Discrete Frames on Finite Dimensional Left Quaternion Hilbert Spaces

M. Khokulan [1], K. Thirulogasanthar [2],* and S. Srisatkunarajah [1]

[1] Department of Mathematics and Statistics, University of Jaffna, Thirunelveli, Jaffna, Sri Lanka; mkhokulan@gmail.com (M.K.); srisatku@yahoo.com (S.S.)

[2] Department of Computer Science and Software Engineering, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, QC H3G 1M8, Canada

* Correspondence: santhar@gmail.com; Tel.: +1-514-931-8731 (ext. 1662)

Abstract: An introductory theory of frames on finite dimensional left quaternion Hilbert spaces is demonstrated along the lines of their complex counterpart.

Keywords: frames; quaternions; quaternion Hilbert spaces

1. Introduction

Duffin and Schaeffer invented frames while working on non-harmonic Fourier series [1]. Later, despite the fact that many others investigated the frame theory, the mechanism brought by Daubechies et al. gave a strong place to frames in harmonic analysis [2,3]. Coherent states of quantum optics and wavelets are subclasses of continuous frames [4]. In the modern era, frames became standard tool in many areas of engineering and physical problems partly because of their success in digital signal processing, particularly one can name the time-frequency analysis [5,6]. In this note we are primarily interested in frames on finite dimensional quaternion Hilbert spaces. There has been a constant surge in finding finite tight frames, largely as a result of several important applications such as internet coding, wireless communication, quantum detection theory, and many more [5,7–10]. There is no fixed class of frames which is appropriate for all physical problems. Each problem is different and a solution to it demands a specific frame. The current nature of technology advancement constantly faces new problems, therefore, the research to find tools to solve them will continue.

Any vector in a separable Hilbert space can be written in terms of an orthonormal basis of the Hilbert space and the so-written expression is unique. Besides, finding an orthonormal basis for a separable Hilbert space is hard, the uniqueness of the expression of vectors, in terms of an orthonormal basis, is undesirable in applications. In fact, this uniqueness limited flexibility in applications. In order to overcome this strain practitioners looked for a substitute. At this juncture, frames appeared as a replacement to orthonormal bases. In a finite dimensional Hilbert space, usually, a frame contains more vectors than any orthonormal basis and this surplus allowed vectors to have enormously many expressions. This flexibility of frames is the key to their success in applications. How flexible a frame should be varies according to the nature of each problem. Flexibility of frames provides design freedom and which permits one to build frames adequate to a particular problem in a way that is not possible by an orthonormal basis [2,4,5,11].

We can only define Hilbert spaces over the fields $\mathbb{R}, \mathbb{C}$ and H, which are the set of all real numbers, the set of all complex numbers, and the set of all quaternions respectively [12]. The fields $\mathbb{R}$ and $\mathbb{C}$ are commutative and associative. Functional analytic properties of real and complex Hilbert spaces well-studied. However, the field of quaternions is non-commutative but associative. Due to this non-commutativity of quaternions, a systematic study of functional analytic properties of quaternionic

Hilbert spaces is not concluded yet. Further, as a result of this non-commutativity we have two types of Hilbert spaces, namely left quaternionic Hilbert space and right quaternionic Hilbert space. The functional analytic properties of the underlying Hilbert space are crucial to the study of frames. In this regard, we shall examine some functional analytic properties of finite dimensional quaternion Hilbert spaces as required for the development of the frame theory.

As far as we know a general theory of discrete frames on quaternionic Hilbert spaces is not available in the literature. In this respect we shall construct discrete frames on finite dimensional left quaternionic Hilbert spaces following the lines of [5]. In the construction of frames on finite dimensional quaternion Hilbert spaces the effect of non-commutativity of quaternions is adaptable, and therefore most of the results can be manipulated from their complex counterparts. Finally, as a scope of the construction, four dimensional quaternions may provide more feasibility in applications than its two dimensional complex counterpart.

2. Quaternion Algebra

In this section we shall define quaternions and some of their properties as needed here. For details one may consult [12–14].

