface into the unit pseudo-Riemannian m-sphere* $S^m_s(1)$ *and* $L = \iota : \psi : M \to \mathbb{E}^{m+1}_s$ *be the composition of* ψ *and the inclusion* $\iota : S^m_s(1) \to \mathbb{E}^{m+1}_s$. *Then either*

(A) *the surface is congruent to an open part of one of the following 18 surfaces:*

(1) *a totally geodesic 2-sphere* $S^2(1) \subset S^m_s(1)$;
(2) *a totally umbilical* S^2 *immersed in* $S^m_s(1) \subset \mathbb{E}^{m+1}_s$ *as*

$$\left(0,\ldots,0, r\sin u, r\cos u\cos v, r\cos u\sin v, \sqrt{1-r^2}\right),\ \ 0<r<1;$$

(3) *a totally umbilical* S^2 *immersed in* $S^m_s(1) \subset \mathbb{E}^{m+1}_s$ *as*

$$\left(\sqrt{r^2-1}, 0,\ldots,0, r\sin u, r\cos u\cos v, r\cos u\sin v\right),\ \ r>1\ \ s\geq 1;$$

(4) *a flat torus* $S^1 \times S^1$ *immersed in* $S^m_s(1) \subset \mathbb{E}^{m+1}_s$ *as*

$$\left(0,\ldots,0, b\cos u, b\sin u, c\cos v, c\sin v, \sqrt{1-b^2-c^2}\right),\ \ b,c>0,\ b^2+c^2\leq 1;$$

(5) *a flat torus* $S^1 \times S^1$ *immersed in* $S^m_s(1) \subset \mathbb{E}^{m+1}_s$ *as*

$$\left(\sqrt{b^2+c^2-1}, 0,\ldots,0, b\cos u, b\sin u, c\cos v, c\sin v\right),\ \ b,c,s>0,\ b^2+c^2>1;$$

(6) *a real projective plane* RP^2 *immersed in* $S^m_s(1) \subset \mathbb{E}^{m+1}_s$ *as*

$$\left(0,\ldots,0, \frac{rvw}{\sqrt{3}}, \frac{ruw}{\sqrt{3}}, \frac{ruv}{\sqrt{3}}, \frac{r(u^2-v^2)}{2\sqrt{3}}, \frac{r}{6}(u^2+v^2-2w^2), \sqrt{1-r^2}\right)$$

with $u^2+v^2+w^2=3$ *and* $0<r\leq 1$;
(7) *a real projective plane* RP^2 *immersed in* $S^m_s(1) \subset \mathbb{E}^{m+1}_s$ *as*

$$\left(\sqrt{r^2-1}, 0,\ldots,0, \frac{rvw}{\sqrt{3}}, \frac{ruw}{\sqrt{3}}, \frac{ruv}{\sqrt{3}}, \frac{r(u^2-v^2)}{2\sqrt{3}}, \frac{r}{6}(u^2+v^2-2w^2)\right)$$

with $u^2+v^2+w^2=3$ *and* $r>1, s\geq 1$;
(8) *a hyperbolic 2-plane* H^2 *immersed in* $S^m_s(1) \subset \mathbb{E}^{m+1}_s$ *as*

$$\left(r\cosh u, 0,\ldots,0, r\sinh u\cos v, r\sinh u\sin v, \sqrt{1+r^2}\right),\ \ r,s>0;$$

(9) *a flat surface* $H^1 \times H^1$ *immersed in* $S^m_s(1) \subset \mathbb{E}^{m+1}_s$ *as*

$$\left(b\cosh u, c\cosh v, 0,\ldots,0, b\sinh u, c\sinh v, \sqrt{1+b^2+c^2}\right),\ \ b,c>0,\ s\geq 2;$$

(10) *a flat surface* $H^1 \times S^1$ *immersed in* $S^m_s(1) \subset \mathbb{E}^{m+1}_s$ *as*

$$\left(b\cosh u, 0,\ldots,0, b\sinh u, c\cos v, c\sin v, \sqrt{1+b^2-c^2)}\right),\ \ b,c,s>0,\ c^2\leq 1+b^2;$$

(11) *a flat surface* $H^1 \times S^1$ *immersed in* $S^m_s(1) \subset \mathbb{E}^{m+1}_s$ *as*

$$\left(\sqrt{c^2-b^2-1}, b\cosh u, 0,\ldots,0, b\sinh u, c\cos v, c\sin v\right),\ \ c^2>1+b^2>1;$$

(12) *a flat surface immersed in* $S^m_s(1) \subset \mathbb{E}^{m+1}_s$ *as*

$$r\left(u^2+v^2+b+\frac{1}{4}, 0,\ldots,0, \frac{\sqrt{1+br^2}}{r}, u, v, u^2+v^2+b-\frac{1}{4}\right),\ \ r,s>0,\ b\geq -r^{-2};$$

(13) *a flat surface immersed in* $S^m_s(1) \subset \mathbb{E}^{m+1}_s$ *as*

$$r\left(u^2+v^2-b+\frac{1}{4}, \frac{\sqrt{br^2-1}}{r}, 0,\ldots,0, u, v, u^2+v^2-b-\frac{1}{4}\right)$$

with $r > 0,\ s \geq 2,\ b > r^{-2}$;

(14) *a flat surface immersed in* $S_s^m(1) \subset \mathbb{E}_s^{m+1}$ *as*

$$r\left(u^2 + b - \frac{3}{4}, 0, \ldots, 0, \frac{\sqrt{1-(1-b+c^2)r^2}}{r}, u, c\cos v, c\sin v, u^2 + b - \frac{5}{4}\right)$$

with $r, s > 0$ *and* $b \geq 1 + c^2 - r^{-2}$;

(15) *a flat surface immersed in* $S_s^m(1) \subset \mathbb{E}_s^{m+1}$ *as*

$$r\left(u^2 + b - \frac{3}{4}, \frac{\sqrt{(1-b+c^2)r^2 - 1}}{r}, 0, \ldots, 0, u, c\cos v, c\sin v, u^2 + b - \frac{5}{4}\right)$$

with $r > 0, s \geq 2$ *and* $b < 1 + c^2 - r^{-2}$;

(16) *a flat surface immersed in* $S_s^m(1) \subset \mathbb{E}_s^{m+1}$ *as*

$$r\left(v^2 - b + \frac{5}{4}, c\cosh u, 0, \ldots, 0, \frac{\sqrt{1+(1-b+c^2)r^2}}{r}, c\sinh u, v, v^2 - b + \frac{3}{4}\right)$$

with $c, r > 0, s \geq 2$ *and* $b \leq 1 + c^2 + r^{-2}$;

(17) *a flat surface immersed in* $S_s^m(1) \subset \mathbb{E}_s^{m+1}$ *as*

$$r\left(v^2 - b + \frac{5}{4}, c\cosh u, \frac{\sqrt{(b-c^2-1)r^2-1}}{r}, 0, \ldots, 0, c\sinh u, v, v^2 - b + \frac{3}{4}\right)$$

with $c, r > 0, s \geq 3$ *and* $b > 1 + c^2 + r^{-2}$;

(18) *a surface of constant negative curvature immersed in* $S_s^m(1) \subset \mathbb{E}_s^{m+1}$ *as*

$$\begin{aligned} r\Bigg(&\sinh\left(\frac{2s}{\sqrt{3}}\right) - \frac{t^2}{3} - \left(\frac{7}{8} + \frac{t^4}{18}\right)e^{\frac{2s}{\sqrt{3}}},\ t + \left(\frac{t^3}{3} - \frac{t}{4}\right)e^{\frac{2s}{\sqrt{3}}},\ \frac{1}{2} + \frac{t^2}{2}e^{\frac{2s}{\sqrt{3}}}, \\ &0, \ldots, 0, t + \left(\frac{t^3}{3} + \frac{t}{4}\right)e^{\frac{2s}{\sqrt{3}}},\ \sinh\left(\frac{2s}{\sqrt{3}}\right) - \frac{t^2}{3} - \left(\frac{1}{8} + \frac{t^4}{18}\right)e^{\frac{2s}{\sqrt{3}}},\ \frac{\sqrt{1+r^2}}{r}\Bigg) \end{aligned}$$

with $r > 0$ *and* $s \geq 3$ *or*

(B) $L = (f_1, \ldots, f_\ell, \phi, f_\ell, \ldots, f_1)$, *where* ϕ *is a surface given by (4), (5) or (9)–(17) from* (A) *and* $f_1, \ldots, f_\ell$ *are polynomials of degree* ≤ 2 *in* u, v *or*

(C) $L = (r, \phi, r)$, *where* $r \in \mathbb{R}^+$ *and* ϕ *is a surface given by (1), (2), (3), (6), (7), (8) or (18) from* (A).

11.2. Special Case: Parallel Spatial Surfaces in S_1^3

For parallel spatial surfaces in a de Sitter space-time S_1^3, Theorem 31 implies the following.

Corollary 2. *If* M *is a parallel spatial surface in* $S_1^3(1) \subset \mathbb{E}_1^4$, *then* M *is congruent to one of the following ten types of surfaces:*

(1) *a totally umbilical sphere* S^2 *given locally by* $(a, b\sin u, b\cos u\cos v, b\cos u\sin v)$, $b^2 - a^2 = 1$;

(2) *a totally umbilical hyperbolic plane* H^2 *given by* $(a\cosh u\cosh v, a\cosh u\sinh v, a\sinh u, b)$ *with* $b^2 - a^2 = 1$;

(3) *a flat surface* $H^1 \times S^1$ *given by* $(a\cosh u, a\sinh u, b\cos v, b\sin v)$ *with* $a^2 + b^2 = 1$.

(4) *a totally umbilical Euclidean* $\mathbb{E}^2$ *plane given by*

$$\frac{1}{\sqrt{c}}\left(u^2 + v^2 - \frac{3}{4}, u^2 + v^2 - \frac{5}{4}, u, v\right);$$

Remark 4. *The surfaces (1), (2) and (4) are totally umbilical; the surfaces (1) with $a = 0$ and (2) with $b = 0$ are totally geodesic; the surfaces (3) and (4) are flat. And the surface (4) is a totally umbilical isometric immersion of $\mathbb{E}^2$ into $S^3_1(c)$.*

12. Parallel Spatial Surfaces in H^m_s

12.1. Classification of Parallel Spatial Surfaces in H^m_s

For parallel spatial surfaces in a pseudo-hyperbolic space H^m_s, we have the following classification theorem also proved in Reference [90].

Theorem 32. *Let $\psi : M \to H^m_s(-1)$ be a parallel immersion of a spatial surface into the pseudo-hyperbolic m-space $H^m_s(-1)$ and let $L = \iota : \psi : M \to \mathbb{E}^{m+1}_{s+1}$ be the composition of ψ and the inclusion $\iota : H^m_s(-1) \to \mathbb{E}^{m+1}_{s+1}$. Then either*

(A) *the surface is congruent to an open part of one of the following 18 surfaces:*

(1) *a totally geodesic $H^2(-1)$ immersed in $H^m_s(-1)$ as $(\cosh u, 0, \ldots, 0, \sinh u \cos v, \sinh u \sin v)$ with $b > 0$;*

(2) *a totally umbilical H^2 immersed in $H^m_s(-1) \subset \mathbb{E}^{m+1}_{s+1}$ as*

$$\left(r\cosh u, 0, \ldots, 0, r\sinh u \cos v, r\sinh u \sin v, \sqrt{r^2-1}\right) \ r > 1;$$

(3) *a totally umbilical H^2 immersed in $H^m_s(-1) \subset \mathbb{E}^{m+1}_{s+1}$ as*

$$\left(r\cosh u, \sqrt{1-r^2}, 0, \ldots, 0, r\sinh u \cos v, r\sinh u \sin v\right), \ s \geq 1, \ 0 < r < 1;$$

(4) *a totally umbilical S^2 immersed in $H^m_s(-1) \subset \mathbb{E}^{m+1}_{s+1}$ as*

$$\left(\sqrt{1+r^2}, 0, \ldots, 0, r\sin u, r\cos u \cos v, r\cos u \sin v\right), \ r > 0;$$

(5) *a flat torus $S^1 \times S^1$ in $H^m_s(-1) \subset \mathbb{E}^{m+1}_{s+1}$ as $(\sqrt{1+b^2+c^2}, 0, \ldots, 0, b\cos u, b\sin u, c\cos v, c\sin v,)$, with $b, c > 0$;*

(6) *a surface of constant positive curvature immersed in $H^m_s(-1) \subset \mathbb{E}^{m+1}_{s+1}$ as*

$$\left(\sqrt{1+r^2}, 0, \ldots, 0, \frac{rvw}{\sqrt{3}}, \frac{ruw}{\sqrt{3}}, \frac{ruv}{\sqrt{3}}, \frac{r(u^2-v^2)}{2\sqrt{3}}, \frac{r}{6}(u^2+v^2-2w^2)\right)$$

with $u^2 + v^2 + w^2 = 3$ and $r > 0$;

(7) *a flat surface $H^1 \times H^1$ in $H^m_s(-1)$ as $\left(b\cosh u, c\cosh v, 0, \ldots, 0, b\sinh u, c\sinh v, \sqrt{b^2+c^2-1}\right)$ with $b, c, s > 0$ and $b^2 + c^2 \geq 1$;*

(8) *a flat surface $H^1 \times H^1$ in $H^m_s(-1)$ as $\left(\sqrt{1-b^2-c^2}, b\cosh u, c\cosh v, 0, \ldots, 0, b\sinh u, c\sinh v\right)$ with $b, c > 0$, $s \geq 2$ and $b^2 + c^2 < 1$;*

(9) *a flat surface $H^1 \times S^1$ in $H^m_s(-1) \subset \mathbb{E}^{m+1}_{s+1}$ as $(b\cosh u, 0, \ldots, 0, b\sinh u, c\cos v, c\sin v, \sqrt{b^2-c^2-1})$ with $b, c > 0$ and $b^2 \geq c^2 + 1$;*

(10) *a flat surface $H^1 \times S^1$ immersed in $H^m_s(-1)$ as $(\sqrt{1-b^2+c^2}, b\cosh u, 0, \ldots, 0, b\sinh u, c\cos v, c\sin v)$ with $b, c, s > 0$ and $b^2 < c^2 + 1$;*

(11) *a flat surface immersed in $H^m_s(-1) \subset \mathbb{E}^{m+1}_{s+1}$ as*

$$r\left(u^2+v^2+b+\frac{1}{4}, 0, \ldots, 0, \frac{\sqrt{br^2-1}}{r}, u, v, u^2+v^2+b-\frac{1}{4}\right), \ \ r > 1, \ b \geq r^{-2};$$

(12) *a flat surface immersed in $H^m_s(-1) \subset \mathbb{E}^{m+1}_{s+1}$ as*

$$r\left(u^2+v^2-b+\frac{1}{4}, \frac{\sqrt{br^2+1}}{r}, 0, \ldots, 0, u, v, u^2+v^2-b-\frac{1}{4}\right), \ \ r, s > 0, \ b \geq -r^{-2};$$

(13) *a flat surface immersed in* $H^m_s(-1) \subset \mathbb{E}^{m+1}_{s+1}$ *as*

$$r\left(u^2+b-\frac{3}{4}, 0, \ldots, 0, \frac{\sqrt{(b-c^2-1)r^2-1}}{r}, c\cos v, c\sin v, u, u^2+b-\frac{5}{4}\right)$$

with $r>0$, *and* $b \geq 1+c^2+r^{-2}$;

(14) *a flat surface immersed in* $H^m_s(-1) \subset \mathbb{E}^{m+1}_{s+1}$ *as*

$$r\left(u^2+b-\frac{3}{4}, \frac{\sqrt{1+(1-b+c^2)r^2}}{r}, 0, \ldots, 0, c\cos v, c\sin v, u, u^2+b-\frac{5}{4}\right)$$

with $r,s>0$ *and* $b<1+c^2+r^{-2}$;

(15) *a flat surface immersed in* $H^m_s(-1) \subset \mathbb{E}^{m+1}_{s+1}$ *as*

$$r\left(v^2+b+\frac{5}{4}, b\cosh u, 0, \ldots, 0, \frac{\sqrt{(1+b+c^2)r^2-1}}{r}, b\sinh u, v, v^2+b+\frac{3}{4}\right)$$

with $b,r>0$, $s \geq 1$ *and* $b \geq r^{-2}-1-c^2$;

(16) *a flat surface immersed in* $H^m_s(-1) \subset \mathbb{E}^{m+1}_{s+1}$ *as*

$$r\left(v^2+b+\frac{5}{4}, b\cosh u, \frac{\sqrt{1-(a+b+c^2)r^2}}{r}, 0, \ldots, 0, b\sinh u, v, v^2+b+\frac{3}{4}\right)$$

with $b,r>0$, $s \geq 2$ *and* $b<r^{-2}-1-c^2$;

(17) *a surface of constant negative curvature immersed in* $H^m_s(-1) \subset \mathbb{E}^{m+1}_{s+1}$ *as*

$$\begin{aligned} r\Bigg(&\sinh\left(\frac{2u}{\sqrt{3}}\right)-\frac{v^2}{3}-\left(\frac{7}{8}+\frac{v^4}{18}\right)e^{\frac{2u}{\sqrt{3}}}, v+\left(\frac{v^3}{3}-\frac{v}{4}\right)e^{\frac{2u}{\sqrt{3}}}, \frac{1}{2}+\frac{v^2}{2}e^{\frac{2u}{\sqrt{3}}}, \\ &0, \ldots, 0, v+\left(\frac{v^3}{3}+\frac{v}{4}\right)e^{\frac{2u}{\sqrt{3}}}, \sinh\left(\frac{2u}{\sqrt{3}}\right)-\frac{v^2}{3}-\left(\frac{1}{8}+\frac{v^4}{18}\right)e^{\frac{2u}{\sqrt{3}}}, \frac{\sqrt{r^2-1}}{r}\Bigg) \end{aligned}$$

with $r \geq 1$ *and* $s \geq 2$;

(18) *a surface of constant negative curvature immersed in* $H^4_2(-1) \subset H^m_s(-1) \subset \mathbb{E}^{m+1}_{s+1}$ *defined as*

$$\begin{aligned} r\Bigg(&\sinh\left(\frac{2u}{\sqrt{3}}\right)-\frac{v^2}{3}-\left(\frac{7}{8}+\frac{v^4}{18}\right)e^{\frac{2u}{\sqrt{3}}}, v+\left(\frac{v^3}{3}-\frac{v}{4}\right)e^{\frac{2u}{\sqrt{3}}}, \frac{1}{2}+\frac{v^2}{2}e^{\frac{2u}{\sqrt{3}}}, \\ &\frac{\sqrt{1-r^2}}{r}, 0, \ldots, 0, v+\left(\frac{v^3}{3}+\frac{v}{4}\right)e^{\frac{2u}{\sqrt{3}}}, \sinh\left(\frac{2u}{\sqrt{3}}\right)-\frac{v^2}{3}-\left(\frac{1}{8}+\frac{v^4}{18}\right)e^{\frac{2u}{\sqrt{3}}}\Bigg) \end{aligned}$$

with $r<1$ *and* $s \geq 3$ *or*

(B) $L=(f_1,\ldots,f_\ell,\phi,f_\ell,\ldots,f_1)$, *where* $f_1,\ldots,f_\ell$ *are polynomials of degree* ≤ 2 *in* u,v *and* ϕ *is a surface given by (5), (7), (8) or (11)–(18) from* (A) *or*

(C) $L=(r,\phi,r)$, *where* r *is a positive number and* ϕ *is a surface given by (1)–(4), (6), (9) or (10) from* (A).

