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A Type of the Shadowing Properties for Generic View Points

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Abstract: We show that if a C^1 generic diffeomorphism of a closed smooth two-dimensional manifold has the average shadowing property or the asymptotic average shadowing property, then it is Anosov. Moreover, if a C^1 generic vector field of a closed smooth three-dimensional manifold has the average shadowing property or the asymptotic average shadowing property, then it satisfies singular Axiom A without cycles.

Keywords: average shadowing; asymptotic average shadowing; generic; chain transitive; Axiom A; singular Axiom A; Anosov

JEL Classification: 37C50; 37D20

1. Introduction

The shadowing property is an important notion to study the stability systems in dynamical systems. Robinson [1] and Sakai [2] proved that a diffeomorphism f of a closed smooth manifold M has the C^1 robustly shadowing property if and only if it is structurally stable. However, Lewowicz [3] constructed examples of transitive diffeomorphisms with the shadowing property that are not Anosov. Thus, we know that the shadowing property does not imply hyperbolic systems. For these view points, Abdenur and Díaz [4] suggested the following problem; *the shadowing property and hyperbolicity are equivalent in C^1 generic sense.* Abdenur and Díaz [4] proved that, for a C^1 generic diffeomorphism $f : M \to M$, if a tame f has the shadowing property, then it is hyperbolic. However, if a C^1 generic diffeomorphism f is not tame, then the above problem is still open.

About this point, we consider a type of the shadowing property and hyperbolicity. The average shadowing property or the asymptotic average shadowing property are different types of the shadowing property. In fact, a Morse–Smale diffeomorphim has the shadowing property. However, the system has sinks or sources, and so it has neither the average shadowing property nor the asymptotic average shadowing property [5,6]. Many results were published about the relation between the shadowing properties and the hyperbolicity (see [1,2,7–18]). Among the results, we introduce some interesting results. Sakai [15] proved that if a diffeomorphism f of a surface has the C^1 robustly average shadowing property, then it is Anosov. Honary and Bahabadi [9] proved that if a diffeomorphism f of a surface has the C^1 robustly asymptotic average shadowing property, then it is Anosov. Lee and Park [10] proved that C^1 generically, if a diffeomorphism f has the average shadowing property or the asymptotic average shadowing property with every periodic points has intersection, then it is Anosov. Arbieto and Ribeiro [7] proved that if a vector field X of a closed smooth three-dimensional manifold has the C^1 robustly average shadowing property or the asymptotic average shadowing property, then it is Anosov. In any dimension, Ribeiro [16] proved that C^1 generically, if a vector field X has the average shadowing property or the asymptotic average shadowing property with every periodic orbits intersect each other, is Anosov. From these results, we show that if a diffeomorphism f or a vector field X has the average shadowing property or the asymptotic average shadowing property, then it is hyperbolic, in C^1 generic sense.

1.1. Diffeomorphisms

Let M be a compact two-dimensional manifold without boundary, and let $\mathrm{Diff}(M)$ be the space of C^1 diffeomorphisms of M. For any $\delta > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}} \subset M$ is a *δ-pseudo orbit* of f if the distance between $f(x_i)$ and x_{i+1} is less than δ for all $i \in \mathbb{Z}$. For a closed f-invariant set Λ, $f \in \mathrm{Diff}(M)$ has the *shadowing property* on Λ if, for every $\epsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$, we can choose a point $y \in M$ with the property $d(f^i(y), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. If $\Lambda = M$, then we say that f has the shadowing property.

Blank [19] introduced the notion of the average shadowing property. For $\delta > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}} \subset M$ is a *δ-average pseudo orbit* of f if there is $K = K(\delta) > 0$ such that, for all $n \geq K$ and $k \in \mathbb{Z}$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

Definition 1. *Let $f \in \mathrm{Diff}(M)$. We say that f has the average shadowing property if for any $\epsilon > 0$ there is a $\delta > 0$ such that every δ-average pseudo orbit $\{x_i\}_{i\in\mathbb{Z}} \subset M$ is ϵ-shadowed in average by the point $z \in M$, that is,*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon \text{ and } \limsup_{n\to-\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon.$$