2.1. Quaternions

Let H denote the field of quaternions. Its elements are of the form $\mathbf{q} = x_0 + x_1 i + x_2 j + x_3 k$, where x_0, x_1, x_2 and x_3 are real numbers, and i, j, k are imaginary units such that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. The quaternionic conjugate of $\mathbf{q}$ is defined to be $\overline{\mathbf{q}} = x_0 - x_1 i - x_2 j - x_3 k$.

2.2. Properties of Quaternions

The quaternion product allows the following properties. For $q, r, s \in H$, we have

(a) $q(rs) = (qr)s$ (associative)
(b) $q(r + s) = qr + qs$
(c)   For each $q \neq 0$, there exists r such that $qr = 1$
(d)   If $qr = qs$ then $r = s$ whenever $q \neq 0$

The quaternion product is not commutative.

3. Frames in Quaternion Hilbert Space

Definition 1. *[12] Let V_H^L is a vector space under left multiplication by quaternionic scalars, where H stands for the quaternion algebra. For $f, g, h \in V_H^L$ and $q \in H$, the inner product*

$$\langle .|. \rangle : V_H^L \times V_H^L \longrightarrow H$$

satisfies the following properties:

(a) $\overline{\langle f|g \rangle} = \langle g|f \rangle$
(b) $\|f\|^2 = \langle f|f \rangle > 0$ *unless* $f = 0$, *a real norm*
(c) $\langle f|g + h \rangle = \langle f|g \rangle + \langle f|h \rangle$
(d) $\langle qf|g \rangle = q \langle f|g \rangle$
(e) $\langle f|qg \rangle = \langle f|g \rangle \overline{q}$.

Assume that the space V_H^L together with $\langle .|. \rangle$ is a separable Hilbert space. Properties of left quaternion Hilbert spaces as needed here can be listed as follows: For $f, g \in V_H^L$ and $p, q \in H$, we have

(a) $pf + qg \in V_H^L$
(b) $p(f + g) = pf + qg$
(c) $(pq)f = p(qf)$
(d) $(p + q)f = pf + qf$.

Proposition 1. *[12] (Schwartz inequality)* $\langle f|g \rangle \langle g|f \rangle \le \|f\|^2 \|g\|^2$ *, for all* $f, g \in V_H^L$

For an enhanced explanation of quaternions and quaternion Hilbert spaces one may consult [12,13] and the many references listed there.

3.1. Some Basic Facts in Left Quaternion Hilbert Space

Definition 2. *(Basis) Let V_H^L be a finite dimensional left quaternion Hilbert space with an inner product $\langle .|. \rangle$. The linearity is assumed in the second entry of the inner product. For V_H^L a sequence $\{e_k\}_{k=1}^m$ in V_H^L is called a basis if the sequence satisfies the following two conditions*

1.   *$V_H^L = $ left span $\{e_k\}_{k=1}^m$;*
2.   *$\{e_k\}_{k=1}^m$ is a linearly independent set.*

As a consequence of this definition, for every $f \in V_H^L$ there exist unique scalar coefficients $\{c_k\}_{k=1}^m \subseteq H$ such that $f = \sum_{k=1}^m c_k e_k$. If $\{e_k\}_{k=1}^m$ is an orthonormal basis, that is $\langle e_k|e_j \rangle = \delta_{kj}$, then $\langle f|e_j \rangle = \left\langle \sum_{k=1}^m c_k e_k | e_j \right\rangle = \sum_{k=1}^m c_k \langle e_k|e_j \rangle = c_j$. Thereby

$$f = \sum_{k=1}^m \langle f|e_k \rangle e_k. \tag{1}$$

We now introduce frames on finite dimensional left quaternion Hilbert spaces. We shall show that the complex treatment adapt to the quaternions as well. In this note *left span* means left span over the quaternion scalar field, H. We shall also prove the functional analytic properties for quaternions as needed here, and these proofs are the adaptation of the proofs of the complex cases given in [15]. The theory of frames offered here, more or less, follows the lines of [5].

