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Oscillation of Fourth-Order Functional Differential Equations with Distributed Delay

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Abstract: In this paper, the authors obtain some new sufficient conditions for the oscillation of all solutions of the fourth order delay differential equations. Some new oscillatory criteria are obtained by using the generalized Riccati transformations and comparison technique with first order delay differential equation. Our results extend and improve many well-known results for oscillation of solutions to a class of fourth-order delay differential equations. The effectiveness of the obtained criteria is illustrated via examples.

Keywords: fourth-order; oscillatory solutions; delay differential equations

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1. Introduction

In this work, we study the oscillation of a fourth-order delay differential equation

$$\left[b(z)\left(y'''(z)\right)^{\gamma}\right]' + \int_c^d q(z,\delta) f(y(h(z,\delta)))\, d(\delta) = 0, \quad z \ge z_0, \tag{1}$$

where γ is a quotient of odd positive integers and we assume that $b(z) \in C([z_0, \infty), \mathbb{R}), b'(z) \ge 0$, $q(z,\delta)$, $h(z,\delta) \in C([z_0,\infty) \times [c,d], \mathbb{R})$, $q(z,\delta)$ is positive, $h(z,\delta)$ is a nondecreasing function in δ, $h(z,\delta) \le z \lim\limits_{t\to\infty} h(z,\delta) = \infty$ and $f \in C(\mathbb{R},\mathbb{R})$ satisfies the following conditions:

$$\begin{aligned} (A_1)\ -f(-xy) &> f(xy) > f(x)f(y) \text{ for all } xy > 0. \\ (A_2)\ \ f(u)/u^{\gamma} &\ge r > 0, \text{ for } u \ne 0. \end{aligned}$$

By a solution of Equation (1), we mean a function $y \in C'''[z_y, \infty)$, $z_y \ge z_\circ$, which has the property $b(z)(y'''(z))^{\gamma} \in C^1[z_y,\infty)$ and satisfies Equation (1) on $[z_y,\infty)$. We consider only those solutions y of Equation (1) which satisfy $\sup\{|y(z)| : z \ge Z\} > 0$, for all $Z > z_y$. We assume that (1) possesses such a solution. A solution of (1) is called oscillatory if it has arbitrarily large zeros on $[z_y,\infty)$ and otherwise it is called to be nonoscillatory. The Equation (1) is said to be oscillatory if all its solutions are oscillatory.

The problem of the oscillation of higher and fourth order differential equations have been widely studied by many authors, who have provided many techniques for obtaining oscillatory criteria for higher and fourth order differential equations. We refer the reader to the related books (see [1–4]) and to the papers (see [5–31]). In what follows, we present some relevant results that have provided the background and the motivation, for the present work.

In the following, we present some related results that served as a motivation for the contents of this paper.

Tunc and Bazighifan [25] studied the oscillatory behavior of the fourth-order nonlinear differential equation with a continuously distributed delay

$$\left[b\left(z\right)y'\left(z\right)\right]''' + \int_c^d q\left(z,\ \tau\right)f\left(y\left(h\left(z,\ \tau\right)\right)\right)d\left(\tau\right) = 0, \quad z \geq z_0.$$

Moaaz et al. [23] have studied the form

$$\left[b\left(z\right)\left(x'''\left(z\right)\right)^{\alpha}\right]' + \int_m^b q\left(z,\ \delta\right)f\left(x\left(h\left(z,\ \delta\right)\right)\right)d\left(\delta\right) = 0.$$

Elabbasy et al. [3] studied the equation

$$\left[b\left(z\right)\left(x'''\left(z\right)\right)^{\kappa}\right]' + q\left(z\right)x^{\kappa}\left(z\right) = 0, \quad y \geq y_0$$

under the conditions

$$\int_{y_0}^{\infty} \frac{1}{b^{\frac{1}{\kappa}}\left(z\right)}dz = \infty.$$

Grace et al. [17] studied the oscillation behavior of the fourth-order nonlinear differential equation

$$\left[r\left(z\right)\left(y'\left(z\right)\right)^{\gamma}\right]''' + q\left(z\right)f\left(y\left(h\left(z\right)\right)\right) = 0, \quad z \geq z_0.$$

Bazighifan et al. [9] and Zhang et al. [32] consider the oscillatory properties of the higher-order differential equation

$$\left[b\left(z\right)\left(y^{(n-1)}\left(z\right)\right)^{\gamma}\right]' + q\left(z\right)y^{\gamma}\left(\tau\left(z\right)\right) = 0,$$

under the conditions

$$\int_{z_0}^{\infty} \frac{1}{b^{\frac{1}{\gamma}}\left(z\right)}dz = \infty \tag{2}$$

and

$$\int_{z_0}^{\infty} \frac{1}{b^{\frac{1}{\gamma}}\left(z\right)}dz < \infty.$$

Our aim in the present paper is to employ the Riccati technique and comparison technique with first order delay differential equation to establish some new conditions for the oscillation of all solutions of (1). Some examples are presented to illustrate our main results.

