Exact Solutions to the Fractional Differential Equations with Mixed Partial Derivatives

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Abstract: In this paper, the solvability of nonlinear fractional partial differential equations (FPDEs) with mixed partial derivatives is considered. The invariant subspace method is generalized and is then used to derive exact solutions to the nonlinear FPDEs. Some examples are solved to illustrate the effectiveness and applicability of the method.

1. Introduction

In recent years, fractional order calculus has been one of the most rapidly developing areas of mathematical analysis. In fact, a natural phenomenon may depend not only on the time instant but also on the previous time history, which can be successfully modeled by fractional calculus. Fractional-order differential equations are naturally related to systems with memory, as fractional derivatives are usually nonlocal operators. Thus fractional differential equations (FDEs) play an important role because of their application in various fields of science, such as mathematics, physics, chemistry, optimal control theory, finance, biology, engineering and so on [1–7].

It is of importance to find efficient methods for solving FDEs. More recently, much attention has been paid to the solutions of FDEs using various methods, such as the Adomian decomposition method (2005) [8], the first integral method (2014) [9], the Lie group theory method (2012, 2015) [10,11], the homotopy analysis method (2016) [12], the inverse differential operational method (2016) [13–15], the F-expansion method (2017) [16], M-Wright transforms (2017) [17], exponential differential operators (2017, 2018) [18,19], and so on. In reality, the finding of exact solutions of the FDEs is hard work and remains a problem.

Recently, investigations have shown that a new method based on the invariant subspace provides an effective tool to find the exact solution of FDEs. This method was initially proposed by Galaktionov and Svirshchevskii (1995, 1996, 2007) [20–22]. The invariant subspace method was developed by Later Gazizov and Kasatkin (2013) [23], Harris and Garra (2013, 2014) [24,25], Sahadevan and Bakkyaraj (2015) [26], and Ouhadan and El Kinani (2015) [27].

In 2016, R. Sahadevan and P. Prakash [28] showed how the invariant subspace method could be extended to time fractional partial differential equations (FPDEs) and could construct their exact solutions.

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \hat{F}[u], \quad \alpha > 0$$

where $\frac{\partial^\alpha}{\partial t^\alpha}(\cdot)$ is a fractional time derivative in the Caputo sense, and $\hat{F}[u]$ is a nonlinear differential operator of order k.

In 2016, S. Choudhary and V. Daftardar-Gejji [29] developed the invariant subspace method for deriving exact solutions of partial differential equations with fractional space and time derivatives.

$$(\lambda_0 \frac{\partial^\alpha}{\partial t^\alpha} + \lambda_1 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} + \cdots + \lambda_m \frac{\partial^{\alpha+m}}{\partial t^{\alpha+m}}) f(x,t) = N(x, f, \frac{\partial^\beta f}{\partial x^\beta}, \frac{\partial^{\beta+1} f}{\partial x^{\beta+1}}, \cdots, \frac{\partial^{\beta+n} f}{\partial x^{\beta+n}})$$

where $N[f]$ is the linear/nonlinear differential operator; $\frac{\partial^{\alpha+j} f}{\partial t^{\alpha+j}}, j = 0, 1, \cdots, m$ and $\frac{\partial^{\beta+i} f}{\partial x^{\beta+i}}, i = 0, 1, \cdots, n$ are Caputo time derivatives and Caputo space derivatives, respectively; $0 < \alpha, \beta \leq 1$ and $\lambda_i \in R$.

In 2017, K.V. Zhukovsky [30] used the inverse differential operational method to obtain solutions for differential equations with mixed derivatives of physical problems.

Motivated by the above results, in this paper, we develop the invariant subspace method for finding exact solutions to some nonlinear partial differential equations with fractional-order mixed partial derivatives (including both fractional space derivatives and time derivatives).

$$(\lambda_0 \frac{\partial^\alpha}{\partial t^\alpha} + \lambda_1 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} + \cdots + \lambda_m \frac{\partial^{\alpha+m_1}}{\partial t^{\alpha+m_1}}) f(x,t) = N(x, f, \frac{\partial^\beta f}{\partial x^\beta}, \frac{\partial^{\beta+1} f}{\partial x^{\beta+1}}, \cdots, \frac{\partial^{\beta+m_2} f}{\partial x^{\beta+m_2}}) + \mu \frac{\partial^\alpha}{\partial t^\alpha}(\frac{\partial^\beta f}{\partial x^\beta})$$

where $f = f(x,t)$, $N[f]$ is a linear/nonlinear differential operator; $\frac{\partial^{\alpha+j} f}{\partial t^{\alpha+j}}, j = 0, 1, \cdots, m_1, m_1 \in N$ and $\frac{\partial^{\beta+i} f}{\partial x^{\beta+i}}, i = 0, 1, \cdots, m_2, m_2 \in N$ are Caputo time derivatives and Caputo space derivatives, respectively; $\frac{\partial^\alpha}{\partial t^\alpha}(\frac{\partial^\beta f}{\partial x^\beta})$ is the Caputo mixed partial derivative of space and time; $k_1 < \alpha \leq k_1 + 1, k_2 < \beta \leq k_2 + 1$, $k_1, k_2 \in N$ and $\lambda_i, \mu \in R$.

Using the invariant subspace method, the FPDEs are reduced to the systems of FDEs that can be solved by familiar analytical methods.

The rest of this paper is organized as follows. In Section 2, the preliminaries and notations are given. In Section 3, we develop the invariant subspace method for solving fractional space and time derivative nonlinear partial differential equations with fractional-order mixed derivatives. In Section 4, illustrative examples are given to explain the applicability of the method. Initial value problems are considered. Finally in Section 5, we give conclusions.

2. Preliminaries and Notation

In this section, we recall some standard definitions and notation.

Definition 1. *(See [7]) The Riemann–Liouville fractional integral of order α and function f is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(x)}{(t-x)^{1-\alpha}} dx, \ t > 0$$

Definition 2. *(See [7]) The Caputo fractional derivative of order α and function f is defined as*

$$\frac{d^\alpha f(t)}{dt^\alpha} = I^{n-\alpha} D^n f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx, & n-1 < \alpha < n \\ f^{(n)}(t), & \alpha = n, n \in N \end{cases}$$

The Riemann–Liouville fractional integral and the Caputo fractional derivative satisfy the following properties [3]:

$$I^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\beta+\alpha}, \quad \alpha > 0, \beta > -1, t > 0$$

$$\frac{d^\alpha t^\beta}{dt^\alpha} = \begin{cases} 0, & \lceil \alpha \rceil = n, \beta \in \{0, 1, 2, \cdots, n-1\} \\ \frac{\Gamma(\beta+1)t^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}, & \lceil \alpha \rceil = n, \beta \in N, and \beta \geq n; or \beta \notin N, and \beta > n-1 \end{cases}$$

$$I^\alpha \left(\frac{d^\alpha f(t)}{dt^\alpha} \right) = f(t) - \sum_{k=0}^{n-1} \frac{d^k f(0)}{dt^k} \frac{t^k}{k!}, \qquad n-1 < \alpha < n, t > 0$$

Definition 3. *(See [7]) A two-parametric Mittag–Leffler function is defined as*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \qquad \alpha, \beta \in C, R(\alpha), R(\beta) > 0$$

noting that $E_{\alpha,1}(z) = E_\alpha(z)$.

Derivatives of the Mittag–Leffler function are given as

$$E_{\alpha,\beta}^{(n)}(z) = \frac{d^n}{dz^n} E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(k+n)!z^k}{k!\Gamma(\alpha k + \alpha n + \beta)}, \qquad n = 0, 1, 2, \cdots$$

$$\frac{d^\gamma}{dt^\gamma} \left(t^{\beta-1} E_{\alpha,\beta}(at^\alpha) \right) = t^{\beta-\gamma-1} E_{\alpha,\beta-\gamma}(at^\alpha), \qquad \alpha, \gamma > 0, a \in R$$

$$\frac{d^\alpha}{dt^\alpha} \left(E_\alpha(at^\alpha) \right) = a E_\alpha(at^\alpha), \qquad \alpha > 0, a \in R$$

The Laplace transform of the αth order Caputo derivative is

$$T\left\{ \frac{d^\alpha f(t)}{dt^\alpha}; s \right\} = s^\alpha \tilde{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \qquad n-1 < \alpha < n, n \in N, R(s) > 0$$

where

$$\tilde{f}(s) = T\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt, \quad s \in R$$

The Laplace transform of the function $t^{\alpha n + \beta - 1} E_{\alpha,\beta}^{(n)}(\pm at^\alpha)$ is as follows [9]:

$$T\{t^{\alpha n + \beta - 1} E_{\alpha,\beta}^{(n)}(\pm at^\alpha); s\} = \frac{n! s^{\alpha-\beta}}{(s^\alpha \mp a)^{n+1}}, \qquad R(s) > |a|^{\frac{1}{\alpha}}, n = 0, 1, 2, \cdots$$

We let I_n be the n-dimensional linear space over R. It is spanned by n linearly independent functions $\varphi_0(x), \varphi_1(x), \cdots, \varphi_{n-1}(x)$:

$$I_n = L\{\varphi_0(x), \varphi_1(x), \cdots, \varphi_{n-1}(x)\} = \{\sum_{i=0}^{n-1} k_i \varphi_i(x) | k_i \in R, i = 0, 1, \cdots, n-1\}$$

We let M be a differential operator; if $M[f] \in I_n, \forall f \in I_n$, then a finite-dimensional linear space I_n is invariant with respect to a differential operator M.