12.2. *A Parallel Spatial Surfaces in* H^4_2

There is a famous minimal immersion of the 2-sphere $S^2(\frac{1}{3})$ of curvature $\frac{1}{3}$ into the unit 4-sphere $S^4(1)$, known as the Veronese surface, which is constructed by using spherical harmonic homogeneous polynomials of degree two defined as

$$\left(\frac{vw}{\sqrt{3}}, \frac{uw}{\sqrt{3}}, \frac{uv}{\sqrt{3}}, \frac{u^2-v^2}{2\sqrt{3}}, \frac{u^2+v^2-2w^2}{6}\right), \quad u^2+v^2+w^2=3. \tag{1}$$

It is well known that the Veronese surface is the only minimal parallel surface lying fully in $S^4(1)$ (see, e.g., References [94–96]). On the other hand, it was also known that there does not exist minimal surface of constant Gauss curvature lying fully in the hyperbolic 4-space $H^4(-1)$ (cf. References [95–97]). Furthermore, it was known from Reference [98] that there exist no minimal spatial parallel surfaces lying fully in $H^4_1(-1)$.

B.-Y. Chen discovered in Reference [99] a minimal immersion of the hyperbolic plane $H^2(-\frac{1}{3})$ of Gauss curvature $-\frac{1}{3}$ into the unit neutral pseudo-hyperbolic 4-space $H^4_2(-1)$ as follows:

The following map $\mathcal{B} : \mathbf{R}^2 \to \mathbb{E}^5_3$:

$$\begin{aligned}\mathcal{B}(s,t) = \Bigg(&\sinh\left(\frac{2s}{\sqrt{3}}\right) - \frac{t^2}{3} - \left(\frac{7}{8} + \frac{t^4}{18}\right) e^{\frac{2s}{\sqrt{3}}}, t + \left(\frac{t^3}{3} - \frac{t}{4}\right) e^{\frac{2s}{\sqrt{3}}}, \\ &\frac{1}{2} + \frac{t^2}{2} e^{\frac{2s}{\sqrt{3}}}, t + \left(\frac{t^3}{3} + \frac{t}{4}\right) e^{\frac{2s}{\sqrt{3}}}, \sinh\left(\frac{2s}{\sqrt{3}}\right) - \frac{t^2}{3} - \left(\frac{1}{8} + \frac{t^4}{18}\right) e^{\frac{2s}{\sqrt{3}}} \Bigg)\end{aligned} \tag{2}$$

was introduced in Reference [99]. It is direct to verify that the position vector field x of $\mathcal{B}$ satisfies $\langle x, x \rangle = -1$ and the induced metric is given by $g = ds^2 + e^{\frac{2s}{\sqrt{3}}} dt^2$. Thus, $\mathcal{B}$ defines an isometric immersion $\psi_{\mathcal{B}} : H^2(-\frac{1}{3}) \to H^4_2(-1)$ of the hyperbolic plane $H^2(-\frac{1}{3})$ of curvature $-\frac{1}{3}$ into $H^4_2(-1)$.

In Reference [99], Chen characterized this parallel immersion $\psi_{\mathcal{B}} : H^2(-\frac{1}{3}) \to H^4_2(-1)$ as the following.

Theorem 33. *Up to rigid motions, the isometric immersion $\psi_{\mathcal{B}} : H^2(-\frac{1}{3}) \to H^4_2(-1)$ defined via (2) is the only minimal parallel spatial surface lying fully in $H^4_2(-1)$.*

Remark 5. *Although our construction of this minimal surface in $H^4_2(-1)$ is quite different from the Veronese surface given by (1), we show in Reference [99] that this parallel surface defined by (2) does share several important geometric properties with Veronese surface.*

12.3. Special Case: Parallel Surfaces in H^3_1

Theorem 32 implies the following classification of parallel surfaces in H^3_1.

Corollary 3. *A parallel spatial surface in $H^3_1(-1) \subset \mathbb{E}^4_2$ is congruent to an open part of one of the following two types of surfaces:*

(i) *a hyperbolic plane H^2 defined by $(a, b\cosh u \cosh v, b \cosh u \sinh v, b \sinh u)$, $a^2 + b^2 = 1$;*
(ii) *a surface $H^1 \times H^1$ defined by $(a \cosh u, b \cosh v, a \sinh u, b \sinh v)$, $a^2 + b^2 = 1$.*

Remark 6. *The surfaces (i) of Corollary 3 are totally umbilical and (i) with $a = 0$ is totally geodesic. Further, the surfaces (ii) are flat and surface (ii) with $a^2 = b^2 = \frac{1}{2}$ is minimal.*

13. Parallel Lorentz Surfaces in Pseudo-Euclidean Spaces

Lorentzian geometry is a vivid field that represents the mathematical foundation of the general theory of relativity, which is probably one of the most successful and beautiful theories of physics. An interesting phenomenon for Lorentzian surfaces in Lorentzian Kaehler surfaces states that Ricci equation is a consequence of the Gauss and the Codazzi equations (see Reference [100]). This indicates that Lorentzian surfaces have many interesting properties which are different from surfaces in Riemannian manifolds. In particular, Lorentzian surfaces in indefinite real space forms behaved differently from surfaces in Riemannian space forms. For instance, the family of minimal surfaces in Euclidean spaces is huge (see, e.g., Chapter 5 of Reference [95]). In contrast, all Lorentzian minimal surfaces in a pseudo-Euclidean m-space $\mathbb{E}^m_s$ were completely classified in Reference [101] (see also Reference [102]) as the following.

Theorem 34. *A Lorentzian surface in a pseudo-Euclidean m-space $\mathbb{E}^m_s$ is minimal if and only if the immersion takes the form*

$$L(x,y) = z(x) + w(y),$$

where z and w are null curves satisfying $\langle z'(x), w'(y)\rangle \neq 0$.

13.1. Classification of Parallel Lorentzian Surfaces in $\mathbb{E}^m_s$

In Reference [103], we have the following classification theorem for parallel Lorentzian surfaces in an arbitrary pseudo-Euclidean space.

Theorem 35. *Let M be a parallel Lorentzian surface into the pseudo-Euclidean m-space $\mathbb{E}^m_s$, $s \geq 1$. Then up to dilations and rigid motions of $\mathbb{E}^m_s$, we have either*

(A) the surface is an open portion of one of the following fifteen types of surfaces:

(1) *a totally geodesic plane $\mathbb{E}^2_1 \subset \mathbb{E}^m_s$ given by $(x,y) \in \mathbb{E}^2_1 \subset \mathbb{E}^m_s$;*
(2) *a totally umbilical de Sitter space S^2_1 in a totally geodesic $\mathbb{E}^3_1 \subset \mathbb{E}^m_s$ given by*

$$(\sinh x, \cosh x \cos y, \cosh x \sin y);$$

(3) *a flat cylinder $\mathbb{E}^1_1 \times S^1$ in a totally geodesic $\mathbb{E}^3_1 \subset \mathbb{E}^m_s$ given by $(x, \cos y, \sin y)$;*
(4) *a flat cylinder $S^1_1 \times \mathbb{E}^1$ in a totally geodesic $\mathbb{E}^3_1 \subset \mathbb{E}^m_s$ given by $(\sinh x, \cosh x, y)$;*
(5) *a flat minimal surface in a totally geodesic $\mathbb{E}^3_1 \subset \mathbb{E}^m_s$ given by*

$$\left(\frac{1}{6}(x-y)^3 + x, \frac{1}{6}(x-y)^3 + y, \frac{1}{2}(x-y)^2\right);$$

(6) *a flat surface $S^1_1 \times S^1$ in a totally geodesic $\mathbb{E}^4_1 \subset \mathbb{E}^m_s$ given by $(a\sinh x, a\cosh x, b\cos y, b\sin y)$, with $a, b > 0$;*
(7) *an anti-de Sitter space H^2_1 in a totally geodesic $\mathbb{E}^3_2 \subseteq \mathbb{E}^m_s$ given by $(\sin x, \cos x\cosh y, \cos x \sinh y)$;*
(8) *a flat minimal surface in a totally geodesic $\mathbb{E}^3_2 \subseteq \mathbb{E}^m_s$ defined by*

$$\left(\frac{a^2x^2}{2}, \frac{x}{2} - \frac{a^4x^2}{6} + y, \frac{x}{2} + \frac{a^4x^2}{6} - y\right),\ a > 0;$$

(9) *a non-minimal flat surface in a totally geodesic $\mathbb{E}^3_2 \subseteq \mathbb{E}^m_s$ defined by*

$$\left(\frac{1}{2b}\cos\left(\frac{\sqrt{2b}}{a}(a^2x+by)\right), \frac{1}{2b}\sin\left(\frac{\sqrt{2b}}{a}(a^2x+by)\right), \frac{a^2x-by}{a\sqrt{2b}}\right),\ a, b > 0;$$

(10) *a non-minimal flat surface in a totally geodesic $\mathbb{E}^3_2 \subseteq \mathbb{E}^m_s$ defined by*

$$\left(\frac{a^2x+by}{a\sqrt{2b}}, \frac{1}{2b}\cosh\left(\frac{\sqrt{2b}}{a}(a^2x-by)\right), \frac{1}{2b}\sinh\left(\frac{\sqrt{2b}}{a}(a^2x-by)\right)\right), a, b > 0;$$

(11) *a flat surface $H^1_1 \times H^1$ in a totally geodesic $\mathbb{E}^4_2 \subset \mathbb{E}^m_s$ given by $(a\sinh x, b\cosh v, a\cosh x, b\sinh y)$ with $a, b > 0$;*
(12) *a marginally trapped flat surface in a totally geodesic $\mathbb{E}^4_2 \subseteq \mathbb{E}^m_s$ defined by*

$$\begin{gathered}\big(a\cos x\cosh y + b\sin x\sinh y, a\sin x\cosh y - b\cos x\sinh y,\\ b\cos x\cosh y - a\sin x\sinh y, b\sin x\cosh y + a\cos x\sinh y\big),\ a, b \in \mathbf{R};\end{gathered}$$

(13) *a marginally trapped flat surface in a totally geodesic $\mathbb{E}^4_2 \subseteq \mathbb{E}^m_s$ given by*

$$\begin{gathered}\big((1+a)\sin y - (x+ay)\cos y, (1+a)\cos y + (x+ay)\sin y,\\ (1-a)\sin y + (x+ay)\cos y, (1-a)\cos y - (x+ay)\sin y\big),\ a \in \mathbf{R};\end{gathered}$$

(14) *a non-minimal flat surface in a totally geodesic* $\mathbb{E}^4_3 \subseteq \mathbb{E}^m_s$ *defined by*

$$\left(\cos\left(\frac{\sqrt{b}(a^3x+by)}{a^{5/2}}\right), \sin\left(\frac{\sqrt{b}(a^3x+by)}{a^{5/2}}\right), \cosh\left(\frac{\sqrt{b}(a^3x-by)}{a^{5/2}}\right), \sinh\left(\frac{\sqrt{b}(a^3x-by)}{a^{5/2}}\right)\right),$$

with $a, b > 0$;

(15) *a non-minimal flat surface in a totally geodesic* $\mathbb{E}^4_3 \subseteq \mathbb{E}^m_s$ *defined by*

$$\left(\frac{\sqrt[4]{\delta^2+\varphi^2}\cos\left(\lambda\left(bx+\sqrt{\delta^2+\varphi^2}y\right)\right)}{\sqrt{2}b\sqrt{\sqrt{\delta^2+\varphi^2}+\delta}}, \frac{\sqrt[4]{\delta^2+\varphi^2}\sin\left(\lambda\left(bx+\sqrt{\delta^2+\varphi^2}y\right)\right)}{\sqrt{2}b\sqrt{\sqrt{\delta^2+\varphi^2}+\delta}},\right.$$
$$\left.\frac{\sqrt[4]{\delta^2+\varphi^2}\cosh\left(\mu\left(bx-sqrt\delta^2+\varphi^2y\right)\right)}{\sqrt{2}b\sqrt{\sqrt{\delta^2+\varphi^2}-\delta}}, \frac{\sqrt[4]{\delta^2+\varphi^2}\sin\left(\mu\left(bx-\sqrt{\delta^2+\varphi^2}y\right)\right)}{\sqrt{2}b\sqrt{\sqrt{\delta^2+\varphi^2}-\delta}}\right)$$

with $\delta, \varphi \neq 0, b > 0$ *and*

$$\lambda = \frac{\sqrt{b\sqrt{\delta^2+\varphi^2}+b\delta}}{\sqrt{\delta^2+\varphi^2}}, \quad \mu = \frac{\sqrt{b\sqrt{\delta^2+\varphi^2}-b\delta}}{\sqrt{\delta^2+\varphi^2}},$$

or

(B) M^2_1 *is a flat surface and the immersion takes the form:* $(f_1, \ldots, f_\ell, \phi(x,y), f_\ell, \ldots, f_1)$, *where* $\phi = \phi(x,y)$ *is given by one of (1), (3)–(6), (8)–(15) and* $f_1, \ldots, f_\ell$ $(\ell \geq 1)$ *are polynomials of degree* ≤ 2 *in* x, y.

13.2. Classification of Parallel Lorentzian Surfaces in $\mathbb{E}^3_1$

Theorem 35 implies the following.

Corollary 4. *A parallel Lorentzian surface in the Minkowski 3-space* $\mathbb{E}^3_1$ *is congruent to an open part of one of the following five types of surfaces:*

(1) *the Lorentzian plane* $E^2_1 : L(u,v) = (u,v,0)$;
(2) *a de Sitter space* $S^2_1 : L(u,v) = a(\sinh u, \cosh u \cos v, \cosh u \sin v), a > 0$;
(3) *a cylinder* $\mathbb{E}^1_1 \times S^1 : L(u,v) = (u, a\cos v, a\sin v),\ a > 0$;
(4) *a cylinder* $S^1_1 \times \mathbb{E}^1 : L(u,v) = (a\sinh u, a\cosh u, v),\ a > 0$;
(5) *the null scroll* $\mathbb{N}^2_1$ *with rulings in the direction of* $(1,1,0)$ *of the null cubic given by* $\alpha(u) = \left(\frac{4}{3}u^3 + u, \frac{4}{3}u^3 - u, 2u^2\right)$.

Remark 7. *The surface (1) is totally geodesic; the surface is totally umbilical but not totally geodesic, all others are flat; the surfaces (1), (3) and (4) are products of parallel curves in totally geodesic subspaces; the surface (5) is flat and minimal but not totally geodesic.*

14. Parallel Surfaces in a Light Cone $\mathcal{LC}$

The *light cone* $\mathcal{LC}$ of a pseudo-Euclidean $(n+1)$-space $\mathbb{E}^{n+1}_s$ is defined by

$$\mathcal{LC}^n_s = \{x \in \mathbb{E}^{n+1}_s : \langle x, x\rangle = 0\}.$$

A curve in a pseudo-Riemannian manifold is called a *null curve* if its velocity vector is a light-like at each point.

14.1. Light Cones in General Relativity

In physics, a *space-time* is a time-oriented 4-dimensional Lorentz manifold. As with any time-oriented space-time, the time-orientation is called the *future* and its negative is called the *past*.

A tangent vector in a future time-cone is called *future-pointing*. Similarly, a tangent vector in the past time-cone is called *past-pointing*.

Light cones play a very important role in general relativity. Since signals and other causal influences cannot travel faster than light, the light cone plays an essential role in defining the concept of causality: for a given event E, the set of events that lie on or inside the past light cone of E would also be the set of all events that could send a signal that would have time to reach E and influence it in some way. Likewise, the set of events that lie on or inside the future light cone of E would also be the set of events that could receive a signal sent out from the position and time of E, so the future light cone contains all the events that could potentially be causally influenced by E. Events which lie neither in the past or future light cone of E cannot influence or be influenced by E in relativity.

14.2. Parallel Surfaces in $\mathcal{L}C^3_1 \subset \mathbb{E}^4_1$

Parallel surfaces in the light cone $\mathcal{L}C^3_1 \subset \mathbb{E}^4_1$ were classified by Chen and J. Van der Veken in Reference [98] as follows.

Theorem 36. *Let M be a parallel surface of $\mathbb{E}^4_1$. If M lies in the light cone $\mathcal{L}C^3_1 \subset \mathbb{E}^4_1$, then M is congruent to an open part of one of the following four types of surfaces:*

(1) *a totally umbilical surface of positive curvature given by* $a(1, \cos u \cos v, \cos u \sin v, \sin u),\ a > 0$;
(2) *totally umbilical surface of negative curvature given by* $a(\cosh u \cosh v, \cosh u \sinh v, \sinh u, 1),\ a > 0$;
(3) *a flat totally umbilical surface given by* $\left(u^2 + v^2 + \frac{1}{4}, u^2 + v^2 - \frac{1}{4}, u, v\right)$;
(4) *a flat surface given by* $a(\cosh u, \sinh u, \cos v, \sin v), a > 0$.

14.3. Parallel Surfaces in $\mathcal{L}C^3_2 \subset \mathbb{E}^4_2$

For parallel surfaces in the light cone $\mathcal{L}C^3_2 \subset \mathbb{E}^4_2$, we have the following result from Reference [98] as well.

Theorem 37. *Let M be a parallel surface of $\mathbb{E}^4_2$. If M lies in the light cone $\mathcal{L}C^3_2 \subset \mathbb{E}^4_2$, then M is congruent to an open part of one of the following eight types of surfaces:*

(1) *a totally umbilical surface of positive curvature given by* $a(\sinh u, 1, \cosh u \cos v, \cosh u \sin v),\ a > 0$;
(2) *a totally umbilical surface of negative curvature given by* $a(\sin u, \cos u \cosh v, 1, \cos u \sinh v), a > 0$;
(3) *a totally umbilical flat surface defined by*

$$\left(u, u^2 + v^2 - \frac{1}{4}, u^2 + v^2 + \frac{1}{4}, v\right);$$

(4) *a flat surface defined by* $a(\sinh u, \cosh v, \cosh u, \sinh v), a > 0$;
(5) *a flat surface defined by* $a(\sin u, \cos u, \cos v, \sin v), a > 0$;
(6) *a flat surface defined by*

$$\begin{aligned} a(&\sinh u \cos v + \sinh u \sin v, \cosh u \sin v - \sinh u \cos v, \\ &\cosh u \cos v - \sinh u \sin v, \cosh u \sin v + \sinh u \cos v),\ a > 0; \end{aligned}$$

(7) *a flat surface defined by* $a(cosv - usinv, sinv + ucosv, cosv + usinv, sinv - ucosv), a > 0$;
(8) *a flat surface defined by* $a(\cosh u - v \sinh u, \sinh u + v \cosh u, \cosh u + v \sinh u, \sinh u - v \cosh u)$ *with* $a > 0$.

15. Parallel Surfaces in De Sitter Space-Time S^4_1

The geometry of 4-dimensional space-time is much more complex than that of 3-dimensional space, due to the extra degree of freedom. Four-dimensional space-times play extremely important roles in the theory of relativity. In physics, space-time is a mathematical model that combines space and time into a single continuum. Space-time is usually interpreted with space being three-dimensional and time playing the role of a fourth dimension. By combining space and time into a single manifold, physicists have significantly simplified a large number of physical theories, as well as described in a more uniform way the workings of the universe at both the super-galactic and subatomic levels.

In recent times, physics and astrophysics have played a central role in shaping the understanding of the universe through scientific observation and experiment. After Kaluza-Klein's theory, the term space-time has taken on a generalized meaning beyond treating space-time events with the normal 3+1 dimensions. It becomes the combination of space and time. Some proposed space-time theories include additional dimensions, normally spatial but there exist some speculative theories that include additional temporal dimensions and even some that include dimensions that are neither temporal nor spatial. How many dimensions are needed to describe the Universe is still a big open question.

15.1. Classification of Parallel Spatial Surfaces in De Sitter Space-Time S^4_1

For parallel spatial surfaces in the de Sitter space-time $S^4_1(1)$, we have the following classification theorem proved by Chen and Van der Veken in Reference [98].