Definition 3. *(Frames) A countable family of elements $\{f_k\}_{k \in I}$ in V_H^L is a frame for V_H^L if there exist constants $A, B > 0$ such that*

$$A \|f\|^2 \le \sum_{k \in I} |\langle f|f_k \rangle|^2 \le B \|f\|^2, \tag{2}$$

for all $f \in V_H^L$.

The real numbers A and B are called frame bounds. The numbers A and B are not unique. The supremum over all lower frame bounds is the *optimal lower frame bound*, and the infimum over all upper frame bounds is the *optimal upper frame bound* [5]. In fact, optimal frame bounds are the frame bounds. A frame is said to be normalized if $\|f_k\| = 1$, for all $k \in I$. In this note we shall only consider finite frames $\{f_k\}_{k=1}^m$, $m \in \mathbb{N}$. With this restriction, Schwartz inequality shows that

$$\sum_{k=1}^m |\langle f|f_k \rangle|^2 \le \sum_{k=1}^m \|f_k\|^2 \|f\|^2, \quad \text{for all } f \in V_H^L. \tag{3}$$

From (3) it is clear that the upper frame condition is always satisfied with $A = \sum_{k=1}^m \|f_k\|^2$. In order for the lower condition in (2) to be satisfied, it is necessary that leftspan$\{f_k\}_{k=1}^m = V_H^L$. Let us see this in the following.

Lemma 1. *Let $\{f_k\}_{k=1}^m$ be a sequence in V_H^L and $W := leftspan\{f_k\}_{k=1}^m$. Then a mapping $\varphi : W \longrightarrow \mathbb{R}$ is continuous if and only if for any sequence $\{x_n\}$ in W which converges to x_0 as $n \to \infty$ then $\varphi(x_n)$ converges to $\varphi(x_0)$ as $n \to \infty$.*

Proof. Similar to the complex case. □

Lemma 2. *Let $\varphi : W \longrightarrow \mathbb{R}$ be a continuous mapping and M is a compact subset of W, then φ assumes a maximum and a minimum at some points of M.*

Proof. Similar to the complex case. □

Proposition 2. *Let $\{f_k\}_{k=1}^m$ be a sequence in V_H^L. Then $\{f_k\}_{k=1}^m$ is a frame for left span $\{f_k\}_{k=1}^m$.*

Proof. From the Schwartz inequality the upper frame condition is satisfied with $B = \sum_{k=1}^m \|f_k\|^2$. Thereby

$$\sum_{k=1}^m |\langle f|f_k\rangle|^2 \le B\|f\|^2. \tag{4}$$

Let $W := leftspan\{f_k\}_{k=1}^m$ and consider the mapping

$$\varphi : W \longrightarrow \mathbb{R}, \ \varphi(f) := \sum_{k=1}^m |\langle f, f_k\rangle|^2.$$

Now we want to prove that φ is continuous. Let $\{g_n\}$ be a sequence in W such that $g_n \longrightarrow g$ as $n \longrightarrow \infty$. Now,

$$
\begin{aligned}
\|\varphi(g_n) - \varphi(g)\| &= \left\| \sum_{k=1}^m |\langle g_n|f_k\rangle|^2 - \sum_{k=1}^m |\langle g|f_k\rangle|^2 \right\| \\
&\le \sum_{k=1}^m \left\| |\langle g_n|f_k\rangle|^2 - |\langle g|f_k\rangle|^2 \right\| \\
&= \sum_{k=1}^m \left\| \langle g_n|f_k\rangle \overline{\langle g_n|f_k\rangle} - \langle g|f_k\rangle \overline{\langle g|f_k\rangle} \right\| \\
&= \sum_{k=1}^m \| \langle g_n|f_k\rangle \langle f_k|g_n\rangle - \langle g|f_k\rangle \langle f_k|g\rangle \| \\
&= \sum_{k=1}^m \| \langle g_n|f_k\rangle \langle f_k|g_n\rangle - \langle g|f_k\rangle \langle f_k|g_n\rangle + \langle g|f_k\rangle \langle f_k|g_n\rangle - \langle g|f_k\rangle \langle f_k|g\rangle \| \\
&= \sum_{k=1}^m \| (\langle g_n|f_k\rangle - \langle g|f_k\rangle) \langle f_k|g_n\rangle + \langle g|f_k\rangle (\langle f_k|g_n\rangle - \langle f_k|g\rangle) \| \\
&= \sum_{k=1}^m \left\| \left(\overline{\langle f_k|g_n\rangle} - \overline{\langle f_k|g\rangle} \right) \langle f_k|g_n\rangle + \langle g|f_k\rangle (\langle f_k|g_n\rangle - \langle f_k|g\rangle) \right\| \\
&= \sum_{k=1}^m \left\| \left(\overline{\langle f_k|g_n\rangle} - \overline{\langle f_k|g\rangle} \right) \langle f_k|g_n\rangle + \langle g|f_k\rangle (\langle f_k|g_n\rangle - \langle f_k|g\rangle) \right\| \\
&= \sum_{k=1}^m \left\| \overline{\langle f_k|g_n - g\rangle} \langle f_k|g_n\rangle + \langle g|f_k\rangle (\langle f_k|g_n - g\rangle) \right\| \\
&\longrightarrow 0 \ \text{as } n \longrightarrow \infty \ \lceil \because g_n \longrightarrow g \text{ as } n \longrightarrow \infty \rceil
\end{aligned}
$$