3. Invariant Subspace Method; Fractional Partial Differential Equations with Fractional-Order Mixed Partial Derivative

The FPDE with fractional-order mixed partial derivative is as follows:

$$(\lambda_0 \frac{\partial^\alpha}{\partial t^\alpha} + \lambda_1 \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} + \cdots + \lambda_{m_1} \frac{\partial^{\alpha+m_1}}{\partial t^{\alpha+m_1}})f = N[f] + \mu \frac{\partial^\alpha}{\partial t^\alpha} \left(\frac{\partial^\beta}{\partial x^\beta} f \right) \tag{1}$$

where

$$f = f(x,t), \; N[f] = N(x, f, \frac{\partial^\beta}{\partial x^\beta} f, \frac{\partial^{\beta+1}}{\partial x^{\beta+1}} f, \cdots, \frac{\partial^{\beta+m_2}}{\partial x^{\beta+m_2}} f)$$

Here, $\frac{\partial^{\alpha+j}f}{\partial t^{\alpha+j}}, j = 0, 1, \cdots, m_1; m_1 \in N$ and $\frac{\partial^{\beta+i}f}{\partial x^{\beta+i}}, i = 0, 1, \cdots, m_2; m_2 \in N$ are Caputo time derivatives and Caputo space derivatives respectively; $\frac{\partial^{\alpha}}{\partial t^{\alpha}}(\frac{\partial^{\beta}}{\partial x^{\beta}}f)$ is the Caputo mixed partial derivative of space and time; $k_1 < \alpha \leq k_1 + 1, k_2 < \beta \leq k_2 + 1$, $k_1, k_2 \in N$ and $\lambda_i, \mu \in R$.

Theorem 1. *Suppose* $I_{n+1} = L\{\varphi_0(x), \varphi_1(x), \cdots, \varphi_n(x)\}$ *is a finite-dimensional linear space, and it is invariant with respect to the operators* $N[f]$ *and* $\frac{\partial^{\beta}}{\partial x^{\beta}}f$; *then FPDE (1) has an exact solution as follows:*

$$f(x, t) = \sum_{i=0}^{n} k_i(t)\varphi_i(x) \tag{2}$$

where $\{k_i(t)\}$ *satisfies the following system of FDEs:*

$$\sum_{j=0}^{m_1} \lambda_j \frac{d^{\alpha+j}k_i(t)}{dt^{\alpha+j}} - \mu \frac{d^{\alpha}\psi_{n+1+i}(k_0(t), k_1(t), \cdots, k_n(t))}{dt^{\alpha}} = \psi_i(k_0(t), k_1(t), \cdots, k_n(t)) \tag{3}$$

Here $i = 0, 1, \cdots, n, \{\psi_0, \psi_1, \cdots, \psi_n\}$ *are the expansion coefficients of* $N[f]$ *with respect to* $\{\varphi_0(x), \varphi_1(x), \cdots, \varphi_n(x)\}$; $\{\psi_{n+1}, \psi_{n+2}, \cdots, \psi_{2n+1}\}$ *are the expansion coefficients of* $\frac{\partial^{\beta}}{\partial x^{\beta}}f$ *with respect to* $\{\varphi_0(x), \varphi_1(x), \cdots, \varphi_n(x)\}$.

Proof. Using Equation (2) and the linearity of Caputo fractional derivatives, we obtain

$$\sum_{j=0}^{m_1} \lambda_j \frac{\partial^{\alpha+j}f(x, t)}{\partial t^{\alpha+j}} = \sum_{j=0}^{m_1} \lambda_j \frac{\partial^{\alpha+j}}{\partial t^{\alpha+j}}(\sum_{i=0}^{n} k_i(t)\varphi_i(x)) = \sum_{i=0}^{n}(\sum_{j=0}^{m_1} \lambda_j \frac{d^{\alpha+j}k_i(t)}{dt^{\alpha+j}})\varphi_i(x) \tag{4}$$

Further, as I_{n+1} is an invariant space under the operator $N[f]$ and $\frac{\partial^{\beta}}{\partial x^{\beta}}f$, there exist $2n + 2$ functions $\psi_0, \psi_1, \cdots, \psi_n; \psi_{n+1}, \psi_{n+2}, \cdots, \psi_{2n+1}$ such that

$$N(\sum_{i=0}^{n} k_i(t)\varphi_i(x)) = \sum_{i=0}^{n} \psi_i(k_0(t), k_1(t), \cdots, k_n(t))\varphi_i(x) \tag{5}$$

$$\frac{\partial^{\beta}}{\partial x^{\beta}}f(x, t) = \sum_{i=0}^{n} \psi_{n+1+i}(k_0(t), k_1(t), \cdots, k_n(t))\varphi_i(x) \tag{6}$$

where $\{\psi_0, \psi_1, \cdots, \psi_n\}$ are the expansion coefficients of $N[f]$ with respect to $\{\varphi_0(x), \varphi_1(x), \cdots, \varphi_n(x)\}$; $\{\psi_{n+1}, \psi_{n+2}, \cdots, \psi_{2n+1}\}$ are the expansion coefficients of $\frac{\partial^{\beta}}{\partial x^{\beta}}f$ with respect to $\{\varphi_0(x), \varphi_1(x), \cdots, \varphi_n(x)\}$.

In view of Equations (2), (5) and (6),

$$\begin{aligned}
&N[f(x, t)] + \mu \frac{\partial^{\alpha}}{\partial t^{\alpha}}(\frac{\partial^{\beta}}{\partial x^{\beta}}f(x, t)) \\
&= \sum_{i=0}^{n} \psi_i(k_0(t), k_1(t), \cdots, k_n(t))\varphi_i(x) + \mu \frac{\partial^{\alpha}}{\partial t^{\alpha}}(\sum_{i=0}^{n} \psi_{n+1+i}(k_0(t), k_1(t), \cdots, k_n(t))\varphi_i(x)) \\
&= \sum_{i=0}^{n} \psi_i(k_0(t), k_1(t), \cdots, k_n(t))\varphi_i(x) + \mu(\sum_{i=0}^{n} \frac{d^{\alpha}\psi_{n+1+i}(k_0(t), k_1(t), \cdots, k_n(t))}{dt^{\alpha}}\varphi_i(x)) \\
&= \sum_{i=0}^{n}(\psi_i(k_0(t), k_1(t), \cdots, k_n(t))\varphi_i(x) + \mu \frac{d^{\alpha}\psi_{n+1+i}(k_0(t), k_1(t), \cdots, k_n(t))}{dt^{\alpha}}\varphi_i(x)))
\end{aligned} \tag{7}$$

Equations (4) and (7) are substituted in Equation (1) to obtain

$$\begin{aligned}
&\sum_{i=0}^{n}(\sum_{j=0}^{m_1} \lambda_j \frac{d^{\alpha+j}k_i(t)}{dt^{\alpha+j}} - \psi_i(k_0(t), k_1(t), \cdots, k_n(t)) - \mu \frac{d^{\alpha}\psi_{n+1+i}(k_0(t), k_1(t), \cdots, k_n(t))}{dt^{\alpha}})\varphi_i(x) \\
&= 0
\end{aligned} \tag{8}$$

Using Equation (8) and the fact that $\varphi_0(x), \varphi_1(x), \cdots, \varphi_n(x)$ are linearly independent, we have the system of FDEs that follows:

$$\sum_{j=0}^{m_1} \lambda_j \frac{d^{\alpha+j}k_i(t)}{dt^{\alpha+j}} - \mu \frac{d^\alpha \psi_{n+1+i}(k_0(t), k_1(t), \cdots, k_n(t))}{dt^\alpha} = \psi_i(k_0(t), k_1(t), \cdots, k_n(t)) \tag{9}$$

where $i = 0, 1, \cdots, n$. □

If FPDE (1) satisfies the conditions of Theorem 1, then FPDE (1) has a particular solution given by Equation (2).

