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A Note on Anosov Homeomorphisms

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Abstract: For an α-expansive homeomorphism of a compact space we give an elementary proof of the following well-known result in topological dynamics: A sufficient condition for the homeomorphism to have the shadowing property is that it has the α-shadowing property for one-jump pseudo orbits (known as the local product structure property). The proof relies on a reformulation of the property of expansiveness in terms of the pseudo orbits of the system.

Keywords: shadowing property; expansive homeomorphism; pseudo orbit

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1. Introduction

In [1] (Theorem 1.2.1) it is proved, among other things, that *Anosov diffeomorphisms* has the *shadowing property*, called *pseudo orbit tracing property* there. In the proof, on [1] (p. 23), the authors only uses the so-called *local product structure property*: if $d(x,y) < \delta$ then $W^s_\varepsilon(x) \cap W^u_\varepsilon(y) \neq \varnothing$ if $\delta > 0$ is chosen small enough for a given $\varepsilon > 0$, and the special (hyperbolic) properties of the metric d coming from the Riemannian structure of the manifold supporting the system [1] ((B), p. 20).

As can be easily checked the first of these two conditions is equivalent to the shadowing property for pseudo orbits with one jump, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every bi-sequence of points of the form

$$\ldots,\ z_{-2} = T^{-2}y,\ z_{-1} = T^{-1}y,\ z_0 = x,\ z_1 = Tx,\ z_2 = T^2x,\ \ldots$$

with $d(x,y) < \delta$, where T denotes the diffeomorphism, there exists a point z such that $d(T^n z, z_n) < \varepsilon$ for all $n \in \mathbb{Z}$.

On the other hand, in [2] (Theorem 5.1) it is shown that for every expansive homeomorphism on a compact space there exists a compatible metric (which we call *hyperbolic metric*) with similar properties to those of the metric d in the case of Anosov diffeomorphisms. Then the proof of the shadowing property for Anosov diffeomorphisms given in [1] carries over the more general case of expansive systems.

In this paper, we give an alternative and elementary proof of this well-known shadowing condition (Proposition 4), not making use of Fathi's hyperbolic metric. Instead we use a reformulation of the property of expansiveness of a system (Proposition 1) which seems interesting in its own right.

2. Terminology and Notation

In this note X denotes a compact metric space with metric d and $T\colon X \to X$ a homeomorphism. The *orbit* of a point $x \in X$ is the bi-sequence $O(x) = (T^n x)_{n\in\mathbb{Z}}$.

Definition 1. *T is said to be* expansive *if there exists a constant $\alpha > 0$, called* expansivity constant, *such that if $x, y \in X$ and $d(T^n x, T^n y) \leq \alpha$ for all $n \in \mathbb{Z}$ then $x = y$.*

Expansive homeomorphisms was introduced in [3] with the name *unestable homeomorphisms*.

Definition 2. *Let $\xi = (x_n)_{n\in\mathbb{Z}}$ be a bi-sequence of elements of X. If $\delta > 0$ and $d(Tx_n, x_{n+1}) < \delta$ for all $n \in \mathbb{Z}$ then ξ is called* δ-pseudo orbit*. We say that ξ has a* jump *at the n-th step if $Tx_{n-1} \neq x_n$. Given $\varepsilon > 0$ a bi-sequence $\eta = (y_n)_{n\in\mathbb{Z}}$ is said to* ε-shadow *ξ if $d(x_n, y_n) < \varepsilon$ for all $n \in \mathbb{Z}$. If in the previous situation $\eta = O(x)$ is the orbit of a point $x \in X$ we simply say that x ε-shadows ξ and that ξ is ε-shadowed (or ε-shadowable).*

Definition 3. *Given $\varepsilon > 0$ we say that T has the* ε-shadowing property *if for some $\delta > 0$ every δ-pseudo orbit is ε-shadowable. We say that T has the* shadowing property *if it has the ε-shadowing property for all $\varepsilon > 0$. If T is expansive and has the shadowing property then it is called* Anosov homeomorphism*.*

3. Rephrasing Expansivity

The following simple result states an equivalent condition for the expansiveness of the system (X, T). This alternative characterization of expansiveness will allow us to give an elementary proof of the shadowing condition in Proposition 4.

Proposition 1. *Let $\alpha > 0$. The following conditions are equivalent.*

(1) T is expansive with expansivity constant α.
(2) For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if} \quad d(x_n, y_n) \le \alpha \text{ for all } n \in \mathbb{Z} \quad \text{then} \quad d(x_n, y_n) < \varepsilon \text{ for all } n \in \mathbb{Z},$$

for every pair of δ-pseudo orbits $(x_n)_{n\in\mathbb{Z}}$ and $(y_n)_{n\in\mathbb{Z}}$ of T.