Thereby $\varphi(g_n)$ converges to $\varphi(g)$ as $n \longrightarrow \infty$. From the Lemma (1), φ is continuous. Since the closed unit ball in W is compact, from the Lemma (2), we can find $g \in W$ with $\|g\| = 1$ such that

$$A := \sum_{k=1}^{m} |\langle g|f_k\rangle|^2 = \inf\left\{\sum_{k=1}^{m} |\langle f|f_k\rangle|^2 : f \in W, \|f\| = 1\right\}.$$

It is clear that $A > 0$ as not all f_k are zero. Now given $f \in W$, $f \neq 0$, we have $\left\|\dfrac{f}{\|f\|}\right\| = 1$, so $\displaystyle\sum_{k=1}^{m} \left|\left\langle \dfrac{f}{\|f\|}\Big|f_k\right\rangle\right|^2 \geq A$. Hence

$$\sum_{k=1}^{m} |\langle f|f_k\rangle|^2 = \sum_{k=1}^{m} \left|\left\langle \frac{f}{\|f\|}\Big|f_k\right\rangle\right|^2 \|f\|^2 \geq A \|f\|^2.$$

Thereby

$$\sum_{k=1}^{m} |\langle f|f_k\rangle|^2 \geq A \|f\|^2. \tag{5}$$

From (4) and (5),

$$A \|f\|^2 \leq \sum_{k=1}^{m} |\langle f|f_k\rangle|^2 \leq B \|f\|^2, \text{ for all } f \in W.$$

Hence $\{f_k\}_{k=1}^{m}$ is a frame for $leftspan\{f_k\}_{k=1}^{m}$. $\square$

Corollary 1. *A family of elements $\{f_k\}_{k=1}^{m}$ in V_H^L is a frame for V_H^L if and only if $leftspan\{f_k\}_{k=1}^{m} = V_H^L$.*

Proof. Suppose that $\{f_k\}_{k=1}^{m}$ is a frame for V_H^L. Then there exist $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k=1}^{m} |\langle f|f_k\rangle|^2 \leq B\|f\|^2, \tag{6}$$

If $V_H^L \neq leftspan\{f_k\}_{k=0}^{\infty}$ then there exists $f \in V_H^L$, $f \neq 0$ such that

$$\langle f|f_k\rangle = 0, \quad k = 1, \cdots m.$$

Hence $\{f_k\}_{k=1}^{m}$ cannot be a frame.
Conversely suppose that $V_H^L = leftspan\{f_k\}_{k=1}^{m}$. From Proposition (2) $\{f_k\}_{k=1}^{m}$ is a frame for $leftspan\{f_k\}_{k=1}^{m}$, thereby $\{f_k\}_{k=1}^{m}$ is a frame for V_H^L. $\square$

From the above corollary it is clear that a frame is an over complete family of vectors in a finite dimensional Hilbert space.

