*axioms*

MDPI

*Article*

# Special Numbers and Polynomials Including Their Generating Functions in Umbral Analysis Methods

**Yilmaz Simsek**

Department of Mathematics, Faculty of Science University of Akdeniz, TR-07070 Antalya, Turkey; ysimsek@akdeniz.edu.tr

Received: 20 January 2018; Accepted: 30 March 2018; Published: 1 April 2018

**Abstract:** In this paper, by applying umbral calculus methods to generating functions for the combinatorial numbers and the Apostol type polynomials and numbers of order $k$, we derive some identities and relations including the combinatorial numbers, the Apostol-Bernoulli polynomials and numbers of order $k$ and the Apostol-Euler polynomials and numbers of order $k$. Moreover, by using $p$-adic integral technique, we also derive some combinatorial sums including the Bernoulli numbers, the Euler numbers, the Apostol-Euler numbers and the numbers $y_1(n, k; \lambda)$. Finally, we make some remarks and observations regarding these identities and relations.

**Keywords:** Apostol-Bernoulli polynomials and numbers; Apostol-Euler polynomials and numbers; Sheffer sequences; Appell sequences; Fibonacci numbers; umbral algebra, $p$-adic integral.

**MSC:** 05A40, 11B68, 11B73, 11B83, 11S80, 26C05, 30B10.

## 1. Introduction

In order to give the results presented in this paper, we use two techniques which are $p$-adic integral technique and the umbral calculus technique. In [1–5], we constructed generating functions for families of combinatorial numbers which are used in counting techniques and problems and also computing negative order of the first and the second kind Euler numbers and other combinatorial sums. In this paper, by applying umbral algebra and umbral analysis methods and their operators to generating functions of the combinatorial numbers and the Apostol type polynomials and numbers, we give many identities and relations including the Fibonacci numbers, the combinatorial numbers, the Apostol-Bernoulli polynomials and numbers of higher order and the Apostol-Euler polynomials and numbers of higher order.

Throughout this paper, we use the following notations, definitions and relations.

Here and in the following, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ be the sets of complex numbers, real numbers, integers, and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We assume $0^0 = 1$.

Moreover, throughout this paper, $\log z$ is tacitly assumed to denote the principal branch of the many-valued function $\log z$ with the imaginary part $(\log z)$ constrained by

$$-\pi < \operatorname{Im}(\log z) \leq \pi$$

(cf. [6–9]).

The Apostol-Bernoulli polynomials $\mathcal{B}_n^{(k)}(x, \lambda)$ of order $k$ are defined by

$$F_{\mathcal{B}}(t, x; \lambda, k) = \left(\frac{t}{\lambda e^t - 1}\right)^k e^{tx} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(k)}(x, \lambda) \frac{t^n}{n!}, \tag{1}$$

where $\lambda$ is an arbitrary (real or complex) parameter and $x \in \mathbb{R}$, and $|t| < 2\pi$ when $\lambda = 1$ and $|t| < \frac{2\pi}{|\log \lambda|}$ when $\lambda \neq 1$. Moreover, $\mathcal{B}_n^{(k)}(\lambda) := \mathcal{B}_n^{(k)}(0, \lambda)$ denote the Apostol-Bernoulli numbers of order. $B_n^{(k)} := B_n^{(k)}(1)$ denote the Bernoulli numbers of order $k$ and also $B_n := B_n^{(1)}$ denote the Bernoulli numbers (cf. see,for details, [6,8–14], and the references cited therein).

The Apostol-Euler polynomials $\mathcal{E}_n^{(k)}(x, \lambda)$ of order $k$ are defined by

$$F_{\mathcal{E}}(t, x; \lambda, k) = \left(\frac{2}{\lambda e^t + 1}\right)^k e^{tx} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(k)}(x, \lambda) \frac{t^n}{n!}, \tag{2}$$

where $\lambda$ is an arbitrary (real or complex) parameter and $x \in \mathbb{R}$, and $|t| < \pi$ when $\lambda = 1$ and $|t| < \frac{\pi}{|\log \lambda|}$ when $\lambda \neq 1$. Moreover, $\mathcal{E}_n^{(k)}(\lambda) := \mathcal{E}_n^{(k)}(0, \lambda)$ denote the Apostol-Euler numbers of order $k$. $E_n^{(k)} := E_n^{(k)}(1)$ denote the Euler numbers of order $k$ and also $E_n := E_n^{(1)}$ denote the Euler numbers (cf. see, for details, [6,8–15], and the references cited therein).

The $\lambda$-array polynomials $S_v^n(x; \lambda)$ are defined by

$$F_S(t, x, v; \lambda) = \frac{(\lambda e^t - 1)^v}{v!} e^{xt} = \sum_{n=0}^{\infty} S_v^n(x; \lambda) \frac{t^n}{n!} \tag{3}$$

where $v \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (cf. [16]). Furthermore,

$$S_2(n, v; \lambda) := S_v^n(0; \lambda)$$

where, as usual, $S_2(n, v; \lambda)$ denote the $\lambda$-Stirling numbers (cf. [8,12]). Substituting $\lambda = 1$ into (3), we have the array polynomials:

$$S_v^n(x) := S_v^n(x; 1)$$

(cf. [16–18] and (Theorem 2 [19])).