Theorem 38. *If M is a parallel spatial surface in $S^4_1(1) \subset \mathbb{E}^5_1$, then M is congruent to one of the following ten types of surfaces:*

(1) *a totally umbilical sphere S^2 given locally by $(c, b\cos u\cos v, b\cos u\sin v, b\sin u, a)$, $a^2+b^2-c^2=1$;*
(2) *a totally umbilical hyperbolic plane H^2 given by $(a\cosh u\cosh v, a\cosh u\sinh v, a\sinh u, b, c)$ with $b^2+c^2-a^2=1$;*
(3) *a torus $S^1\times S^1$ given by $(a, b\cos u, b\sin u, c\cos v, c\sin v)$ with $b^2+c^2-a^2=1$;*
(4) *a flat surface $H^1\times S^1$ given by $(b\cosh u, b\sinh u, c\cos v, c\sin v, a)$ with $a^2+c^2-b^2=1$;*
(5) *a totally umbilical flat surface defined by*

$$\left(u^2+v^2+a^2+\frac{1}{4}, u^2+v^2+a^2-\frac{1}{4}, u, v, \sqrt{1+a^2}\right);$$

(6) *a flat surface defined by*

$$\left(v^2-\frac{3}{4}+a^2, a\cos u, a\sin u, v, v^2-\frac{5}{4}+a^2\right), \ a>0;$$

(7) *a flat surface defined by*

$$\frac{1}{\sqrt{1+a^2}}\left(u^2+v^2-\frac{3}{4}, u^2+v^2-\frac{5}{4}, u, v, a\right), \ a\in\mathbf{R};$$

(8) *a marginally trapped flat surface defined by $\frac{1}{2}\left(2u^2-1, 2u^2-2, 2u, \sin v, \cos v\right)$;*
(9) *a marginally trapped flat surface defined by*

$$\left(\frac{b}{\sqrt{4-b^2}}, \frac{\cos u}{\sqrt{2-b}}, \frac{\sin u}{\sqrt{2-b}}, \frac{\cos v}{\sqrt{2+b}}, \frac{\sin v}{\sqrt{2+b}}\right); \ |b|<2;$$

(10) *a marginally trapped flat surface defined by*

$$\left(\frac{\cosh u}{\sqrt{b-2}}, \frac{\sinh u}{\sqrt{b-2}}, \frac{\cos v}{\sqrt{2+b}}, \frac{\sin v}{\sqrt{2+b}}, \frac{b}{\sqrt{b^2-4}}\right); \ b>2.$$

For parallel spatial surface in $S^3_1(1) \subset \mathbb{E}^4_1$, Theorem 38 implies the following.

Corollary 5. *If M is a parallel spatial surface in $S^3_1(c) \subset \mathbb{E}^4_1$, $c>0$, then M is congruent to one of the following four types of surfaces:*

(1) *a totally umbilical sphere* S^2 *given locally by* $(a, b\sin u, b\cos u\cos v, b\cos u\sin v)$ *with* $b^2 - a^2 = c^{-1}$;
(2) *a totally umbilical Euclidean* $\mathbb{E}^2$ *plane given by* $\frac{1}{\sqrt{c}}\left(u^2+v^2-\frac{3}{4}, u^2+v^2-\frac{5}{4}, u, v\right)$;
(3) *a totally umbilical hyperbolic plane* H^2 *given by* $(a\cosh u\cosh v, a\cosh u\sinh v, a\sinh u, b)$, *with* $b^2 - a^2 = c^{-1}$;
(4) *a flat surface* $H^1 \times S^1$ *given by* $(a\cosh u, a\sinh u, b\cos v, b\sin v)$ *with* $a^2 + b^2 = c^{-1}$.

15.2. Classification of Parallel Lorentzian Surfaces in De Sitter Space-Time S_1^4

For parallel Lorentzian surfaces in $S_1^4(1)$, we also have the following result from Reference [98].

Theorem 39. *If M is a parallel Lorentzian surface in* $S_1^4(1) \subset \mathbb{E}_1^5$, *then M is congruent to an open part of one of the following two types of surfaces:*

(1) *a totally umbilical de Sitter space* S_1^2 *in* $S_1^4(1)$ *given by* $(a\sinh u, a\cosh u\cos v, a\cosh u\sin v, b, 0)$ *with* $a^2 + b^2 = 1$;
(2) *a flat surface* $S_1^1 \times S^1$ *given by* $(a\sinh u, a\cosh u, b\cos v, b\sin v, 0)$, $a^2 + b^2 = 1$.

Conversely, each surface defined above is a Lorentzian parallel surface in $S_1^4(1)$

16. Parallel Surfaces in Anti-De Sitter Space-Time H_1^4

16.1. Classification of Parallel Spatial Surfaces in H_1^4

Parallel surfaces in the anti-de Sitter space-time $H_1^4(-1)$ were also classified by Chen and Van der Veken in Reference [98].

Theorem 40. *If M is a parallel spatial surface in* $H_1^4(-1) \subset \mathbb{E}_2^5$, *then M is congruent to one of the following ten types of surfaces:*

(1) *a totally umbilical sphere* S^2 *given locally by* $(a, c, b\sin u, b\cos u\cos v, b\cos u\sin v)$, $a^2 - b^2 + c^2 = 1$;
(2) *a totally umbilical hyperbolic plane* H^2 *given locally by* $(a, b\cosh u\cosh v, b\cosh u\sinh v, b\sinh u, c)$ *with* $a^2 + b^2 - c^2 = 1$;
(3) *flat surface* $H^1 \times S^1$ *given by* $(a, b\cosh u, b\sinh u, c\cos v, c\sin v)$ *with* $a^2 + b^2 - c^2 = 1$;
(4) *a flat surface* $H^1 \times H^1$ *given by* $(b\cosh u, c\cosh v, b\sinh u, c\sinh v, a)$ *with* $b^2 + c^2 - a^2 = 1$;
(5) *a totally umbilical flat surface defined by*

$$\left(\sqrt{1-a^2}, u^2+v^2+a^2+\frac{1}{4}, u^2+v^2+a^2-\frac{1}{4}, u, v\right),\ a \in (0,1);$$

(6) *a flat surface defined by*

$$\left(a, b\left(u^2+v^2-\frac{3}{4}\right), b\left(u^2+v^2-\frac{5}{4}\right), bu, bv\right),\ a^2 = 1 + b^2 > 1;$$

(7) *a flat surface defined by*

$$\left(v^2+\frac{5}{4}-a^2, a\cosh u, a\sinh u, v, v^2+\frac{3}{4}-a^2\right),\ a \neq 0;$$

(8) *the marginally trapped flat surface defined by*

$$\left(u^2+1, \frac{1}{2}\cosh v, u, \frac{1}{2}\sinh v, u^2+\frac{1}{2}\right);$$

(9) *a marginally trapped flat surface defined by*

$$\left(\frac{\cosh u}{\sqrt{2-b}}, \frac{\cosh v}{\sqrt{2+b}}, \frac{\sinh u}{\sqrt{2-b}}, \frac{\sinh v}{\sqrt{2+b}}, \frac{b}{\sqrt{4-b^2}}\right),\ |b| < 2;$$

(10) *a flat marginally trapped surface defined by*

$$\left(\frac{b}{\sqrt{b^2-4}}, \frac{\cosh v}{\sqrt{b+2}}, \frac{\sinh u}{\sqrt{b+2}}, \frac{\cos u}{\sqrt{b-1b}}, \frac{\sin u}{\sqrt{b-2}}\right),\ b>2.$$

Conversely, each surface of the ten types given above is spatial and parallel.

For parallel spatial surfaces in $H_1^3(-1)$, Theorem 40 implies the following.

Corollary 6. *If M is a parallel spatial surface in $H_1^3(-1) \subset \mathbb{E}_1^4$, $c>0$, then M is congruent to one of the following two types of surfaces:*

(1) *a hyperbolic plane H^2 defined by $(a, b\cosh u\cosh v, b\cosh u\sinh v, b\sinh u)$, $a^2+b^2=1$;*
(2) *a surface $H^1 \times H^1$ defined by $(a\cosh u, b\cosh v, a\sinh u, b\sinh v), a^2+b^2=1$.*

16.2. Classification of Parallel Lorentzian Surfaces in Anti-De Sitter Space-Time H_1^4

Parallel Lorentzian surfaces in $H_1^4(-1)$ were classified by Chen and J. Van der Veken in Reference [98] as follows.

Theorem 41. *If M is a parallel Lorentzian surface in $H_1^4(-1) \subset \mathbb{E}_2^5$, then M is congruent to one of the following twelve types of surfaces:*

(1) *a totally umbilical de Sitter space S_1^2 given by $(c, a\sinh u\cos v, a\cosh u\cos v, a\cosh u\sin b, b)$ with $c^2-a^2-b^2=1$;*
(2) *a totally umbilical anti-de Sitter space H_1^2 given by $(a\sin u, a\cos u\cosh v, a\cos u\sinh v, 0, b)$ with $a^2-b^2=1$;*
(3) *a flat surface $S_1^1 \times H^1$ given by $(c, a\sinh u, a\cosh u\cos v, a\cosh u\sin v, b)$ with $c^2-a^2-b^2=1$;*
(4) *a flat surface $H_1^1 \times S^1$ given by $(a\cos u, a\sin u, b\cos v, b\sin v, c)$ with $a^2+b^2-c^2=1$;*
(5) *a flat surface $S_1^1 \times S^1$ given by $(a, b\sinh u, b\cosh u, c\cos v, c\sin v)$ with $a^2-b^2-c^2=1$;*
(6) *a totally umbilical flat surface defined by $\left(u^2-v^2-\frac{5}{4}, au, av, a\left(u^2-v^2-\frac{3}{4}\right), b\right)$ with $a^2-b^2=1$;*
(7) *a flat surface defined by*

$$\left(a\cos v-\frac{a(u-v)}{2}\sin v, a\sin v+\frac{a(u-v)}{2}\cos v, \frac{a(u-v)}{2}\sin v, \frac{a(u-v)}{2}\cos v, b\right),\ a\in\mathbf{R};$$

(8) *a flat surface defined by*

$$\left(a\cosh v-\frac{a(u+v)}{2}\sinh v, \frac{a(u+v)}{2}\cosh v, a\sinh v-\frac{a(u+v)}{2}\cosh v, \frac{a(u+v)}{2}\sinh v, b\right)$$

with $a^2-b^2=1$;
(9) *a surface defined by*

$$\begin{gathered}(a\cos u\cosh v - a\tan k\sin u\sinh v, a\sec k\sin u\cosh v,\\ a\cos u\sinh v - a\tan k\sin u\cosh v, a\sec k\sin u\sinh v, b),\end{gathered}$$

with $a^2-b^2=1, \cos k\neq 0$;
(10) *a surface defined by*

$$\left(\frac{b^2(u^2-k^2-1)-1}{2b^2k}, u, \frac{\cos bv}{b}, \frac{\sin bv}{b}, \frac{b^2(u^2+k^2-1)-1}{2b^2k}\right),\ b,k\neq 1;$$

(11) *a surface defined by*

$$\left(\frac{-a^2(v^2+k^2+1)+1}{2a^2k}, \frac{\sinh au}{a}, \frac{\cosh au}{a}, v, \frac{a^2(k^2-v^2-1)-1}{2a^2k}\right),\ a,k \neq 1;$$

(12) *a surface defined by*

$$\left(\frac{(u-v)^4}{24k} + \frac{u^2-v^2-k^2-1}{2k}, \frac{1}{6}(u-v)^3+u, \frac{1}{2}(u-v)^2,\right.$$
$$\left.\frac{1}{6}(u-v)^3+v, \frac{(u-v)^4}{24k} + \frac{u^2-v^2+k^2-1}{2k}\right),\ k \neq 0.$$

16.3. Special Case: Parallel Lorentzian Surfaces in H_1^3

For parallel Lorentzian surfaces in $H_1^3(-1)$, Theorem 41 implies the following.

Corollary 7. *If M is a parallel Lorentzian surface in $H_1^3(-1) \subset \mathbb{E}_1^4$, then M is congruent to one of the following eight types of surfaces:*

(1) *a de Sitter space S_1^2 defined by $(a, b\sinh u, b\cosh u \sin v, b\cosh u\cos v)$ with $a^2-b^2=1$;*
(2) *the surface $\left(u^2-v^2-\frac{5}{4}, u, v, u^2-v^2-\frac{3}{4}\right)$;*
(3) *an anti-de Sitter space H_1^2 defined by $(a\sin u, a\cos u\cosh v, a\cos u\sinh v, b)$ with $a^2-b^2=1$;*
(4) *a surface $S_1^1 \times H^1$ defined by $(a\sinh u, b\cosh v, a\cosh u, b\sinh v)$ with $b^2-a^2=1$;*
(5) *a surface $H_1^1 \times S^1$ defined by $(a\cos u, a\sin u, b\cos v, b\sin v)$ with $a^2-b^2=1$;*
(6) *a surface defined by*

$$\big(\cos u\cosh v - \tan k\sin u\sinh v, \sec k\sin u\cosh v,$$
$$\cos u\sinh v - \tan k\sin u\cosh v, \sec k\sin u\sinh v\big),\quad \cos k \neq 0;$$

(7) *the surface defined by*

$$\left(\cos v - \frac{u-v}{2}\sin v, \sin v + \frac{u-v}{2}\cos v, \frac{u-v}{2}\sin v, \frac{u-v}{2}\cos v\right);$$

(8) *the surface defined by*

$$\left(\cosh v - \frac{u+v}{2}\sinh v, \frac{u+v}{2}\cosh v, \sinh v - \frac{u+v}{2}\cosh v, \frac{u+v}{2}\sinh v\right).$$

17. Parallel Spatial Surfaces in S_2^4

17.1. Four-Dimensional Manifolds with Neutral Metrics

The metrics of neutral signature $(--++)$ appear in many geometric and physics problems in the last 25 years. It has been realized that the theory of integrable systems and the techniques from the Seiberg–Witten theory can be successfully used to study Kaehler–Einstein and self-dual metrics as well as the self-dual Yang–Mills equations in neutral signature. Riemannian manifolds with neutral signature are of special interest since it retains many interesting parallels with Riemannian geometry. Such parallels are particularly evident in four dimensions, where Hodge's star operator is involutory for both positive-definite and neutral signatures. Both signatures possess the decomposition of two-forms into self-dual and anti-self-dual parts without the need to complexify as in the Lorentzian case.

As an interplay between indefiniteness and parallels with Riemannian geometry for neutral signature, the curvature decomposition in four dimensions for the two signatures allows one to deduce a neutral analogue of the Thorpe–Hitchin inequality for compact Einstein 4-manifolds (cf., e.g., Reference [104]). Also, the development of the geometry of neutral signature in the work of H. Ooguri and C. Vafa [105] showed that neutral signature arises naturally in string theory as well.

Para-Kaehler manifolds provide further interesting examples of metrics of neutral signature. Such manifolds play some important roles in super-symmetric field theories as well as in string theory (see, for instance, References [106–109]).

17.2. Classification of Parallel Lorentzian Surfaces in S^4_2

Complete classification of parallel Lorentzian surfaces in neutral pseudo-sphere $S^4_2(1)$ was obtained by Chen in Reference [110] as follows.

Theorem 42. *There exist 24 families of parallel Lorentzian surfaces in the neutral pseudo 4-sphere $S^4_2(1) \subset \mathbb{E}^5_2$:*

(1) *a totally geodesic de Sitter space-time $S^2_1(1) \subset S^4_2(1) \subset \mathbb{E}^5_2$;*

(2) *a flat surface in a totally geodesic $S^3_1(1) \subset S^4_2(1)$ defined by*

$$\left(\sqrt{a^2+b^2-1}, a\sinh u, a\cosh u, b\cos v, b\sin v\right), \ a,b>0, a^2+b^2\geq 1;$$

(3) *a flat surface defined by*

$$\Big(a\cos u\sinh v + b\sin u\cosh v, \sqrt{a^2+b^2}\sin u\sinh v, \sqrt{a^2+b^2}\sin u\cosh v,$$
$$a\cos u\cosh v + b\sin u\sinh v, \sqrt{1-a^2}\,\Big), \ a\in(0,1];$$

(4) *a flat surface defined by* $\left(a\cos u, a\sin u, b\cos v, b\sin v, \sqrt{1+a^2-b^2}\right), \ a,b>0, \ b^2\leq 1+a^2;$

(5) *a flat surface defined by*

$$\left(ku, pu^2+\frac{(1-b^2)\varphi}{k^2}-\frac{k^2}{4\varphi}, b\sin v, b\cos v, pu^2+\frac{(1-b^2)\varphi}{k^2}+\frac{k^2}{4\varphi}\right), \ b,k,p,\varphi\neq 0;$$

(6) *a flat surface defined by* $\left(\sqrt{b^2-a^2-1}, a\cosh u, a\sinh u, b\cos v, b\sin v\right), \ a,b>0, \ b^2\geq 1+a^2;$

(7) *a flat surface defined by*

$$\left(pu^2+\frac{(b^2-1)\varphi}{k^2}+\frac{k^2}{4\varphi}, b\sinh v, b\cosh v, ku, pu^2+\frac{(b^2-1)\varphi}{k^2}-\frac{k^2}{4\varphi}\right), \ b,k,p,\varphi\neq 0;$$

(8) *a flat surface given by* $\left(a\cosh u, b\sinh v, a\sinh u, b\cosh v, \sqrt{1+a^2-b^2}\right), a,b>0, b^2\leq 1+a^2;$

(9) *a marginally trapped surface of constant curvature one defined by*

$$\left(\frac{xy}{x+y}, \frac{2}{x+y}, \frac{x-y}{x+y}, \frac{2+xy}{x+y}, 0\right), \ x+y\neq 0;$$

(10) *a flat surface defined by* $(x+xy, y-xy, x-y+xy, 1+xy, 0);$

(11) *a surface of positive curvature c^2 defined by*

$$\left(\frac{xy-c^2}{c^2(x+y)}, \frac{2\sqrt{1-c^2}\,y}{c^2(x+y)}, \frac{xy+c^2}{c^2(x+y)}, \frac{c^2(x+y)-2y}{c^2(x+y)}, 0\right), \ c\in(0,1), \ x+y\neq 0;$$

(12) *a surface of positive curvature c^2 defined by*

$$\left(0, \frac{xy-c^2}{c^2(x+y)}, \frac{xy+c^2}{c^2(x+y)}, \frac{c^2(x+y)-2y}{c^2(x+y)}, \frac{2\sqrt{c^2-1}\,y}{c^2(x+y)}\right), \ c>1, \ x+y\neq 0;$$

(13) *a surface of negative curvature $-c^2$ defined by*

$$\frac{1}{c}\Big(\cosh u - \sinh u \tanh v, \sinh u \tanh v, \sinh u - \cosh u \tanh v, \sqrt{1+c^2}, 0\Big),\ c > 0;$$

(14) *a flat surface defined by*

$$\bigg(\frac{1+8c^2+2v}{4c}\cos u + \frac{1+v}{2c}\sin u, \frac{4c^2-1}{4c}\cos u + \Big(c + \frac{v}{2c}\Big)\sin u,$$
$$\Big(\frac{1}{4c} + 2c + \frac{v}{2c}\Big)\cos u + \frac{v\sin u}{2c}, \frac{4c^2+1}{4c}\cos u + \frac{1+2c^2+v}{2c}\sin u, 0\bigg),\ c > 0;$$

(15) *a flat surface defined by*

$$\Big(e^u - \frac{(2c-v)e^{-u}}{8c}, \frac{ve^u}{4} - \frac{e^{-u}}{2c}, e^u + \frac{(2c-v)e^{-u}}{8c}, \frac{ve^u}{4} + \frac{e^{-u}}{2c}, 0\Big),\ c > 0;$$

(16) *a flat surface defined by*

$$\Big(x + \frac{y}{2} + \frac{2c^2y^3}{3}, xy + \frac{c^2y^4}{6}, x - \frac{y}{2} + \frac{2c^2y^3}{3}, cy^2, 1 + xy + \frac{c^2y^4}{6}\Big),\ c > 0;$$

(17) *a flat surface defined by*

$$\Big(av\sinh u + b\cosh u, av\cosh u, av\cosh u + b\sinh u, av\sinh u, \sqrt{1+b^2}\Big),\ a, b \neq 0;$$

(18) *a flat surface defined by* $(a\sin u - bv\cos u, a\cos u + bv\cos u, bv\cos u, bv\sin u, \sqrt{1+a^2}),\ a, b \neq 0;$

(19) *a flat surface defined by*

$$\Big(v\cos u + \frac{\sin u}{c}, v\sin u - \frac{\cos u}{c}, v\cos u - \frac{\sin u}{c}, v\sin u + \frac{\cos u}{c}, 1\Big),\ c > 0;$$

(20) *a flat surface defined by*

$$\bigg(\cos u\cos v - \frac{\sin u\sin v}{c}, \cos u\sin v + \frac{\sin u\cos v}{c}, \cos u\cos v + \frac{\sin u\sin v}{c},$$
$$\cos u\sin v - \frac{\sin u\cos v}{c}, 1\bigg),\ c > 0;$$

(21) *a flat surface defined by*

$$\bigg(e^v\cos u + \frac{e^{-v}\sin u}{c}, e^{-v}\cos u - \frac{e^v\sin u}{c}, e^v\cos u - \frac{e^{-v}\sin u}{c}, e^{-v}\cos u + \frac{e^v\sin u}{c}, 1\bigg),\ c > 0;$$

(22) *a flat surface defined by* $(e^u + ae^{-u}v, e^u v - ae^{-u}, e^u - ae^{-u}v, e^u v + ae^{-u}, 1),\ a \neq 0;$

(23) *a flat surface defined by* $(e^u - ae^{-u}, e^v + ae^{-v}, e^u + ae^{-u}, e^v - ae^{-v}, 1),\ a \neq 0;$

(24) *a flat surface defined by* $(a\cosh u\cos v, a\cosh u\sin v, a\sinh u, \cos v, a\sinh u\sin v, \sqrt{1+a^2}),\ a > 0.$

Conversely, every parallel immersion $L : M \to S^4_2(1) \subset \mathbb{E}^5_2$ of a Lorentzian surface M into the pseudo 4-sphere $S^4_2(1)$ is congruent to an open portion a surface obtained from one of 24 families of surfaces described above.