3.2. Frame Operator in Left Quaternion Hilbert Space

Frame Operators

Consider now a left quaternion Hilbert space, V_H^L with a frame $\{f_k\}_{k=1}^{m}$ and define a linear mapping

$$T : H^m \longrightarrow V_H^L, \ T\{c_k\}_{k=1}^{m} = \sum_{k=1}^{m} c_k f_k, \ c_k \in H. \tag{7}$$

T is called the *synthesis operator* or *pre-frame operator*. The adjoint of T

$$T^* : V_H^L \longrightarrow H^m, \ \text{by } T^*f = \{\langle f|f_k\rangle\}_{k=1}^{m} \tag{8}$$

is called the *analysis operator.* By composing T with its adjoint we obtain the *frame operator*

$$S : V_H^L \longrightarrow V_H^L, \ Sf = TT^*f = \sum_{k=1}^{m} \langle f | f_k \rangle f_k. \tag{9}$$

Note that in terms of the frame operator, for $f \in V_H^L$

$$\langle Sf | f \rangle \ = \ \left\langle \sum_{k=1}^{m} \langle f | f_k \rangle f_k | f \right\rangle = \sum_{k=1}^{m} \langle f | f_k \rangle \langle f_k | f \rangle = \sum_{k=1}^{m} |\langle f | f_k \rangle|^2.$$

That is,

$$\langle Sf | f \rangle = \sum_{k=1}^{m} |\langle f | f_k \rangle|^2, \ f \in V_H^L. \tag{10}$$

A frame $\{f_k\}_{k=1}^{m}$ is tight if we can choose $A = B$ in the Definition (3), in this case (2) gives

$$\sum_{k=1}^{m} |\langle f | f_k \rangle|^2 = A \|f\|^2, \ \text{for all} \ f \in V_H^L.$$

Thereby $\langle Sf | f \rangle = A \|f\|^2$, for all $f \in V_H^L$.

Proposition 3. *Let $\{f_k\}_{k=1}^{m}$ be a tight frame for V_H^L with the frame bound A. Then $S = AI$ (where I is the identity operator on V_H^L), and*

$$f = \frac{1}{A} \sum_{k=1}^{m} \langle f | f_k \rangle f_k, \ \text{for all} \ f \in V_H^L.$$

Proof. The frame operator S is given by

$$S : V_H^L \longrightarrow V_H^L; \quad Sf = \sum_{k=1}^{m} \langle f | f_k \rangle f_k.$$

Let $f \in V_H^L$, then

$$\langle Sf | f \rangle \ = \ \left\langle \sum_{k=1}^{m} \langle f | f_k \rangle f_k | f \right\rangle = \sum_{k=1}^{m} |\langle f | f_k \rangle|^2.$$

Since the frame $\{f_k\}_{k=1}^{m}$ is tight for V_H^L,

$$\langle Sf | f \rangle = A \|f\|^2, \ \text{for all} \ f \in V_H^L.$$

Now,

$$\langle Sf | f \rangle \ = \ A \|f\|^2 = A \langle f | f \rangle = A \langle If | f \rangle = \langle AIf | f \rangle.$$

Thereby $\langle Sf | f \rangle = \langle AIf | f \rangle$, for all $f \in V_H^L$. Hence $S = AI$. Since $\{f_k\}_{k=1}^{m}$ is a frame for V_H^L, from Corollary (1),

$$V_H^L = leftspan \{f_k\}_{k=1}^{m}. \tag{11}$$

Therefore for given $f \in V_H^L$, there exists $c_k \in H$ such that

$$f = \sum_{k=1}^{m} c_k f_k. \tag{12}$$

Now define $c_k = \langle f | g_k \rangle$ and $g_k = \dfrac{1}{A} f_k$, here $g_k \in V_H^L$. Then (12) becomes

$$f = \sum_{k=1}^{m} \langle f | g_k \rangle f_k = \sum_{k=1}^{m} \left\langle f | \frac{1}{A} f_k \right\rangle f_k = \sum_{k=1}^{m} \langle f | f_k \rangle \overline{\left(\frac{1}{A}\right)} f_k = \frac{1}{A} \sum_{k=1}^{m} \langle f | f_k \rangle f_k \quad \text{as } A \text{ is real.}$$