We begin with the following lemmas.

Lemma 1. *(See [5], Lemma 2.1) Let $\beta \geq 1$ be a ratio of two numbers. Then*

$$A^{\frac{\beta+1}{\beta}} - (A-B)^{\frac{\beta+1}{\beta}} \leq \frac{B^{\frac{1}{\beta}}}{\beta}\left[(1+\beta)A - B\right], \quad AB \geq 0,$$

and

$$Uy - Vy^{\frac{\beta+1}{\beta}} \leq \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\frac{U^{\beta+1}}{V^{\beta}},\ V > 0.$$

Lemma 2. *(See [20], Lemma 1.1) If the function z satisfies $y^{(i)} > 0$, $i = 0, 1, ..., n$, and $y^{(n+1)} < 0$, then*

$$\frac{y\left(z\right)}{z^n/n!} \geq \frac{y'\left(z\right)}{z^{n-1}/\left(n-1\right)!}.$$

Lemma 3. *(See [9], Lemma 1.1) Let* $x \in \left(C^n\left[x_0, \infty\right], \mathbb{R}^+\right)$ *and assume that* $x^{(n)}$ *is of fixed sign and not identically zero on a subray of* $\left[x_\circ, \infty\right]$. *If moreover,* $x\left(z\right) > 0,\ x^{(n-1)}\left(z\right) x^{(n)}\left(z\right) \le 0$ *and* $\lim_{z\to\infty} x\left(z\right) \neq 0$*, then, for every* $\lambda \in (0, 1)$*, there exists* $z_\lambda \ge z_\circ$ *such that*

$$x\left(z\right) \ge \frac{\lambda}{(n-1)} z^{n-1}\left|x^{(n-1)}\left(z\right)\right|, \text{ for } z \in [z_\lambda,\ \infty).$$

Lemma 4. *(See [33], Lemma 2.1) Assume that (2) holds and* y *is an eventually positive solution of (1). Then there are the following two possible cases eventually:*

$$\begin{array}{ll} (C_1) & y\left(z\right) > 0,\ y'\left(z\right) > 0, y''\left(z\right) > 0, y'''\left(z\right) > 0,\ y^{(4)}\left(z\right) > 0,\ \left(b\left(z\right)\left(y'''\left(z\right)\right)^\gamma\right) < 0, \\ (C_2) & y\left(z\right) > 0,\ y'\left(z\right) > 0, y''\left(z\right) < 0, y'''\left(z\right) > 0,\ \left(b\left(z\right)\left(y'''\left(z\right)\right)^\gamma\right) < 0, \end{array}$$

2. Main Results

In this section, we shall establish some oscillation criteria for Equation (1). For convenience, we denote

$$\pi\left(z\right) := \int_{z_0}^{\infty} \frac{1}{b^{\frac{1}{\gamma}}\left(s\right)} ds,$$

$$\vartheta'_+\left(z\right) := \max\left\{0,\ \vartheta'\left(z\right)\right\} \text{ and } Q\left(z\right) = \int_a^b q\left(z, \delta\right) d\left(\delta\right).$$

Theorem 1. *Let (2) holds and assume that for some constant* $\lambda_0 \in (0,\ 1)$*, the differential equation*

$$x'\left(z\right) + Q\left(z\right) f\left(\frac{\lambda_0^3 h\left(z, \delta\right)}{6b^{1/\gamma}\left(h\left(z, \delta\right)\right)}\right) f\left(x^{1/\gamma}\left(h\left(z, \delta\right)\right)\right) = 0, \tag{3}$$

is oscillatory. If there exists a positive function $\vartheta \in C\left([z_0, \infty)\right)$ *such that*

$$\int_{z_0}^{\infty}\left(\psi^*\left(s\right) - \frac{1}{4}\vartheta\left(s\right)\left(\phi^*\left(s\right)\right)^2\right) ds = \infty, \tag{4}$$

where

$$\begin{aligned} \psi^*\left(z\right) &:= \vartheta\left(z\right)\left[\int_z^{\infty}\left[\frac{r}{b\left(v\right)}\int_v^{\infty}\int_c^d q\left(z,\ \delta\right)\frac{h^\gamma\left(s, \delta\right)}{s^\gamma} d\left(\delta\right) ds\right]^{1/\gamma} dv - \frac{1 + b^{\frac{1}{\gamma}}\left(z\right)}{\pi^2(z)}\right], \\ \phi^*\left(z\right) &:= \frac{\vartheta'_+\left(z\right)}{\vartheta\left(z\right)} + \frac{2}{\pi(z)}, \end{aligned}$$

for some $\mu \in (0, 1)$*, then every solution of (1) is oscillatory.*

Proof. Assume that (1) has a nonoscillatory solution y. Without loss of generality, we can assume that $y\left(z\right) > 0$. From Lemma 4, we get that there exist y two possible cases for $z \ge z_1$, where $z_1 \ge z_0$ is sufficiently large.