17.3. Classification of Parallel Lorentzian Surfaces in H^4_2

Complete classification of parallel Lorentzian surfaces in neutral pseudo-hyperbolic 4-space $H^4_2(-1) \subset \mathbb{E}^5_3$ was obtained by Chen in Reference [111], in which he proved that there exist 53 families of parallel Lorentzian surfaces in neutral pseudo hyperbolic 4-space $H^4_2(-1)$.

Among the 53 families we have: one family of totally geodesic anti-de Sitter space-time; one family of marginally trapped surfaces of curvature one; one family of untrapped flat surfaces; one family of untrapped surfaces of positive curvature; one family of untrapped surfaces of negative curvature; two families of trapped surfaces of negative curvature; two families of flat minimal surfaces; 7 families of untrapped flat surfaces; 8 families of marginally trapped flat surfaces; 9 families of flat surface which can be either trapped or untrapped; and 20 families of trapped flat surfaces.

Conversely, every parallel Lorentzian surface in $H^4_2(-1)$ is congruent to an open portion of a surface obtained from one of the 53 families.

18. Parallel Spatial Surfaces in S^4_3 and in H^4_3

Parallel Lorentzian surfaces in $S^4_3(1)$ and in $H^4_3(-1)$ were completely classified by Chen in Reference [112].

18.1. Classification of Parallel Spatial Surfaces in S^4_3

Chen proved in Reference [112] that there are 21 families of parallel Lorentzian surfaces in $S^4_3(1) \subset \mathbb{E}^5_3$. Among the 21 families, we have: the totally geodesic de Sitter space-time $S^2_1(1) \subset S^4_3(1)$; one family of minimal flat surfaces in $S^4_3(1)$; a totally umbilical flat surfaces lying in a totally geodesic $S^3_2(1) \subset S^4_2(1)$; one family of totally umbilical de Sitter space $S^2_1(c^2)$ in a totally geodesic $S^3_2(1) \subset S^4_2(1)$; one family of totally umbilical anti-de Sitter space $H^2_1(-c^2)$ lying in a totally geodesic $S^3_2(1) \subset S^4_2(1)$; four families of CMC flat surfaces lying in a totally geodesic $S^3_2(1) \subset S^4_2(1)$; and 12 families of flat minimal surfaces.

Conversely, every parallel Lorentzian surface in $S^4_3(1) \subset \mathbb{E}^5_3$ is congruent to an open portion of a surface obtained from one of the 21 families.

18.2. Classification of Parallel Spatial Surfaces in H^4_3

For parallel Lorentzian surfaces in $H^4_3(-1) \subset \mathbb{E}^5_4$, Chen proved in Reference [112] the following classification theorem.

Theorem 43. *There are six families of parallel Lorentzian surfaces in $H^4_3(-1) \subset \mathbb{E}^5_4$:*

(1) A totally geodesic anti-de Sitter space $H^2_1(-1) \subset H^4_3(-1)$;
(2) A flat minimal surface in a totally geodesic $H^3_2(-1) \subset H^4_3(-1)$ defined by

$$\frac{1}{\sqrt{2}}\left(\sin\left(ax+\frac{y}{a}\right), \cos\left(ax+\frac{y}{a}\right), \cosh\left(ax-\frac{y}{a}\right), \sinh\left(ax-\frac{y}{a}\right), 0\right), \quad a>0;$$

(3) A totally umbilical anti-de Sitter space $H^2_1(-c^2)$ in a totally geodesic $H^3_2(-1) \subset H^4_3(-1)$ given by

$$\frac{1}{c}\left(0, \sqrt{c^2-1}, \tanh\left(\frac{cx+cy}{\sqrt{2}}\right), \sinh(\sqrt{2}cy)\tanh\left(\frac{cx+cy}{\sqrt{2}}\right)-\cosh(\sqrt{2}cy),\right.$$
$$\left.\sinh(\sqrt{2}cy)-\cosh(\sqrt{2}cy)\tanh\left(\frac{cx+cy}{\sqrt{2}}\right)\right), \quad c>1;$$

(4) A CMC flat surface in a totally geodesic $H_2^3(-1)$ given by

$$\Bigg(\frac{\sqrt{\sqrt{1+b^2}-b}}{\sqrt{2}\sqrt[4]{1+b^2}}\cos\Big(\frac{\sqrt{\sqrt{1+b^2}+b}(a^2x+\sqrt{1+b^2}y)}{a}\Big),$$
$$\frac{\sqrt{\sqrt{1+b^2}-b}}{\sqrt{2}\sqrt[4]{1+b^2}}\sin\Big(\frac{\sqrt{\sqrt{1+b^2}+b}(a^2x+\sqrt{1+b^2}y)}{a}\Big),$$
$$\frac{\sqrt{\sqrt{1+b^2}+b}}{\sqrt{2}\sqrt[4]{1+b^2}}\cosh\Big(\frac{\sqrt{\sqrt{1+b^2}-b}(a^2x-\sqrt{1+b^2}y)}{a}\Big),$$
$$\frac{\sqrt{\sqrt{1+b^2}+b}}{\sqrt{2}\sqrt[4]{1+b^2}}\sin\Big(\frac{\sqrt{\sqrt{1+b^2}-b}(a^2x-\sqrt{1+b^2}y)}{a}\Big)\Bigg),\quad a,b,c>0;$$

(5) A non-minimal flat surface given by

$$\frac{1}{\sqrt{2(1+b^2)}}\left(\sqrt{2}b,\cos\left(kx+\frac{k^3}{\gamma^2}y\right),\sin\left(kx+\frac{k^3}{\gamma^2}y\right),\cosh\left(kx-\frac{k^3}{\gamma^2}y\right),\sinh\left(kx-\frac{k^3}{\gamma^2}y\right)\right)$$

with $k=\sqrt[4]{(1+b^2)\gamma^2},\ b,\gamma>0$;

(6) A non-minimal flat surface given by

$$\Bigg(\frac{b\varphi}{\sqrt{\delta^2+(1+b^2)\varphi^2}},\frac{\sqrt{\sqrt{1+b^2}(\delta^2+\varphi^2)-b\delta\sqrt{\delta^2+\varphi^2}}}{\sqrt{2}\sqrt[4]{1+b^2}\sqrt{\delta^2+(1+b^2)\varphi^2}}\cos\big(\lambda(\sqrt{1+b^2}x+\sqrt{\delta^2+\varphi^2}y\big),$$
$$\frac{\sqrt{\sqrt{1+b^2}(\delta^2+\varphi^2)-b\delta\sqrt{\delta^2+\varphi^2}}}{\sqrt{2}\sqrt[4]{1+b^2}\sqrt{\delta^2+(1+b^2)\varphi^2}}\sin\big(\lambda(\sqrt{1+b^2}x+\sqrt{\delta^2+\varphi^2}y\big),$$
$$\frac{\sqrt{\sqrt{1+b^2}(\delta^2+\varphi^2)+b\delta\sqrt{\delta^2+\varphi^2}}}{\sqrt{2}\sqrt[4]{1+b^2}\sqrt{\delta^2+(1+b^2)\varphi^2}}\cosh\big(\mu(\sqrt{1+b^2}x-\sqrt{\delta^2+\varphi^2}y\big),$$
$$\frac{\sqrt{\sqrt{1+b^2}(\delta^2+\varphi^2)+b\delta\sqrt{\delta^2+\varphi^2}}}{\sqrt{2}\sqrt[4]{1+b^2}\sqrt{\delta^2+(1+b^2)\varphi^2}}\sinh\big(\mu(\sqrt{1+b^2}x-\sqrt{\delta^2+\varphi^2}y\big)\Bigg)$$

with $\delta,\varphi\neq 0, b>0$ *and*

$$\lambda=\frac{\sqrt{\sqrt{1+b^2}\sqrt{\delta^2+\varphi^2}+b\delta}}{\sqrt{\delta^2+\varphi^2}},\quad \mu=\frac{\sqrt{\sqrt{1+b^2}\sqrt{\delta^2+\varphi^2}-b\delta}}{\sqrt{\delta^2+\varphi^2}}.$$

Conversely, every parallel Lorentzian surface in $H_3^4(-1)$ is congruent to an open portion of one of the six families of surfaces described above.

19. Parallel Lorentz Surfaces in $\mathbb{C}_1^2$, CP_1^2 and CH_1^2

19.1. Hopf Fibrations

Let $\mathbb{C}^n=\{(z_1,\dots,z_n):z_1,\dots,z_n\in\mathbf{C}\}$ be the complex n-space. If $\mathbb{C}^n$ endows with the metric given by the real part of the Hermitian form

$$b_{j,n}((z_1,\dots,z_n),(w_1,\dots,w_n))=-\sum_{k=1}^{j}\bar{z}_kw_k+\sum_{k=j+1}^{n}\bar{z}_kw_k, \tag{3}$$

then we obtain a flat indefinite Kaehler manifold of complex index j, denoted by $\mathbb{C}^n_j$. In particular, $\mathbb{C}^n_1$ is a flat Lorentzian Kaehler manifold.

For any real number $c > 0$, the differentiable manifold

$$S^{2n+1}_2(c) = \{z \in \mathbb{C}^{n+1}_1 : b_{1,n+1}(z,z) = 1/c\},$$

with the induced metric, is an indefinite real space form of constant sectional curvature $c > 0$. The *Hopf-fibration*: $\pi : S^{2n+1}_2(c) \to CP^n_1(4c) : z \mapsto z \cdot \mathbb{C}^*$ with $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is a submersion and there is a unique Lorentzian Kaehler metric on $CP^n_1(4c)$ such that π is a Riemannian submersion. The space $CP^n_1(4c)$ equipped with this metric is a Lorentzian Kaehler manifold of positive holomorphic sectional curvature $4c$.

Similarly, for any real number $c < 0$, the differentiable manifold

$$H^{2n+1}_3(c) = \{z \in \mathbb{C}^{n+1}_2 : b_{2,n+1}(z,z) = 1/c\},$$

with the induced metric, is an indefinite real space form of constant sectional curvature $c < 0$. The Hopf-fibration: $\pi : H^{2n+1}_3(c) \to \mathbb{C}H^n_1(4c) : z \mapsto z \cdot \mathbb{C}^*$ is a submersion and there is a unique Lorentzian Kaehler metric on $CH^n_1(4c)$ such that π is a Riemannian submersion. The space $CH^n_1(4c)$ equipped with this metric is a Lorentzian Kaehler manifold of negative holomorphic sectional curvature $4c$.

The manifolds $\mathbb{C}^n_1$, $CP^n_1(4c)$ and $CH^n_1(4c)$ are called *complex Lorentzian space forms*. The Riemann curvature tensor of a complex Lorentzian space form of constant holomorphic sectional curvature $4c$ takes the form

$$\tilde{R}(X,Y) = c(X \wedge Y + JX \wedge JY - 2\langle JX, Y\rangle J),$$

where X and Y are arbitrary tangent vectors at an arbitrary point and $\wedge$ is defined by

$$(X \wedge Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y.$$

Remark 8. *The mapping*

$$\psi : \mathbb{C}^3_1 \to \mathbb{C}^3_2 : (z_1, z_2, z_3) \mapsto (z_3, z_2, z_1)$$

maps $S^5_2(c)$ to $H^5_3(-c)$ and, via the Hopf-fibrations, it induces a conformal mapping with factor -1 between $CP^2_1(4c)$ and $CH^2_1(-4c)$.

19.2. Classification of Parallel Lorentzian Surface in $\mathbb{C}^2_1$

For parallel Lorentzian surface in $\mathbb{C}^2_1$, we have the following result from Reference [73] by Chen, Dillen and Van der Veken.

Theorem 44. *A parallel Lorentzian surface M in $\mathbb{C}^2_1$ is isometric to an open part of one of the following nine types of surfaces:*

(1) *a Lorentzian totally geodesic surface;*
(2) *a Lorentzian product of parallel curves;*
(3) *a complex circle, given by $(a+ib)\big(\cos(x+iy), \sin(x+iy)\big)$ with $a, b \in \mathbf{R}$, $(a,b) \neq (0,0)$;*
(4) *a B-scroll over the null cubic in $\mathbb{E}^3_1 \subseteq \mathbb{C}^2_1$;*
(5) *a B-scroll over the null cubic in $\mathbb{E}^3_2 \subseteq \mathbb{C}^2_1$;*
(6) *a surface given by*

$$\frac{e^{-iy}}{\sqrt{2}}\big(i(1+a) - x - ay, i(1-a) + x + ay\big), \text{ with } a \in \mathbf{R};$$

(7) *a surface with light-like mean curvature vector given by $(q(x,y), x, y, q(x,y))$ with $q(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$ and $a, b, c, d, e, f \in \mathbf{R}$;*

(8) *a totally umbilical de Sitter space S^2_1 in $\mathbb{E}^3_1 \subseteq \mathbb{C}^2_1$, given by $a(0, \sinh x, \cosh x \cos y, \cosh x \sin y)$ with $a \in \mathbf{R} \setminus \{0\}$;*

(9) *a totally umbilical anti-de Sitter space H^2_1 in $\mathbb{E}^3_2 \subseteq \mathbb{C}^2_1$ given by $a(\sin x, \cos x \cosh y, \cos x \sinh y, 0)$ with $a \in \mathbf{R} \setminus \{0\}$.*

Conversely, each of the surfaces listed above is a Lorentzian surface with parallel second fundamental form in $\mathbb{C}^2_1$.

19.3. Classification of Parallel Lorentzian Surface in CP^2_1

First we mention the following result from Reference [73].

Lemma 5. *Every parallel Lorentzian surface in $CP^2_1(4)$ and in $CH^2_1(-4)$ is Lagrangian.*

The next classification of parallel Lorentzian surface in CP^2_1 was obtained by Chen, Dillen and Van der Veken in Reference [73].

Theorem 45. *Let M be a Lorentzian surface in $CP^2_1(4)$ with parallel second fundamental form. Then there are two possibilities:*

(I) *M is an open part of the totally geodesic, Lagrangian surface $RP^2_1(1) \subseteq CP^2_1(4)$.*

(II) *M is flat and the immersion is congruent to $\pi \circ L$, where $\pi : S^5_2(1) \to CP^2_1(4)$ is the Hopf-fibration and $L : M^2_1 \to S^5_2(1) \subseteq \mathbb{C}^3_1$ is locally one of the following twelve maps:*

$$(1)\quad L = \frac{1}{\sqrt{3}}\left(\sqrt{2}e^{\frac{i}{2}x}\sinh\left(\frac{\sqrt{3}}{2}y\right),\ \sqrt{2}e^{\frac{i}{2}x}\cosh\left(\frac{\sqrt{3}}{2}y\right),\ e^{-ix}\right);$$

$$(2)\quad L = \left(\frac{e^{\frac{i}{2}(2x+y+\sqrt{1+4a}y)}}{(1+4a)^{1/4}},\ \frac{e^{\frac{i}{2}(2x+y-\sqrt{1+4a}y)}}{(1+4a)^{1/4}},\ e^{iy}\right),\quad a > -\frac{1}{4};$$

$$(3)\quad L = \left(\frac{(2-ie^{-\sqrt{4a-1}y})e^{ix+\frac{1}{2}(i+\sqrt{4a-1})y}}{2\sqrt[4]{4a-1}}, \frac{(2+ie^{-\sqrt{4a-1}y})e^{ix+\frac{1}{2}(i+\sqrt{4a-1})y}}{2\sqrt[4]{4a-1}},\ e^{iy}\right),\quad a > \frac{1}{4};$$

$$(4)\quad L = \frac{1}{\sqrt{2}}\left(e^{i(x+\frac{y}{2})}(1+iy),\ e^{i(x+\frac{y}{2})}(1-iy),\ \sqrt{2}e^{iy}\right);$$

$$(5)\quad L = \left(\frac{\sqrt{a(2-a-b)}\, e^{i(bx+\frac{(1-b)y}{a(2-a-b)})}}{\sqrt{(a-b)(a+2b-2)}},\ \frac{\sqrt{b(2-a-b)}e^{i(ax+\frac{(1-a)y}{b(2-a-b)})}}{\sqrt{(a-b)(2a+b-2)}}, \frac{\sqrt{ab}\, e^{i((2-a-b)x+\frac{a+b-1}{ab}y)}}{\sqrt{(2a+b-2)(a+2b-2)}}\right)$$

with $a > b > 2-a-b > 0$ or $0 > a > b > 2-a-b$;

$$(6)\quad L = \left(\frac{\sqrt{b(a+b-2)}e^{i(ax+\frac{(1-a)y}{b(2-a-b)})}}{\sqrt{(a-b)(2a+b-2)}}, \frac{\sqrt{a(a+b-2)}e^{i(bx+\frac{(1-b)y}{a(2-a-b)})}}{\sqrt{(a-b)(a+2b-2)}}, \frac{\sqrt{ab}\, e^{i((2-a-b)x+\frac{a+b-1}{ab}y)}}{\sqrt{(2a+b-2)(a+2b-2)}}\right)$$

with $a > b > 0$ and $a+b > 2$;

$$(7)\quad L = \left(\frac{\sqrt{-ab}\, e^{i((2-a-b)x+\frac{a+b-1}{ab}y)}}{\sqrt{(2a+b-2)(a+2b-2)}}, \frac{\sqrt{b(2-a-b)}\, e^{i(ax+\frac{(1-a)y}{b(2-a-b)})}}{\sqrt{(a-b)(2a+b-2)}}, \frac{\sqrt{a(a+b-2)}e^{i(bx+\frac{(1-b)y}{a(2-a-b)})}}{\sqrt{(a-b)(a+2b-2)}}\right),$$

with $a > 0 > b > 2-a-b$;

$$(8)\quad L = \left(\left(\frac{2i\sqrt{(2a-1)(1-a)}}{2-3a} + \frac{2a^2(a-1)x+(2a-1)y}{2a\sqrt{(2a-1)(1-a)}}\right)e^{i(ax+\frac{y}{2a})},\right.$$

$$\left.\frac{(2a^2(a-1)x+(2a-1)y)e^{i(ax+\frac{y}{2a})}}{2a\sqrt{(2a-1)(1-a)}},\ \frac{a\, e^{i(2(1-a)x+\frac{2a-1}{a^2}y)}}{3a-2}\right),\ a \in (\tfrac{1}{2}, 1) \setminus \{\tfrac{2}{3}\};$$

$$(9)\quad L = \left(\frac{(2a^2(a-1)x+(2a-1)y)e^{i(ax+\frac{y}{2a})}}{2a\sqrt{(2a-1)(a-1)}}, \left(\frac{2a^2(a-1)x+(2a-1)y}{2a\sqrt{(2a-1)(a-1)}} + \frac{2i\sqrt{(2a-1)(a-1)}}{3a-2}\right.\right.$$

$$\left.\left.\times e^{i(ax+\frac{y}{2a})},\ \frac{a\, e^{i(2(1-a)x+\frac{2a-1}{a^2}y)}}{3a-2}\right)\right),\ a \in \mathbf{R} \setminus ([\tfrac{1}{2}, 1] \cup \{0\});$$