We consider the following FPDE:

$$
\begin{aligned}
(\lambda_1 \tfrac{\partial^\alpha}{\partial t^\alpha} + \lambda_2 \tfrac{\partial^{2\alpha}}{\partial t^{2\alpha}} + \cdots + \lambda_{m_1} \tfrac{\partial^{m_1\alpha}}{\partial t^{m_1\alpha}}) f &= N[f] + \mu \tfrac{\partial^\alpha}{\partial t^\alpha}(\tfrac{\partial^\beta}{\partial x^\beta} f) \\
&= N[x, f, \tfrac{\partial^\beta}{\partial x^\beta} f, \tfrac{\partial^{2\beta}}{\partial x^{2\beta}} f, \cdots, \tfrac{\partial^{m_2\beta}}{\partial x^{m_2\beta}} f] + \mu \tfrac{\partial^\alpha}{\partial t^\alpha}(\tfrac{\partial^\beta}{\partial x^\beta} f)
\end{aligned}
\tag{10}
$$

where $f = f(x, t), N[f]$ is a linear/nonlinear differential operator; $\frac{\partial^{j\alpha} f}{\partial t^{j\alpha}}, j = 1, 2, \cdots, m_1; m_1 \in N$ and $\frac{\partial^{i\beta} f}{\partial x^{i\beta}}, i = 1, 2, \cdots, m_2; m_2 \in N$ are Caputo time derivatives and Caputo space derivatives, respectively; $k_1 < \alpha \le k_1 + 1, k_2 < \beta \le k_2 + 1$, $k_1, k_2 \in N$ and $\lambda_i, \mu \in R$.

Theorem 2. *Suppose $I_n = L\{\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)\}$ is a finite-dimensional linear space, and it is invariant with respect to the operator $N[f]$ and $\frac{\partial^\beta}{\partial x^\beta} f$; then FPDE (10) has an exact solution as follows:*

$$f(x, t) = \sum_{i=1}^{n} k_i(t)\varphi_i(x) \tag{11}$$

where $\{k_i(t)\}$ satisfy the following system of FDEs:

$$\sum_{j=1}^{m_1} \lambda_j \frac{d^{j\alpha}k_i(t)}{dt^{j\alpha}} - \mu \frac{d^\alpha \psi_{n+i}(k_1(t), k_2(t), \cdots, k_n(t))}{dt^\alpha} = \psi_i(k_1(t), k_2(t), \cdots, k_n(t)) \tag{12}$$

Here, $i = 1, 2, \cdots, n, \{\psi_1, \psi_2, \cdots, \psi_n\}$ are the expansion coefficients of $N[f]$ with respect to $\{\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)\}$; $\{\psi_{n+1}, \psi_{n+2}, \cdots, \psi_{2n}\}$ are the expansion coefficients of $\frac{\partial^\beta}{\partial x^\beta} f$ with respect to $\{\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)\}$.

Proof. Using Equation (11) and the linearity of Caputo fractional derivatives, we obtain

$$\sum_{j=1}^{m_1} \lambda_j \frac{\partial^{j\alpha} f(x, t)}{\partial t^{j\alpha}} = \sum_{j=1}^{m_1} \lambda_j \frac{\partial^{j\alpha}}{\partial t^{j\alpha}}(\sum_{i=1}^{n} k_i(t)\varphi_i(x)) = \sum_{i=1}^{n}(\sum_{j=1}^{m_1} \lambda_j \frac{d^{j\alpha}k_i(t)}{dt^{j\alpha}})\varphi_i(x) \tag{13}$$

Further, as I_n is an invariant space under the operator $N[f]$ and $\frac{\partial^\beta}{\partial x^\beta} f$, there exist $2n$ functions $\psi_1, \psi_2, \cdots, \psi_n; \psi_{n+1}, \psi_{n+2}, \cdots, \psi_{2n}$ such that

$$N(\sum_{i=1}^{n} k_i(t)\varphi_i(x)) = \sum_{i=1}^{n} \psi_i(k_1(t), k_2(t), \cdots, k_n(t))\varphi_i(x) \tag{14}$$

$$\frac{\partial^\beta}{\partial x^\beta} f(x, t) = \sum_{i=1}^{n} \psi_{n+i}(k_1(t), k_2(t), \cdots, k_n(t))\varphi_i(x) \tag{15}$$

where $\{\psi_1, \psi_2, \cdots, \psi_n\}$ are the expansion coefficients of $N[f]$ with respect to $\{\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)\}$; $\{\psi_{n+1}, \psi_{n+2}, \cdots, \psi_{2n}\}$ are the expansion coefficients of $\frac{\partial^\beta}{\partial x^\beta} f$ with respect to $\{\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)\}$.

In view of Equations (11), (14) and (15),

$$
\begin{aligned}
&N[f(x,t)] + \mu \frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\beta}{\partial x^\beta} f(x,t)\right) \\
&= \sum_{i=1}^{n}\left(\psi_i(k_1(t),k_2(t),\cdots,k_n(t))\varphi_i(x) + \mu \frac{d^\alpha \psi_{n+i}(k_1(t),k_2(t),\cdots,k_n(t))}{dt^\alpha}\varphi_i(x)\right)
\end{aligned}
\tag{16}
$$

Equations (13) and (16) are substituted into Equation (10), to obtain

$$
\begin{aligned}
&\sum_{i=1}^{n}\left(\sum_{j=1}^{m_1}\lambda_j\frac{d^{j\alpha}k_i(t)}{dt^{j\alpha}} - \psi_i(k_1(t),k_2(t),\cdots,k_n(t)) - \mu\frac{d^\alpha \psi_{n+i}(k_1(t),k_2(t),\cdots,k_n(t))}{dt^\alpha}\right)\varphi_i(x) \\
&= 0
\end{aligned}
\tag{17}
$$

Using Equation (17) and the fact that $\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)$ are linearly independent, we have the system of FDEs as follows:

$$
\sum_{j=1}^{m_1}\lambda_j\frac{d^{j\alpha}k_i(t)}{dt^{j\alpha}} - \mu\frac{d^\alpha \psi_{n+i}(k_1(t),k_2(t),\cdots,k_n(t))}{dt^\alpha} = \psi_i(k_1(t),k_2(t),\cdots,k_n(t))
\tag{18}
$$

here $i = 0,1,\cdots,n$. $\square$

Remark: Theorems 1 and 2 in [27] are special cases of our results for $\mu = 0$.

4. Illustrative Examples

In this section, we give several examples to illustrate Theorems 1 and 2.

Example 1. *The fractional diffusion equation is as follows:*

$$
\frac{\partial^\alpha f}{\partial t^\alpha} = C\frac{\partial^\beta f}{\partial x^\beta} + \mu\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\beta}{\partial x^\beta}f\right)
\tag{19}
$$

where $C = constant$.

Diffusion is a process in which molecules move around until they are evenly spread out in the area. For $\alpha > 1$, the phenomenon is referred to as super-diffusion, and for $\alpha = 1$, it is called normal diffusion, whereas $\alpha < 1$ describes subdiffusion.

We consider two cases of Equation (19): case 1: $\alpha \in (0,1], \beta \in (1,2]$; case 2: $\alpha \in (1,2], \beta \in (1,2]$.

Case 1: $\alpha \in (0,1], \beta \in (1,2]$.

The subspace $I_2 = L\{1,x^\beta\}$ is invariant under $N[f] = C\frac{\partial^\beta f}{\partial x^\beta}$ and $\frac{\partial^\beta}{\partial x^\beta}f$ as

$$
N[k_0 + k_1 x^\beta] = Ck_1\Gamma(\beta+1) \in I_2
$$

$$
\frac{\partial^\beta(k_0 + k_1 x^\beta)}{\partial x^\beta} = k_1\Gamma(\beta+1) \in I_2
$$

It follows from Theorem 1 applied to Equation (19) that Equation (19) has the exact solution that follows:

$$
f(x,t) = k_0(t) + k_1(t)x^\beta
\tag{20}
$$

where $k_0(t)$ and $k_1(t)$ satisfy the system of FDEs as follows:

$$
\frac{d^\alpha k_0(t)}{dt^\alpha} - \frac{d^\alpha k_1(t)}{dt^\alpha}\mu C\Gamma(\beta+1) = k_1(t)C\Gamma(\beta+1)
\tag{21}
$$

$$
\frac{d^\alpha k_1(t)}{dt^\alpha} = 0
\tag{22}
$$

Solving the above FDE (22), we obtain

$$k_1(t) = b \tag{23}$$

Substituting Equation (23) into Equation (21), we obtain

$$\frac{d^\alpha k_0(t)}{dt^\alpha} = bC\Gamma(\beta + 1) \tag{24}$$

Then

$$k_0(t) = a + \frac{bC\Gamma(\beta + 1)}{\Gamma(\alpha + 1)}t^\alpha \tag{25}$$

Substituting Equations (23) and (25) into Equation (19), we obtain Equation (19) with the solution as follows:

$$f(x, t) = a + \frac{bC\Gamma(\beta + 1)}{\Gamma(\alpha + 1)}t^\alpha + bx^\beta \tag{26}$$

where a and b are arbitrary constants.