Assume that we have Case 1 and $\left(A_1\right)$ hold. From Lemma 3, we have

$$y\left(h\left(z, \delta\right)\right) \ge \frac{\lambda h^3\left(z, \delta\right)}{6b^{1/\gamma}\left(z\right)}\left(b^{1/\gamma}\left(z\right) y''' h\left(z, \delta\right)\right), \tag{5}$$

for every $\lambda \in (0,\ 1)$. Using (5) in Equation (1), we see that

$x(z) = b(z)\,[y'''(z)]^{\gamma}$ is a positive solution of the differential inequality

$$x'(z) + Q(z)\, f\left(\frac{\lambda_0^3 h(z,\delta)}{6b^{1/\gamma}(h(z,\delta))}\right) f\left(x^{1/\gamma}(h(z,\delta))\right) \leq 0.$$

By Theorem 1 in [34], we conclude that the corresponding Equation (1) also has a positive solution. This is a contradiction.

Assume that we have Case 2 and (A_2) hold. From Lemma 2, we get that $y(z) \geq zy'(z)$, by integrating this inequality from $h(z,\delta)$ to z, we get

$$y(h(z,\delta)) \geq \frac{h(z,\delta)}{z} y(z).$$

Hence, we have

$$f(y(h(z,\delta))) \geq r\frac{h^{\gamma}(s,\delta)}{s^{\gamma}} y^{\gamma}(z).$$

Integrating (1) from z to u and using $y'(z) > 0$, we obtain

$$\begin{aligned} b(u)\left(y'''(u)\right)^{\gamma} - b(z)\left(y'''(z)\right)^{\gamma} &= -\int_z^u \int_c^d q(z,\ \delta)\, f(y(h(z,\ \delta)))\, d(\delta) \\ &\leq -ry^{\gamma}(z) \int_z^u \int_c^d q(z,\ \delta)\, \frac{h^{\gamma}(s,\delta)}{s^{\gamma}} d(\delta)\, ds. \end{aligned}$$

Letting $u \to \infty$, we see that

$$b(z)\left(y'''(z)\right)^{\gamma} \geq ry^{\gamma}(z) \int_z^{\infty} \int_c^d q(z,\ \delta)\, \frac{h^{\gamma}(s,\delta)}{s^{\gamma}} d(\delta)\, ds,$$

and so,

$$y'''(z) \geq y(z) \left[\frac{r}{b(z)} \int_z^{\infty} \int_c^d q(z,\ \delta)\, \frac{h^{\gamma}(s,\delta)}{s^{\gamma}} d(\delta)\, ds\right]^{1/\gamma}.$$

Integrating again from z to ∞, we get

$$y''(z) \leq -y(z) \int_z^{\infty} \left[\frac{r}{b(v)} \int_v^{\infty} \int_c^d q(z,\ \delta)\, \frac{h^{\gamma}(s,\delta)}{s^{\gamma}} d(\delta)\, ds\right]^{1/\gamma} dv. \tag{6}$$

Now, we define

$$w(z) = \vartheta(z) \left[\frac{y'(z)}{y(z)} + \frac{1}{\pi(z)}\right].$$

Then $w(z) > 0$ for $z \geq z_1$. By differentiating and using (6), we find

$$w'(z) = \frac{\vartheta'(z)}{\vartheta(z)} w(z) + \vartheta(z) \frac{y''(z)}{y(z)} - \vartheta(z) \frac{(y'(z))^2}{y^2(z)} - \frac{\vartheta(z)}{b^{\frac{1}{\gamma}}(z)\,\pi^2(z)}$$

Thus, we obtain

$$w'(z) = \frac{\vartheta'(z)}{\vartheta(z)} w(z) + \vartheta(z) \frac{y''(z)}{y(z)} - \vartheta(z) \left[\frac{w(z)}{\rho(z)} - \frac{1}{\pi(z)}\right]^2 - \frac{\vartheta(z)}{b^{\frac{1}{\gamma}}(z)\,\pi^2(z)}. \tag{7}$$

Using Lemma 1 with $A = \frac{w(z)}{\rho(z)}$, $B = \frac{1}{\pi(z)}$ and $\gamma = 1$, we get

$$\left[\frac{w(z)}{\vartheta(z)} - \frac{1}{\pi(z)}\right]^2 \geq \left(\frac{w(z)}{\vartheta(z)}\right)^2 - \frac{1}{\pi(z)}\left(\frac{2w(z)}{\vartheta(z)} - \frac{1}{\pi(z)}\right). \tag{8}$$