Proof. $(1 \Rightarrow 2)$ Suppose that the thesis is not true. Then, there exists $\varepsilon > 0$ such that for every $k \in \mathbb{N}$ one can find $1/k$-pseudo orbits $(x_n^k)_{n\in\mathbb{Z}}$ and $(y_n^k)_{n\in\mathbb{Z}}$ of T satisfying $d(x_n^k, y_n^k) \le \alpha$ for all $n \in \mathbb{Z}$ but $d(x_{n_k}^k, y_{n_k}^k) \ge \varepsilon$ for a suitable $n_k \in \mathbb{Z}$. Changing the indexing of the pseudo orbits if necessary it can be assumed that $n_k = 0$ for all $k \in \mathbb{N}$. As X is compact it can be also assumed that $x_0^k \to x$ and $y_0^k \to y$ for some $x, y \in X$. It is easy to see that then $x_n^k \to T^n x$ and $y_n^k \to T^n y$ for all $n \in \mathbb{Z}$ (the pseudo orbits converge pointwise to actual orbits). However, now, as $d(x_n^k, y_n^k) \le \alpha$ for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}$ we have $d(T^n x, T^n y) \le \alpha$ for all $n \in \mathbb{Z}$, and as $d(x_0^k, y_0^k) \ge \varepsilon$ for al $k \in \mathbb{N}$ we get $d(x, y) \ge \varepsilon$, so that $x \neq y$. This contradicts that α in an expansivity constant and the proof finishes.

$(2 \Rightarrow 1)$ Suppose $x, y \in X$ verifies $d(T^n x. T^n y) \le \alpha$ for all $n \in \mathbb{Z}$, and note that $(T^n x)_{n\in\mathbb{Z}}$ and $(T^n y)_{n\in\mathbb{Z}}$ are δ-pseudo orbits for every $\delta > 0$. Therefore, by the hypothesis, for every $\varepsilon > 0$ we have $d(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{Z}$, that is, $(T^n x)_{n\in\mathbb{Z}} = (T^n y)_{n\in\mathbb{Z}}$. Then $x = y$ and hence α is an expansivity constant. □

For future reference we recall from [4] (Theorem 5) the following basic property of expansive homeomorphisms on compact spaces known as *uniform expansivity*.

Proposition 2. *Let $\alpha > 0$. The following conditions are equivalent.*

(1) T is expansive with expansivity constant α.
(2) For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $x, y \in X$

$$\text{if} \quad d(T^n x, T^n y) \le \alpha \text{ for all } |n| \le N \quad \text{then} \quad d(x, y) < \varepsilon.$$

We also recall the following easy result that can be found in [5] (Lemma 8).

Proposition 3. *T has the shadowing property if and only if T has the shadowing property for pseudo orbits with a finite number of jumps.*

4. The Shadowing Condition

As pointed out in the Introduction the next is a known result that can be proved with the techniques in [1] (p. 23) replacing the metric coming from the Riemannian structure in that argument by Fathi's hyperbolic metric [2] (Theroem 5.1).

Proposition 4. *If T is expansive with expansivity constant $\alpha > 0$ then the following conditions are equivalent.*

(1) T has the shadowing property.
(2) There exists $\delta > 0$ such that every one-jump δ-pseudo orbit is α-shadowed.

Proof. Clearly we only need to prove that the last statement implies the first one. By Proposition 3 it is enough to show that for every $\varepsilon > 0$ there exists $\rho > 0$ such that all ρ-pseudo orbits with a finite number of jumps are ε-shadowed. To do that it is sufficient to find a $\rho > 0$ corresponding only to $\varepsilon = \alpha$, because by Proposition 1 for any $\varepsilon > 0$ taking a smaller value of ρ, more precisely choosing $\rho \leq \delta$ where δ is given by the cited proposition, we have that to α-shadow a ρ-pseudo orbit is equivalent to ε-shadow it.