It is clearly verified that the subspace $I_3 = L\{1, x^\beta, x^{2\beta}\}$ is invariant under $N[f] = C\frac{\partial^\beta f}{\partial x^\beta}$ and $\frac{\partial^\beta}{\partial x^\beta}f$ as

$$N[k_0 + k_1 x^\beta + k_2 x^{2\beta}] = Ck_1\Gamma(\beta + 1) + Ck_2\frac{\Gamma(2\beta + 1)}{\Gamma(\beta + 1)}x^\beta \in I_3$$

$$\frac{\partial^\beta(k_0 + k_1 x^\beta + k_2 x^{2\beta})}{\partial x^\beta} = k_1\Gamma(\beta + 1) + k_2\frac{\Gamma(2\beta + 1)}{\Gamma(\beta + 1)}x^\beta \in I_3$$

We let Equation (19) have the exact solution that follows:

$$f(x, t) = k_0(t) + k_1(t)x^\beta + k_2(t)x^{2\beta} \tag{27}$$

where $k_0(t), k_1(t)$ and $k_2(t)$ satisfy the system of FDEs as follows:

$$\frac{d^\alpha k_0(t)}{dt^\alpha} - \mu\frac{d^\alpha k_1(t)}{dt^\alpha}\Gamma(\beta + 1) = C\Gamma(\beta + 1)k_1(t) \tag{28}$$

$$\frac{d^\alpha k_1(t)}{dt^\alpha} - \mu\frac{d^\alpha k_2(t)}{dt^\alpha}\frac{\Gamma(2\beta + 1)}{\Gamma(\beta + 1)} = \frac{C\Gamma(2\beta + 1)}{\Gamma(\beta + 1)}k_2(t) \tag{29}$$

$$\frac{d^\alpha k_2(t)}{dt^\alpha} = 0 \tag{30}$$

Equation (30) implies that $k_2(t) = a_2$. Thus Equation (29) takes the form

$$\frac{d^\alpha k_1(t)}{dt^\alpha} = \frac{a_2 C\Gamma(2\beta + 1)}{\Gamma(\beta + 1)}$$

which has the following solution:

$$k_1(t) = a_1 + \frac{a_2 C\Gamma(2\beta + 1)}{\Gamma(\beta + 1)}t^\alpha$$

Similarly, Equation (28) yields

$$k_0(t) = a_0 + \frac{a_1 C\Gamma(\beta + 1)^2 + a_2\mu C\Gamma(2\beta + 1)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)}t^\alpha + \frac{a_2 C^2\Gamma(2\beta + 1)}{\Gamma(2\alpha + 1)}t^{2\alpha}$$

Thus, Equation (19) has the following solution:

$$
\begin{aligned}
f(x,t) = \ & (a_0 + \tfrac{a_1 C \Gamma(\beta+1)^2 + a_2 \mu C \Gamma(2\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} t^\alpha + \tfrac{a_2 C^2 \Gamma(2\beta+1)}{\Gamma(2\alpha+1)} t^{2\alpha}) \\
& + (a_1 + \tfrac{a_2 C \Gamma(2\beta+1)}{\Gamma(\beta+1)} t^\alpha) x^\beta + a_2 x^{2\beta}
\end{aligned}
\tag{31}
$$

where a_0, a_1 and a_2 are arbitrary constants.

It can be easily verified that $I_2 = L\{1, E_\beta(x^\beta)\}$ is also an invariant subspace with respect to $N[f] = C \frac{\partial^\beta f}{\partial x^\beta}$ and $\frac{\partial^\beta}{\partial x^\beta} f$, as

$$
N[k_0 + k_1 E_\beta(x^\beta)] = C \frac{\partial^\beta}{\partial x^\beta}(k_0 + k_1 E_\beta(x^\beta)) = C k_1 E_\beta(x^\beta) \in I_2
$$

$$
\frac{\partial^\beta (k_0 + k_1 E_\beta(x^\beta))}{\partial x^\beta} = k_1 E_\beta(x^\beta) \in I_2
$$

We consider the exact solution of the form

$$
f(x,t) = k_0(t) + k_1(t) E_\beta(x^\beta)
$$

where $k_0(t)$ and $k_1(t)$ satisfy the following system of FDEs:

$$
\frac{d^\alpha k_0(t)}{dt^\alpha} = 0
\tag{32}
$$

$$
\frac{(1 - \mu C) d^\alpha k_1(t)}{dt^\alpha} = C k_1(t)
\tag{33}
$$

Clearly, $k_0(t) = a$. Solving Equation (33) with the Laplace transform method, we obtain the following:

If $\mu C \neq 1$,

$$
k_1(t) = b E_\alpha \left(\frac{C}{1 - \mu C} t^\alpha \right)
$$

Thus Equation (19) has the exact solution that follows:

$$
f(x,t) = a + b E_\alpha \left(\frac{C}{1 - \mu C} t^\alpha \right) E_\beta(x^\beta)
\tag{34}
$$

where a and b are arbitrary constants.

We find that Equations (26), (31) and (34) are distinct particular solutions of Equation (19) under distinct invariant subspaces. Subspace $I_{n+1} = L\{1, x^\beta, x^{2\beta}, \cdots, x^{n\beta}\}, n \in N$ is invariant under $N[f] = C \frac{\partial^\beta f}{\partial x^\beta}$ and $\frac{\partial^\beta}{\partial x^\beta} f$, as

$$
\begin{aligned}
N[k_0 + k_1 x^\beta + \cdots + k_n x^{n\beta}] & = C k_1 \Gamma(\beta+1) + \tfrac{C k_2 \Gamma(2\beta+1)}{\Gamma(\beta+1)} x^\beta + \cdots + \tfrac{C k_n \Gamma(n\beta+1)}{\Gamma((n-1)\beta+1)} x^{(n-1)\beta} \\
& \in I_{n+1}
\end{aligned}
$$

Thus we obtain infinitely many invariant subspaces for Equation (19), which in turn yield infinitely many particular solutions.

Case 2: $\alpha \in (1,2], \beta \in (1,2]$.

Clearly, subspace $I_3 = L\{1, x^\beta, x^{2\beta}\}$ is an invariant subspace under $N[f] = C \frac{\partial^\beta f}{\partial x^\beta}$ and $\frac{\partial^\beta}{\partial x^\beta} f$, as

$$
N[k_0 + k_1 x^\beta + k_2 x^{2\beta}] = C k_1 \Gamma(\beta+1) + C k_2 \frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)} x^\beta \in I_3
$$

$$\frac{\partial^\beta (k_0 + k_1 x^\beta + k_2 x^{2\beta})}{\partial x^\beta} = k_1 \Gamma(\beta + 1) + k_2 \frac{\Gamma(2\beta + 1)}{\Gamma(\beta + 1)} x^\beta \in I_3$$

