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New Bell–Sheffer Polynomial Sets

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Abstract: In recent papers, new sets of Sheffer and Brenke polynomials based on higher order Bell numbers, and several integer sequences related to them, have been studied. The method used in previous articles, and even in the present one, traces back to preceding results by Dattoli and Ben Cheikh on the monomiality principle, showing the possibility to derive explicitly the main properties of Sheffer polynomial families starting from the basic elements of their generating functions. The introduction of iterated exponential and logarithmic functions allows to construct new sets of Bell–Sheffer polynomials which exhibit an iterative character of the obtained shift operators and differential equations. In this context, it is possible, for every integer r, to define polynomials of higher type, which are linked to the higher order Bell-exponential and logarithmic numbers introduced in preceding papers. Connections with integer sequences appearing in Combinatorial analysis are also mentioned. Naturally, the considered technique can also be used in similar frameworks, where the iteration of exponential and logarithmic functions appear.

Keywords: Sheffer polynomials; generating functions; monomiality principle; shift operators; combinatorial analysis

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1. Introduction

In recent articles [1,2], new sets of Sheffer [3] and Brenke [4] polynomials, based on higher order Bell numbers [2,5–7], have been studied. Furthermore, several integer sequences associated [8] with the considered polynomials sets both of exponential [9,10] and logarithmic type have been introduced [1].

It is worth noting that exponential and logarithmic polynomials have been recently studied in the multidimensional case [11–13].

In this article, new sets of Bell–Sheffer polynomials are considered and some particular cases are analyzed.

It is worth noting that the Sheffer A-type 0 polynomial sets have been also approached with elementary methods of linear algebra (see, e.g., [14–16] and the references therein).

Connection with umbral calculus has been recently emphasized (see, e.g., [17,18] and the references therein).

2. Sheffer Polynomials

The Sheffer polynomials $\{s_n(x)\}$ are introduced [3] by means of the exponential generating function [19] of the type:

$$A(t)\exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x)\,\frac{t^n}{n!}\,, \tag{1}$$

where

$$\begin{aligned} A(t) &= \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, \qquad (a_0 \neq 0), \\ H(t) &= \sum_{n=0}^{\infty} h_n \frac{t^n}{n!}, \qquad (h_0 = 0). \end{aligned} \tag{2}$$

According to a different characterization (see [20], p. 18), the same polynomial sequence can be defined by means of the pair $(g(t), f(t))$, where $g(t)$ is an invertible series and $f(t)$ is a delta series:

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} g_n \frac{t^n}{n!}, \qquad (g_0 \neq 0), \\ f(t) &= \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, \qquad (f_0 = 0,\ f_1 \neq 0). \end{aligned} \tag{3}$$

Denoting by $f^{-1}(t)$ the compositional inverse of $f(t)$ (i.e., such that $f\left(f^{-1}(t)\right) = f^{-1}\left(f(t)\right) = t$), the exponential generating function of the sequence $\{s_n(x)\}$ is given by

$$\frac{1}{g[f^{-1}(t)]} \exp\left(x f^{-1}(t)\right) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \tag{4}$$

so that

$$A(t) = \frac{1}{g[f^{-1}(t)]}, \qquad H(t) = f^{-1}(t). \tag{5}$$

When $g(t) \equiv 1$, the Sheffer sequence corresponding to the pair $(1, f(t))$ is called the associated Sheffer sequence $\{\sigma_n(x)\}$ for $f(t)$, and its exponential generating function is given by

$$\exp\left(x f^{-1}(t)\right) = \sum_{n=0}^{\infty} \sigma_n(x) \frac{t^n}{n!}. \tag{6}$$

A list of known Sheffer polynomial sequences and their associated ones can be found in [21].