To find $\rho > 0$ such that every ρ-pseudo orbit with a finite number of jumps is α-shadowed, let $\delta > 0$ be as in the statement of this proposition, that is, such that

$$\textit{every } \delta\textit{-pseudo orbit with one jump is } \alpha\textit{-shadowed.} \tag{1}$$

By Proposition 1 (with $\varepsilon = \alpha/2$) we can take a smaller δ to also guarantee that

$$\textit{if a } \delta\textit{-pseudo orbit } \xi \ \alpha\textit{-shadows a } \delta\textit{-pseudo orbit } \eta \textit{ then } \xi \ \alpha/2\textit{-shadows } \eta. \tag{2}$$

Obviously we can also require that $\delta \leq \alpha$. For this δ there exists $N \in \mathbb{N}$ such that

$$\textit{if } d(T^n x, T^n y) \leq \alpha \textit{ for all } |n| \leq N \text{ then } d(x,y) < \delta, \tag{3}$$

for all $x, y \in X$, according to Proposition 2. Finally, as T is uniformly continuous we can take $\rho > 0$, $\rho \leq \delta$, such that any segment of length $2N+1$ of a ρ-pseudo orbit, say $x_0, \dots, x_{2N+1}$, is δ-shadowed by its first element x_0, that is,

$$\textit{if } d(Tx_n, x_{n+1}) < \rho,\ 0 \leq n \leq 2N, \textit{ then } d(T^n x_0, x_n) < \delta,\ 0 \leq n \leq 2N+1. \tag{4}$$

We will prove that this ρ works by induction in the number of jumps in the ρ-pseudo orbits. If a ρ-pseudo orbit has only one jump, as $\rho \leq \delta$ we know by condition (1) that it can be α-shadowed. Assume now that $\xi = (x_n)_{n\in\mathbb{Z}}$ is a ρ-pseudo orbit with $k \geq 2$ jumps. Indices can be arranged so that the last jump takes place in the step from x_{2N} to x_{2N+1}, so that $(x_n)_{n>2N}$ is a segment of a true orbit. By condition (4) we can replace $(x_n)_{n=0}^{2N}$ by $(T^n x_0)_{n=0}^{2N}$ in ξ getting a δ-pseudo orbit

$$\xi' = (x_n)_{n<0} \sqcup (T^n x_0)_{0\leq n\leq 2N} \sqcup (x_n)_{n>2N},$$

where we denote $(x_n)_{n\in I} \sqcup (x_n)_{n\in J} = (x_n)_{n\in I\cup J}$ if $I, J \subseteq \mathbb{Z}$ are disjoint sets of indices. We have that ξ' α-shadows ξ because $\delta \leq \alpha$. Note that on one hand the bi-sequence given by

$$\eta = (x_n)_{n<0} \sqcup (T^n x_0)_{n\geq 0}$$

is a ρ-pseudo orbit with less than k jumps, then by the inductive hypothesis there exists $y \in X$ that α-shadows η. By condition (2) we know that in fact η is $\alpha/2$-shadowed by y. On the other hand consider

$$\zeta = (T^n x_0)_{n\leq 2N} \sqcup (x_n)_{n>2N}$$

which is a δ-pseudo orbit with one jump. Then by condition (1) there exists $z \in X$ that α-shadows ζ. Again condition (2) implies that ζ is $\alpha/2$-shadowed by z.

Now, as the segment of orbit $(T^n x_0)_{0 \le n \le 2N}$ is in both sequences η and ζ we have that the corresponding segments of the orbits of y and z verifies $d(T^n y, T^n z) < \alpha$ for $0 \le n \le 2N$. Hence, by condition (3) we have that $d(T^N y, T^N z) < \delta$. Consequently

$$\tau = (T^n y)_{n<N} \sqcup (T^n z)_{n \ge N}$$

is a one-jump δ-pseudo orbit that $\alpha/2$-shadows ξ'. A new application of condition (1) gives an element $w \in X$ that α-shadows τ. Finally, as w α-shadows τ, τ $\alpha/2$-shadows ξ' and ξ' α-shadows ξ, we obtain by repeated application of condition (2) that w α-shadows ξ, and we are done. $\square$

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References

1. Aoki, N.; Hiraide, K. *Topological Theory of Dynamical Systems*; Elsevier: North-Holland, The Netherlands, 1994.
2. Fathi, A. Expansiveness, hyperbolicity and Hausdorff dimension. *Commun. Math. Phys.* **2005**, *126*, 249–262. [CrossRef]
3. Utz, W.R. Unstable homeomorphisms. *Proc. Am. Math. Soc.* **1950**, *1*, 769–774. [CrossRef]
4. Bryant, B.F. Expansive self-homeomorphisms of a compact metric space. *Am. Math. Mon.* **1962**, *69*, 386–391. [CrossRef]
5. Walters, P. On the pseudo orbit tracing property and its relationship to stability. *Struct. Attractors Dyn. Syst.* **1978**, *668*, 231–244.