In (cf. Equation (8) [1]), we defined the combinatorial numbers $y_1(n, k; \lambda)$ by means of the following generating function:

$$F_{y_1}(t, k; \lambda) = \frac{1}{k!} (\lambda e^t + 1)^k = \sum_{n=0}^{\infty} y_1(n, k; \lambda) \frac{t^n}{n!} \tag{4}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$.

By using (4), we have

$$y_1(n, k; \lambda) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} j^n \lambda^j \tag{5}$$

where $n \in \mathbb{N}_0$ (cf. Equation (9) [1]).

Relationships between the $\lambda$-array polynomials $S_v^n(x; \lambda)$ and the numbers $y_1(n, k; \lambda)$ and the Stirling numbers of the second kind $S_2(n, k)$ are given below, respectively:

$$S_k^n(0; \lambda) = S_2(n, v; \lambda) = (-1)^k y_1(n, k; -\lambda)$$

and

$$S_2(n, k) = (-1)^k y_1(n, k; -1) \tag{6}$$

(cf. [1,17,20–25]).

The Fibonacci numbers $F_j$ are defined by the following generating function

$$\frac{t}{1 - t - t^2} = \sum_{n=0}^{\infty} F_n t^n$$

(cf. (p. 229. [26])). We need the following well-known formulas for the Fibonacci numbers. Let $\lambda = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Let $j \in \mathbb{N}$, we have

$$\lambda^j = \lambda F_j + F_{j-1} \tag{7}$$

and

$$\beta^j = \beta F_j + F_{j-1}$$

(cf. (p. 78, Lemma 5.1. [26])). Using the above identities, one easily derives the following Binet's formula:

$$F_j = \frac{\lambda^j - \beta^j}{\lambda - \beta}.$$

Substituting $-n$ with $n \in \mathbb{N}$, into the above equation, we easily have

$$F_{-n} = (-1)^{n+1} F_n$$

(cf. (p. 84 [26])).

### *1.1. p-Adic Integrals*

In the last section, we will give some combinatorial sums with $p$-adic integrals technique. Hence, let us give definitions of these integrals and a few properties of them.

Let $f(x) \in C^1(\mathbb{Z}_p \to \mathbb{K})$, a set of continuous derivative functions, and $\mathbb{K}$ is a field with a complete valuation.

The Volkenborn integral (the bosonic $p$-adic integral) is defined by

$$\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{8}$$

where $\mu_1(x) = \mu_1\left(x + p^N\mathbb{Z}_p\right)$ is the Haar distribution on $\mathbb{Z}_p$:

$$\mu_1\left(x + p^N\mathbb{Z}_p\right) = \frac{1}{p^N},$$

(cf. [27,28]). On the other hand, the $p$-adic fermionic integral is defined by

$$\int\limits_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x f(x) \tag{9}$$

where

$$\mu_{-1}(x) = \mu_{-1}\left(x + p^N\mathbb{Z}_p\right) = \frac{(-1)^x}{p^N}$$

(cf. [29]).

The Bernoulli numbers and the Euler numbers are related to the following $p$-adic integrals representations, respectively,

$$B_n = \int\limits_{\mathbb{Z}_p} x^n d\mu_1(x), \tag{10}$$

(cf. [27,28]) and

$$E_n = \int\limits_{\mathbb{Z}_p} x^n d\mu_{-1}(x), \tag{11}$$

(cf. [27]).

### *1.2. Umbral Algebra and Calculus*

Throughout this section, we use the notations and definitions of the Roman's book (cf. [13]). Let $\mathbb{P} = \mathbb{C}[x]$ be the algebra of polynomials in the single variable $x$ over the field of complex numbers. Let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. Let $\langle L \mid p(x)\rangle$ be the action of a linear functional $L$ on a polynomial $p(x)$. Let $\mathcal{F}$ denote the algebra of formal power series

$$f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!},$$

(cf. [13]). Furthermore, for all $n \in \mathbb{N}_0$, one has

$$\langle f(t) \mid x^n \rangle = a_n \tag{12}$$

and also

$$\langle f(t)g(t) \mid p(x)\rangle = \langle f(t) \mid g(t)p(x)\rangle, \tag{13}$$

where $f(t)$, $g(t)$ are in $\mathcal{F}$ (cf. [13]).