Gu [5] introduced the asymptotic average shadowing property. A sequence $\{x_i\}_{i\in\mathbb{Z}} \subset M$ is an *asymptotically average pseudo orbit* of f if

$$\lim_{n\to\infty} \frac{1}{2n+1}\sum_{i=-n}^{n} d(f(x_i), x_{i+1}) = 0.$$

An asymptotically pseudo orbit $\{x_i\}_{i\in\mathbb{Z}} \subset M$ is said to be *asymptotically shadowed in average* by the point z in M if

$$\lim_{n\to\infty} \frac{1}{2n+1}\sum_{i=-n}^{n} d(f^i(z), x_i) = 0.$$

Definition 2. *Let $f \in \mathrm{Diff}(M)$. We say that f has the asymptotic average shadowing property if every asymptotically average pseudo orbit $\{x_i\} \subset M$ of f can be asymptotically shadowed on average by some point in M.*

Remark 1. *In [20], the authors proved that the asymptotic average shadowing property implies the average shadowing property for continuous surjective topological dynamical systems. For convenience, we will provide information on the results of the asymptotic average shadowing property.*

A point $x \in M$ is *chain recurrent* if, for any $\eta > 0$, there is a η-pseudo orbit $\{x_i\}_{i=0}^{n} (n \geq 1)$ such that $x_0 = x$ and $x_n = x$. Let $\mathcal{CR}(f)$ be the set of all chain recurrent points of f. An f-invariant closed set $C(f) \subset M$ is *chain transitive* if, for any $\delta > 0$ and $x, y \in C(f)$, there is a δ-pseudo orbit $\{x_i\}_{i=0}^{n} (n \geq 1) \subset C(f)$ such that $x_0 = x$ and $x_n = y$. We say that a diffeomorphism f is *chain transitive* if a chain transitive set $C(f) = M$.

Lemma 1. *Let $f \in \mathrm{Diff}(M)$. If f has one of the following,*

(a) the average shadowing property (see [6]),
(b) the asymptotic average shadowing property (see [5]),

then f is chain transitive.

A point $p \in M$ is *periodic* if there is $n > 0$ such that $f^n(p) = p$. Let $P(f)$ be the set of all periodic points of f. We say that $p \in P(f)$ is a *sink* if all the eigenvalues of the derivative of p, that is, $D_p f^{\pi(p)}$, are less than one, and $p \in P(f)$ is a *source* if all the eigenvalues of $D_p f^{\pi(p)}$ are greater than one, where $\pi(p)$ is the period of p.

Lemma 2. (Ref. [21] (Lemma 2.1)) *If $C(f)$ is a chain transitive set of f, then $C(f)$ has neither sinks nor sources.*

A subset $\mathcal{R} \subset \text{Diff}(M)$ is said to be *residual* if it contains a countable intersection of open and dense subsets of $\text{Diff}(M)$. A dynamic property is called C^1 *generic* if it holds in a residual subset of $\text{Diff}(M)$. A closed f-invariant set $\Lambda \subset M$ is *transitive* if there is $x \in \Lambda$ such that $\omega(x) = \Lambda$, where $\omega(x)$ is the omega limit set of x. Crovisier [22] proved that for C^1 generic $f \in \text{Diff}(M)$, a chain transitive set $C(X)$ is a transitive set Λ. Then, according to Pugh's closing lemma, there is a sequence of periodic orbit P_n of f such that $\lim_{n\to\infty} P_n \to \Lambda$. If f is chain transitive, then we can consider the periodic point in M. The result can be extended vector fields.

A closed f-invariant set $\Lambda \subset M$ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exists $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \leq \lambda^n \text{ and } \|D_x f^{-n}|_{E^u_x}\| \leq \lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. We say that f is *Anosov* if $\Lambda = M$ in the above notion. We say that f is *Axiom A* if the nonwandering set $\Omega(f)$ is the closure of $P(f)$ and is hyperbolic. According to Mañé [23], we have the following.

Theorem 1. (Ref. [24] (Theorem(Mañé)) *There is a residual subset $\mathcal{R} \subset \text{Diff}(M)$ such that for $f \in \mathcal{R}$ satisfies one of the following:*

(a) f is Axiom A without cycles;
(b) f has infinitely many sinks or sources.

From Theorem 1, we have the following theorem, which is a main result of the paper.

Theorem 2. *There is a residual subset $\mathcal{R} \subset \text{Diff}(M)$ such that for $f \in \mathcal{R}$, if f has the following;*

(a) the asymptotic average shadowing property, or
(b) the average shadowing property,

then f is Anosov.