We look for the exact solution that follows:

$$f(x,t) = k_0(t) + k_1(t) x^\beta + k_2(t) x^{2\beta}$$

where $k_0(t), k_1(t)$ and $k_2(t)$ are unknown functions to be determined; $k_0(t), k_1(t)$ and $k_2(t)$ satisfy the system of FDEs as follows:

$$\frac{d^\alpha k_0(t)}{dt^\alpha} - \mu \frac{d^\alpha k_1(t)}{dt^\alpha} \Gamma(\beta + 1) = C\Gamma(\beta + 1) k_1(t) \tag{35}$$

$$\frac{d^\alpha k_1(t)}{dt^\alpha} - \mu \frac{d^\alpha k_2(t)}{dt^\alpha} \frac{\Gamma(2\beta + 1)}{\Gamma(\beta + 1)} = \frac{C\Gamma(2\beta + 1)}{\Gamma(\beta + 1)} k_2(t) \tag{36}$$

$$\frac{d^\alpha k_2(t)}{dt^\alpha} = 0 \tag{37}$$

Solving Equations (35)–(37), we obtain

$$k_2(t) = d_1 + d_2 t$$

$$k_1(t) = b_1 + b_2 t + \frac{C d_1 \Gamma(2\beta + 1)}{\Gamma(\beta + 1)\Gamma(\alpha + 1)} t^\alpha + \frac{C d_2 \Gamma(2\beta + 1)}{\Gamma(\beta + 1)\Gamma(\alpha + 2)} t^{\alpha + 1}$$

$$k_0(t) = a_1 + a_2 t + \frac{C b_1 \Gamma(\beta + 1) + \mu C d_2 \Gamma(2\beta + 1)}{\Gamma(\alpha + 1)} t^\alpha + \frac{C b_2 \Gamma(\beta + 1) + \mu C d_2 \Gamma(2\beta + 1)}{\Gamma(\alpha + 2)} t^{\alpha + 1} + \frac{C^2 d_1 \Gamma(2\beta + 1)}{\Gamma(2\alpha + 1)} t^{2\alpha}$$
$$+ \frac{C^2 d_2 \Gamma(2\beta + 1)}{\Gamma(2\alpha + 2)} t^{2\alpha + 1}$$

Then, we obatin the exact solution of Equation (19) as

$$f(x,t) = \left(a_1 + a_2 t + \frac{C b_1 \Gamma(\beta + 1) + \mu C d_2 \Gamma(2\beta + 1)}{\Gamma(\alpha + 1)} t^\alpha + \frac{C b_2 \Gamma(\beta + 1) + \mu C d_2 \Gamma(2\beta + 1)}{\Gamma(\alpha + 2)} t^{\alpha + 1} + \frac{C^2 d_1 \Gamma(2\beta + 1)}{\Gamma(2\alpha + 1)} t^{2\alpha}\right.$$
$$\left. + \frac{C^2 d_2 \Gamma(2\beta + 1)}{\Gamma(2\alpha + 2)} t^{2\alpha + 1}\right) + \left(b_1 + b_2 t + \frac{C d_1 \Gamma(2\beta + 1)}{\Gamma(\beta + 1)\Gamma(\alpha + 1)} t^\alpha + \frac{C d_2 \Gamma(2\beta + 1)}{\Gamma(\beta + 1)\Gamma(\alpha + 2)} t^{\alpha + 1}\right) x^\beta + (d_1 + d_2 t) x^{2\beta}$$

where a_1, a_2, b_1, b_2, d_1 and d_2 are arbitrary constants.

When α and β are other numbers, we can similarly obtain the exact solution of Equation (19).

Next, we find the closed-form solutions of FPDEs satisfying initial conditions using the invariant subspace method.

Example 2. *We have the following FPDE with the initial condition as follows:*

$$\frac{\partial^\alpha f}{\partial t^\alpha} = \left(\frac{\partial^\beta f}{\partial x^\beta}\right)^2 - f\left(\frac{\partial^\beta f}{\partial x^\beta}\right) + \mu \frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\beta}{\partial x^\beta} f\right), \alpha, \beta \in (0,1] \tag{38}$$

$$f(x,0) = 3 + \frac{5}{2} E_\beta(x^\beta) \tag{39}$$

The subspace $I_2 = L\{1, E_\beta(x^\beta)\}$ is invariant under $N[f] = \left(\frac{\partial^\beta f}{\partial x^\beta}\right)^2 - f\left(\frac{\partial^\beta f}{\partial x^\beta}\right)$ and $\frac{\partial^\beta}{\partial x^\beta} f$, as

$$N[k_0 + k_1 E_\beta(x^\beta)] = (k_1 E_\beta(x^\beta))^2 - (k_0 + k_1 E_\beta(x^\beta)) k_1 E_\beta(x^\beta) = -k_0 k_1 E_\beta(x^\beta) \in I_2$$

$$\frac{\partial^\beta (k_0 + k_1 E_\beta(x^\beta))}{\partial x^\beta} = k_1 E_\beta(x^\beta) \in I_2$$

We consider the exact solution that follows:

$$f(x,t) = k_0(t) + k_1(t)E_\beta(x^\beta) \tag{40}$$

where $k_0(t)$ and $k_1(t)$ are unknown functions to be determined.

By substituting Equation (40) into Equation (38) and equating coefficients of different powers of x, we obtain the following system of FDEs:

$$\frac{d^\alpha k_0(t)}{dt^\alpha} = 0 \tag{41}$$

$$(1-\mu)\frac{d^\alpha k_1(t)}{dt^\alpha} = -k_0(t)k_1(t) \tag{42}$$

We obtain $k_0(t) = a$, and Equation (42) takes the following form:
If $\mu \neq 1$,

$$\frac{d^\alpha k_1(t)}{dt^\alpha} = \frac{a}{\mu-1}k_1(t)$$

Then using the Laplace transform technique, we obtain

$$s^\alpha \tilde{k}_1(s) - s^{\alpha-1}k_1(0) = \frac{a}{\mu-1}\tilde{k}_1(s)$$

$$\tilde{k}_1(s) = \frac{s^{\alpha-1}}{s^\alpha - \frac{a}{\mu-1}}k_1(0)$$

Using the inverse Laplace transform, we obtain

$$k_1(t) = k_1(0)E_\alpha\left(\frac{a}{\mu-1}t^\alpha\right)$$

which leads to the exact solution of Equation (38) that follows:

$$f(x,t) = a + bE_\alpha\left(\frac{a}{\mu-1}t^\alpha\right)E_\beta(x^\beta)$$

where a and b are arbitrary constants.

Thus the exact solution of Equation (38) along with the initial condition of Equation (39) is

$$f(x,t) = 3 + \frac{5}{2}E_\alpha\left(\frac{3}{\mu-1}t^\alpha\right)E_\beta(x^\beta)$$

Example 3. *The fractional wave equation is used as an example to model the propagation of diffusive waves in viscoelastic solids. We considered the fractional wave equation with a constant absorption term as follows:*

$$\frac{\partial^{2\alpha} f}{\partial t^{2\alpha}} = \frac{\partial^\beta}{\partial x^\beta}\left(f\frac{\partial^\beta f}{\partial x^\beta}\right) - 1 + \mu\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\beta}{\partial x^\beta}f\right), \alpha, \beta \in (0,1] \tag{43}$$

$$f(x,0) = e + \frac{x^\beta}{\Gamma(\beta+1)}, f_t(x,0) = 1 - x^\beta \tag{44}$$

Clearly, the subspace $I_2 = L\{1, x^\beta\}$ is invariant under $N[f] = \frac{\partial^\beta}{\partial x^\beta}\left(f\frac{\partial^\beta f}{\partial x^\beta}\right) - 1$ and $\frac{\partial^\beta}{\partial x^\beta}f$, as

$$N[k_0 + k_1 x^\beta] = \frac{\partial^\beta}{\partial x^\beta}\left((k_0 + k_1 x^\beta)k_1\Gamma(\beta+1)\right) - 1 = k_1^2\Gamma^2(\beta+1) - 1 \in I_2$$

$$\frac{\partial^\beta(k_0 + k_1 x^\beta)}{\partial x^\beta} = k_1\Gamma(\beta+1) \in I_2$$

By an application of Theorem 2, we know that Equation (43) has the exact solution as follows:

$$f(x,t) = k_0(t) + k_1(t)x^\beta$$

where $k_0(t)$ and $k_1(t)$ satisfy the system of FDEs as follows:

$$\frac{d^{2\alpha}k_0(t)}{dt^{2\alpha}} - \mu\frac{d^\alpha k_1(t)}{dt^\alpha}\Gamma(\beta+1) = k_1^2(t)\Gamma^2(\beta+1) - 1 \tag{45}$$

$$\frac{d^{2\alpha}k_1(t)}{dt^{2\alpha}} = 0 \tag{46}$$

Solving Equations (45) and (46) we obtain the following:

Case 1: when $0 < \alpha \le \frac{1}{2}$:

$$k_1(t) = b_1$$

$$k_0(t) = a_1 + \frac{b_1^2\Gamma^2(\beta+1) - 1}{\Gamma(2\alpha+1)}t^{2\alpha}$$

Thus Equation (43) has the exact solution that follows:

$$f(x,t) = (a_1 + \frac{b_1^2\Gamma^2(\beta+1) - 1}{\Gamma(2\alpha+1)}t^{2\alpha}) + b_1 x^\beta$$

where a_1 and b_1 are arbitrary constants.

By the initial conditions of Equation (44), we obtain $a_1 = e$ and $b_1 = \frac{1}{\Gamma(\beta+1)}$.