3. New Bell–Sheffer Polynomial Sets

We introduce, for shortness, the following compact notation.
Put, by definition:

$$\begin{aligned} &E_0(t) := \exp(t) - 1, \\ &E_1(t) := E_0(E_0(t)) = \exp(\exp(t) - 1) - 1 \\ &\cdots\cdots\cdots \\ &E_r(t) := E_0(E_{r-1}(t)) = \exp(\ldots\exp(\exp(t) - 1) - 1)\cdots - 1 \qquad [(r+1) - \text{ times } \ \exp], \end{aligned}$$

and in a similar way:

$$\begin{aligned} &\Lambda_0(t) := \log(t+1) \\ &\Lambda_1(t) := \Lambda_0(\Lambda_0(t)) = \log(\log(t+1) + 1) \\ &\cdots\cdots\cdots \\ &\Lambda_r(t) := \Lambda_0(\Lambda_{r-1}(t)) = \log\left(\log\left(\ldots\left(\log(t+1) + 1\right)\ldots\right) + 1\right), \qquad [(r+1) - \text{ times } \ \log]. \end{aligned}$$

Remark 1. *Note that, for all integers r, k, h,*

$$E_r(\Lambda_r(t)) = t\,, \qquad \Lambda_r(E_r(t)) = t\,,$$

$$(\text{if } k > h) \qquad E_k(\Lambda_h(t)) = E_{k-h-1}(t)\,, \quad E_h(\Lambda_k(t)) = \Lambda_{k-h-1}(t)\,,$$

$$(\text{if } k > h) \qquad \Lambda_k(E_h(t)) = \Lambda_{k-h-1}(t)\,, \quad \Lambda_h(E_k(t)) = E_{k-h-1}(t)\,,$$

$$e^{E_r(t)} = E_{r+1}(t) + 1\,, \qquad e^{\Lambda_r(t)} = \Lambda_{r-1}(t) + 1\,.$$

Remark 2. *Note that the coefficients of the Taylor expansion of $E_1(t)$ are given by the Bell numbers $b_n = b_n^{[1]}$*

$$E_1(t) = \sum_{n=1}^{\infty} b_n^{[1]} \frac{t^n}{n!}\,,$$

and, in general, the coefficients of the Taylor expansion of $E_r(t)$ are given by the higher order Bell numbers $b_n^{[r]}$

$$E_r(t) = \sum_{n=1}^{\infty} b_n^{[r]} \frac{t^n}{n!}\,.$$

The higher order Bell numbers, also known as higher order exponential numbers, have been considered in [5,7,22], and used in [2] in the framework of Brenke and Sheffer polynomials.

Remark 3. *Note that the coefficients of the Taylor expansion of $\Lambda_0(t)$ are given by the logarithmic numbers $l_n^{[1]} = (-1)^{n-1}(n-1)!$*

$$\Lambda_0(t) = \sum_{n=1}^{\infty} l_n^{[1]} \frac{t^n}{n!} = \sum_{n=1}^{\infty} (-1)^{n-1}(n-1)!\, \frac{t^n}{n!}\,,$$

and, in general, the coefficients of the Taylor expansion of $\Lambda_{r-1}(t)$ are given by the higher order logarithmic numbers $l_n^{[r]}$

$$\Lambda_{r-1}(t) = \sum_{n=1}^{\infty} l_n^{[r]} \frac{t^n}{n!}\,.$$

The higher order logarithmic numbers, which are the counterpart of the higher order Bell (exponential) numbers, have been considered in [1], and used there in the framework of new sets of Sheffer polynomials.

3.1. The Polynomials $\mathcal{E}_k^{(1)}(x)$

Therefore, we consider the Sheffer polynomials, defined through their generating function, by putting

$$\begin{aligned} &A(t) = E_1(t) + 1 = e^{E_0(t)}, \qquad H(t) = E_0(t)\,, \\ &G(t,x) = \exp\left[(1+x)E_0(x)\right] = \sum_{k=0}^{\infty} \mathcal{E}_k^{(1)}(x)\, \frac{t^k}{k!}\,. \end{aligned} \tag{7}$$

3.2. Recurrence Relation for the $\mathcal{E}_k^{(1)}(x)$

Theorem 1. *For any $k \geq 0$, the polynomials $\mathcal{E}_k^{(1)}(x)$ satisfy the following recurrence relation:*

$$\mathcal{E}_{k+1}^{(1)}(x) = \sum_{h=0}^{k} \binom{k}{h} (1+x)\, \mathcal{E}_h^{(1)}(x)\,. \tag{8}$$