Proof. Let $f \in \mathcal{R}$ have the asymptotic average shadowing property or the average shadowing property. According to Lemma 1, f is chain transitive. By Lemma 2, f has neither sinks nor sources. According to Theorem 1, f satisfies Axiom A without cycles. Finally, we show that $\Omega(f) = M$. Since f is Axiom A without cycles, we have the nonwandering set $\Omega(f)$ is hyperbolic and $\Omega(f) = \mathcal{CR}(f)$. Since f is chain transitive, $\mathcal{CR}(f) = M$ and $\mathcal{CR}(f)$ is hyperbolic, and so f is Anosov. □

In the introduction, we mention that the asymptotic average shadowing property or the average shadowing property is not equal to the shadowing property. However, C^1 generically, in three-dimensional cases, if a diffeomorphism f has the asymptotic average shadowing property or the average shadowing property, then it has the shadowing property.

Corollary 1. *For C^1 generic $f \in \text{Diff}(M)$, if f has the asymptotic average shadowing property or the average shadowing property, then it has the shadowing property.*

Proof. Let $f \in \mathcal{R}$ have the average shadowing property or the asymptotic average shadowing property. According to Theorem 2, f is Anosov. By [25], f has the shadowing property. □

1.2. Vector Fields

Let M be a compact three-dimensional manifold without boundaries. Denote by $\mathfrak{X}(M)$ the set of C^1-vector fields on M endowed with the C^1-topology. Let $X \in \mathfrak{X}(M)$. The flow of X will be denoted by $X^t, t \in \mathbb{R}$. A point $x \in M$ is *singular* of X if $X^t(x) = x$ for all $t \in \mathbb{R}$. Denote by $Sing(X)$ the set of all singular points of X. A point $p \in M$ is *periodic* if there is $\pi(p) > 0$ such that $X^{\pi(p)}(p) = p$, where $\pi(p)$ is the prime period of p. Let $Per(X)$ be the set of all closed orbits of X. Let $Crit(X) = Sing(X) \cup Per(X)$. It is clear that $Crit(X) \subset \Omega(X)$, where $\Omega(X)$ is the set of all nonwandering points of X. For any $\delta > 0$, a sequence $\{(x_i, t_i) : x_i \in M, t_i \geq 1$ and $i \in \mathbb{Z}\}$ is a δ-pseudo orbit of X if $d(X^{t_i}(x_i), x_{i+1}) < \delta$ for any $i \in \mathbb{Z}$. For vector fields, the average shadowing property was introduced by Gu et al. [26] and the asymptotic average shadowing property was introduced by Gu [27]. We say that a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ is a *reparametrization* of $\mathbb{R}$ if h is increasing and $h(0) = 0$. Denote by Rep($\mathbb{R}$) the set of reparametrizations of $\mathbb{R}$. For $\epsilon > 0$, we define Rep(ϵ) as follows:

$$\mathrm{Rep}(\epsilon) = \{h \in \mathrm{Rep}(\mathbb{R}) : \left|\frac{h(t)}{t} - 1\right| < \epsilon\}.$$

For any $\delta > 0$, a sequence $\{(x_i, t_i) : x_i \in M, t_i \geq 1,$ and $i \in \mathbb{Z}\}$ is a *δ-average pseudo orbit* of X if there is a $T > 0$ such that for any $n \geq T$ and $k \in \mathbb{Z}$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} d(X^{t_{i+k}}(x_{i+k}), x_{i+k+1})dt < \delta.$$

A δ-average pseudo orbit $\{(x_i, t_i)\}_{i \in \mathbb{Z}} \subset M$ is *positively shadowed on average* by the orbit of X through a point $z \in M$ if there exists $h \in \mathrm{Rep}(\epsilon)$ with $h(0) = 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}} d(X^{h(t)}(z), X^{t-s_i}(x_i))dt < \epsilon,$$

where $s_0 = 0$, and $s_n = \sum_{i=0}^{n-1} t_i (i \in \mathbb{N})$. Analogously, we define *negatively shadowed on average*.

