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On One Problems of Spectral Theory for Ordinary Differential Equations of Fractional Order

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Abstract: The present paper is devoted to the spectral analysis of operators induced by fractional differential equations and boundary conditions of Sturm-Liouville type. It should be noted that these operators are non-self-adjoint. The spectral structure of such operators has been insufficiently explored. In particular, a study of the completeness of systems of eigenfunctions and associated functions has begun relatively recently. In this paper, the completeness of the system of eigenfunctions and associated functions of one class of non-self-adjoint integral operators corresponding boundary value problems for fractional differential equations is established. The proof is based on the well-known Theorem of M.S. Livshits on the spectral decomposition of linear non-self-adjoint operators, as well as on the sectoriality of the fractional differentiation operator. The results of Dzhrbashian-Nersesian on the asymptotics of the zeros of the Mittag-Leffler function are used.

Keywords: Mittag-Leffler function; spectrum; eigenvalue; fractional derivative

1. Introduction

The present paper is devoted to the spectral analysis of operators induced by the fractional differential equations and boundary conditions of Sturm-Liouville type. It should be noted that these operators are non-self-adjoint. The spectral structure of such operators has been insufficiently explored. In particular, a study of the completeness of systems of eigenfunctions and associated functions has begun relatively recently. In this paper, the completeness of the system of eigenfunctions and associated functions of one class of non-self-adjoint integral operators corresponding boundary value problems for fractional differential equations is established. The proof is based on the well-known Theorem of M.S. Livshits on the spectral decomposition of linear non-self-adjoint operators, as well as on the sectoriality of the fractional differentiation operator. The results of Dzhrbashian-Nersesian on the asymptotics of the zeros of the Mittag-Leffler function are used.

2. Results

Reference [1] studied the operator in the space $L_2(0,1)$

$$-A_\rho u = \int_0^1 G(x,t)u(t)\,dt = \frac{1}{\Gamma(\rho^{-1})}\left[\int_0^x (x-t)^{\frac{1}{\rho}-1}u(t)dt - \int_0^1 x^{\frac{1}{\rho}-1}(1-t)^{\frac{1}{\rho}-1}u(t)\,dt\right],$$

which was first considered in References [2,3], where $0<\rho<2$ and

$$G(x,t) = \begin{cases} \dfrac{(1-t)^{\frac{1}{\rho}-1}x^{\frac{1}{\rho}-1} - (x-t)^{\frac{1}{\rho}-1}}{\Gamma(\rho^{-1})}, & 0 \le t \le x \le 1 \\ \dfrac{(1-t)^{\frac{1}{\rho}-1}x^{\frac{1}{\rho}-1}}{\Gamma(\rho^{-1})}, & 0 \le x \le t \le 1 \end{cases}$$

is the Green function of the following problem S (with $\lambda = 0$):

$$\frac{1}{\Gamma(n-\rho^{-1})}\frac{d^n}{dx^n}\int\limits_0^x (x-s)^{n-\rho^{-1}-1}u(s)ds + \lambda\, u = 0,$$

$(n-1 \le \rho^{-1} < n,\ n = [\rho^{-1}]+1$, where $[\rho^{-1}]$ is the integer part of $\rho^{-1})$

$$u(0) = 0, u'(0) = 0, \cdots, u^{(n-2)}(0) = 0, u(1) = 0.$$

In particular, in this paper, we provide very important proof of the completeness of the system of eigenfunctions and associated functions in $L_2(0,1)$ of the operator A_ρ for $1 < \rho < 2$ based on fact that the operator of fractional differentiation is sectorial and for $0 < \rho < 1$ (this fact plays a main role in solving boundary value problems for advection-diffusion equation of fractional order by the method of separation of variables [4] since we can write out both the exact solution in the form of an infinite series by eigenfunctions and the approximate solution replacing the infinite sums by sums of the first n terms), a proof based on the well-known Livshits theorem [5] (researching of case for $1 < \rho < 2$ published in this paper firstly):

Theorem 1 (Livshits). *If $K(x,y)$ $(a \le x, y \le b)$ – is a limited kernel and "real part" $\frac{1}{2}(K+K^*)$ of it is non-negative kernel, then the inequality is hold*

$$\sum_{j=1}^{\infty} Re(\frac{1}{\lambda_j}) \le \int\limits_a^b ReK(t,t)dt,$$

where λ_j – is the characteristic numbers of kernel K. The system of main functions of the kernel K is complete in domain of values of the integral operator Kf if and only if, when there is an equal sign in inequality above.

In his paper [6] M. M. Dzhrbashian wrote, that "the question about the completeness of the systems of eigenfunctions of the operator A_ρ or a finer question about whether these systems compose a basis in $L_2(0,1)$, has a certain interest but its solving is apparently associated with significant analytic difficulties". The questions of the completeness of the systems of eigenfunctions and associated functions for similar problems were studied by A. V. Agibalova in [7,8]. Undoubtedly, we shall note the fundamental results of M. M. Malamud and L. L. Oridoroga [9–12] obtained in this direction. In [13,14] (see also [2,15]), using the theorem of Matsaev and Palant, it was established that the system of eigenfunctions of the operator A_ρ is complete in $L_2(0,1)$. And this fact used by M. Ali, S. Aziz and S.A. Malik in their paper [16].