Proof. Differentiating $G(t,x)$ with respect to t, we have

$$\frac{\partial G(t,x)}{\partial t} = G(t,x)\, e^t\, (1+x)\,, \tag{9}$$

and therefore

$$\sum_{k=0}^{\infty} (1+x)\, \mathcal{E}_k^{(1)}(x)\, \frac{t^k}{k!} \sum_{k=0}^{\infty} \frac{t^k}{k!} = \sum_{k=0}^{\infty} \mathcal{E}_{k+1}^{(1)}(x) \frac{t^k}{k!}\,,$$

i.e.,

$$\sum_{k=0}^{\infty} \sum_{h=0}^{k} \binom{k}{h} (1+x)\, \mathcal{E}_h^{(1)}(x)\, \frac{t^k}{k!} = \sum_{k=0}^{\infty} \mathcal{E}_{k+1}^{(1)}(x) \frac{t^k}{k!}$$

so that the recurrence relation (8) follows. □

3.3. Generating Function's PDEs

Theorem 2. *The generating function* $(7)_2$ *satisfies the homogeneous linear PDEs:*

$$(1-e^{-t})\, \frac{\partial G(t,x)}{\partial t} = (1+x)\, \frac{\partial G(t,x)}{\partial x}\,, \tag{10}$$

$$\frac{\partial G(t,x)}{\partial t} = \frac{\partial G(t,x)}{\partial x} + G(t,x)(1+xe^t)\,, \tag{11}$$

$$\frac{\partial G(t,x)}{\partial t} = (1+x) \left[\frac{\partial G(t,x)}{\partial x} + G(t,x)\right]\,. \tag{12}$$

Proof. Differentiating $G(t,x)$ with respect to x, we have

$$\frac{\partial G(t,x)}{\partial x} = G(t,x)\,(e^t - 1)\,. \tag{13}$$

By taking the ratio between the members of Equations (9) and (13), we find Equation (10). The other ones easily follows by elementary algebraic manipulations. □

3.4. Shift Operators

We recall that a polynomial set $\{p_n(x)\}$ is called quasi-monomial if and only if there exist two operators $\hat{P}$ and $\hat{M}$ such that

$$\hat{P}\,(p_n(x)) = n p_{n-1}(x)\,, \qquad \hat{M}\,(p_n(x)) = p_{n+1}(x)\,, \qquad (n = 1, 2, \ldots). \tag{14}$$

$\hat{P}$ is called the *derivative* operator and $\hat{M}$ the *multiplication* operator, as they act in the same way of classical operators on monomials.

This definition traces back to a paper by Steffensen [23], recently improved by Dattoli [24] and widely used in several applications.

Ben Cheikh [25] proved that every polynomial set is quasi-monomial under the action of suitable derivative and multiplication operators. In particular, in the same article (Corollary 3.2), the following result is proved.

Theorem 3. *Let $(p_n(x))$ denote a Boas–Buck polynomial set, i.e., a set defined by the generating function*

$$A(t)\psi(xH(t)) = \sum_{n=0}^{\infty} p_n(x)\,\frac{t^n}{n!}\,, \tag{15}$$

where

$$\begin{aligned} A(t) &= \sum_{n=0}^{\infty} a_n t^n\,, \qquad (a_0 \neq 0)\,, \\ \psi(t) &= \sum_{n=0}^{\infty} \gamma_n t^n\,, \qquad (\gamma_n \neq 0 \quad \forall n)\,, \end{aligned} \tag{16}$$

with $\psi(t)$ not a polynomial, and lastly

$$H(t) = \sum_{n=0}^{\infty} h_n\, t^{n+1}\,, \qquad (h_0 \neq 0)\,. \tag{17}$$

Let $\sigma \in \Lambda^{(-)}$ the lowering operator defined by

$$\sigma(1) = 0\,, \qquad \sigma(x^n) = \frac{\gamma_{n-1}}{\gamma_n}\, x^{n-1}\,, \qquad (n = 1, 2, \ldots). \tag{18}$$