Definition 3. *Let $X \in \mathfrak{X}(M)$. We say that X has the average shadowing property if, for every $\epsilon > 0$, there is a $\delta > 0$ such that every δ-average pseudo orbit $\{(x_i, t_i)\}_{i\in\mathbb{Z}} \subset M$ of X can be positively and negatively shadowed in average by the orbit of X through some point in M.*

A sequence $\{(x_i, t_i) : x_i \in M, t_i \geq 1$ and $i \in \mathbb{Z}\}$ is an *asymptotically average pseudo orbit* of X if

$$\lim_{n\to\infty} \frac{1}{2n+1}\sum_{i=-n}^{n} d(X^{t_i}(x_i), x_{i+1}) = 0.$$

An asymptotically average pseudo orbit $\{(x_i, t_i)\}_{i\in\mathbb{Z}}$ is *positively asymptotically shadowed on average* by the orbit of X through a point $z \in M$ if there is $h \in \mathrm{Rep}(\epsilon)$ with $h(0) = 0$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}\int_{s_i}^{s_{i+1}} d(X^{h(t)}(z), X^{t-s_i}(x_i))dt = 0,$$

where $s_0 = 0$, and $s_n = \sum_{i=0}^{n-1} t_i (i \in \mathbb{N})$. Analogously, we define *negatively asymptotically shadowed on average*.

Definition 4. *Let $X \in \mathfrak{X}(M)$. We say that X has the asymptotic average shadowing property if every asymptotically average pseudo orbit $\{(x_i, t_i)\}_{i\in\mathbb{Z}} \subset M$ of X can be positively and negatively asymptotically shadowed on average by some point in M.*

A closed X^t-invariant set Λ is *transitive* if $\overline{Orb(x)} = \Lambda$ for some $x \in \Lambda$, where $\overline{A}$ is the closure of set A. A compact X^t-invariant $C(X) \subset M$ is *chain transitive* if, for any $x, y \in C(X)$ and $\eta > 0$, there is a η-pseudo orbit $\{(x_i, t_i) : t_i \geq 1, \text{ for } 0 \leq i \leq n\}(n \geq 1) \subset C(X)$ with $x_0 = x, x_n = y$. Note that if a flow is transitive, then it is chain transitive. However, the converse is not true in general. A point $x \in M$ is *chain recurrent* if, for any $\eta > 0$, there is a finite η-pseudo orbit from x to x. Let $\mathcal{CR}(X)$ be the set of all chain recurrent points of X.

Lemma 3. *Let $X \in \mathfrak{X}(M)$. If X has one of the following,*

(a) the average shadowing property (see [26]),
(b) the asymptotic average shadowing property (see [27]),

then X is chain transitive.

A periodic point p is a *sink* if the eigenvalues of $D_p f$ have absolute values of less than one, where $D_p f$ is the derivative of p and f is the Poincaré map of X. A *source* is sink for a vector field $-X$.

Lemma 4. (Ref. [7] (Lemma 6)) *If $C(X)$ is a chain transitive set of X, then $C(X)$ has neither sinks nor sources.*

A compact X^t invariant set Λ is called *hyperbolic* for X^t if there are constants $C > 0, \lambda > 0$ and a splitting $T_x M = E^s_x \oplus \langle X(x) \rangle \oplus E^u_x$ such that the tangent flow $DX^t : TM \to TM$ leaves the invariant continuous splitting and

$$\|DX^t|_{E^s_x}\| \leq Ce^{-\lambda t} \quad \text{and} \quad \|DX^{-t}|_{E^u_x}\| \leq Ce^{-\lambda t}$$

for $t > 0$ and $x \in \Lambda$, where $\langle X(x) \rangle$ is the subspace generated by X. We say that $X \in \mathfrak{X}(M)$ is *Anosov* if $M = \Lambda$ in the above notion.

We say that Λ is *partially hyperbolic* if there is an invariant splitting $T_\Lambda M = E^s \oplus E^c$ and constants $C > 0, \lambda > 0$ such that

(i) $\|DX^t|_{E^s_x}\| \leq Ce^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$,
(ii) E^s dominates E^c, that is, $E^s_x \neq 0, E^c_x \neq 0$ and $\|DX^t|_{E^s_x}\| \cdot \|DX^{-t}|_{E^c_{X_t(x)}}\| \leq Ce^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.

In the above definition, we say that the central bundle E^c is *volume expanding* if the constants $C > 0$ and $\lambda > 0$ satisfy $|J(DX^t|_{E^c_x})| \geq Ce^{\lambda t}$, for all $x \in \Lambda$ and $t > 0$, where $J(\cdot)$ is the Jacobian.