As noted above, in this paper, a similar result was obtained using the well-known Livshchits theorem [5]. The following proof of the completeness of the system of eigenfunctions is simpler than the previously presented proofs, which makes the results of this paper very significant.

Next, we need one definition.

Definition 1. *If a series of s-numbers [17] of the completely continuous operator is convergent, that is, $\sum\limits_{k=1}^{\infty} s_k(A) < \infty$ then such operator called as trace-class operator.*

Lemma 1. *Let $0 < \rho < 2$, then the operator A_ρ is trace-class and*

$$sp(A_\rho) = \frac{\Gamma(\rho^{-1})}{\Gamma(2\rho^{-1})}.$$

Proof of Lemma 1. To find the trace spA_ρ of the operator A_ρ, let's rewrite A_ρ as $A_\rho u = A_1 u - A_0 u$ where

$$A_0u = \frac{1}{\Gamma(\rho^{-1})}\int\limits_0^x (x-t)^{\frac{1}{\rho}-1}u(t)dt,$$

$$A_1u = \frac{1}{\Gamma(\rho^{-1})}\int\limits_0^1 x^{\frac{1}{\rho}-1}(1-t)^{\frac{1}{\rho}-1}u(t)dt.$$

Clearly, for $0<\rho<2$, the operators A_0 and A_1 are trace class. Hence

$$spA_\rho = sp(A_1-A_0) = sp(A_1)-sp(A_0).$$

Moreover, it is clear that $sp(A_0)=0$. Thus

$$spA_\rho = sp(A_1).$$

Since operator A_1 is one-dimensional, it is easy to find a trace. Consider the equation

$$u(x) - \frac{\lambda}{\Gamma(\rho^{-1})}\int\limits_0^1 x^{\frac{1}{\rho}-1}(1-t)^{\frac{1}{\rho}-1}u(t)dt = 0$$

The Fredhold determinant

$$d(\lambda) = |1-\lambda K_{11}|,$$

where

$$K_{11} = \frac{1}{\Gamma(\rho^{-1})}\int\limits_0^1 t^{\frac{1}{\rho}-1}(1-t)^{\frac{1}{\rho}-1}dt = \frac{\Gamma(2-\nu)}{\Gamma(4-2\nu)}\,(\nu = 2-\rho^{-1}).$$

From above follow that

$$sp(A_1) = \frac{\Gamma(2-\nu)}{\Gamma(4-2\nu)}$$

which proves the Lemma 1.

□

Remark 1. *Of course, for $\rho > 1/2$, nuclearity of the operator A_ρ follows from well-known Dzhrbaschian-Nersisian lemma ([18], p. 142).*

Lemma 2 (Dzhrbaschian-Nersisian)**.** *1. All zeros of functions $E_\rho(z;\mu) = \sum\limits_{n=0}^{\infty}\frac{z^n}{\Gamma(\mu+n/\rho)}$ (where $\rho > \frac{1}{2}, \rho \neq 1; Im\mu = 0$) with largest absolute values, are prime.*

2. The following asymptotic formulas are valid

$$\gamma_k^{\pm} = e^{\pm i\frac{\pi}{2\rho}}(2\pi k)^{1/\rho}\left(1+O(\frac{logk}{k})\right),\ k\to\infty,$$

and the fact that the value λ_j is an eigenvalue of the operator A_ρ if and only if $E_\rho(\lambda_j;\frac{1}{\rho})=0$.

Now we give the main result of paper.

Theorem 2. *The system of eigenfunctions and associated functions of the operator A_ρ, where $0<\rho<1$, is complete in $L_2(0,1)$.*

Proof of Theorem 2. We denote the kernel of A_ρ as $K(x,y)$. In [13] the authors have proved that this kernel is non-negative by the following way: Let us rewrite A_ρ as

$$A_\rho u = \frac{1}{\Gamma(\rho^{-1})}\left[\int_0^1 (x-xt)^{\frac{1}{\rho}-1}u(t)dt - \int_0^x (x-t)^{\frac{1}{\rho}-1}u(t)\,dt\right].$$

Clearly, for $\rho < 1$, the kernel of A_ρ is non-negative.

By the same way, we may show that the kernel $K^*(x,y)$ for adjoint operator

$$A_\rho^* u = \frac{1}{\Gamma(\rho^{-1})}\left[\int_0^1 (t-xt)^{\frac{1}{\rho}-1}u(x)dx - \int_x^1 (t-x)^{\frac{1}{\rho}-1}u(x)\,dx\right]$$

is non-negative too. Thus $\frac{1}{2}(K+K^*)$ is non-negative. Let us show that the following expression holds

$$\sum_{j=1}^{\infty} Re(\frac{1}{\lambda_j}) = \int_0^1 ReK(t,t)dt.$$

If $\lambda_j = \alpha_j + i\beta_j$ is eigenvalue of the operator A_ρ, then complex conjugate $\overline{\lambda}_j = \alpha_j - i\beta_j$ is eigenvalue of the operator A_ρ too. Thus

$$spA_\rho = \sum_{j=1}^{\infty}\frac{1}{\lambda_j} = \sum_{j=1}^{\infty} Re(\frac{1}{\lambda_j}).$$