From (1), (7) and (8), we obtain

$$\begin{aligned} w'(z) &\leq \frac{\vartheta'(z)}{\vartheta(z)} w(z) - \vartheta(z) \int_z^\infty \left[\frac{r}{b(v)} \int_v^\infty \int_c^d q(z,\ \delta) \frac{h^\gamma(s,\delta)}{s^\gamma} d(\delta)\, ds \right]^{1/\gamma} dv \\ &\quad - \vartheta(z) \left[\left(\frac{w(z)}{\vartheta(z)} \right)^2 - \frac{1}{\pi(z)} \left(\frac{2w(z)}{\vartheta(z)} - \frac{1}{\pi(z)} \right) \right] - \frac{\vartheta(z)}{b^{\frac{1}{\gamma}}(z)\, \pi^2(z)}. \end{aligned}$$

This implies that

$$\begin{aligned} w'(z) &\leq \left(\frac{\vartheta'_+(z)}{\vartheta(z)} + \frac{2}{\pi(z)} \right) w(z) - \frac{1}{\vartheta(z)} w^2(z) \\ &\quad - \vartheta(z) \left[\int_z^\infty \left[\frac{r}{b(v)} \int_v^\infty \int_c^d q(z,\ \delta) \frac{h^\gamma(s,\delta)}{s^\gamma} d(\delta)\, ds \right]^{1/\gamma} dv - \frac{1 + b^{\frac{1}{\gamma}}(z)}{\pi^2(z)} \right]. \end{aligned} \tag{9}$$

Thus, inequality (9) yields

$$w'(z) \leq -\psi^*(z) + \phi^*(z)\, w(z) - \frac{1}{\vartheta(z)} w^2(z). \tag{10}$$

Applying the Lemma 1 with $U = \phi^*(z)$, $V = \frac{1}{\vartheta(z)}$, $\gamma = 1$ and $y = w$, we get

$$w'(z) \leq -\psi^*(z) + \frac{1}{4} \vartheta(z) (\phi^*(z))^2. \tag{11}$$

Integrating from z_1 to z, we get

$$\int_{z_1}^z \left(\psi^*(s) - \frac{1}{4} \vartheta(s) (\phi^*(s))^2 \right) ds \leq w(z_1),$$

this contradicts (4). □

Corollary 1. *Let (1.2) holds. If*

$$\liminf_{z\to\infty} \int_{h(z,\delta)}^z Q(z) f\left(\frac{\lambda_0^3 h(z,\delta)}{6b^{1/\gamma}(h(z,\delta))} \right) ds > \frac{1}{e} \tag{12}$$

and (4) hold for some constant $\lambda \in (0,1)$, then every solution of (1) is oscillatory.

3. Example

In this section, we give the following examples to illustrate our main results.

Example 1. *Consider the fourth-order differential equation*

$$\left(\mathbf{z}^3 \left(y'''(z) \right)^3 \right)' + \int_0^1 z^{-3} \delta y^3 \left(\frac{z-\delta}{3} \right) d\delta = 0. \tag{13}$$

We note that $\gamma = 3$, $b(z) = \mathbf{z}^3$, $q(z,\delta) = z^{-3}\delta$, $c = 0$, $d = 1$, $h(z,c) = \frac{z}{3}$, $\pi(z) = \infty$ and $Q(z) = \frac{1}{2z^3}$. If we choose $\vartheta(z) = r(z) = 1$, then it easy to see that the condition (4) holds. Hence, by Theorem 1, every solution of Equation (13) is oscillatory.

Example 2. *Consider the following differential equation of fourth order*

$$\left(y'''(z)\right)' + \int_0^1 \left(\nu \backslash z^4\right) \delta y\left(\frac{z-\delta}{2}\right) d\delta = 0, \tag{14}$$

where $\nu > 0$ is a constant. Let

$$\gamma = 1,\ b(z) = 1,\ c = 0,\ d = 1,\ h(z,c) = \frac{z}{2},\ q(z,\delta) = \left(\nu \backslash z^4\right)\delta.$$

Then, we get

$$Q(t) = \int_0^1 q(z,\ \delta)\, d\delta = \frac{\nu}{2z^4}.$$

If we now set $k = 1$, thus, by Corollary 1, every solution of Equation (14) is oscillatory, provided $\nu > \frac{144}{\lambda_0}$.

Remark 1. *Our results supplement and improve the results obtained in [3].*

4. Conclusions

The results of this paper are presented in a form which is essentially new and of a high degree of generality. In this paper, using the generalized Riccati transformations and comparison technique, we offer some new sufficient conditions which ensure that any solution of Equation (1) oscillates under the condition (2). Further, we can consider the case of $h(z,\delta) \geq z$ in the future work.

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