(10) $$L = \frac{e^{\frac{i}{12}(8x+9y)}}{24}\Big(1+(8x-9y)^2+432iy,\ 2(8x-9y+12i), 1-(8x-9y)^2-432iy\Big);$$

(11) $$L = \left(\frac{\sqrt{1-a}\, e^{i(ax+\frac{(a^2-b^2-a)y}{2(a-1)(a^2+b^2)})}}{b\sqrt{2a-1}\sqrt{(3a-2)^2+b^2}}\Big(2b(1-2a)\cosh\big(bx+\frac{b(2a-1)y}{2(a-1)(a^2+b^2)}\big)\right.$$

$$+i(3a^2-b^2-2a)\sinh\big(bx+\frac{b(2a-1)y}{2(a-1)(a^2+b^2)}\big)\Big),$$

$$\frac{\sqrt{1-a}\sqrt{a^2+b^2}\, e^{i(ax+\frac{(a^2-b^2-a)y}{2(a-1)(a^2+b^2)})}}{b\sqrt{2a-1}}\sinh\big(bx+\frac{b(2a-1)y}{2(a-1)(a^2+b^2)}\big),$$

$$\left.\frac{\sqrt{a^2+b^2}\, e^{i(2(1-a)x+\frac{2a-1}{a^2+b^2}y)}}{\sqrt{(3a-2)^2+b^2}}\right), \text{ with } a \in (\tfrac{1}{2}, 1) \text{ and } b \in \mathbf{R}\setminus\{0\};$$

(12) $$L = \left(\sqrt{\frac{a-1}{2a-1}}\frac{e^{i(ax+\frac{(a^2-b^2-a)y}{2(a-1)(a^2+b^2)})}}{b\sqrt{(3a-2)^2+b^2}}\Big(2b(1-2a)\sinh\big(bx+\frac{b(2a-1)y}{2(a-1)(a^2+b^2)}\big)\right.$$

$$+i(3a^2-b^2-2a)\cosh\big(bx+\frac{b(2a-1)y}{2(a-1)(a^2+b^2)}\big)\Big),$$

$$\sqrt{\frac{a-1}{2a-1}}\frac{\sqrt{a^2+b^2}\, e^{i(ax+\frac{(a^2-b^2-a)y}{2(a-1)(a^2+b^2)})}}{b}\cosh\big(bx+\frac{b(2a-1)y}{2(a-1)(a^2+b^2)}\big),$$

$$\left.\frac{\sqrt{a^2+b^2}\, e^{i(2(1-a)x+\frac{2a-1}{a^2+b^2}y)}}{\sqrt{(3a-2)^2+b^2}}\right), \text{ with } a \in \mathbf{R}\setminus[\tfrac{1}{2}, 1] \text{ and } b \in \mathbf{R}\setminus\{0\}.$$

19.4. Classification of Parallel Lorentzian Surface in CH^2_1

It follows from Remark 8 that one obtains immediately the classification of parallel Lorentzian surfaces in $CH^2_1(-4)$ from Theorem 45 via the mapping:

$$\psi : \mathbb{C}^3_1 \to \mathbb{C}^3_2 : (z_1, z_2, z_3) \mapsto (z_3, z_2, z_1)$$

since ψ gives rise to a conformal mapping with factor -1 between $CP^2_1(4)$ and $CH^2_1(-4)$. Hence, besides totally geodesic Lagrangian surface $RH^2_1(-1) \subset CH^2_1(-4)$, there are twelve families of flat parallel Lorentzian surfaces in $CH^2_1(-4)$.

20. Parallel Surfaces in Warped Product $I \times_f R^n(c)$

20.1. Basics on Robertson–Walker Space-Times

In the theory of general relativity, a *Robertson–Walker space-time* is a warped product

$$L^4_1(c, f) = (I \times R^3(c), g), \ \ g = -dt^2 + f^2(t)g_c,$$

of an open interval I and a Riemannian 3-manifold $(R^3(c), g_c)$ of constant curvature c, while the warping function f describes the expanding or contracting of our Universe (cf. References [113,114]).

A Robertson–Walker space-time possesses two relevant geometrical features. On one hand, its fibers have constant curvature. Hence, the space-time is *spatially homogeneous*. On the other hand, it has a time-like vector field $K = f(t)\partial_t$ which satisfies $\nabla_X K = f'(t)X$ for any X. In particular, we have $\mathcal{L}_K g = 2f'g$, where $\mathcal{L}_K$ is the Lie derivative along K. Hence, the canonical time-like vector field K is a conformal vector field. These properties of K show a certain symmetry on $L^n_1(c,f)$.

One may also consider a higher dimensional Robertson–Walker space-time as

$$L^n_1(c,f) := (I \times R^{n-1}(c), g), \;\; g = -dt^2 + f^2(t) g_c,$$

where $R^{n-1}(c)$ is a Riemannian $(n-1)$-manifold of constant curvature c for $n > 5$.

A *rest space* or a *space-like slice* in $L^n_1(c,f)$ is a space-like hypersurface given by t constant. Thus, a rest space in $L^n_1(c,f)$ is a fiber

$$S(t_0) = \{t_0\} \times_{f(t_0)} R^n(c), \;\; t_0 \in I.$$

Hence a rest space $S(t_0)$ in $L^n_1(c,f)$ is an $(n-1)$-manifold of constant curvature whose metric tensor is given by $f^2(t_0)g_k$.

A pseudo-Riemannian submanifold N of a Robertson–Walker space-time $L^n_1(c,f)$ is called *transverse* if it is contained in a *rest space* $S(t_0)$ for some $t_0 \in I$. A pseudo-Riemannian submanifold N of $L^n_1(c,f)$ is called a *$\mathcal{H}$-submanifold* if the tangent field $\frac{\partial}{\partial t}$, known as the *comoving observer field*, is tangent to N at each point on N.

20.2. Parallel Submanifolds of Robertson–Walker Space-Times

For parallel submanifolds of $L^n_1(c,f)$, we have the next classification result from References [114,115].

Theorem 46. *If a Robertson–Walker space-time $L^n_1(c,f)$ does not contain any open subsets of constant curvature, then a k-dimensional pseudo-Riemannian submanifold of $L^n_1(c,f)$ is a parallel submanifold if and only if it is one of the following:*

(a) A transverse submanifold lying in a rest space $S(t_0)$ of $L^n_1(c,f)$ as a parallel submanifold.
(b) An $\mathcal{H}$-submanifold which is locally a warped product $I \times_f P^{k-1}$, where I is an open interval and P^{k-1} is a submanifold of $R^{n-1}(c)$. Further,

(b.1) if $f' \neq 0$ on I, then $I \times_f P^{k-1}$ is totally geodesic in $L^m_1(k,f)$;
(b.2) if $f' = 0$ on I, then P^{k-1} is a parallel submanifold of $R^{n-1}(c)$.

A similar result holds for submanifolds in a warped product $I \times_f R^{n-1}(c)$ with the Riemannian warped product metric $g = dt^2 + f^2(t)g_c$ (cf. References [116,117]).

21. Thurston's Eight Three-Dimensional Model Geometries

The uniformization theorem for 2-dimensional surfaces says that every simply-connected Riemann surface is conformally equivalent to one of the three Riemann surfaces: the open unit disk, the complex plane or the Riemann sphere. This result implies that every Riemann surface admits a Riemannian metric of constant curvature.

Roughly speaking, for closed 3-manifolds W. Thurston's Geometrization Conjecture states that every closed 3-manifold can be decomposed in a canonical way into pieces that each have one of eight types of geometric structure locally (see Reference [118]). In 2005, G. Perelman [119] provided a proof of Thurston's geometrization conjecture via Ricci flow with surgery.

The eight Thurston's 3-dimensional model geometries are the following.

(1) *Euclidean geometry $\mathbb{E}^3$.*
(2) *Spherical geometry S^3.*
(3) *Hyperbolic geometry H^3.*
(4) *The geometry of $S^2 \times \mathbb{R}$.*

(5) *The geometry of* $H^2 \times \mathbb{R}$.
(6) *The geometry* $\widetilde{SL_2}(\mathbb{R})$. The 3-dimensional Lie group of all 2×2 real matrices with determinant one is denoted by $SL_2(\mathbb{R})$; and $\widetilde{SL_2}(\mathbb{R})$ denotes its universal covering. $\widetilde{SL_2}(\mathbb{R})$ is a unimodular Lie group with a special left invariant metric. Examples of these manifolds in this geometry include the manifold of unit vectors of the tangent bundle of a hyperbolic surface and, more generally, the Brieskorn homology spheres.
(7) *Nil geometry* Nil_3. The group Nil_3 is a 3-dimensional unimodular Lie group with a special left invariant metric consisting of real matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

under multiplication. This group, also known as the *Heisenberg group*, is nilpotent.
(8) *Sol geometry* Sol_3. This group Sol_3 has the least symmetry of all the eight geometries as the identity component of the stabilizer of a point is trivial.

We mentioned earlier in Section 1 that the complete classification of parallel surfaces in $\mathbb{E}^3$ was obtained by V. F. Kagan; the complete classifications of parallel surfaces in S^3 and in H^3 were given in Sections 5.4 and 5.5, respectively; the classifications of parallel surfaces in $S^2 \times \mathbb{R}$ and in $H^2 \times \mathbb{R}^3$ were given in Sections 24.1 and 25.1.

In this section, we will deal the classification of parallel surfaces in $Sol_3, \widetilde{SL_2}(\mathbb{R})$ and Nil_3 in Sections 22.2, 22.4 and 22.5, respectively.

22. Parallel Surfaces in Three-Dimensional Lie Groups

22.1. Milnor's Classification of 3-Dimensional Unimodular Lie Groups

A Lie group G is called *unimodular* if its left-invariant Haar measure is also right-invariant. In Reference [120], J. Milnor provides an infinitesimal reformulation of unimodularity for 3-dimensional Lie groups. We recall it briefly as follows:

Let $\mathfrak{g}$ be a 3-dimensional oriented Lie algebra equipped with an inner product $\langle\ ,\rangle$. Define the vector product operation $\times : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ as the skew-symmetric bilinear map which is uniquely determined by the following three conditions:

(a) $\langle X, X \times Y\rangle = \langle Y, X \times Y\rangle = 0$,
(b) $|X \times Y|^2 = \langle X, X\rangle \langle Y, Y\rangle - \langle X, Y\rangle^2$,
(c) if X and Y are linearly independent, then $\det(X, Y, X \times Y) > 0$,

for all $X, Y \in \mathfrak{g}$. The Lie-bracket $[\,\cdot\,,\,\cdot\,]$ on $\mathfrak{g}$ is a skew-symmetric bilinear map. By comparing these two operations, one obtains a linear endomorphism $L_{\mathfrak{g}}$ which is uniquely determined by the formula

$$[X, Y] = L_{\mathfrak{g}}(X \times Y), \quad X, Y \in \mathfrak{g}.$$

If G is an oriented 3-dimensional Lie group equipped with a left-invariant Riemannian metric, then the metric induces an inner product on the Lie algebra $\mathfrak{g}$. With respect to the orientation on $\mathfrak{g}$ induced from G, the endomorphism field $L_{\mathfrak{g}}$ is uniquely determined.

J. Milnor proved in Reference [120] that the unimodularity of G is characterized as follows.

Theorem 47. *Let G be an oriented 3-dimensional Lie group with a left-invariant Riemannian metric. Then G is unimodular if and only if the endomorphism $L_{\mathfrak{g}}$ is self-adjoint with respect to the metric.*

If G is a 3-dimensional unimodular Lie group with a left-invariant metric, then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{g}$ such that

$$[e_1, e_2] = c_3 e_3, \quad [e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_2 e_2, \quad c_i \in \mathbb{R}.$$

Milnor obtained the following Table 1 classification of 3-dimensional unimodular Lie groups.

Table 1. Three-dimensional unimodular Lie groups classified by J. Milnor.

(c_1, c_2, c_3)	**Simply-Connected Lie Group**	**Property**
$(+,+,+)$	$SU(2)$	Compact and simple
$(+,+,-)$	$\widetilde{SL}(2,\mathbb{R})$	Non-compact and simple
$(+,+,0)$	$\tilde{E}(2)$	Solvable
$(+,-,0)$	$E(1,1)$	Solvable
$(+,0,0)$	Heisenberg group	Nilpotent
$(0,0,0)$	$(\mathbb{E}^3,+)$	Abelian

Here $E(1,1)$ denotes the the group of orientation-preserving rigid motions of Minkowski plane, $E(2)$ denotes the group of orientation-preserving rigid motions of Euclidean plane and $\tilde{E}(2)$ is the universal covering of $E(2)$.

22.2. Parallel Surfaces in the Motion Group $E(1,1)$

Let $E(1,1)$ be the motion group of the Minkowski plane:

$$E(1,1) = \left\{ \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{e}(1,1)$ is given by $\mathfrak{e}(1,1) = \left\{ \begin{pmatrix} w & 0 & u \\ 0 & -w & v \\ 0 & 0 & 0 \end{pmatrix} : u, v, w \in \mathbb{R} \right\}$.

Consider the basis

$$F_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of $\mathfrak{e}(1,1)$. Then the left-translated vector fields of $\{F_1, F_2, F_3\}$ are given by

$$f_1 = e^z \frac{\partial}{\partial x}, \quad f_2 = e^{-z} \frac{\partial}{\partial y}, \quad f_3 = \frac{\partial}{\partial z}.$$

The dual coframe field is $\omega^1 = e^{-z} dx$, $\omega^2 = e^z dy$, $\omega^3 = dz$.

Now we take the following left-invariant vector fields u_1, u_2, u_3:

$$u_1 = \frac{1}{\sqrt{2}}(-f_1 + f_2), \quad u_2 = \frac{1}{\sqrt{2}}(f_1 + f_2), \quad u_3 = f_3.$$

This left-invariant frame field satisfies the relations $[u_1, u_2] = 0$, $[u_2, u_3] = u_1$, $[u_3, u_1] = -u_2$.

We equip $E(1,1)$ with a left-invariant Riemannian metric such that $\{e_1, e_2, e_3\}$, with $e_i = u_i / \lambda_i$, is orthonormal, where $\lambda_1, \lambda_2, \lambda_3$ are positive constants. The resulting Riemannian metric is

$$g_{(\lambda_1,\lambda_2,\lambda_3)} = \frac{\lambda_1^2}{2}(-\omega^1 + \omega^2)^2 + \frac{\lambda_2^2}{2}(\omega^1 + \omega^2)^2 + \lambda_3^2 (\omega^3)^2.$$

V. Patrangenaru proved the following result in Reference [121].

Theorem 48. *A left-invariant metric on $E(1,1)$ is isometric to one of the metrics $g_{(\lambda_1,\lambda_2,\lambda_3)}$ with $\lambda_1 \geq \lambda_2 > 0$ and $\lambda_3 = \frac{1}{\lambda_1\lambda_2}$.*

22.3. Parallel Surfaces in Sol_3

If we put $g(\lambda_1,\lambda_2) = g_{(\lambda_1,\lambda_2,\frac{1}{\lambda_1\lambda_2})}$, then the homogeneous 3-manifold $Sol_3 = (E(1,1), g_{(1,1)})$ is one of Thurston's eight model spaces. Hence, Sol_3 has a natural 2-parametric deformation family $\{(E(1,1), g(\lambda_1,\lambda_2)) \mid \lambda_1 \geq \lambda_2 > 0\}$.

In Reference [122], J. Inoguchi and J. Van der Veken classified parallel surfaces in $Sol_3 = (E(1,1), g(\lambda_1,\lambda_2))$ as follows.

Theorem 49. *Let M be a parallel surface in $Sol_3 = (E(1,1), g(\lambda_1,\lambda_2))$. Then M is one of the following:*

(a) an integral surface of the distribution spanned by $\{\partial/\partial x, \partial/\partial y\}$,
(b) an integral surface of the distribution spanned by $\{\partial/\partial x, \partial/\partial z\}$ or $\{\partial/\partial y, \partial/\partial z\}$,

The latter case only occurring if $\lambda_1 = \lambda_2$. Moreover, the surfaces described in (a) *are flat and minimal but not totally geodesic and the surfaces in* (b) *are totally geodesic and have constant Gaussian curvature $-\lambda_1^4$.*

22.4. Parallel Surfaces in the Motion Group $E(2)$

The Euclidean motion group $E(2)$ is given by the following matrix group:

$$E(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{R},\ \theta \in S^1 \right\}.$$

The universal covering group of $E(2)$ is $\mathbb{R}^3$ with multiplication

$$(x,y,z)\cdot(x',y',z') = (x + x'\cos z - y'\sin z\,, y + x'\sin z + y'\cos z\,, z + z').$$

Take positive constants λ_1, λ_2 and λ_3 and a left-invariant frame

$$e_1 = \frac{1}{\lambda_2}\left(-\sin z\frac{\partial}{\partial x} + \cos z\frac{\partial}{\partial y}\right), \quad e_2 = \frac{1}{\lambda_3}\frac{\partial}{\partial z}, \quad e_3 = \frac{1}{\lambda_1}\left(\cos z\frac{\partial}{\partial x} + \sin z\frac{\partial}{\partial y}\right).$$

Then this frame satisfies the commutation relations: $[e_1,e_2] = c_1e_3$, $[e_2,e_3] = c_2e_1$, $[e_3,e_1] = 0$, with $c_1 = \frac{\lambda_1}{\lambda_2\lambda_3}$ and $c_2 = \frac{\lambda_2}{\lambda_1\lambda_3}$.

The left-invariant Riemannian metric determined by the condition with orthonormal $\{e_1,e_2,e_3\}$ is given by

$$g_{(\lambda_1,\lambda_2,\lambda_3)} = \lambda_1^2(\cos z\,dx + \sin z\,dy)^2 + \lambda_2^2(-\sin z\,dx + \cos z\,dy)^2 + \lambda_3^2\,dz^2.$$

We have the following result on $\widetilde{E(2)}$ from Reference [121].