A compact X^t-invariant Λ is *attracting* if there is a neighborhood U of Λ such that $\Lambda = \bigcap_{t \geq 0} X^t(U)$. A set Λ is an *attractor* if it is a transitive attracting set.

We say that Λ is *singular hyperbolic* if every singularity in Λ is hyperbolic and it is partially hyperbolic with a volume expanding central bundle. A *singular hyperbolic attractor* is an attractor that is also a singular hyperbolic set for X, and a *singular hyperbolic repeller* is an attractor that is a singular hyperbolic set for $-X$. We say that X satisfies*Axiom A* if the nonwandering set $\Omega(X) = \overline{Crit(X)}$ is hyperbolic. Note that if X satisfies Axiom A, then we know that $\Omega(X) = \Lambda_1 \cup \cdots \cup \Lambda_n$, where each Λ_i is a hyperbolic basic set of X (see [28])). A collection of basic sets $\Lambda_{i_1}, \ldots, \Lambda_{i_k}$ of X is called a *cycle*, if there exist points $x_n \in \Omega(X)(1 \leq n \leq k)$ such that $\alpha(x_n) \in \Lambda_{i_n}$ and $\omega(x_n) \in \Lambda_{i_{n+1}}(k + 1 \equiv 1)$. An Axiom A vector field X satisfies the *no-cycle condition* if there are no cycles among the basic sets of X. A vector field X is called *singular Axiom A* if there is a finite disjoint decomposition $\Omega(X) = \Lambda_1 \cup \cdots \cup \Lambda_n$, where each Λ_i is a hyperbolic basic set, a singular hyperbolic attractor or a singular hyperbolic repeller, $i = 1, \ldots, n$.

A subset $\mathcal{G} \subset \mathfrak{X}(M)$ is called *residual* if it contains a countable intersection of open and dense subsets of $\mathfrak{X}(M)$. A dynamic property is called C^1 *generic* if it holds in a residual subset of $\mathfrak{X}(M)$. The following is found in ([29] (Thereom A)).

Theorem 3. *There is a residual subset $\mathcal{G} \subset \mathfrak{X}(M)$ such that for $X \in \mathcal{G}$, X satisfies one of the following:*

(a) *X is singular Axiom A without cycles;*
(b) *X has infinitely many sinks or sources.*

According to Theorem 3, we have the following result, which is a main result in the paper and is a vector field version of Theorem 2.

Theorem 4. *There is a residual subset $\mathcal{G} \subset \mathfrak{X}(M)$ such that, for $X \in \mathcal{G}$, if X has the following:*

(a) *asymptotic average shadowing property, or*
(b) *average shadowing property,*

then X is singular Axiom A without cycles.

Proof. Let $X \in \mathcal{G}$ have the average shadowing property or the asymptotic average shadowing property. According to Lemma 3, X is chain transitive. Since X is chain transitive, according to Lemma 4, X has neither sinks nor sources. Since $X \in \mathcal{G}$, by Theorem 3, X is singular Axiom A without cycles. □

Corollary 2. *There is a residual subset $\mathcal{G} \subset \mathfrak{X}(M)$ such that, for $X \in \mathcal{G}$, if $Sing(X) = \emptyset$ and X has the average shadowing property or the asymptotic average shadowing property, then X is Anosov.*

Proof. Let $X \in \mathcal{G}$ have the average shadowing property or the asymptotic average shadowing property. As in the proof of Theorem 4, X is singular Axiom A without cycles. Since $Sing(X) = \emptyset$, X is Axiom A without cycles. Since X is chain transitive, the chain recurrence set $\mathcal{CR}(X) = M$. Since X is Axiom A without cycles and $Sing(X) = \emptyset$, by ([30] (Theorem A)), $\mathcal{CR}(X) = \Omega(X)$. Thus, X is Anosov. □

The following is similar to the proof Corollary 1. Thus, we omit the proof.

Corollary 3. *There is a residual subset $\mathcal{G} \subset \mathfrak{X}(M)$ such that, for $X \in \mathcal{G}$, if $Sing(X) = \emptyset$ and X has the average shadowing property or the asymptotic average shadowing property, then X has the shadowing property.*

Remark 2. *In Corollary 3, for C^1 generic $X \in \mathfrak{X}(M)$, if $Sing(X) \neq \emptyset$, then the result is still open (see [17]). Thus, we will consider that, C^1 generically, if $Sing(X) \neq \emptyset$, then does X have the shadowing property?*

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