For $p(x) \in \mathbb{P}$, as a *linear functional*, we have

$$\left\langle e^{yt} \mid p(x)\right\rangle = p(y). \tag{14}$$

and as a *linear operator*, we have

$$e^{yt}p(x) = p(x+y) \tag{15}$$

(cf. [13]). The Sheffer polynomials for pair $(g(t), f(t))$, where $g(t)$ must be invertible and $f(t)$ must be delta series. The Sheffer polynomials for pair $(g(t), t)$ is the Appell polynomials or Appell sequences for $g(t)$. The Appell polynomials are defined by means of the following generating function

$$\sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k = \frac{1}{g(t)} e^{xt}, \tag{16}$$

(cf. [13]). Some properties of the Appell polynomials are given as follows.

$$s_n(x) = g(t)^{-1} x^n, \tag{17}$$

(p. 86, Theorem 2.5.5 [13]), derivative formula

$$ts_n(x) = ns_{n-1}(x) \tag{18}$$

(cf. p. 86, Theorem 2.5.6 [13]); and see also [6,30,31]).

We summarize the results presented in this paper as follows:

In Section 2, by applying the umbral algebra and umbral calculus methods to generating functions of the special numbers and polynomials, we derive some identities and relations including the numbers $y_1(n,k;\lambda)$, combinatorial sums, the Fibonacci numbers, Apostol-Bernoulli type numbers and polynomials, and the Apostol-Euler type numbers and polynomials. Finally, we give some remarks and observations.

In Section 3, by using the $p$-adic integrals, we give many combinatorial sums related to the Bernoulli numbers, the Euler numbers, the Apostol-Euler numbers and the numbers $y_1(n,k;\lambda)$.

## 2. Identities Including the Numbers $y_1(n,k;\lambda)$, Combinatorial Sums, and Apostol-Euler Type Numbers and Polynomials

In this section, by using the umbral algebra and umbral calculus methods, we derive many identities and relations containing the numbers $y_1(n,k;\lambda)$, combinatorial sums, the Fibonacci numbers, Apostol-Bernoulli type numbers and polynomials, and the Apostol-Euler type numbers and polynomials.

**Theorem 1.**

$$\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^{j}\sum_{v=0}^{m}\binom{m}{v}x^{m-v}j^{v}=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^{j}(x+j)^{m}$$

*or*

$$\sum_{v=0}^{m}\frac{(m)_{v}}{v!}x^{m-v}y_{1}(v,k;\lambda)=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^{j}(x+j)^{m} \tag{19}$$

**Proof.** By applying linear operators in (15) and (18) to (4), respectively, we obtain

$$\frac{1}{k!}\left(\lambda e^{t}+1\right)^{k}x^{m}=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^{j}(x+j)^{m} \tag{20}$$

and

$$\begin{aligned}\frac{1}{k!}\left(\lambda e^{t}+1\right)^{k}x^{m} &= \sum_{n=0}^{\infty}y_{1}(n,k;\lambda)\frac{1}{n!}t^{n}x^{m} \\ &= \begin{cases} 0, & n>m \\ y_{1}(n,k;\lambda), & n=m \\ \sum_{v=0}^{m}\frac{(m)_{v}}{v!}x^{m-v}y_{1}(v,k;\lambda) & n<m. \end{cases}\end{aligned} \tag{21}$$

Combining (20) with (21), we get the desired results. □

**Remark 1.** *Substituting $x=0$ into (19), we arrive at (5).*

By applying the action of a linear operator $\left(\lambda e^{t}+1\right)^{k}$ to the Apostol-Euler polynomial $\mathcal{E}_{n}^{(a)}(x,\lambda)$, we obtain the following result.

**Theorem 2.**

$$\sum_{j=0}^{k}\binom{k}{j}\lambda^{j}\mathcal{E}_{n}^{(a)}(x+j,\lambda)=\sum_{j=0}^{k}\binom{k}{j}\sum_{v=0}^{j}(-1)^{j-v}\binom{j}{v}2^{v}\mathcal{E}_{n}^{(a-v)}(x,\lambda). \tag{22}$$

**Proof.** By applying the action of a linear operator $\left(\lambda e^{t}+1\right)^{k}$ to the Apostol-Euler polynomial $\mathcal{E}_{n}^{(a)}(x,\lambda)$, we obtain

$$\left(\lambda e^{t}+1\right)^{k}\mathcal{E}_{n}^{(a)}(x,\lambda)=\sum_{j=0}^{k}\binom{k}{j}\lambda^{j}e^{tj}\mathcal{E}_{n}^{(a)}(x,\lambda). \tag{23}$$

Applying linear operators in (15) to the above equation, we have

$$\left(\lambda e^{t}+1\right)^{k}\mathcal{E}_{n}^{(a)}(x,\lambda)=\sum_{j=0}^{k}\binom{k}{j}\lambda^{j}\mathcal{E}_{n}^{(a)}(x+j,\lambda). \tag{24}$$

Combining the following relation with (23)

$$\mathcal{E}_{n}^{(a)}(x,\lambda)=\left(\frac{2}{\lambda e^{t}+1}\right)^{a}x^{n}$$

(cf. p. 101 [13]), we have

$$\left(\lambda e^{t}+1\right)^{k}\mathcal{E}_{n}^{(a)}\left(x,\lambda\right)=\sum_{j=0}^{k}\left(\begin{array}{c}k\\j\end{array}\right)\sum_{v=0}^{j}(-1)^{j-v}\left(\begin{array}{c}j\\v\end{array}\right)2^{v}\left(\frac{2}{\lambda e^{t}+1}\right)^{a-v}x^{n}.$$