Proposition 1. *A left-invariant metric on $\widetilde{E(2)}$ is isometric to one of the metrics $g_{(\lambda_1,\lambda_2,\lambda_3)}$ with $\lambda_1 > \lambda_2 > 0$ and $\lambda_3 = \frac{1}{\lambda_1\lambda_2}$ or $\lambda_1 = \lambda_2 = \lambda_3 = 1$. In particular, $\widetilde{E(2)}$ with metric $g_{(1,1,1)}$ is isometric to Euclidean 3-space $\mathbb{E}^3$.*

J. Inoguchi and J. Van der Veken classified parallel surfaces in $\widetilde{E(2)}$ (see Reference [122]) as follows.

Theorem 50. *The only parallel surfaces in $\widetilde{E(2)}$ are integral surfaces of the distribution spanned by $\{\partial/\partial x, \partial/\partial y\}$. These surfaces are flat and minimal but not totally geodesic.*

22.5. Parallel Surfaces in $SU(2)$

The group $SU(2)$ is diffeomorphic to S^3, since

$$SU(2) = \left\{ \begin{pmatrix} x_0 + \sqrt{-1}x_3 & -x_2 + \sqrt{-1}x_1 \\ x_2 + \sqrt{-1}x_1 & x_0 - \sqrt{-1}x_3 \end{pmatrix} : x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \right\}.$$

From Reference [121], we have the following proposition which describes all possible left-invariant metrics on $SU(2)$.

Proposition 2. *Any left-invariant metric on $SU(2)$ is isometric to one of the following metrics $g_{(\lambda_1,\lambda_2,\lambda_3)}$ with $\lambda_i \in \mathbb{R}$ and $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$:*

$$g_{(\lambda_1,\lambda_2,\lambda_3)} = \frac{4}{\lambda_2\lambda_3}\sigma_1^2 + \frac{4}{\lambda_3\lambda_1}\sigma_2^2 + \frac{4}{\lambda_1\lambda_2}\sigma_3^2$$

on the unit three-sphere $S^3(1) = \{(x_0, x_1, x_2, x_3) \in \mathbb{E}^4 : x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$, where

$$\begin{aligned} \sigma_1 &= -x_1 dx_0 + x_0 dx_1 - x_3 dx_2 + x_2 dx_3, \\ \sigma_2 &= -x_2 dx_0 + x_3 dx_1 - x_0 dx_2 + x_1 dx_3, \\ \sigma_3 &= -x_3 dx_0 + x_2 dx_1 - x_1 dx_2 + x_0 dx_3. \end{aligned}$$

The dimension $d(\lambda_1,\lambda_2,\lambda_3)$ of the isometry group of $(SU(2), g_{(\lambda_1,\lambda_2,\lambda_3)})$ is

$$d(\lambda_1,\lambda_2,\lambda_3) = \begin{cases} 3 & \text{if} \quad \lambda_1 > \lambda_2 > \lambda_3, \\ 4 & \text{if} \quad \lambda_1 = \lambda_2 > \lambda_3 \text{ or } \lambda_1 > \lambda_2 = \lambda_3, \\ 6 & \text{if} \quad \lambda_1 = \lambda_2 = \lambda_3. \end{cases}$$

Let $\mathfrak{su}(2)$ denote the Lie algebra of $SU(2)$. Take the following quaternionic basis $\{i, j, k\}$ of $\mathfrak{su}(2)$:

$$i = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \; j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; k = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

We denote the left-translated vector fields of i, j, k by E_1, E_2, E_3. Then the commutation relations of$\{E_1, E_2, E_3\}$ are given by $[E_1, E_2] = 2E_3$, $[E_2, E_3] = 2E_1$, $[E_3, E_1] = 2E_2$. Choose strictly positive real constants λ_1, λ_2, λ_3 and define

$$e_1 = \frac{1}{\lambda_2\lambda_3} i, \; e_2 = \frac{1}{\lambda_3\lambda_1} j, \; e_3 = \frac{1}{\lambda_1\lambda_2} k.$$

Then $[e_1, e_2] = c_3 e_3$, $[e_2, e_3] = c_1 e_1$, $[e_3, e_1] = c_2 e_2$ with $c_1 = 2/\lambda_1^2, c_2 = 2/\lambda_2^2, c_3 = 2/\lambda_3^2$. The left-invariant metric $\overline{g}_{(c_1,c_2,c_3)}$, defined by the condition that $\{e_1, e_2, e_3\}$ is an orthonormal basis, is

$$\overline{g}_{(c_1,c_2,c_3)} = 4\left(\frac{1}{c_2c_3}\omega_1^2 + \frac{1}{c_1c_3}\omega_2^2 + \frac{1}{c_1c_2}\omega_3^2\right),$$

where $\{\omega_1, \omega_2, \omega_3\}$ is the dual coframe field of $\{E_1, E_2, E_3\}$.

The following result from Reference [121] describes all left-invariant metrics on $SU(2)$

Proposition 3. *A left-invariant metric on $SU(2)$ is isometric to one of the metrics $\overline{g}_{(c_1,c_2,c_3)}$ with $c_1, c_2, c_3 \geq 0$. Moreover, the dimension of the isometry group is greater or equal to 4 if and only if at least two of the parameters c_i coincide.*

The next non-existence result was proved by J. Inoguchi and J. Van der Veken in Reference [123].

Theorem 51. *There are no parallel surfaces in $SU(2)$ equipped with a left-invariant metric with 3-dimensional isometry group.*

22.6. Parallel Surfaces in the Real Special Linear Group $Sl(2,\mathbb{R})$

The group $SL(2,\mathbb{R})$ is defined as the following subgroup of $GL(2,\mathbb{R})$:

$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}.$$

This group is isomorphic to the following subgroup of $GL(2,\mathbb{C})$:

$$SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha| - |\beta|^2 = 1 \right\}$$

via the isomorphism $SL(2,\mathbb{R}) \to SU(1,1) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}$.

The Lie algebra of $SU(1,1)$ is explicitly given by

$$\mathfrak{su}(1,1) = \left\{ \begin{pmatrix} iu & v - iw \\ v + iw & -iu \end{pmatrix} : u, v, w \in \mathbb{R} \right\}.$$

We take the following split-quaternionic basis of the Lie algebra su(1, 1):

$$\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j}' = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbf{k}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Denote the left-translated vector fields of $\{\mathbf{j}', \mathbf{k}', \mathbf{i}\}$ by $\{E_1, E_2, E_3\}$ and choose strictly positive real constants $\lambda_1, \lambda_2, \lambda_3$ and define

$$e_1 = \frac{1}{\lambda_2\lambda_3} E_1, \quad e_2 = \frac{1}{\lambda_1\lambda_3} E_2, \quad e_3 = \frac{1}{\lambda_1\lambda_2} E_3.$$

Then we have $[e_1, e_2] = c_3 e_3$, $[e_2, e_2] = c_1 e_1$, $[e_3, e_1] = c_2 3 e_2$ with $c_1 = 2/\lambda_1^2, c_2 = 2/\lambda_2^2$ and $c_2 = -2/\lambda_3^2$.

The left-invariant Riemannian metric $g(c_1, c_2, c_3)$ by the condition that $\{e_1, e_2, e_3\}$ is an orthonormal basis is

$$g(c_1, c_2, c_3) = 4\left(-\frac{1}{c_2 c_3}\omega_1^2 - \frac{1}{c_1 c_3}\omega_2^2 + \frac{1}{c_1 c_2}\omega_3^2 \right),$$

where $\{\omega_1, \omega_2, \omega_3\}$ is the dual coframe field of $\{E_1.E_2, E_3\}$. This three-parameter family of Riemannian metrics exhausts all left-invariant metrics on $SL(2,\mathbb{R})$ as shown in the next proposition from Reference [121].

Proposition 4. *Any left-invariant metric on $SU(1,1)$ is isometric to one of the metrics $g(c_1, c_2, c_3)$ with $c_3 < 0 < c_2 \leq c_1$. Moreover, this metric gives rise to an isometry group of dimension 4 if and only if $c_1 = c_2$.*

Consider $SL(2,\mathbb{R})$ equipped with a left-invariant metric such that the dimension of the isometry group is only 3. With the notations given above, we have that $c_1 > c_2 > 0 > c_3$. The following result was proved by J. Inoguchi and J. Van der Veken in Reference [123].

Theorem 52. *Consider $SL(2,\mathbb{R})$ equipped with a left-invariant metric with $c_1 > c_2 > 0 > c_3$. Parallel surfaces only occur if $c_2 = c_1 + c_3$. Moreover, they are integral surfaces of the distribution spanned by $\{\cos\theta\, e_1 + \sin\theta\, e_3, e_2\}$, where θ is a constant, satisfying $\tan^2\theta = -c_3/c_1$. These surfaces are totally geodesic and of constant Gaussian curvature given by $c_1c_3 < 0$.*

M. Belkhelfa, F. Dillen and J. Inoguchi classified parallel surfaces in $SL(2,\mathbb{R})$ with 4-dimensional isometry group in Reference [124] as follows.

Theorem 53. *The only parallel surfaces in the real special linear group $SL(2,\mathbb{R})$ are rotational surfaces of constant mean curvature. The generating curve is a Riemannian circle. Furthermore such surfaces are flat.*

22.7. Parallel Surfaces in Non-Unimodular Three-Dimensional Lie Groups

Let G be a non-unimodular 3-dimensional Lie group with a left-invariant metric. Then the *unimodular kernel* $\mathfrak{u}$ of the Lie algebra $\mathfrak{g}$ of G is defined by $\mathfrak{u} = \{X \in \mathfrak{g} : \operatorname{Tr}\operatorname{ad}(X) = 0\}$, where $\operatorname{ad} : \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ is a homomorphism defined by $\operatorname{ad}(X)Y = [X,Y]$. Then $\mathfrak{u}$ is an ideal of $\mathfrak{g}$ containing the ideal $[\mathfrak{g},\mathfrak{g}]$.

On $\mathfrak{g}$, we can take an orthonormal basis $\{e_1,e_2,e_3\}$ such that (a) $\langle e_1, X\rangle = 0$, $X \in \mathfrak{u}$ and (b) $\langle [e_1,e_2],[e_1,e_3]\rangle = 0$. The commutation relations of this basis are given by

$$[e_1,e_2] = ae_2 + be_3,\ [e_2,e_3] = 0,\ [e_1,e_3] = ce_2 + de_3,$$

with $a + d \neq 0$ and $ac + bd = 0$. Under a suitable homothetic change of the metric, we may assume that $a + d = 2$. Then the constants a, b, c and d are represented as

$$a = 1+\xi,\ b = (1+\xi)\eta,\ c = -(1-\xi)\eta,\ d = 1-\xi,$$

where (ξ,η) satisfies the condition $\xi,\eta \geq 0$ and $\xi^2 + \eta^2 \neq 0$.

The next was also proved by J. Inoguchi and J. Van der Veken in Reference [123].

Proposition 5. *The non-unimodular Lie group G is locally symmetric if and only if $\xi = 0$ or $(\xi,\eta) = (1,0)$.*

Since parallel surfaces in $H^2 \times \mathbb{R}$ are not classified yet up to this stage (see Sections 24.1 and 25.1), we shall restrict our attention to such surfaces in the non-unimodular Lie groups satisfying $\xi \notin \{0,1\}$.

The following theorem of J. Inoguchi and J. Van der Veken from Reference [123] provides the classification of parallel surfaces in the corresponding Lie groups.

Theorem 54. *Let G be a non-unimodular Lie group with structure constants (ξ,η). Assume that $\xi \notin \{0,1\}$. Then the only parallel surfaces in G are:*

(1) Integral surfaces of the distributions spanned by $\{e_1,e_2\}$, respectively $\{e_1,e_3\}$. These surfaces are totally geodesic and of constant negative curvature $-(1+\xi)^2$, respectively $-(1-\xi)^2$.
(2) Integral surfaces of the distribution spanned by $\{e_2,e_3\}$. These surfaces are flat and of constant mean curvature 1.

The former case only occurs when $\eta = 0$.

22.8. Parallel Surfaces in the Heisenberg Group Nil_3

The following classification theorem of parallel surfaces in the Heisenberg group Nil_3 was proved by M. Belkhelfa, F. Dillen and J. Inoguchi in Reference [125].

Theorem 55. *The only parallel surfaces in the Heisenberg group Nil_3 are open parts of vertical planes and vertical round cylinders.*

Remark 9. *The oscillator group was introduced and first studied by R. F. Streater in Reference [126] and owes its name to the fact that its Lie algebra coincides with the one generated by the differential operators associated to the harmonic oscillator problem. Generalizing this construction, oscillator groups have been defined in any even dimension greater or equal to four. Since their introduction, the oscillator groups have been intensively studied from several different points of view, both in differential geometry and in mathematical physics. Beside direct extensions with Euclidean groups, the oscillator groups are the only simply connected non-Abelian solvable Lie groups admitting a bi-invariant Lorentzian metric.*

In Reference [127] G. Calvaruso and J. Van der Veken obtained the complete classification and explicitly description of totally geodesic and parallel hypersurfaces of four-dimensional oscillator groups equipped with a one-parameter family of left-invariant Lorentzian metrics.

23. Parallel Surfaces in Three-Dimensional Lorentzian Lie Groups

Homogeneous Lorentzian 3-spaces (N, g) were classified by G. Calvaruso in Reference [128]. Unless they are symmetric, they are Lie groups equipped with left-invariant Lorentzian metrics.

23.1. Three-Dimensional Lorentzian Lie Groups

G. Calvaruso in Reference [128] classified 3-dimensional simply connected, complete homogeneous Lorentzian manifold as the following theorem.

Theorem 56. *Let (N, g) be a 3-dimensional connected, simply connected, complete homogeneous Lorentzian manifold. If (N, g) is not symmetric, then $N = G$ is a 3-dimensional Lie group and g is left-invariant. Moreover, there exists a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$, with e_3 time-like, such that the Lie algebra of G is one of the following seven types.*

(1) *Type $\mathfrak{g}_1$:*

$$[e_1, e_2] = \alpha e_1 - \beta e_3, \ \ [e_1, e_3] = -\alpha e_1 - \beta e_2, \ \ [e_2, e_3] = \beta e_1 + \alpha e_2 + \alpha e_3, \ \ \alpha \neq 0.$$

In this case, $G = O(1, 2)$ or $G = SL(2, \mathbb{R})$ if $\beta \neq 0$, while $G = E(1, 1)$ if $\beta = 0$.

(2) *Type $\mathfrak{g}_2$:*

$$[e_1, e_2] = \gamma e_2 - \beta e_3, \ \ [e_1, e_3] = -\beta e_2 + \gamma e_3, \ \ [e_2, e_3] = \alpha e_1, \ \ \gamma \neq 0.$$

In this case, $G = O(1, 2)$ or $G = SL(2, \mathbb{R})$ if $\alpha \neq 0$, while $G = E(1, 1)$ if $\alpha = 0$.

(3) *Type $\mathfrak{g}_3$:*

$$[e_1, e_2] = -\gamma e_3, \ \ [e_1, e_3] = -\beta e_2, \ \ [e_2, e_3] = \alpha e_1.$$

The following Table 2 lists all the Lie groups G which admit a Lie algebra $\mathfrak{g}_3$, taking into account the different possibilities for α, β and γ:

Table 2. Lie groups G with Lie algebra of type $\mathfrak{g}_3$.

G	α	β	γ
$O(1,2)$ or $SL(2,\mathbb{R})$	+	+	+
$O(1,2)$ or $SL(2,\mathbb{R})$	+	−	−
$SO(3)$ or $SU(2)$	+	+	−
$E(2)$	+	+	0
$E(2)$	+	0	−
$E(1,1)$	+	−	0
$E(1,1)$	+	0	+
H_3	+	0	0
H_3	0	0	−
$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	0	0	0

(4) *Type* $\mathfrak{g}_4$:

$$[e_1, e_2] = -e_2 + (2\eta - \beta)e_3, \quad [e_1, e_3] = -\beta e_2 + e_3, \quad [e_2, e_3] = \alpha e_1, \quad \eta = \pm 1.$$

The following Table 3 describes all Lie groups G admitting a Lie algebra $\mathfrak{g}_4$*:*

Table 3. Lie groups G with Lie algebra of type $\mathfrak{g}_4$.

G	α	β
$O(1,2)$ or $SL(2,\mathbb{R})$	$\neq 0$	$\neq \eta$
$E(1,1)$	0	$\neq \eta$
$E(1,1)$	< 0	η
$E(2)$	> 0	η
H_3	0	η

(5) *Type* $\mathfrak{g}_5$:

$$[e_1, e_2] = 0, \quad [e_1, e_3] = \alpha e_1 + \beta e_2, \quad [e_2, e_3] = \gamma e_1 + \delta e_2, \quad \alpha + \delta \neq 0, \ \alpha\gamma + \beta\delta = 0.$$

(6) *Type* $\mathfrak{g}_6$:

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3, \ [e_2, e_3] = 0, \quad \alpha + \delta \neq 0, \ \alpha\gamma - \beta\delta = 0.$$

(7) *Type* $\mathfrak{g}_7$:

$$[e_1, e_2] = -\alpha e_1 - \beta e_2 - \beta e_3, \quad [e_1, e_3] = \alpha e_1 + \beta e_2 + \beta e_3, \quad [e_2, e_3] = \gamma e_1 + \delta e_2 + \delta e_3,$$

with $\alpha + \delta \neq 0$, $\alpha\gamma = 0$.

Lie algebras of types $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_3$ *and* $\mathfrak{g}_4$ *correspond to unimodular groups, whereas Lie algebras of types* $\mathfrak{g}_5$, $\mathfrak{g}_6$ *and* $\mathfrak{g}_7$ *correspond to non-unimodular groups.*

G. Calvaruso determined in Reference [129] those 3-dimensional Lorentzian Lie groups (G, g) which have constant sectional curvature and which are symmetric.

By a *3-dimensional Lorentzian Lie group* G_i we mean a connected, simply connected 3-dimensional Lie group G equipped with a left-invariant Lorentzian metric g and having Lie algebra $\mathfrak{g}_i$.

23.2. Classification of Parallel Surfaces in Three-Dimensional Lorentzian Lie Groups

Let (N, g) be a 3-dimensional homogeneous Lorentzian manifold and M is a surface in N. We denote by ξ a fixed normal vector field on the surface with $\langle \xi, \xi \rangle = \varepsilon$. Here, either $\varepsilon = -1$ or $\varepsilon = 1$, according to the surface being either Riemannian or Lorentzian, respectively. We call ξ an *ε-unit normal vector field.*

Parallel surfaces in 3-dimensional Lorentzian Lie groups were classified by G. Calvaruso and J. Van der Veken in Reference [130]. More precisely, under the notations of Theorem 56, they proved the following.

Theorem 57. *Let M be a parallel surface in a 3-dimensional Lorentzian Lie group* G_1*. Then* $\beta = 0$, $\xi = e_1 + be_2 + be_3$ *and the vector fields* $E_1 = (be_1 - e_2)/\sqrt{1+b^2}$ *and* $E_2 = (be_1 + b^2 e_2 + (1+b^2)e_3)/\sqrt{1+b^2}$ *form a pseudo-orthonormal basis for the tangent plane at every point. Moreover, the function b satisfies* $E_1(b) = E_2(b)$ *and*

$$E_1\left(\frac{E_1 b}{\sqrt{1+b^2}} - \frac{2b}{1+b^2}\alpha\right) + 2\left(\frac{E_1 b}{\sqrt{1+b^2}} - \frac{2b}{1+b^2}\alpha\right)\left(\frac{b}{\sqrt{1+b^2}}E_1 b - \frac{\alpha}{\sqrt{1+b^2}}\right) = 0.$$

The surface is flat and parallel. Moreover, it is totally geodesic in the case that $E_1 b = E_2 b = 2b\alpha/\sqrt{1+b^2}$.