Hence the exact solution of Equations (40) and (41) is

$$f(x,t) = e + \frac{x^\beta}{\Gamma(\beta+1)}$$

Case 2: when $\frac{1}{2} < \alpha \le 1$:

$$k_1(t) = b_1 + b_2 t$$

$$k_0(t) = a_1 + a_2 t + \mu b_2\frac{\Gamma(\beta+1)}{\Gamma(\alpha+2)}t^{\alpha+1} + \frac{b_1^2\Gamma^2(\beta+1) - 1}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{2b_1 b_2\Gamma^2(\beta+1)}{\Gamma(2\alpha+2)}t^{2\alpha+1} + \frac{2b_2^2\Gamma^2(\beta+1)}{\Gamma(2\alpha+3)}t^{2\alpha+2}$$

Thus Equation (43) has the exact solution that follows:

$$f(x,t) = (a_1 + a_2 t + \mu b_2\frac{\Gamma(\beta+1)}{\Gamma(\alpha+2)}t^{\alpha+1} + \frac{b_1^2\Gamma^2(\beta+1)-1}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{2b_1 b_2\Gamma^2(\beta+1)}{\Gamma(2\alpha+2)}t^{2\alpha+1} + \frac{2b_2^2\Gamma^2(\beta+1)}{\Gamma(2\alpha+3)}t^{2\alpha+2})$$
$$+ (b_1 + b_2 t)x^\beta$$

where a_1, a_2, b_1 and b_2 are arbitrary constants.

Substituting the initial conditions of Equation (44), we obtain $a_1 = e, a_2 = 1, b_1 = \frac{1}{\Gamma(\beta+1)}$ and $b_2 = -1$.

Thus the exact solution of Equations (43) and (44) is

$$f(x,t) = (e + t - \mu\frac{\Gamma(\beta+1)}{\Gamma(\alpha+2)}t^{\alpha+1} - \frac{2\Gamma(\beta+1)}{\Gamma(2\alpha+2)}t^{2\alpha+1} + \frac{2\Gamma^2(\beta+1)}{\Gamma(2\alpha+3)}t^{2\alpha+2}) + (\frac{1}{\Gamma(\beta+1)} - t)x^\beta$$

We consider the following fractional generalization of the wave equation with a constant absorption term:

$$\frac{\partial^{\alpha+1}f}{\partial t^{\alpha+1}} = \frac{\partial^\beta}{\partial x^\beta}(f\frac{\partial^\beta f}{\partial x^\beta}) - 1 + \mu\frac{\partial^\alpha}{\partial t^\alpha}(\frac{\partial^\beta}{\partial x^\beta}f), \alpha, \beta \in (0,1] \tag{47}$$

We know that the subspace $I_2 = L\{1, x^\beta\}$ is invariant from the above. In view of Theorem 1, Equation (47) has the exact solution that follows:

$$f(x,t) = k_0(t) + k_1(t)x^\beta$$

where $k_0(t)$ and $k_1(t)$ satisfy the system of FDEs as follows:

$$\frac{d^{\alpha+1}k_0(t)}{dt^{\alpha+1}} - \mu \frac{d^\alpha k_1(t)}{dt^\alpha}\Gamma(\beta+1) = k_1^2(t)\Gamma^2(\beta+1) - 1 \tag{48}$$

$$\frac{d^{\alpha+1}k_1(t)}{dt^{\alpha+1}} = 0 \tag{49}$$

Solving the system of FDEs (48) and (49), we obtain

$$k_1(t) = b_1 + b_2 t$$

$$k_0(t) = a_1 + a_2 t + \frac{b_1^2\Gamma^2(\beta+1)-1}{\Gamma(\alpha+2)}t^{\alpha+1} + \frac{\mu b_2\Gamma(\beta+1)}{2}t^2 + \frac{2b_1 b_2\Gamma^2(\beta+1)}{\Gamma(\alpha+3)}t^{\alpha+2} + \frac{2b_2^2\Gamma^2(\beta+1)}{\Gamma(\alpha+4)}t^{\alpha+3}$$

Therefore Equation (47) has the exact solution that follows:

$$f(x,t) = \left(a_1 + a_2 t + \frac{b_1^2\Gamma^2(\beta+1)-1}{\Gamma(\alpha+2)}t^{\alpha+1} + \frac{\mu b_2\Gamma(\beta+1)}{2}t^2 + \frac{2b_1 b_2\Gamma^2(\beta+1)}{\Gamma(\alpha+3)}t^{\alpha+2} + \frac{2b_2^2\Gamma^2(\beta+1)}{\Gamma(\alpha+4)}t^{\alpha+3}\right)$$
$$+ (b_1 + b_2 t)x^\beta$$

where a_1, a_2, b_1 and b_2 are arbitrary constants.

Example 4. *The Korteweg–de Vries (KdV) equation describes the evolution in time of long, unidirectional, nonlinear shallow water waves. We considered the fractional KdV equation that follows:*

$$\frac{\partial^\alpha f}{\partial t^\alpha} = \frac{\partial^\beta f}{\partial x^\beta}\left(\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta}{\partial x^\beta}\left(\frac{f^2}{2}\right)\right)\right) + \mu\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\beta}{\partial x^\beta}f\right), \alpha, \beta \in (0,1] \tag{50}$$

$I_3 = L\{1, x^\beta, x^{2\beta}\}$ is an invariant subspace under $N[f] = \frac{\partial^\beta f}{\partial x^\beta}\left(\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta}{\partial x^\beta}\left(\frac{f^2}{2}\right)\right)\right)$ and $\frac{\partial^\beta}{\partial x^\beta}f$, as

$$N[k_0 + k_1 x^\beta + k_2 x^{2\beta}] = k_1 k_2\Gamma(3\beta+1) + \frac{k_2^2\Gamma(4\beta+1)}{2\Gamma(\beta+1)}x^\beta \in I_3$$

$$\frac{\partial^\beta(k_0 + k_1 x^\beta + k_2 x^{2\beta})}{\partial x^\beta} = k_1\Gamma(\beta+1) + \frac{k_2\Gamma(2\beta+1)}{\Gamma(\beta+1)}x^\beta \in I_3$$

We consider an exact solution that follows:

$$f(x,t) = k_0(t) + k_1(t)x^\beta + k_2(t)x^{2\beta}$$

where $k_0(t), k_1(t)$ and $k_2(t)$ are unknown functions to be determined. It follows from Theorem 1 applied to Equation (47) that $k_0(t), k_1(t)$ and $k_2(t)$ satisfy the FDEs as follows:

$$\frac{d^\alpha k_0(t)}{dt^\alpha} - \mu\frac{d^\alpha k_1(t)}{dt^\alpha}\Gamma(\beta+1) = \Gamma(3\beta+1)k_1(t)k_2(t) \tag{51}$$

$$\frac{d^\alpha k_1(t)}{dt^\alpha} - \mu\frac{d^\alpha k_2(t)}{dt^\alpha}\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)} = \frac{\Gamma(4\beta+1)}{2\Gamma(\beta+1)}k_2^2(t) \tag{52}$$

$$\frac{d^\alpha k_2(t)}{dt^\alpha} = 0 \tag{53}$$

Solving Equation (53), we obtain $k_2(t) = c$.

Hence Equation (52) has the form

$$\frac{d^\alpha k_1(t)}{dt^\alpha} = \frac{c^2 \Gamma(4\beta + 1)}{2\Gamma(\beta + 1)}$$

We obtain

$$k_1(t) = b + \frac{c^2 \Gamma(4\beta + 1)}{2\Gamma(\beta + 1)\Gamma(\alpha + 1)} t^\alpha$$

Similarly, we obtain

$$k_0(t) = a + \frac{2bc\Gamma(3\beta + 1) + \mu c^2 \Gamma(4\beta + 1)}{2\Gamma(\alpha + 1)} t^\alpha + \frac{c^3 \Gamma(4\beta + 1)\Gamma(3\beta + 1)}{2\Gamma(\beta + 1)\Gamma(2\alpha + 1)} t^{2\alpha}$$

Thus the exact solution of Equation (50) is

$$f(x,t) = (a + \frac{2bc\Gamma(3\beta+1)+\mu c^2\Gamma(4\beta+1)}{2\Gamma(\alpha+1)} t^\alpha + \frac{c^3\Gamma(4\beta+1)\Gamma(3\beta+1)}{2\Gamma(\beta+1)\Gamma(2\alpha+1)} t^{2\alpha}) + (b + \frac{c^2\Gamma(4\beta+1)}{2\Gamma(\beta+1)\Gamma(\alpha+1)} t^\alpha) x^\beta$$
$$+ cx^{2\beta}$$

where a, b and c are arbitrary constants.