After some elementary calculation in the above equation, we have

$$\left(\lambda e^{t}+1\right)^{k}\mathcal{E}_{n}^{(a)}\left(x,\lambda\right)=\sum_{j=0}^{k}\left(\begin{array}{c}k\\j\end{array}\right)\sum_{v=0}^{j}(-1)^{j-v}\left(\begin{array}{c}j\\v\end{array}\right)2^{v}\mathcal{E}_{n}^{(a-v)}\left(x,\lambda\right). \tag{25}$$

Combining (24) and (25), we arrive at the desired result. □

Substituting $\lambda=k=1$ into (22), we arrive at the following corollary, which was proved by Roman (p. 103, Equation (4.2.11) [13]), see also (cf. [32]).

**Corollary 1.**

$$2E_{n}^{(a-1)}(x)=E_{n}^{(a)}(x+1)+E_{n}^{(a)}(x)$$

We assume that, $\lambda\neq 1$ and $a\in\mathbb{N}$, we have the following well-known relationships between the polynomials $\mathcal{B}_{n}^{(a)}\left(x,\lambda\right)$ and $\mathcal{E}_{n}^{(a)}\left(x,-\lambda\right)$:

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n}^{(a)}\left(x,-\lambda\right)\frac{t^{n}}{n!} &= \left(\frac{2}{-\lambda e^{t}+1}\right)^{a}e^{tx}\\ &= \left(-\frac{2}{t}\right)^{a}\sum_{n=0}^{\infty}\mathcal{B}_{n}^{(a)}\left(x,\lambda\right)\frac{t^{n}}{n!}.\end{aligned}$$

Therefore

$$(n)_{a}\,\mathcal{E}_{n-a}^{(a)}\left(x,-\lambda\right)=(-2)^{a}\,\mathcal{B}_{n}^{(a)}\left(x,\lambda\right)$$

or

$$\mathcal{E}_{n}^{(a)}\left(x,\lambda\right)=\frac{(-2)^{a}}{(n+a)_{a}}\mathcal{B}_{n+a}^{(a)}\left(x,-\lambda\right).$$

Substituting the above relation into (22), we get the following result.

**Theorem 3.**

$$\begin{aligned}&\sum_{j=0}^{k}\left(\begin{array}{c}k\\j\end{array}\right)\lambda^{j}\frac{(-2)^{a}}{(n+a)_{a}}\mathcal{B}_{n+a}^{(a)}\left(x+j,-\lambda\right) \\ =&\sum_{j=0}^{k}\left(\begin{array}{c}k\\j\end{array}\right)\sum_{v=0}^{j}(-1)^{j+a}\left(\begin{array}{c}j\\v\end{array}\right)\frac{2^{a}}{(n+a-v)_{a-v}}\mathcal{B}_{n+a-v}^{(a-v)}\left(x,-\lambda\right).\end{aligned} \tag{26}$$

Setting $k=1$ in (26), we get the following corollary.

**Corollary 2.**

$$\mathcal{B}_{n+a}^{(a)}\left(x,-\lambda\right)+\lambda\mathcal{B}_{n+a}^{(a)}\left(x+1,-\lambda\right)=-\left(n+a\right)\mathcal{B}_{n+a-1}^{(a-1)}\left(x,-\lambda\right). \tag{27}$$

**Remark 2.** *Another proof of the Equation (27) is given by Dere et al. [6] and see also (cf. [32]).*

**Remark 3.** *Substituting $n+a=m$ and $\lambda=-1$ into (27), we get*

$$B_{m}^{(a)}(x+1)=B_{m}^{(a)}(x)+mB_{m-1}^{(a-1)}(x)$$

*(cf. p. 95, Equation (4.2.6) [13]).*

The following theorem was proved in (cf. [1]). Here, we give a proof different from that in (cf. [1]).

**Theorem 4.** *Let $n$ and $k$ be nonnegative integers. Then we have*

$$y_1(n,k;\lambda) = \frac{2^k}{k!}\mathcal{E}_n^{(-k)}(0,\lambda).$$

**Proof.** Using (12), we obtain

$$y_1(n,k;\lambda) = \frac{1}{k!}\left\langle \left(\lambda e^t+1\right)^k \mid x^n \right\rangle. \tag{28}$$

From the above equation, we have

$$\begin{aligned} y_1(n,k;\lambda) &= \frac{2^k}{k!}\left\langle \left(\frac{\lambda e^t+1}{2}\right)^k \mid x^n \right\rangle \\ &= \frac{2^k}{k!}\left\langle t^0 \mid \left(\frac{\lambda e^t+1}{2}\right)^{-k} x^n \right\rangle \\ &= \frac{2^k}{k!}\mathcal{E}_n^{(-k)}(0,\lambda). \end{aligned}$$