Theorem 58. *Let M be a parallel surface in a three-dimensional Lorentzian Lie group G_2. Then one of the following statements holds.*

(a) *M is an integral surface of the distribution spanned by $\{e_2, e_3\}$. This case only occurs if $\alpha = 0$ and M is parallel, flat and minimal but not totally geodesic.*
(b) *M is an integral surface of the distribution spanned by $\{e_1, ce_2 + be_3\}$, where b and c are real constants satisfying $b^2 - c^2 = \varepsilon = \pm 1$, $bc = -\varepsilon\beta/(2\gamma)$. This case only occurs if $\alpha = 2\beta$ and M is totally geodesic.*

Theorem 59. *Let M be a parallel surface in a non-symmetric three-dimensional Lorentzian Lie group G_3. Then one of the following statements holds.*

(a) *M is an integral surface of the distribution spanned by $\{e_2, e_3\}$. This case only occurs if $\gamma = 0$ and M is flat and minimal but not totally geodesic.*
(b) *M is an integral surface of the distribution spanned by $\{e_2, e_3\}$. This case only occurs if $\alpha = 0$ and M is flat and minimal but not totally geodesic.*
(c) *M is an integral surface of the distribution spanned by $\{e_1, e_3\}$. This case only occurs if $\beta = 0$ and M is flat and minimal but not totally geodesic.*
(d) *M is an integral surface of the distribution spanned by $\{E_1 = e_1, E_2 = ce_2 + be_3\}$, where b and c are functions on M satisfying $b^2 - c^2 = \varepsilon$ and $E_1 b = \beta c$, $E_1 c = \beta b$, $E_2 b = k_1 \varepsilon c$, $E_2 c = k_1 \varepsilon b$, for some real constant k_1. This case only occurs if $\beta = \gamma$ and M is flat.*
(e) *M is an integral surface of the distribution spanned by $\{ce_2 + be_3, e_1\}$. Here, b and c are real constants satisfying $b^2 = \gamma\varepsilon/(\gamma - \beta)$, $c^2 = \beta\varepsilon/(\gamma - \beta)$. This case only occurs if $\alpha = \beta + \gamma$ and $\beta \neq \gamma$ and M is totally geodesic.*
(f) *M is an integral surface of the distribution spanned by $\{E_1 = ce_1 + ae_3, E_2 = e_2\}$, where a and c are functions on the surface satisfying $a^2 - c^2 = \varepsilon$ and $E_1 a = k_2 \varepsilon c$, $E_1 c = k_2 \varepsilon a$, $E_2 a = -\alpha c$, $E_2 c = -\alpha a$, for some real constant k_2. This case only occurs if $\alpha = \gamma$ and M is flat.*
(g) *M is an integral surface of the distribution spanned by $\{ce_1 + ae_3, e_2\}$. Here, a and c are real constants satisfying $a^2 = -\gamma\varepsilon/(\alpha - \gamma)$, $c^2 = -\alpha\varepsilon/(\alpha - \gamma)$. This case only occurs if $\beta = \alpha + \gamma$ and $\alpha \neq \gamma$ and M is totally geodesic.*
(h) *M is an integral surface of the distribution spanned by $\{E_1 = be_1 - ae_2, E_2 = e_3\}$, where a and b are functions satisfying $a^2 + b^2 = 1$ and*

$$E_1 a = \frac{k_3 b}{b^2 - a^2}, \quad E_1 b = -\frac{k_3 a}{b^2 - a^2}, \quad E_2 a = \frac{b\alpha}{b^2 - a^2}, \quad E_2 b = -\frac{a\alpha}{b^2 - a^2},$$

for some real constant k_3. This case only occurs if $\alpha = \beta$ and M is flat.
(i) *M is an integral surface of the distribution spanned by $\{be_1 - ae_2, e_3\}$, where a and b are constants satisfying $a^2 = -\beta/(\alpha - \beta)$, $b^2 = \alpha/(\alpha - \beta)$. This case only occurs if $\gamma = \alpha + \beta$ and $\alpha \neq \beta$ and M is totally geodesic.*

Theorem 60. *Let M be a parallel surface in a non-symmetric three-dimensional Lorentzian Lie group G_4. Then one of the following statements holds.*

(a) *M is an integral surface of the distribution spanned by $\{e_2, e_3\}$. This case only occurs if $\alpha = 0$. M is parallel, flat and minimal but not totally geodesic.*
(b) *M is an integral surface of the distribution spanned by $\{e_1, ce_2 + be_3\}$, where b and c are constants satisfying $b^2 - c^2 = \varepsilon$ and $\beta b^2 + 2bc + (\beta - 2\eta)c^2 = 0$. M is totally geodesic and has constant Gaussian curvature $G = -\varepsilon(\beta - \eta)$.*

Theorem 61. *Let M be a parallel surface in a non-symmetric three-dimensional Lorentzian Lie group G_5. Then M is one of the surfaces listed below.*

(a) *M is an integral surface of the distribution spanned by e_1 and e_2. M is flat but not totally geodesic.*
(b) *M is an integral surface of the distribution spanned by e_2 and e_3. This case only occurs if either $\beta = \gamma = 0$ or $\gamma = \delta = 0$. In the first case, M is totally geodesic and has constant Gaussian curvature $K = -\delta^2 \leq 0$. In the second case, M is flat and minimal but not necessarily totally geodesic.*
(c) *M is an integral surface of the distribution spanned by e_1 and e_3. This case only occurs if either $\alpha = \beta = 0$ or $\beta = \gamma = 0$. In the first case, M is flat and minimal but not necessarily totally geodesic. In the second case, M is totally geodesic and has constant Gaussian curvature $K = \alpha^2 \geq 0$.*
(d) *M is an integral surface of the distribution spanned by $\{E_1 = e_1, E_2 = ce_2 + be_3\}$, where b and c are functions satisfying $b^2 - c^2 = \varepsilon$ and $E_1 b = E_1 c = 0$, $E_2 b = c(k_1 - c\delta)$, $E_2 c = b(k_1 - c\delta)$, for some real constant k_1. This case only occurs if $\alpha = \beta = 0$ and M is flat.*
(e) *M is an integral surface of the distribution spanned by $\{E_1 = ce_1 + ae_3, E_2 = e_2\}$, where a and c are functions satisfying $a^2 - c^2 = \varepsilon$ and $E_1 a = -\varepsilon c(a^2 c\alpha - k_2)$, $E_1 c = -\varepsilon a(a^2 c\alpha - k_2)$, $E_2 a = E_2 c = 0$, for some real constant k_2. This case only occurs if $\gamma = \delta = 0$ and M is flat.*

Theorem 62. *Let M be a parallel surface in a three-dimensional Lorentzian Lie group G_6. Then, one of the following statements holds.*

(a) *M is an integral surface of the distribution spanned by e_1 and e_2. This case only occurs if either $\alpha = \beta = 0$ or $\beta = \gamma = 0$. In the first case, M is parallel, flat and minimal but not necessarily totally geodesic. In the second case, M is totally geodesic.*
(b) *M is an integral surface of the distribution spanned by e_2 and e_3. M is parallel and flat but not necessarily totally geodesic.*
(c) *M is an integral surface of the distribution spanned by e_1 and e_3. This case only occurs if either $\beta = \gamma = 0$ or $\gamma = \delta = 0$. In the first case, M is totally geodesic. In the second case, M is parallel, flat and minimal but not necessarily totally geodesic.*
(d) *M is an integral surface of the distribution spanned by $\{E_1 = ce_1 + ae_3, E_2 = e_2\}$, where a and c are functions satisfying $a^2 - c^2 = \varepsilon$ and $E_1 a = c(k_1 - \delta a)$, $E_1 c = a(k_1 - \delta a)$, $E_2 a = E_2 c = 0$ for some real constant k_1. This case only occurs if $\alpha = \beta = 0$ and M is parallel and flat.*
(e) *M is an integral surface of the distribution spanned by $\{E_1 = be_1 - ae_2, E_2 = e_3\}$, where a and b are functions satisfying $a^2 + b^2 = 1$ and $E_1 a = b(k_2 + \alpha a)$, $E_1 b = -a(k_2 + \alpha b)$, $E_2 a = E_2 c = 0$ for some real constant k_2. This case only occurs if $\gamma = \delta = 0$ and M is parallel and flat.*

Theorem 63. *Let M be a parallel surface in a non-symmetric three-dimensional Lorentzian Lie group G_7. Then M is one of surfaces listed below.*

(a) *M is an integral surface of the distribution spanned by $\{e_2, e_3\}$. This case only occurs if either $\beta = \gamma = 0$ or $\gamma = \delta = 0$. In the first case, M is totally geodesic. In the second case, M is parallel and flat but not necessarily totally geodesic.*
(b) *M is an integral surface of the distribution spanned by $\{E_1 = e_1, E_2 = ce_2 + be_3\}$, where b and c are functions satisfying $b^2 - c^2 = \varepsilon$ and $E_1 b = E_1 c = 0$, $E_2 b = c((b-c)\delta - k_1)$, $E_2 c = b((b-c)\delta - k_1)$ for some real constant k_1. This case only occurs if $\alpha = \beta = 0$. M is flat but not necessarily totally geodesic.*
(c) *M is an integral surface of the distribution spanned by $E_1 = (be_1 - e_2)/\sqrt{1+b^2}$ and $E_2 = (be_1 + b^2 e_2 + (1+b^2)e_3)/\sqrt{1+b^2}$, where b is a function satisfying $E_1(b) = E_2(b)$ and*

$$E_1\left(\frac{E_1 b}{\sqrt{1+b^2}} + \frac{b(\alpha-\delta)}{1+b^2}\right) + 2\left(\frac{E_1 b}{\sqrt{1+b^2}} + \frac{b(\alpha-\delta)}{1+b^2}\right)\left(\frac{bE_1 b}{\sqrt{1+b^2}} - \frac{\delta}{\sqrt{1+b^2}}\right) = 0.$$

The surface is flat and parallel. Moreover, it is totally geodesic in the special case that $E_1 b = E_2 b = b(\delta - \alpha)/\sqrt{1+b^2}$.

24. Parallel Surfaces in Reducible Three-Spaces

24.1. Classification of Parallel Surfaces in Reducible Three-Spaces

Parallel submanifolds of the a Robertson–Walker space-time $I \times_f R^n(c)$ have been treated in Section 20. In Reference [131], G. Calvaruso and J. Van der Veken studied parallel surfaces in 3-dimensional reducible spaces $\mathbb{M}^2 \times \mathbb{E}^1$. More precisely, they proved the following results.

Theorem 64. *Let M be a parallel surface in a reducible 3-dimensional Riemannian manifold* $\mathbb{M}^2 \times \mathbb{E}^1$*. Then one of the following three cases holds:*

(1) M is isometric to an open portion of a surface of type $\mathbb{M}^2 \times \{t_0\}$ *for some* $t_0 \in \mathbf{R}$*;*
(2) M is isometric to an open portion of a surface of type $\gamma \times \mathbb{E}^1$*, where* γ *is a curve of constant geodesic curvature in M;*
(3) $\mathbb{M}^2 \times \mathbb{E}^1$ *is flat and M is isometric to an open portion of a standard sphere* $S^2 \subset \mathbb{E}^3$*.*

The following is a consequence of Theorem 64.

Corollary 8. *The pair* $(S^2, \mathbb{E}^3)$ *is the only proper parallel surface in a reducible Riemannian 3-space.*

For parallel surfaces in a reducible 3-dimensional Lorentzian manifold, G. Calvaruso and J. Van der Veken obtained the following.

Theorem 65. *Let M be a parallel surface in a reducible 3-dimensional Lorentzian manifold* $\mathbb{M}^2_1 \times \mathbb{E}^1$ *(respectively* $\mathbb{M}^2 \times \mathbb{E}^1_1$*). Then one of the following holds.*

(1) M is isometric to an open portion of a surface of type $\mathbb{M}^2_1 \times \{t_0\}$ *(respectively* $\mathbb{M}^2 \times \{t_0\}$*) for some real number* t_0*.*
(2) M is isometric to an open portion of a surface of type $\gamma \times \mathbb{E}^1$ *(respectively* $\gamma \times \mathbb{E}^1_1$*) where* γ *is a non-degenerate curve of constant geodesic curvature in* $\mathbb{M}^2_1$ *(respectively* $\mathbb{M}^2$*).*
(3) The ambient space is flat and M is isometric to an open portion of one of the following surfaces: (a) a hyperbolic plane H^2*; (b) an indefinite sphere* S^2_1*; (c) the null scroll* N^2_1*.*

As a consequence of Theorem 65, G. Calvaruso and J. Van der Veken obtained the following.

Corollary 9. *The pairs* $(H^2, \mathbb{E}^3_1)$*,* $(S^2_1, \mathbb{E}^3_1)$ *and* $(N^2_1, \mathbb{E}^3_1)$ *are the only proper parallel surfaces in a reducible Lorentzian 3-space.*

24.2. Parallel Surfaces in Walker Three-Manifolds

A particularly interesting class of pseudo-Riemannian manifolds are ones which admit a parallel null vector field. The study of such metrics in the 3-dimensional Lorentzian setting was initiated by M. Chaichi, E. García-Río and M. E. Vázquez-Abal in Reference [132]. W. Batat and S. J. Hall named such manifolds as *Walker manifolds* in Reference [133].

Complete classification of parallel surfaces of an arbitrary reducible 3-manifold, both in Riemannian and Lorentzian was obtained by G. Calvaruso and J. Van der Veken in Reference [131]. It turns out that the Euclidean space $\mathbb{E}^3$ and the Minkowski space $\mathbb{E}^3_1$ are the only cases admitting parallel surfaces which are non-trivial, in the sense that they do not reflect the reducibility of the space itself. Since the reducibility of a pseudo-Riemannian manifold corresponds to the existence of a parallel non-null vector field, it is natural to study parallel surfaces in a Lorentzian 3-manifold which admits a parallel null vector field, that is, in a Walker 3-manifold. G. Calvaruso and J. Van der Veken provided in Reference [134] a complete classification of parallel surfaces in Walker 3-manifolds.

In Reference [133], W. Batat and S. J. Hall proved that totally umbilical non-degenerate surfaces in a Walker 3-manifold with metric $g = \epsilon dx^2 + f(x,y)dy^2 + 2dtdy$ where $\epsilon = \pm 1$ and satisfying

$f_{xx} \neq 0$ are either one of a totally geodesic family described by G. Calvaruso and J. Van der Veken in Reference [134] or the ambient manifold must be locally conformally flat (here the surface can also be totally geodesic).

25. Bianchi–Cartan–Vranceanu Spaces

25.1. Basics on Bianchi–Cartan–Vranceanu Spaces

The simply-connected homogeneous 3-manifolds are classified according to the dimension of their isometry group which is equal to 3, 4 or 6. If it is 6, one obtains the real space forms. The Bianchi–Cartan–Vranceanu spaces are homogeneous Riemannian 3-manifolds with isometry group of dimension 4 or 6. Such spaces, denoted by $\widetilde{\mathcal{M}}^3(\lambda, \mu)$, are given by a two-parameter family of Riemannian 3-manifolds $(\mathcal{M}, g_{\lambda,\mu})$ where the underlying 3-manifolds $\widetilde{\mathcal{M}}^3$ are $\mathbb{R}^3$ if $\mu \geq 0$; and

$$\widetilde{\mathcal{M}}^3 = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < -\frac{1}{\mu} \right\} \quad \text{if} \quad \mu < 0.$$

The metrics $\tilde{g}_{\lambda,\mu}$ on $\widetilde{\mathcal{M}}^3$ are given by

$$g_{\lambda,\mu} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{\{1 + \mu(x^2 + y^2)\}^2} + \left(\mathrm{d}z + \frac{\lambda(y\mathrm{d}x - x\mathrm{d}y)}{2\{1 + \mu(x^2 + y^2)\}} \right)^2. \tag{4}$$

The 2-parameter family $\tilde{g}_{\lambda,\mu}$ is called the *Bianchi–Cartan–Vranceanu metrics*. The metrics above are defined over the whole 3-space $\mathbb{R}^3$ for $\mu > 0$ and over the region $x^2 + y^2 < -1/\mu$ for $\mu < 0$.

Consider the following Riemannian surface with constant Gaussian curvature 4μ:

$$\widetilde{\mathcal{M}}^2(\mu) = \left(\left\{ (x, y) \in \mathbb{R}^2 : 1 + \mu(x^2 + y^2) > 0 \right\} , \frac{dx^2 + dy^2}{(1 + \mu(x^2 + y^2))^2} \right).$$

Then the mapping

$$\pi : \widetilde{\mathcal{M}}^3(\lambda, \mu) \to \widetilde{\mathcal{M}}^2(\mu) : (x, y, z) \mapsto (x, y)$$

is a Riemannian submersion, referred to as the *Hopf-fibration*. For $\mu = 4\lambda^2 \neq 0$, this mapping coincides with the “classical” Hopf-fibration $\pi : S^3(\mu) \to S^2(4\mu)$.

In the following, by a *Hopf-cylinder* we mean the inverse image of a curve in $\widetilde{\mathcal{M}}^2(\mu)$ under π. By a *leaf* of the Hopf-fibration, we mean a surface which is everywhere orthogonal to the fibres.

The family of Bianchi–Cartan–Vranceanu spaces $\widetilde{\mathcal{M}}^3(\lambda, \mu)$ includes six of the eight Thurston’s 3-dimensional geometries except Sol_3 and the hyperbolic space H^3. The family of the Riemannian metrics given by (4) includes all 3-dimensional homogeneous metrics whose group of isometries has dimension 4 or 6, except for those with negative constant curvature.

For two given real numbers λ, μ, the Bianchi–Cartan–Vranceanu space $\widetilde{\mathcal{M}}^3(\lambda, \mu)$ is the following 3-spaces (cf., e,g., References [135–137]).

(1) If $\lambda = \mu = 0$, it is the Euclidean 3-space.
(2) If $\lambda = 0$, $\mu \neq 0$, it is the product of real line and a surface of constant curvature 4λ.
(3) If $\lambda \neq 0$, $\lambda^2 = 4\mu$, it is a space of positive constant curvature.
(4) If $\lambda \neq 0$, $\mu > 0$, it is $SU(2) \setminus \{\infty\}$.
(5) If $\lambda \neq 0$, $\mu < 0$, it is $\widetilde{SL}_2(\mathbb{R})$ with a left-invariant metric.
(6) If $\lambda \neq 0$, $\mu = 0$, it is the Heisenberg group Nil_3 with a left-invariant metric.

25.2. B-Scrolls

For every γ in the unit 3-sphere $S^3(1)$, one can define the Frenet frame $\{T, N, B\}$ provided the geodesic curvature κ does not vanish. The *B-scroll* of a curve γ in the unit 3-sphere $S^3(1)$ is the surface described by moving the geodesic through $\gamma(s)$ in the direction of spherical binormal $B(s)$

along γ. A curve in $S^3(1)$ of constant geodesic curvature and constant torsion ± 1 is called a *twisted spherical spiral*. The B-scroll of a twisted spherical spiral has parallel second fundamental form (cf., Reference [138]), so it is a parallel surface in $S^3(1)$.

If γ is a closed curve in $S^2(\frac{1}{2})$, then the Hopf cylinder $\pi^{-1}(\gamma)$ is called a *Hopf torus*. A B-scroll of a twisted spherical spiral is a Hopf cylinder (torus) over a curve with constant curvature in $S^2(\frac{1}{2})$ (cf. Reference [125]).

25.3. Parallel Surfaces in Bianchi–Cartan–Vranceanu Spaces

If $4\mu = \lambda^2$, then $\widetilde{\mathcal{M}}^3(\lambda,\mu)$ is a real space form whose parallel surfaces are already known. In the next theorem, M. Belkhelfa, F. Dillen and J. Inoguchi [125] classified parallel surfaces in Bianchi–Cartan–Vranceanu spaces $\widetilde{\mathcal{M}}^3(\lambda,\mu)$ with $4\mu \neq \lambda^2$.