Example 5. *The fractional version of the nonlinear heat equation is as follows:*

$$\frac{\partial^\alpha f}{\partial t^\alpha} = \frac{\partial^\beta f}{\partial x^\beta}(f \frac{\partial^\beta}{\partial x^\beta}) + \mu \frac{\partial^\alpha}{\partial t^\alpha}(\frac{\partial^\beta}{\partial x^\beta} f), \alpha, \beta \in (0, 1] \tag{54}$$

Clearly, the subspace $I_2 = L\{1, x^\beta\}$ is invariant under $N[f] = \frac{\partial^\beta f}{\partial x^\beta}(f \frac{\partial^\beta}{\partial x^\beta})$ and $\frac{\partial^\beta}{\partial x^\beta} f$, as

$$N[k_0 + k_1 x^\beta] = \frac{\partial^\beta}{\partial x^\beta}((k_0 + k_1 x^\beta)k_1 \Gamma(\beta + 1)) = k_1^2 \Gamma^2(\beta + 1) \in I_2$$

$$\frac{\partial^\beta (k_0 + k_1 x^\beta)}{\partial x^\beta} = k_1 \Gamma(\beta + 1) \in I_2$$

It follows from Theorem 1 that we consider the exact solution of Equation (54) as follows:

$$f(x,t) = k_0(t) + k_1(t)x^\beta$$

such that

$$\frac{d^\alpha k_0(t)}{dt^\alpha} - \mu \frac{d^\alpha k_1(t)}{dt^\alpha}\Gamma(\beta + 1) = k_1^2(t)\Gamma^2(\beta + 1) \tag{55}$$

$$\frac{d^\alpha k_1(t)}{dt^\alpha} = 0 \tag{56}$$

Solving Equations (55) and (56), we obtain

$$k_1(t) = b$$

$$k_0(t) = a + \frac{b^2 \Gamma^2(\beta + 1)}{\Gamma(\alpha + 1)} t^\alpha$$

We obtain an exact solution as follows:

$$f(x,t) = (a + \frac{b^2 \Gamma^2(\beta + 1)}{\Gamma(\alpha + 1)} t^\alpha) + bx^\beta$$

where a and b are arbitrary constants.

Next, we consider the integer-order differential equations in [30]. We can obtain some new different solutions using the invariant subspace method.

Example 6. *The modified hyperbolic heat conduction equation with the mixed derivative term is as follows ([30]):*

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} + \kappa^2 f - \varepsilon \frac{\partial^2}{\partial t \partial x} f \tag{57}$$

$$f(x,0) = g(x), f(x,\infty) < \infty \tag{58}$$

where $\varepsilon, \kappa = const.$

Clearly, the subspace $I_2 = L\{1, x\}$ is an invariant subspace under $N[f] = \frac{\partial^2 f}{\partial x^2} + \kappa^2 f$ and $\frac{\partial}{\partial x} f$, as

$$N[k_0 + k_1 x] = \kappa^2 k_0 + \kappa^2 k_1 x \in I_2$$

$$\frac{\partial(k_0 + k_1 x)}{\partial x} = k_1 \in I_2$$

We let the exact solution be as follows:

$$f(x,t) = k_0(t) + k_1(t)x$$

where $k_0(t), k_1(t)$ are unknown functions to be determined, and $k_0(t)$ and $k_1(t)$ satisfy the system of differential equations as follows:

$$\frac{d^2 k_0(t)}{dt^2} + \varepsilon \frac{dk_1(t)}{dt} = \kappa^2 k_0(t) \tag{59}$$

$$\frac{d^2 k_1(t)}{dt^2} = \kappa^2 k_1(t) \tag{60}$$

Solving Equation (60), we obtain

$$k_1(t) = b_1 e^{-\kappa t} + b_2 e^{\kappa t}$$

Hence Equation (59) has the form

$$\frac{d^2 k_0(t)}{dt^2} - \kappa^2 k_0(t) = b_1 \varepsilon \kappa e^{-\kappa t} + b_2 \varepsilon \kappa e^{\kappa t}$$

We obtain

$$k_0(t) = a_1 e^{-\kappa t} + a_2 e^{\kappa t} - \frac{a_1 \varepsilon}{2} t e^{-\kappa t} - \frac{a_2 \varepsilon}{2} t e^{\kappa t}$$

Then, we obtain the exact solution of Equation (57) as

$$f(x,t) = a_1 e^{-\kappa t} + a_2 e^{\kappa t} - \frac{a_1 \varepsilon}{2} t e^{-\kappa t} - \frac{a_2 \varepsilon}{2} t e^{\kappa t} + (b_1 e^{-\kappa t} + b_2 e^{\kappa t})x \tag{61}$$

where a_1, a_2, b_1 and b_2 are arbitrary constants.

Substituting the conditions of Equation (58) into Equation (61), we obtain $a_2 = b_2 = 0$ and $a_1 + b_1 x = g(x)$.

When $g(x)$ has linear dependence on x, Equations (57) and (58) have the partial solution

$$f(x,t) = a_1 e^{-\kappa t} - \frac{a_1 \varepsilon}{2} t e^{-\kappa t} + b_1 x e^{-\kappa t}$$

where $a_1 + b_1 x = g(x)$.

When $g(x)$ is not linearly dependent on x, Equations (57) and (58) do not have the form of the solution given by Equation (61).

The subspace $I_{n+1} = L\{1, x, x^2, \cdots, x^n\}$, $n \in N$ is invariant under $N[f] = \frac{\partial^2 f}{\partial x^2} + \kappa^2 f$ and $\frac{\partial}{\partial x} f$, as

$$N[k_0 + k_1 x + \cdots + k_n x^n]$$
$$= (\kappa^2 k_0 + 2k_2) + \cdots + (\kappa^2 k_{n-2} + n(n-1)k_n)x^{n-2} + \kappa^2 k_{n-1} x^{n-1} + \kappa^2 k_n x^n$$
$$\in I_{n+1}$$

$$\frac{\partial(k_0 + k_1 x + \cdots + k_n x^n)}{\partial x} = k_1 + k_2 x + \cdots + k_{n-1} x^{n-1} \in I_{n+1}$$

Thus we obtain infinitely many invariant subspaces for Equation (57), which in turn yield infinitely many solutions. If $g(x)$ is a polynomial, we can obtain an exact solution of Equations (57) and (58).

Example 7. *The Fokker–Planck equation is the following ([30]):*

$$\frac{\partial^2 f}{\partial t^2} = \alpha \frac{\partial^2 f}{\partial x^2} + \beta x \frac{\partial f}{\partial x} - \varepsilon \frac{\partial^2}{\partial t \partial x} f \tag{62}$$

where $\alpha, \beta, \kappa = const.$

Clearly, the subspace $I_2 = L\{1, x\}$ is an invariant subspace under $N[f] = \alpha \frac{\partial^2 f}{\partial x^2} + \beta x \frac{\partial f}{\partial x}$ and $\frac{\partial}{\partial x} f$, as

$$N[k_0 + k_1 x] = \beta k_1 x \in I_2$$

$$\frac{\partial(k_0 + k_1 x)}{\partial x} = k_1 \in I_2$$

We suppose the exact solution that follows:

$$f(x, t) = k_0(t) + k_1(t)x$$

where $k_0(t)$ and $k_1(t)$ are unknown functions to be determined; $k_0(t)$ and $k_1(t)$ satisfy the system of differential equations as follows:

$$\frac{d^2 k_1(t)}{dt^2} = \beta k_1(t) \tag{63}$$

$$\frac{d^2 k_0(t)}{dt^2} = -\varepsilon \frac{dk_1(t)}{dt} \tag{64}$$

Case 1: when $\beta > 0$:

$$k_1(t) = b_1 e^{-\sqrt{\beta}t} + b_2 e^{\sqrt{\beta}t}$$

$$k_0(t) = a_1 + a_2 t + \frac{\varepsilon b_1}{\sqrt{\beta}} e^{-\sqrt{\beta}t} - \frac{\varepsilon b_2}{\sqrt{\beta}} e^{\sqrt{\beta}t}$$

Thus Equation (62) has the exact solution that follows:

$$f(x, t) = (a_1 + a_2 t + \frac{\varepsilon b_1}{\sqrt{\beta}} e^{-\sqrt{\beta}t} - \frac{\varepsilon b_2}{\sqrt{\beta}} e^{\sqrt{\beta}t}) + (b_1 e^{-\sqrt{\beta}t} + b_2 e^{\sqrt{\beta}t})x$$

where a_1, a_2, b_1 and b_2 are arbitrary constants.