So, taking to account lemma 1, we obtain that the system of eigenfunctions and associated functions of the operator A_ρ for $0 < \rho < 1$, is complete in $L_2(0,1)$. □

Remark 2. *For $(\frac{1}{\rho}-1) > 0$ the kernel of the operator A_ρ is continuous. Therefore, as it was showed by Lalesko [5], the Fredholm determinant of this kernel is whole function of zero kind. In this case [5],*

$$\sum_{i=1}^{\infty}\frac{1}{\lambda_j} = \int_0^1 K(t,t)dt,$$

that is, the equation

$$\sum_{j=1}^{\infty} Re(\frac{1}{\lambda_j}) = \int_0^1 ReK(t,t)dt$$

we can get by the obvious way.

Theorem 3. *The system of eigenfunctions and associated functions of the operator A_ρ, where $1 < \rho < 2$, is complete in $L_2(0,1)$.*

Proof of Theorem 3. For $1 < \rho < 2$ the kernel of the operator A_ρ is not fixed-sign, thus we cannot use the Livshits theorem, used above. To prove the formulated theorem, let us consider the value of the the form $(A_\rho u, \overline{u})$ [19]. Let us introduce the following designation

$$A_\rho u = \frac{1}{\Gamma(\rho^{-1})}\left[\int_0^1 (x-xt)^{\frac{1}{\rho}-1}u(t)dt - \int_0^x (x-t)^{\frac{1}{\rho}-1}u(t)\,dt\right] = v(x).$$

So,

$$(A_\rho u, \overline{u}) = (v, D_{0x}^{1/\rho} v) = \int_0^1 v(x)[D_{0x}^{1/\rho} v]dx = \int_0^\varepsilon v(x)[D_{0x}^{1/\rho} v]dx + \int_\varepsilon^1 v(x)[D_{0x}^{1/\rho} v]dx$$

where

$$D_{0x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}},$$

$n = [\alpha] + 1$, $[\alpha]$ is the integer part of α, called the operator of fractional differentiation in the Sturm-Liouville sense of order α. As was mentioned in Reference [19] (see also the references therein), the study of forms

$$\int_\varepsilon^1 v(x)[D_{0x}^{1/\rho} v]dx$$

was provided in the paper and there, in particular, were established the values of those forms lying in $|arg\lambda| < \frac{\pi\rho}{2}$. Clearly, for small values ε, the operator A_ρ is sectorial. Since the operator A_ρ is sectorial and a trace-class operator, by Lidskii's Theorem [20] the system of eigenfunctions and associated functions of A_ρ are complete in $L_2(0,1)$. □

Corollary 1. *Since the operator A_ρ does not generate any associated functions [21], we prove the completeness of system*

$$\chi_n(x) = x^{\frac{1}{\rho}-1} E_\rho(\lambda_n x^{\frac{1}{\rho}}; \frac{1}{\rho})$$

in $L_2(0,1)$ (but the system of these eigenfunctions, unfortunately, is not orthogonal, therefore, for solving inverse problems, and in Reference [16] the corresponding biorthogonal system was used).

By the same method, we can provide spectral analysis of the operator

$$A_\rho^{[\alpha^{-1},\rho]} u = \frac{1}{\Gamma(\rho^{-1})} \int_0^1 x^{\frac{1}{\rho}-1}(1-t)^{\alpha-1} u(t)dt - \frac{1}{\Gamma(\rho^{-1})} \int_0^x (x-t)^{\frac{1}{\rho}-1} u(t)dt,$$

considered in Reference [13] (and see the references therein).

Theorem 4. *Let $0 < \rho < 2$, $\alpha < \frac{1}{\rho}$. Then, the system of eigenfunctions and associated functions of the operator $A_\rho^{[\alpha^{-1},\rho]}$ is complete in $L_2(0,1)$.*

Proof. We carry out the proof of Theorem 4 in the same way as the proof of Theorem 3. It can easily be shown that the kernel $M(x,t)$ of the operator $A_\rho^{[\alpha^{-1},\rho]}$ is non-negative. Elementary calculations show that the kernel $M^*(x,t)$ of the operator adjoint to the operator $A_\rho^{[\alpha^{-1},\rho]}$ will be non-negative too. Thus $\frac{1}{2}(M + M^*)$ will be non-negative too. The fact that

$$\sum_{j=1}^{\infty} Re(\frac{1}{\mu_j}) = \int_0^1 ReM(t,t)dt$$

where μ_j are eigenvalues of the operator $A_\rho^{[\alpha^{-1},\rho]}$, shown in the same way as in Theorem 2. □

3. Conclusions

In the present paper, by way of the Livshits Theorem we provide proof of the completeness of the eigenfunctions and associated functions of the operators, generated by the ordinary differential expressions of the fractional order and boundary conditions of Sturm-Liouville type.

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