Therefore, we arrive at the desired result. □

**Theorem 5.**

$$\frac{2^k}{k!}\sum_{j=0}^{n}\binom{n}{j}\mathcal{E}_j^{(a)}(\lambda)\,\mathcal{E}_{n-j}^{(-k)}(\lambda) = \sum_{j=0}^{n}\binom{n}{j}\mathcal{E}_j^{(a)}(\lambda)\,y_1(n-j,k;\lambda)$$

**Proof.** We set

$$\begin{aligned} \frac{1}{k!}\left\langle \left(\lambda e^t+1\right)^k \mid \mathcal{E}_n^{(a)}(\lambda) \right\rangle &= \frac{1}{k!}\left\langle \left(\lambda e^t+1\right)^k \mid \left(\sum_{j=0}^{n}\binom{n}{j}\mathcal{E}_j^{(a)}(\lambda)\right) x^{n-j} \right\rangle \\ &= \frac{1}{k!}\sum_{j=0}^{n}\binom{n}{j}\mathcal{E}_j^{(a)}(\lambda)\left\langle \left(\lambda e^t+1\right)^k \mid x^{n-j} \right\rangle \end{aligned}$$

Combining the above equation with (28), we get

$$\frac{1}{k!}\left\langle \left(\lambda e^t+1\right)^k \mid \mathcal{E}_n^{(a)}(\lambda) \right\rangle = \frac{1}{k!}\sum_{j=0}^{n}\binom{n}{j}\mathcal{E}_j^{(a)}(\lambda)\,y_1(n-j,k;\lambda). \tag{29}$$

On the other hand

$$\begin{aligned} &\frac{1}{k!}\sum_{j=0}^{n}\binom{n}{j}\mathcal{E}_j^{(a)}(\lambda)\left\langle \left(\lambda e^t+1\right)^k \mid x^{n-j} \right\rangle \qquad (30) \\ =\ &\frac{2^k}{k!}\sum_{j=0}^{n}\binom{n}{j}\mathcal{E}_j^{(a)}(\lambda)\left\langle t^0 \mid \left(\frac{\lambda e^t+1}{2}\right)^{-k} x^{n-j} \right\rangle. \end{aligned}$$

Therefore, combining (29) with (30), we arrive at the desired result. □

**Theorem 6.**

$$y_1(m,k;\lambda) = \frac{1}{2k!}\sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix}\frac{\lambda^j}{2}\left(E_m(j+1)+E_m(j)\right).$$

**Proof.** We set the following functional equation

$$F_{y1}(t,k;\lambda) = \frac{1}{2k!}\sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix}\lambda^j\left(F_{\mathcal{E}}(t,j+1;1,k)+F_{\mathcal{E}}(t,j;1,1)\right).$$

By combining the above equation with (4) and (2), we get

$$\sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\frac{1}{2k!}\sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix}\lambda^j\left(E_n(j+1)+E_n(j)\right)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the desired result. □

**Theorem 7.** *Let $m \in \mathbb{N}$. Then we have*

$$S_2(m-1,k;\lambda) = \frac{1}{mk!}\sum_{j=0}^{k}(-1)^{k-j}\begin{pmatrix} k \\ j \end{pmatrix}\lambda^j\left(\mathcal{B}_m(j+1,\lambda)-\mathcal{B}_m(j,\lambda)\right).$$

**Proof.** We also set the following functional equation

$$F_S(t,0,k;\lambda) = \frac{1}{tk!}\sum_{j=0}^{k}(-1)^{k-j}\begin{pmatrix} k \\ j \end{pmatrix}\lambda^j\left(F_{\mathcal{B}}(t,j+1;\lambda,1)-F_{\mathcal{B}}(t,j;\lambda,1)\right).$$

By combining the above equation with (1) and (3), we get

$$\begin{aligned} &\sum_{m=0}^{\infty} S_2(m,k;\lambda)\frac{t^m}{m!} \\ = &\sum_{m=0}^{\infty}\left(\frac{1}{tk!}\sum_{j=0}^{k}(-1)^{k-j}\begin{pmatrix} k \\ j \end{pmatrix}\lambda^j\left(\mathcal{B}_m(j+1,\lambda)-\mathcal{B}_m(j,\lambda)\right)\right)\frac{t^m}{m!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation yields the desired result. □

**Theorem 8.** *Let $2\lambda = 1+\sqrt{5}$. Then we have*

$$\sum_{j=0}^{n}\begin{pmatrix} n \\ j \end{pmatrix}\mathcal{E}_j^{(a)}(\lambda)\,y_1(n-j,k;\lambda) = \frac{1}{k!}\sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix}\left(\lambda F_j + F_{j-1}\right)\mathcal{E}_n^{(a)}(j,\lambda).$$

**Proof.** We define the following functional equation:

$$F_{y1}(t,k;\lambda)\,F_{\mathcal{E}}(t,0;\lambda,a) = \frac{1}{k!}\sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix}\lambda^j\left(F_{\mathcal{E}}(t,j;\lambda,a)\right).$$

By combining the above equation with (4), (2), and (7), we get

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\left(\begin{array}{c} n \\ j \end{array}\right)\mathcal{E}_{j}^{(a)}(\lambda)\, y_1(n-j,k;\lambda)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\left(\begin{array}{c} k \\ j \end{array}\right)(\lambda F_j+F_{j-1})\,\mathcal{E}_{n}^{(a)}(j,\lambda)\frac{t^n}{n!},$$

where $\lambda=\frac{1+\sqrt{5}}{2}$. Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the desired result. □

## 3. Combinatorial Sums via $p$-Adic Integral

In this section, by using the $p$-adic integrals, we derive some combinatorial sums containing the Bernoulli numbers, the Euler numbers, the Apostol-Euler numbers and the numbers $y_1(n,k;\lambda)$.