Theorem 66. *Let $\widetilde{\mathcal{M}}^3(\lambda,\mu)$ be a Bianchi–Cartan–Vranceanu space with $4\mu \neq \lambda^2$.*

(1) If $\lambda \neq 0$, then the only parallel surfaces in $\widetilde{\mathcal{M}}^3(\lambda,\mu)$ are Hopf cylinders over curves with constant curvature in $\widetilde{\mathcal{M}}^2(\mu)$.
(2) If $\lambda = 0$, then the only parallel surfaces in $\widetilde{\mathcal{M}}^3(\lambda,\mu)$ with $\mu \neq 0$ are totally geodesic leaves and Hopf cylinders over circles with constant geodesic curvature in $\widetilde{\mathcal{M}}^2(\mu)$.

26. Parallel Surfaces in Homogeneous Three-Spaces

26.1. Homogeneous Three-Spaces

A Riemannian manifold M is said to be *homogeneous* if for any two points p and q of M there exists an isometry of M which carries p into q. It is clear that these spaces are a natural generalization of real space forms. A parallel submanifold is called *proper parallel* it is non-totally geodesic. In dimension 3, the classification of these spaces is well known as follows.

Theorem 67. *Let M^3 be a simply connected homogeneous Riemannian manifold with isometry group $I(M^3)$, that is, $I(M^3)$ acts transitively on M^3. Then $\dim I(M^3) \in \{3,4,6\}$ and moreover:*

(i) if $\dim I(M^3) = 6$, then M^3 is a real space form of constant sectional curvature c, that is, Euclidean space $\mathbb{E}^3$, hyperbolic space $H^3(c)$ or a three-sphere $S^3(c)$,
(ii) if $\dim I(M^3) = 4$, then M^3 is a Bianchi–Cartan–Vranceanu space (different from $\mathbb{E}^3$ and $S^3(c)$), that is, a Riemannian product $H^2(c) \times \mathbb{R}$ or $S^2(c) \times \mathbb{R}$ or one of following Lie groups, equipped with a left-invariant metric yielding a four-dimensional isometry group: the special unitary group $SU(2)$, the universal covering of the special linear group $\widetilde{SL}(2,\mathbb{R})$ or the Heisenberg group Nil_3,
(iii) if $\dim I(M^3) = 3$, then M^3 is a general three-dimensional Lie group with left-invariant metric.

26.2. Classification of Parallel Surfaces in Homogeneous Three-Spaces

In References [123,139], J. Inoguchi and J. Van der Veken classified parallel surfaces in homogeneous 3-spaces in the next two theorems.

For totally geodesic surfaces in a 3-dimensional homogeneous Riemannian manifold, we have:

Theorem 68. *Let (M^3, g) be a 3-dimensional homogeneous Riemannian manifold. Then M^3 admits totally geodesic surfaces if and only if M^3 is locally isometric to one of thefollowing spaces:*

(1) a real space form $S^3, \mathbb{E}^3$ or H^3,
(2) a Riemannian product space $S^2 \times \mathbb{R}$ or $H^2 \times \mathbb{R}$,
(3) $SL(2,\mathbb{R})$ with a left-invariant metric determined by the condition $c_2 = c_1 + c_3$ or equivalently $\mu_2 = 0$,
(4) the Minkowski motion group $E(1,1)$ with Riemannian 4-symmetric metric, including the model space Sol_3,
(5) a non-unimodular Lie group with structure constants (ξ,η) satisfying $\xi \notin \{0,1\}$ and $\eta = 0$.

For proper parallel surfaces in a 3-dimensional homogeneous Riemannian manifold, J. Inoguchi and J. Van der Veken [123] proved the following.

Theorem 69. *Let (M^3, g) be a 3-dimensional homogeneous Riemannian manifold. Then M^3 admits proper parallel surfaces if and only if M^3 is locally isometric to one of the following spaces:*

(1) a real space form $S^3, \mathbb{E}^3$ or H^3,
(2) a Bianchi–Cartan–Vranceanu space,
(3) the Minkowski motion group $E(1,1)$ with any left-invariant metric, including the model space Sol_3,
(4) the Euclidean motion group $E(2)$ with any left-invariant metric,
(5) a non-unimodular Lie group with structure constants (ξ, η) satisfying $\xi \notin \{0, 1\}$.

27. Parallel Surfaces in Symmetric Lorentzian Three-Spaces

Symmetric spaces are one of the most important topics in Riemannian geometry. In the Lorentzian setting, their study goes back to the work of M. Cahen and N. Wallach [140] in the 1970s.

27.1. Symmetric Lorentzian Three-Spaces

It is well known that the curvature of a 3-dimensional pseudo-Riemannian manifold (N, g) is completely determined by the Ricci tensor, denoted by Ric, defined for any point $p \in N$ and any $X, Y \in T_pN$ by

$$Ric(X,Y)_p = \sum_{i=1}^{3} \varepsilon_i g(R(X, e_i)Y, e_i), \tag{5}$$

where R is the Riemann curvature tensor, $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of T_pN and $\varepsilon_i = g_p(e_i, e_i) = \pm 1$ for all i. Throughout this section, if not stated otherwise, we shall assume that e_3 is *time-like*, that is, $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = 1$.

Due to the symmetries of the curvature tensor, the Ricci tensor Ric is symmetric [113]. Thus, the *Ricci operator* Q, defined by $g(QX, Y) = Ric(X, Y)$, is self-adjoint. In the Riemannian case, there always exists an orthonormal basis diagonalizing Q but in the Lorentzian case four different cases can occur [113] and there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 time-like, such that Q takes one of the following canonical forms, called *Segre types*:

$$\text{Segre type } \{11,1\} : \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad \text{Segre type } \{1z\bar{z}\} : \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix}, \; c \neq 0,$$

$$\text{Segre type } \{21\} : \begin{pmatrix} a & 0 & 0 \\ 0 & b & \eta \\ 0 & -\eta & b - 2\eta \end{pmatrix}, \; \eta = \pm 1, \quad \text{Segre type } \{3\} : \begin{pmatrix} b & a & -a \\ a & b & 0 \\ a & 0 & b \end{pmatrix}, \; a \neq 0.$$

When (N, g) is homogeneous, the Ricci operator Q has the same Segre type at any point $p \in N$ and has constant eigenvalues.

G. Calvaruso studied homogeneous Lorentzian 3-manifolds (N^3, g) in References [128,129]. For symmetric ones, he proved that 3-dimensional symmetric spaces can only occur for some Segre types of the Ricci operator Q. More precisely, he proved the following:

(I) For Segre type $\{11,1\}$, (N, g) is symmetric if and only if

  (i) $a = b = c$. Then (N, g) is an Einstein manifold and hence it has constant sectional curvature. If N is connected and simply connected, then (N, g) is isometric to one of the Lorentzian space forms: either S^3_1, $\mathbb{R}^3_1$ or H^3_1.
  (ii) $a = b \neq c$. Then N is reducible as a direct product $M^2 \times \mathbb{R}^1$, where M^2 is a Riemannian surface of constant curvature. If N is connected and simply connected, (N, g) is then isometric to either $S^2 \times \mathbb{R}$ or $H^2 \times \mathbb{R}$.

(iii) $a \neq b = c$. Then N is reducible as a direct product $\mathbb{R} \times M_1^2$, where M_1^2 is a Lorentzian surface of constant sectional curvature. When N is connected and simply connected, (N, g) is isometric to either $\mathbb{R} \times S_1^2$ or $\mathbb{R} \times H_1^2$.

(II) For Segre type $\{21\}$, (N, g) is symmetric if and only if $a - b = \eta$ and, with respect to a suitable pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$, the Levi Civita connection of (N, g) is completely described by

$$\begin{array}{lll} \nabla_{e_1} e_1 = Ae_2 - Ae_3, & \nabla_{e_2} e_1 = Be_2 - Be_3, & \nabla_{e_3} e_1 = Ce_2 - Ce_3, \\ \nabla_{e_1} e_2 = -Ae_1, & \nabla_{e_2} e_2 = -Be_1, & \nabla_{e_3} e_2 = -Ce_1, \\ \nabla_{e_1} e_3 = -Ae_1, & \nabla_{e_2} e_3 = -Be_1, & \nabla_{e_3} e_3 = -Ce_1, \end{array} \tag{6}$$

where A, B, C are smooth functions. Put $u = e_2 - e_3$. Then $\nabla_{e_i} u = 0$ for all i, that is, u is a *parallel null vector field*. Three-dimensional symmetric spaces admitting a parallel null vector field were described in Reference [132] in terms of local coordinates. In fact, a three-dimensional locally symmetric Lorentzian manifold (N, g), having a parallel null vector field, admits local coordinates (t, x, y) such that, with respect to the local frame field $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$, the Lorentzian metric g and the Ricci operator are respectively given by

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & -\varepsilon\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7}$$

where $\varepsilon = \pm 1$, $u = \frac{\partial}{\partial t}$ and

$$f(x, y) = x^2\alpha + x\beta(y) + \xi(y), \tag{8}$$

for a constant $\alpha \in \mathbb{R}$ and functions β, ξ (cf. Reference [132]). It is easy to build a (local) pseudo-orthonormal frame field from $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ and to check that, apart from the flat case $\alpha f \neq 0$, the Ricci operator Q described by (7) is of degenerate Segre type $\{21\}$, with $\lambda = 0$ as the only Ricci eigenvalue, of multiplicity three, associated to a 2-dimensional eigenspace.

(III) For either Segre type $\{1z\bar{z}\}$ or Segre type $\{3\}$, (N, g) is never symmetric.

Therefore, we have the following classification result from Reference [129].

Theorem 70. *A connected, simply connected three-dimensional symmetric Lorentzian space (N, g) is either*

(i) a Lorentzian space form S_1^3, $\mathbb{R}_1^3$ or H_1^3 or

(ii) a direct product $\mathbb{R} \times S_1^2$, $\mathbb{R} \times H_1^2$, $S^2 \times \mathbb{R}_1^1$ or $H^2 \times \mathbb{R}_1^1$ or

(iii) a space with a Lorentzian metric g locally described by (7)–(8).

27.2. Classification of Parallel Surfaces in Symmetric Lorentzian Three-Spaces

Three-dimensional Lorentzian manifolds admitting a parallel null vector field were first studied in Reference [132], in which the attention was focused on local properties. G. Calvaruso and J. Van der Veken described in Reference [141] a global model carrying a metric described by (7) and (8) as follows.

First they showed that the curvature components with respect to the pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$ for which (6) holds and then apply (5) to obtain its Ricci components. Since the Ricci operator must be of degenerate Segre type $\{21\}$ (that is, with $a = b - \eta$), standard calculations lead to the following system of partial differential equations:

$$\begin{cases} -e_1(B) + e_2(A) + e_1(C) - e_3(A) - (B-C)^2 = b - \eta, \\ -e_1(B) + e_2(A) - A^2 - B^2 + BC = b, \\ e_1(C) - e_3(A) + A^2 - C^2 + BC = b - 2\eta, \\ e_2(C) - e_3(B) + A(B-C) = 0, \\ -e_1(B) + e_2(A) - A^2 - B^2 + BC = \eta. \end{cases} \tag{9}$$

System (9) implies $a = b - \eta = 0$ (which also follows from (7) and (8)) and the remaining equations reduce to

$$\begin{cases} e_1(B) - e_2(A) = -A^2 - B^2 + BC - \eta, \\ e_1(C) - e_3(A) = -A^2 + C^2 - BC - \eta, \\ e_2(C) - e_3(B) = A(C-B). \end{cases} \tag{10}$$

Then they proved that, for any smooth function ω, with respect to the following new pseudo-orthonormal frame field

$$e_1' = e_1 + \omega e_2 - \omega e_3,\ e_2' = -\omega e_1 + (1 - \frac{\omega^2}{2})e_2 + \frac{\omega^2}{2}e_3,\ e_3' = -\omega e_1 - \frac{\omega^2}{2}e_2 + (1 + \frac{\omega^2}{2})e_3, \tag{11}$$

the Ricci operator still keeps the same components than with respect to $\{e_1, e_2, e_3\}$. It follows from (6) and (11) that, with respect to $\{e_1', e_2', e_3'\}$, the Levi Civita connection satisfies

$$\begin{array}{lll} \nabla_{e_1'} e_1' = A'e_2' - A'e_3', & \nabla_{e_2'} e_1' = B'e_2' - B'e_3', & \nabla_{e_3'} e_1' = C'e_2' - C'e_3', \\ \nabla_{e_1'} e_2' = -A'e_1', & \nabla_{e_2'} e_2' = -B'e_1', & \nabla_{e_3'} e_2' = -C'e_1', \\ \nabla_{e_1'} e_3' = -A'e_1', & \nabla_{e_2'} e_3' = -B'e_1', & \nabla_{e_3'} e_3' = -C'e_1', \end{array}$$

where

$$\begin{aligned} A' &= A + e_1\omega, \\ B' &= A\omega + \omega e_1\omega - (1 - \tfrac{\omega^2}{2})B - (1 - \tfrac{\omega^2}{2})e_2\omega - \tfrac{\omega^2}{2}C - \tfrac{\omega^2}{2}e_3\omega, \\ C' &= A\omega + \omega e_1\omega + \tfrac{\omega^2}{2}B + \tfrac{\omega^2}{2}e_2\omega - (1 + \tfrac{\omega^2}{2})C - (1 + \tfrac{\omega^2}{2})e_3\omega. \end{aligned}$$

Thus, by choosing ω to be a solution of the system of differential equations

$$A + e_1\omega = k, \quad de_2\omega - e_3\omega = C - B,$$

where k is a real constant, we can always specify the pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$ in such a way that $A = k$ and $B = C$. In this case, system of Equation (10) reduces to

$$e_1B = -k^2 - \eta, \quad e_2B - e_3B = 0, \tag{12}$$

and the Lie brackets $[e_i, e_j]$ are easily determined as follows:

$$[e_1, e_2] = [e_1, e_3] = -ke_1 - B(e_2 - e_3), \quad [e_2, e_3] = 0. \tag{13}$$

With the notations given in Section 27.1, G. Calvaruso and J. Van der Veken proved the following theorem (see Reference [141]).

Theorem 71. *Let (N, g) be a connected, simply connected 3-dimensional Lorentzian manifold. Then the necessary and sufficient condition for (N, g) to be symmetric and to have a Ricci operator of (degenerate) Segre* *type* $\{21\}$*, is the existence of a global pseudo-orthonormal frame field* $\{e_1, e_2, e_3\}$*, with* e_3 *time-like, a real constant* k *and a smooth function* B*, satisfying* (12) and (13)*.*

The following classification of parallel surfaces in a symmetric Lorentzian 3-space was also obtained by G. Calvaruso and J. Van der Veken in Reference [141].

Theorem 72. *Let* M *be a parallel surface in a symmetric Lorentzian three-space* $(\tilde{N}, \tilde{g})$ *carrying a parallel null vector field, described by* (12) and (13)*. Then* M *is an integral surface of the distribution spanned by* $\{e_2, e_3\}$*, on which* B *is constant. Moreover,* M *is always flat and* M *is totally geodesic if and only if* $B = 0$ *on it. If* B *is non-constant on all integral surfaces of the distribution spanned by* $\{e_2, e_3\}$*, then* $(\tilde{N}, \tilde{g})$ *does not admit any parallel surfaces.*

Remark 10. *For pseudo-Riemannian 3-manifolds with prescribed distinct constant Ricci eigenvalues, see Reference [142].*

28. Three Natural Extensions of Parallel Submanifolds

28.1. Submanifolds with Parallel Mean Curvature Vector

One natural extension of the class of parallel submanifolds ($\bar{\nabla} h = 0$) is the class of submanifolds with parallel mean curvature vector, that is, $\bar{\nabla}(\mathrm{Tr}\, h) = 0$ or equivalently $DH = 0$. Trivially, both minimal submanifolds and parallel submanifolds have parallel mean curvature vector automatically. Further, a hypersurface of any Riemannian manifold has parallel mean curvature vector if and only if it has constant mean curvature.

Euclidean hypersurfaces with constant mean curvature are important since they are critical points of some natural functionals. In fact, a hypersurface of constant mean curvature in a Euclidean space is a solution to a variational problem. With respect to any volume-preserving variation of a domain D in a Euclidean space the mean curvature of $M = \partial D$ is constant if and only if the volume of M is critical, where ∂D is the boundary of D.

The condition of submanifolds to have parallel mean curvature vector in higher dimensional Euclidean spaces is very interesting as well since it is equivalent to a critical points of being variational problem; namely, their Gauss maps are harmonic maps (see Reference [143]).

During the last 50 years, there are many research done on submanifolds with parallel mean curvature vector. Among others, for submanifolds with parallel mean curvature vector in real space forms see References [144–151]; for surfaces with parallel mean curvature vector in complex space forms see References [152–157]; for surfaces with parallel mean curvature vector in indefinite space forms see References [158–163]; for surfaces with parallel mean curvature vector in homogeneous spaces or symmetric spaces see References [164,165]; for surfaces with parallel mean curvature vector in Sasakian space forms see Reference [166]; and for surfaces with parallel mean curvature vector in reducible manifolds see References [167–170]. For general references of submanifolds with parallel mean curvature vector see Reference [171].

28.2. Higher Order Parallel Submanifolds

Higher order parallel submanifolds, that is, submanifolds that satisfy $\bar{\nabla}^k h = 0$ for some positive integer k, were first studied by D. Del-Pezzo in Reference [172] and then investigated by several authors after Del-Pezzo (see J. A. Schouten and D. J. Struik's 1938 book [173] for details). This research topic was renewed in late 1980s by F. Dillen, V. Mirzoyan and Ü. Lumiste. Since, this interesting research topic has been studied by several differential geometers.

Among others, for higher order parallel submanifolds in real space forms see References [138,174–179]; for higher order parallel surfaces in three-dimensional homogeneous spaces see Reference [139]; for higher order parallel surfaces in Bianchi–Cartan–Vranceanu spaces see Reference [136]; for higher order parallel surfaces in the Heisenberg group see Reference [180]; and for higher order parallel submanifolds of a complex space form see Reference [181]. For some further results on higher order parallel submanifolds see Ü Lumiste's 2000 survey article [2].

28.3. Semi-Parallel Submanifolds

The notion of semi-parallel submanifolds was introduced in 1985 by J. Deprez in Reference [182]. A submanifold M of a Riemannian manifold N is called *semi-parallel* if its second fundamental form h satisfies

$$\tilde{R}(X,Y)\cdot h = (\bar{\nabla}_X\bar{\nabla}_Y - \bar{\nabla}_Y\bar{\nabla}_X - \bar{\nabla}_{[X,Y]})h = 0,$$

where $\tilde{R}$ is the Riemann curvature tensor of N. Obviously, parallel submanifolds are semi-parallel. Hence, semi-parallel submanifolds are natural extensions of parallel submanifolds as well.

In Reference [182], J. Deprez applying the work of E. Backes [183] on Euclidean Jordan triple systems to prove that totally geodesic surfaces are the only minimal semi-parallel surfaces in a Euclidean space. Furthermore, he proved in Reference [184] that every semi-parallel submanifolds of a Euclidean space is intrinsically a semi-symmetric Riemannian manifold. By a semi-symmetric Riemannian manifold (M,g) we mean that the Riemann curvature curvature tensor of (M,g) satisfies the condition $R\cdot R = 0$, where the first tensor R acts on the second one as a derivation. In Reference [182], Deprez also classified semi-parallel surfaces in a Euclidean space. Since then many articles were devoted to the study of semi-parallel submanifolds.

Among others, for semi-parallel submanifolds in real space forms see References [184–193]; for semi-parallel submanifolds of indefinite space forms see References [194,195]; for semi-parallel submanifolds in Kaehler manifolds see [196–199]; for semi-parallel submanifolds in reducible spaces see Reference [117]; for manifold with semi-parallel geodesic spheres or semi-parallel tubes see Reference [200]; for semi-parallel submanifolds in contact metric manifolds see References [201,202]; and for semi-parallel submanifolds in other Riemannian manifolds see References [203–205]. For some further results on semi-parallel submanifolds see Reference [2].

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