Case 2: when $\beta = 0$:

$$k_1(t) = b_1 + b_2 t$$

$$k_0(t) = -\frac{\varepsilon b_2}{2} t^2 + a_1 + a_2 t$$

Thus Equation (62) has the exact solution that follows:

$$f(x,t) = (-\frac{\varepsilon b_2}{2}t^2 + a_1 + a_2 t) + (b_1 + b_2 t)x$$

where a_1, a_2, b_1 and b_2 are arbitrary constants.

Case 3: when $\beta < 0$:

$$k_1(t) = b_1 \cos \sqrt{-\beta}t + b_2 \sin \sqrt{-\beta}t$$

$$k_0(t) = a_1 + a_2 t - \frac{\varepsilon b_1}{\sqrt{-\beta}} \sin \sqrt{-\beta}t + \frac{\varepsilon b_2}{\sqrt{-\beta}} \cos \sqrt{-\beta}t$$

Thus Equation (62) has the exact solution that follows:

$$f(x,t) = (a_1 + a_2 t - \frac{\varepsilon b_1}{\sqrt{-\beta}} \sin \sqrt{-\beta}t + \frac{\varepsilon b_2}{\sqrt{-\beta}} \cos \sqrt{-\beta}t) + (b_1 \cos \sqrt{-\beta}t + b_2 \sin \sqrt{-\beta}t)x$$

where a_1, a_2, b_1 and b_2 are arbitrary constants.

The subspace $I_{n+1} = L\{1, x, x^2, \cdots, x^n\}, n \in N$ is invariant under $N[f] = \alpha \frac{\partial^2 f}{\partial x^2} + \beta x \frac{\partial f}{\partial x}$ and $\frac{\partial}{\partial x}f$, as

$$N[k_0 + k_1 x + \cdots + k_n x^n]$$
$$= \alpha k_2 + (\alpha k_3 + \beta k_1)x + \cdots + (\beta k_{n-1}(n-1))x^{n-1} + \beta k_n n x^n$$
$$\in I_{n+1}$$

$$\frac{\partial(k_0 + k_1 x + \cdots + k_n x^n)}{\partial x} = k_1 + k_2 x + \cdots + k_{n-1}x^{n-1} \in I_{n+1}$$

Thus we obtain infinitely many invariant subspaces for Equation (62), which in turn yield infinitely many solutions.

5. Conclusions

The present article develops the invariant subspace method for solving certain fractional space and time derivative nonlinear partial differential equations with fractional-order mixed partial derivatives. Using the invariant subspace method, FPDEs are reduced to systems of FDEs; then they are solved by known analytic methods. In general, FPDEs admit more than one invariant subspace, each of which that has the exact solution. In fact, FPDEs admit infinitely many invariant subspaces. The invariant subspace method is used to derive closed-form solutions of fractional space and time derivative nonlinear partial differential equations with fractional-order mixed partial derivatives along with certain kinds of initial conditions. Thus, the invariant subspace method represents an effective and powerful tool for exact solutions of a wide class of linear/nonlinear FPDEs.

The bases of invariant subspaces usually are orthogonal polynomials, Mittag–Leffler functions, trigonometric functions, and so on. What kinds of spaces are the invariant subspaces of one FPDE? At present we can only try one by one. Although we have found some invariant subspaces of the equations examples above, are there any more invariant subspaces of the equations? We hope to find a simple discriminant method for finding the correct invariant subspaces for FPDEs.

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References

1. Oldham, K.B., Spanier, J. *The Fractional Calculus*; Academic Press: New York, NY, USA, 1974.
2. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; Wiley: New York, NY, USA, 1993.
3. Diethelm, K. *The Analysis of Fractional Differential Equations*; Springer: New York, NY, USA, 2010.

4.   Kilbas, A.A.; Trujillo, J.J.; Srivastava, H.M. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.

5.   Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific: Singapore, 2000.

6.   Ansari, A.; Askari, H. On fractional calculus of A2n+1(x) function. *Appl. Math. Comput.* **2014**, *232*, 487–497.

7.   Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1998.

8.   Daftardar-Gejji, V.; Jafari, H. Adomian decomposition: A tool for solving a system of fractional differential equations. *J. Math. Anal. Appl.* **2005**, *301*, 508–518.

9.   Eslami, M.; Vajargah, B.F.; Mirzazadeh, M.; Biswas, A. Applications of first integral method to fractional partial differential equations. *Indian J. Phys.* **2014**, *88*, 177–184.

10.   Sahadevan, R.; Bakkyaraj, T. Invariant analysis of time fractional generalized Burgers and Korteweg-de Vries equations. *J. Math. Anal. Appl.* **2012**, *2*, 341–347.

11.   Ouhadan, A.; El Kinani, E.H. Lie symmetry analysis of some time fractional partial differential equations. *Int. J. Mod. Phys. Conf. Ser.* **2015**, *38*, 1560075.

12.   Bakkyaraj, T.; Sahadevan, R. Approximate analytical solution of two coupled time fractional nonlinear Schrodinger equations. *Int. J. Appl. Comput. Math.* **2016**, *2*, 113–135.

13.   Zhukovsky, K.V.; Srivastava, H.M. Operational solution of non-integer ordinary and evolution-type partial differential equations. *Axioms* **2016**, *29*, doi:10.3390/axioms5040029.

14.   Zhukovsky, K.V. The Operational Solution of Fractional-Order Differential Equations as well as Black-Scholes and Heat-Conduction Equations. *Mosc. Univ. Phys. Bull.* **2016**, *71*, 237–244.

15.   Zhukovsky, K.V. Operational method of solution of linear non-integer ordinary and partial differential equations. *SpringerPlus* **2016**, *5*, 119, doi:10.1186/s40064-016-1734-3.

16.   Pandir, Y.; Huseyin Duzgun H. New Exact Solutions of Time Fractional Gardner Equation by Using New Version of F-Expansion Method. *Commun. Theor. Phys.* **2017**, *67*, 9–14.

17.   Moslehi, L.; Ansari, A. On M-Wright transforms and time-fractional diffusion equations. *Integral Transforms Spec. Funct.* **2017**, *28*, 113–129.

18.   Aghili, A.; Aghili, J. Exponential differential operators for singular integral equations and space fractional Fokker-Planck equation. *Bol. Soc. Parana. Mat.* **2018**, *36*, 223–233.

19.   Aghili, A. Fractional Black-Scholes equation. *Int. J. Financ. Eng.* **2017**, *4*, 1750004.

20.   Galaktionov, V.A. Invariant subspaces and new explicit solutions to evolution equations with quadratic nonlinearities. *Proc. R. Soc. Edinb. Sect. A Math.* **1995**, *125*, 225–246.

21.   Svirshchevskii, S.R. Invariant linear spaces and exact solutions of nonlinear evolution equations. *J. Nonlinear Math. Phys.* **1996**, *3*, 164–169.

22.   Galaktionov, V.A.; Svirshchevskii, S.R. *Exact Solutions and Invariant Subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics*; Chapman and Hall/CRC: London, UK, 2007.

23.   Gazizov, R.K.; Kasatkin, A.A. Construction of exact solutions for fractional order differential equations by invariant subspace method. *Comput. Math. Appl.* **2013**, *66*, 576–584.

24.   Harris, P.A.; Garra, R. Analytic solution of nonlinear fractional Burgers-type equation by invariant subspace method. *Nonlinear Stud.* **2013**, *20*, 471–481.

25.   Harris, P.A.; Garra,R. Nonlinear time-fractional dispersive equations. *arXiv* **2014**, arXiv:1410.8085v1.

26.   Sahadevan, R.; Bakkyaraj, T. Invariant subspace method and exact solutions of certain nonlinear time fractional partial differential equations. *Fract. Calc. Appl. Anal.* **2015**, *18*, 146–162.

27.   Ouhadan, A.; El Kinani, E.H. Invariant subspace method and fractional modified Kuramoto-Sivashinsky equation. *arXiv* **2015**, arXiv:1503.08789v1.

28.   Sahadevan, R. Prakash, P. Exact solution of certain time fractional nonlinear partial differential equations. *Nonlinear Dyn.* **2016**, *85*, 659–673.

29.   Choudhary, S. Daftardar-Gejji, V. Invariant Subspace Method: A tool for solving fractional partial differential equations. *arXiv* **2016**, arXiv:1609.04209v1.

30.   Zhukovsky, K.V. Operational solution for some types of second order differential equations and for relevant physical problems. *J. Math. Anal. Appl.* **2016**, *446*, 628–647.