Put

$$\sigma^{-1}(x^n) = \frac{\gamma_{n+1}}{\gamma_n}\, x^{n+1} \quad (n = 0, 1, 2, \ldots). \tag{19}$$

Denoting, as before, by $f(t)$ the compositional inverse of $H(t)$, the Boas–Buck polynomial set $\{p_n(x)\}$ is quasi-monomial under the action of the operators

$$\hat{P} = f(\sigma)\,, \qquad \hat{M} = \frac{A'[f(\sigma)]}{A[f(\sigma)]} + xD_x H'[f(\sigma)]\sigma^{-1}\,, \tag{20}$$

where prime denotes the ordinary derivatives with respect to t.

Note that, in our case, we are dealing with a Sheffer polynomial set, so that since we have $\psi(t) = e^t$, the operator σ defined by Equation (16) simply reduces to the derivative operator D_x. Furthermore, we have:

$$\begin{aligned} &f(t) = H^{-1}(t) = \Lambda_0(t)\,, \\ &\frac{A'(t)}{A(t)} = e^t\,, \qquad H'(t) = e^t\,, \end{aligned}$$

and, consequently,

$$f(\sigma) = \Lambda_0(D_x)\,, \qquad \frac{A'[\Lambda_0(D_x)]}{A[\Lambda_0(D_x)]} = D_x + 1\,,$$

$$H'[f(\sigma)] = H'[\Lambda_0(D_x)] = D_x + 1\,.$$

Theorem 4. *The Bell–Sheffer polynomials* $\{\mathcal{E}_k^{(1)}(x)\}$ *are quasi-monomial under the action of the operators*

$$\begin{aligned} \hat{P} &= \Lambda_0(D_x) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{D_x^k}{k}\,, \\ \hat{M} &= (1+x)(D_x+1)\,. \end{aligned} \tag{21}$$

3.5. Differential Equation for the $\mathcal{E}_k^{(1)}(x)$

According to the results of monomiality principle [24], the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}\, p_n(x) = n\, p_n(x)\,. \tag{22}$$

In the present case, recalling Equation (22), we have

Theorem 5. *The Bell–Sheffer polynomials* $\{\mathcal{E}_k^{(1)}(x)\}$ *satisfy the differential equation*

$$(1+x)\, \sum_{k=1}^{n} (-1)^{k+1} \left[\frac{D_x^{k+1} + D_x^k}{k!} \right] \mathcal{E}_n^{(1)}(x) = n\, \mathcal{E}_n^{(1)}(x)\,. \tag{23}$$

Proof. Equation (22), by using Equation (21), becomes

$$\begin{aligned} \hat{M}\hat{P}\, \mathcal{E}_n^{(1)}(x) &= (1+x)\,(D_x+1)\,\Lambda_0(D_x)\,\mathcal{E}_n^{(1)}(x) = \\ &= (1+x)\,(D_x+1)\, \sum_{k=1}^{n} (-1)^{k+1} \frac{D_x^k}{k!}\, \mathcal{E}_n^{(1)}(x) = n\, \mathcal{E}_n^{(1)}(x)\,, \end{aligned}$$

i.e.,

$$(1+x)\, \sum_{k=1}^{\infty} (-1)^{k+1} \left[\frac{D_x^{k+1} + D_x^k}{k!} \right] \mathcal{E}_n^{(1)}(x) = n\, \mathcal{E}_n^{(1)}(x)\,,$$

and, furthermore, for any fixed n, the last series expansion reduces to a finite sum, with upper limit $n-1$, when it is applied to a polynomial of degree n because the last not vanishing term (for $k = n-1$) contains the derivative of order n. □

3.6. First Few Values of the $\mathcal{E}_k^{(1)}(x)$

Here, we show the first few values for the Bell–Sheffer polynomials $\mathcal{E}_k^{(1)}(x)$, defined by the generating function (7)$_2$

$$\begin{aligned}
&\mathcal{E}_0^{(1)}(x) = 1,\\
&\mathcal{E}_1^{(1)}(x) = x + 1,\\
&\mathcal{E}_2^{(1)}(x) = x^2 + 3x + 2,\\
&\mathcal{E}_3^{(1)}(x) = x^3 + 6x^2 + 10x + 5,\\
&\mathcal{E}_4^{(1)}(x) = x^4 + 10x^3 + 31x^2 + 37x + 15,\\
&\mathcal{E}_5^{(1)}(x) = x^5 + 15x^4 + 75x^3 + 160x^2 + 151x + 52.
\end{aligned}$$

Further values can be easily achieved by using Wolfram Alpha© (2009, Wolfram Research, Champaign, IL, USA).