**Theorem 9.**

$$B_n=\frac{k!}{2^k}\sum_{j=0}^{n}\left(\begin{array}{c} n \\ j \end{array}\right)y_1(n-j,k;\lambda)\sum_{v=0}^{j}\left(\begin{array}{c} j \\ v \end{array}\right)\mathcal{E}_{v}^{(k)}(\lambda)\, B_{j-v}.$$

**Proof.** Combining (2) with (4), we set the following functional equation:

$$F_{y1}(t,k;\lambda)\, F_{\mathcal{E}}(t,x;\lambda,k)=\frac{2^k}{k!}e^{tx}.$$

By using the above equation, we get

$$\sum_{n=0}^{\infty}\mathcal{E}_{n}^{(k)}(x,\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}y_1(n,k;\lambda)\frac{t^n}{n!}=\frac{2^k}{k!}\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\left(\begin{array}{c} n \\ j \end{array}\right)y_1(n-j,k;\lambda)\sum_{v=0}^{j}\left(\begin{array}{c} j \\ v \end{array}\right)x^{j-v}\mathcal{E}_{v}^{(k)}(\lambda)\frac{t^n}{n!}=\frac{2^k}{k!}\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the following relation:

$$\sum_{j=0}^{n}\left(\begin{array}{c} n \\ j \end{array}\right)y_1(n-j,k;\lambda)\sum_{v=0}^{j}\left(\begin{array}{c} j \\ v \end{array}\right)x^{j-v}\mathcal{E}_{v}^{(k)}(\lambda)=\frac{2^k}{k!}x^n. \tag{31}$$

By applying the Volkenborn integral to (31), we get

$$\sum_{j=0}^{n}\left(\begin{array}{c} n \\ j \end{array}\right)y_1(n-j,k;\lambda)\sum_{v=0}^{j}\left(\begin{array}{c} j \\ v \end{array}\right)\mathcal{E}_{v}^{(k)}(\lambda)\int_{\mathbb{Z}_p}x^{j-v}d\mu_1(x)=\frac{2^k}{k!}\int_{\mathbb{Z}_p}x^n d\mu_1(x).$$

Combining the above equation with (10), we arrive at the desired result. □

**Remark 4.** *Replacing $x$ by $k$ and $\lambda$ by $\lambda^2$, the Equation (31) is reduced to the following relation:*

$$\sum_{j=0}^{n}\left(\begin{array}{c} n \\ j \end{array}\right)y_1\left(n-j,k;\lambda^2\right)\mathcal{E}_{j}^{(k)}\left(k,\lambda^2\right)=\frac{2^k}{k!}k^n. \tag{32}$$

*Since*

$$\lambda^k 2^n\mathcal{E}_{n}^{(k)}(k,\lambda^2)=\sum_{m=0}^{n}\left(\begin{array}{c} n \\ m \end{array}\right)k^m E_{n-m}^{*(k)}(\lambda)$$

*where* $E_n^{*(k)}(\lambda)$ *denote the Apostol-type Euler numbers of the second kind of order* $k$ (*cf.* [25,33]), *the Equation* (32) *yields*

$$\sum_{j=0}^{n}\binom{n}{j}\frac{y_1(n-j,k;\lambda^2)}{2^j}\sum_{m=0}^{j}\binom{j}{m}k^m E_{j-m}^{*(k)}(\lambda)=\frac{(2\lambda)^k}{k!}k^n.$$

*By combining the above equation with the following identity*

$$\sum_{m=0}^{n}\binom{n}{m}2^m y_1\left(m,k;\lambda^2\right)E_{n-m}^{*(k)}(\lambda)=\frac{(2\lambda)^k}{k!}k^n$$

(*cf.* [33]), *we get the following combinatorial sums*

$$\sum_{j=0}^{n}\binom{n}{j}\frac{y_1(n-j,k;\lambda^2)}{2^j}\sum_{m=0}^{j}\binom{j}{m}k^m E_{j-m}^{*(k)}(\lambda)=\sum_{m=0}^{n}\binom{n}{m}2^m y_1\left(m,k;\lambda^2\right)E_{n-m}^{*(k)}(\lambda).$$

**Theorem 10.**

$$E_n=\frac{k!}{2^k}\sum_{j=0}^{n}\binom{n}{j}y_1(n-j,k;\lambda)\sum_{v=0}^{j}\binom{j}{v}\mathcal{E}_v^{(k)}(\lambda)E_{j-v}.$$