Remark 4. *Note that the numerical values $\mathcal{E}_k^{(1)}(0)$ of the considered Bell–Sheffer polynomials*

$$(1, 1, 2, 5, 15, 52, 203, 877, \dots)$$

appears in the Encyclopedia of Integer Sequences [8] under A000110: Bell or exponential numbers: number of ways to partition a set of n labeled elements.

The same sequence also appears under A164864, A164863, A276723, A276724, A276725, A276726, A287278, A287279, A287280.

4. Iterated Bell–Sheffer Polynomial Sets

Here, we iterate the procedure introduced in Section 3, by considering the Sheffer polynomial sets defined by putting

$$\begin{aligned}
&A(t) = E_2(t) + 1 = e^{E_1(t)}, \qquad H(t) = E_1(t)\,,\\
&G(t, x) = \exp\left[(1 + x)E_1(x)\right] = \sum_{k=0}^{\infty} \mathcal{E}_k^{(2)}(x)\,\frac{t^k}{k!}\,.
\end{aligned} \tag{24}$$

We find:

$$f(t) = H^{-1}(t) = \Lambda_1(t)\,,$$

$$\frac{A'(t)}{A(t)} = H'(t) = [E_1(t) + 1]\,e^t = [E_1(t) + 1]\,[E_0(t) + 1]\,,$$

and, consequently,

$$\begin{aligned}
&f(\sigma) = \Lambda_1(D_x)\,, \qquad H'[f(\sigma)] = H'[\Lambda_1(D_x)] = [D_x + 1]\ [\Lambda_0(D_x) + 1]\ ,\\
&\frac{A'[\Lambda_1(D_x)]}{A[\Lambda_1(D_x)]} = [E_1(\Lambda_1(D_x)) + 1]\,e^{\Lambda_1(D_x)} = (D_x + 1)\ [\Lambda_0(D_x) + 1]\ .
\end{aligned}$$

Theorem 6. *The Bell–Sheffer polynomials $\{\mathcal{E}_k^{(2)}(x)\}$ are quasi-monomial under the action of the operators*

$$\begin{aligned}
&\hat{P} = \Lambda_1(D_x)\,,\\
&\hat{M} = (1 + x)\,(D_x + 1)\ [\Lambda_0(D_x) + 1]\ .
\end{aligned} \tag{25}$$

4.1. Differential Equation for the $\mathcal{E}_k^{(2)}(x)$

According to the results of monomiality principle [24,26], the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}\, p_n(x) = n\, p_n(x)\,. \tag{26}$$

In the present case, recalling Equation (19), we have

Theorem 7. *The Bell–Sheffer polynomials $\{\mathcal{E}_k^{(2)}(x)\}$ satisfy the differential equation*

$$(1+x)\,(D_x+1)\,[\Lambda_0(D_x)+1]\,\Lambda_1(D_x)\,\mathcal{E}_n^{(2)}(x) = n\,\mathcal{E}_n^{(2)}(x)\,. \tag{27}$$

4.2. First Few Values of the $\mathcal{E}_k^{(2)}(x)$

Here, we show the first few values for the Bell–Sheffer polynomials $\mathcal{E}_k^{(2)}(x)$, defined by the generating function $(7)_2$

$$\begin{aligned}
&\mathcal{E}_0^{(2)}(x) = 1,\\
&\mathcal{E}_1^{(2)}(x) = x+1,\\
&\mathcal{E}_2^{(2)}(x) = x^2+4x+3,\\
&\mathcal{E}_3^{(2)}(x) = x^3+9x^2+20x+12,\\
&\mathcal{E}_4^{(2)}(x)) = x^4+16x^3+74x^2+119x+60,\\
&\mathcal{E}_5^{(2)}(x) = x^5+25x^4+200x^3+635x^2+817x+358.
\end{aligned}$$

Further values can be easily achieved by using Wolfram Alpha©.