**Proof.** By applying the fermionic $p$-adic integral to (31), we have

$$\sum_{j=0}^{n}\binom{n}{j}y_1(n-j,k;\lambda)\sum_{v=0}^{j}\binom{j}{v}\mathcal{E}_v^{(k)}(\lambda)\int_{\mathbb{Z}_p}x^{j-v}d\mu_{-1}(x)=\frac{2^k}{k!}\int_{\mathbb{Z}_p}x^n d\mu_{-1}(x).$$

Combining the above equation with (11), we arrive at the desired result. □

**Theorem 11.**

$$\sum_{j=0}^{n}\binom{n}{j}y_1(n-j,k;\lambda)\sum_{v=0}^{j}\binom{j}{v}\frac{\mathcal{E}_v^{(k)}(\lambda)}{j+1-v}=\frac{2^k}{(n+1)k!}.$$

**Proof.** Integrate Equation (31) with respect to $x$ from 0 to 1, we obtain

$$\sum_{j=0}^{n}\binom{n}{j}y_1(n-j,k;\lambda)\sum_{v=0}^{j}\binom{j}{v}\mathcal{E}_v^{(k)}(\lambda)\int_0^1 x^{j-v}dx=\frac{2^k}{k!}\int_0^1 x^n dx.$$

After some calculations, we get the desired result. □

**Theorem 12.**

$$\sum_{j=0}^{n}\binom{n}{j}y_1(j,k;\lambda)\left(B_{n-j}-E_{n-j}\right)=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^j\left(B_n(j)-E_n(j)\right).$$

**Proof.** Setting

$$e^{tx}F_{y1}(t,k;\lambda)=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^j e^{(x+j)t}.$$

Combining (4), we have

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}x^{n-j}y_1(j,k;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^j(x+j)^n\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the following relation:

$$\sum_{j=0}^{n} \binom{n}{j} x^{n-j} y_1(j,k;\lambda) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda^j (x+j)^n. \tag{33}$$

By applying the bosonic $p$-adic integral to (33) and combining with (10), we have

$$\sum_{j=0}^{n} \binom{n}{j} B_{n-j} y_1(j,k;\lambda) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda^j B_n(j). \tag{34}$$

By applying the fermionic $p$-adic integral to (33) and combining with (11), we obtain

$$\sum_{j=0}^{n} \binom{n}{j} E_{n-j} y_1(j,k;\lambda) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda^j E_n(j). \tag{35}$$

Subtracting both sides of Equations (34) and (35), after some elementary calculations, we arrive at the desired result. □

**Theorem 13.**

$$\sum_{j=0}^{n} \binom{n}{j} \frac{y_1(j,k;\lambda)}{n+1-j} = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda^j \frac{(j+1)^{n+1} - j^{n+1}}{n+1}.$$

**Proof.** Integrate Equation (33) with respect to $x$ from 0 to 1, we obtain

$$\sum_{j=0}^{n} \binom{n}{j} y_1(j,k;\lambda) \int_0^1 x^{n-j} dx = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda^j \int_0^1 (x+j)^n dx.$$

After some calculations, we get the desired result. □

**Acknowledgments:** The present investigation was supported by the Scientific Research Project Administration of Akdeniz University. The author would like to thank to all referees for their valuable comments.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References


1. Simsek, Y. New families of special numbers for computing negative order Euler numbers and related numbers and polynomials. To appear in *Appl. Anal. Discrete Math.* **2018**, *12*(1), *https://arxiv.org/pdf/1604.05601.pdf*.
2. Simsek, Y. Analysis of the Bernstein basis functions: an approach to combinatorial sums involving binomial coefficients and Catalan numbers. *Math. Method Appl. Sci.* **2015**, *38*, 3007–3021.
3. Simsek, Y. Identities and relations related to combinatorial numbers and polynomials. *Proc. Jangjeon Math. Soc.* **2017**, *20*, 127–135.
4. Simsek, Y. Apostol type Daehee numbers and polynomials. *Adv. Stud. Contemp. Math.* **2016**, *26*, 555–566.
5. Simsek, Y. Construction of some new families of Apostol-type numbers and polynomials via Dirichlet character and $p$-adic $q$-integrals. *Turk. J. Math.* **2018**, *42*, 557–577, doi:10.3906/mat-1703-114.
6. Dere, R.; Simsek, Y.; Srivastava, H.M. A unified presentation of three families of generalized Apostol type polynomials based upon the theory of the umbral calculus and the umbral algebra. *J. Number Theory* **2013**, *133*, 3245–3263.
7. Srivastava, H.M.; Kurt, B; Simsek, Y. Some families of Genocchi type polynomials and their interpolation functions. *Integral Transforms Spec. Funct.* **2012**, *24*, 919–938.
8. Srivastava, H.M. Some generalizations and basic (or $q$-) extensions of the Bernoulli, Euler and Genocchi polynomials. *Appl. Math. Inform. Sci.* **2011**, *5*, 390–444.