Remark 5. *Note that the numerical values $\mathcal{E}_k^{(2)}(0)$ of the considered Bell–Sheffer polynomials*

$$(1, 1, 3, 12, 60, 358, 2471, 19302, \dots)$$

appear in the Encyclopedia of Integer Sequences under A000258: McLaurin coefficients of the function $E_2(x)$.

5. The General Case

In general, by putting

$$\begin{aligned}
&A(t) = E_r(t)+1 = e^{E_{r-1}(t)}, \qquad H(t) = E_{r-1}(t)\,,\\
&G(t,x) = \exp\left[(1+x)E_{r-1}(t)\right] = \sum_{k=0}^{\infty} \mathcal{E}_k^{(r)}(x)\,\frac{t^k}{k!}\,,
\end{aligned} \tag{28}$$

we find:

$$f(t) = H^{-1}(t) = \Lambda_{r-1}(t)\,,$$

$$\frac{A'(t)}{A(t)} = H'(t) = \prod_{\ell=1}^{r-1}\left[E_\ell(t)+1\right]e^t = \prod_{\ell=0}^{r-1}\left[E_\ell(t)+1\right],$$

and, consequently,

$$f(\sigma) = \Lambda_{r-1}(D_x)\,, \qquad \frac{A'[\Lambda_{r-1}(D_x)]}{A[\Lambda_{r-1}(D_x)]} = \prod_{\ell=0}^{r-1} [E_\ell(\Lambda_{r-1}(D_x)) + 1]\,,$$

$$H'[f(\sigma)] = H'[\Lambda_{r-1}(D_x)] = \prod_{\ell=0}^{r-1} [E_\ell(\Lambda_{r-1}(D_x)) + 1]\,.$$

Recalling Remark 3.1, we find

$$E_\ell(\Lambda_{r-1}(D_x)) = \Lambda_{r-\ell-2}(D_x)\,,$$

$$\prod_{\ell=0}^{r-1} [E_\ell(\Lambda_{r-1}(D_x)) + 1] = (D_x + 1) \prod_{k=0}^{r-2} [\Lambda_k(D_x) + 1]\,,$$

so that we have the theorem:

Theorem 8. *The Bell–Sheffer polynomials* $\{\mathcal{E}_k^{(r)}(x)\}$ *are quasi-monomial under the action of the operators*

$$\begin{aligned} \hat{P} &= \Lambda_{r-1}(D_x)\,, \\ \hat{M} &= (1+x)\ (D_x + 1) \prod_{k=0}^{r-2} [\Lambda_k(D_x) + 1]\,. \end{aligned} \tag{29}$$

Differential Equation for the $\mathcal{E}_k^{(r)}(x)$

According to the results of monomiality principle [24], the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}\, p_n(x) = n\, p_n(x)\,. \tag{30}$$

In the present case, recalling Equation (19), we have

Theorem 9. *The Bell–Sheffer polynomials* $\{\mathcal{E}_k^{(r)}(x)\}$ *satisfy the differential equation*

$$(1+x)\ (D_x + 1) \prod_{k=0}^{r-2} [\Lambda_k(D_x) + 1]\, \Lambda_{r-1}(D_x)\, \mathcal{E}_n^{(r)}(x) = n\, \mathcal{E}_n^{(r)}(x)\,. \tag{31}$$

6. Conclusions

By introducing iterated exponential and logarithmic functions, we have shown how to construct new sets of Bell-Sheffer polynomials which exhibit an iterative character. We have found their main properties by using the monomiality property and a general result by Y. Ben Cheikh which gives explicitly the derivative and multiplication operators for polynomials of Sheffer type. The tools we used are internal to Sheffer's polynomial theory and do not use external techniques. In our opinion the demonstrated properties (in particular the differential equations for polynomials of higher order) could hardly be achieved by other methods.

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