9. Srivastava, H.M.; Choi, J. *Zeta and q-Zeta Functions and Associated Series and Integrals*; Elsevier Science Publishers: Amsterdam, The Netherlands; London, UK; New York, NY, USA, 2012.
10. Luo, Q.-M.; Srivastava, H.M. Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials. *J. Math. Anal. Appl.* **2005**, *308*, 290–302.
11. Luo, Q.-M.; Srivastava, H.M. Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials. *Comput. Math. Appl.* **2006**, *51*, 631–642.
12. Luo, Q.-M.; Srivastava, H.M. Some generalizations of the Apostol-Genocchi polynomials and the Stirling numbers of the second kind. *Appl. Math. Comput.* **2011**, *217*, 5702–5728.
13. Roman, S. *The Umbral Calculus*; Dover Publication Inc.: New York, 2005.
14. Srivastava, H.M.; Kim, T; Simsek, Y. $q$-Bernoulli numbers and polynomials associated with multiple $q$-zeta functions and basic $L$-series. *Russ. J. Math. Phys.* **2005**, *12*, 241–268.
15. Khan, N.U.; Usman, T.; Choi, J. A new generalization of Apostol type Laguerre-Genocchi polynomials. *C. R. Acad. Sci. Paris Ser. I* **2017**, doi:10.1016/j.crma.2017.04.010
16. Simsek, Y. Generating functions for generalized Stirling type numbers, array type polynomials, Eulerian type polynomials and their applications. *Fixed Point Theory Appl.* **2013**, *2013*, 1–28.
17. Bayad, A.; Simsek, Y.; Srivastava, H.M. Some array type polynomials associated with special numbers and polynomials. *Appl. Math. Comput.* **2014**, 244, 149–157.
18. Cakic, N.P.; Milovanovic, G.V. On generalized Stirling numbers and polynomials. *Math. Balk.* **2004**, *18*, 241–248.
19. Simsek, Y. *Interpolation Function of Generalized q-Bernstein Type Polynomials and Their Application*; Curves and Surfaces 2011, LNCS 6920; Boissonnat, J.-D.; Chenin, P.; Cohen, A.; Gout, C.; Lyche, T.; Mazure, M.-L.; Schumaker,L.L., Eds.; Springer-Verlag: Berlin/Heidelberg, Germany, 2012; pp. 647–662.
20. Bona, M. *Introduction to Enumerative Combinatorics*; The McGraw-Hill Companies Inc.: New York, NY, USA, 2007.
21. Riordan, J. *Introduction to Combinatorial Analysis*; Princeton University Press: Princeton, NJ, USA, 1958.
22. Simsek, Y. On parametrization of the $q$-Bernstein Basis functions and Their Applications, *J. Inequal. Spec. Funct.* **2017**, *8*, 158–169.
23. Spivey M.Z. Combinatorial Sums and Finite Differences. *Discrete Math.* **2007**, *307*, 3130–3146.
24. Yuluklu, E.; Simsek, Y.; Komatsu, T. Identities Related to Special Polynomials and Combinatorial Numbers. *Filomat* **2017**, *31*, 4833–4844.
25. Simsek, Y. Computation methods for combinatorial sums and Euler type numbers related to new families of numbers. *Math. Method Appl. Sci.* **2017**, *40*, 2347–2361.
26. Koshy, T. *Fibonacci and Lucas Numbers with Applications; A Wiley-Interscience Publication*; John Wiley & Sons, Inc.: New York, NY, USA; Chichester, UK; Weinheim, Germany; Brisbane, Australial; Singapore, Toronto, ON, Canada, 2001.
27. Kim, T. $q$-Volkenborn integration. *Russ. J. Math. Phys.* **2002**, *19*, 288–299.
28. Schikhof, W.H. U*ltrametric Calculus: An Introduction to p-adic Analysis*; Cambridge Studies in Advanced Mathematics 4; Cambridge University Press: Cambridge, UK, 1984.
29. Kim, T. $q$-Euler numbers and polynomials associated with $p$-adic $q$-integral and basic $q$-zeta function. *Trend Math. Inf. Cent. Math. Sci.* **2006**, *9*, 7–12.
30. Dattoli, G.; Migliorati, M.; Srivastava, H.M. Sheffer polynomials, monomiality principle, algebraic methods and the theory of classical polynomials. *Math. Comput. Model.* **2007**, 45, 1033–1041.
31. Dere, R.; Simsek, Y. Genocchi polynomials associated with the Umbral algebra. *Appl. Math. Comput.* **2011**, *218*, 756–761.
32. Komatsu, T.; Simsek, Y. Identities related to the Stirling numbers and modified Apostol-type numbers on Umbral Calculus.*Miskolc Math. Notes* **2017**, *18*(2), 905–916, doi:10.18514/MMN.2017.1458.
33. Kucukoglu, I.; Simsek, Y. Identities and derivative formulas for the combinatorial and Apostol-Euler type numbers by their generating functions. preprint.