Hence

$$f = \frac{1}{A} \sum_{k=1}^{m} \langle f | f_k \rangle f_k, \quad \text{for all } f \in V_H^L.$$

$\square$

Definition 4. *[12] Let V_H^L be any left quaternion Hilbert space. A mapping $S : V_H^L \longrightarrow V_H^L$ is said to be left-linear if,*

$$S(\alpha f + \beta g) = \alpha S(f) + \beta S(g),$$

for all $f, g \in V_H^L$ and $\alpha, \beta \in H$.

Definition 5. *[12] A linear operator $S : V_H^L \longrightarrow V_H^L$ is said to be bounded if,*

$$\|Sf\| \leq K \|f\|,$$

for some constant $K \geq 0$ and all $f \in V_H^L$.

Definition 6. *[12] (Adjoint operator) Let $S : V_H^L \longrightarrow V_H^L$ be a bounded linear operator on a left quaternion Hilbert space V_H^L. We define its adjoint to be the operator $S^* : V_H^L \longrightarrow V_H^L$ that has the property*

$$\langle f | Sg \rangle = \langle S^* f | g \rangle, \text{ for all } f, g \in V_H^L. \tag{13}$$

Lemma 3. *The adjoint operator S^* of a bounded linear operator is linear and bounded.*

Proof. Linearity can easily be verified and every linear operator on a finite dimensional quaternion Hilbert space is bounded. $\square$

Definition 7. *[12] (Self-adjoint operator) Let V_H^L be a left quaternion Hilbert space. A bounded linear operator S on V_H^L is called self-adjoint, if $S = S^*$.*

Lemma 4. *Let $S : V_H^L \longrightarrow V_H^L$ and $T : V_H^L \longrightarrow V_H^L$ be bounded linear operators on V_H^L. Then for any $f, g \in V_H^L$, we have*

(a) $\langle Sf | g \rangle = \langle f | S^* g \rangle$.
(b) $(S + T)^* = S^* + T^*$.
(c) $(ST)^* = T^* S^*$.
(d) $(S^*)^* = S$.
(e) $I^* = I$, *where I is an identity operator on V_H^L.*
(f) *If S is invertible then $(S^{-1})^* = (S^*)^{-1}$.*

Proof. It is straightforward. $\square$

Lemma 5. *Let U_H^L and V_H^L be finite dimensional left quaternion Hilbert spaces and $S : U_H^L \longrightarrow V_H^L$ be a linear mapping then $\ker S$ is a subspace of U_H^L.*

Proof. Easy to verify. $\square$

Lemma 6. *Let U_H^L, V_H^L be finite dimensional left quaternion Hilbert spaces and $S : U_H^L \longrightarrow V_H^L$ be a linear mapping then*

$$dimR_S + dimN_S = dimU_H^L,$$

where $R_S :=$ image of S, $N_S :=$ ker S.

Proof. Proof from the complex theory can easily be adapted. □

Lemma 7. *Let $S : U_H^L \longrightarrow V_H^L$ be a linear mapping. S is one to one if and only if $N_S = \{0\}$.*

Proof. Easy to verify. □

Lemma 8. *Let U_H^L, V_H^L are finite dimensional left quaternion Hilbert spaces with same dimension. Let $S : U_H^L \longrightarrow V_H^L$ be a linear mapping. If S is one to one then S is onto.*

Proof. Easy to verify. □

Lemma 9. *(Pythagoras' law) Suppose that f and g is an arbitrary pair of orthogonal vectors in the left quaternion Hilbert space V_H^L. Then we have Pythagoras' formula*

$$\|f + g\|^2 = \|f\|^2 + \|g\|^2. \tag{14}$$

Proof. Straightforward. □

Lemma 10. *Let $T : H^m \longrightarrow V_H^L$ be a linear mapping and $T^* : V_H^L \longrightarrow H^m$ be its adjoint operator. Then $N_T = R_{T^*}^\perp$, where $N_T :=$ ker T and $R_{T^*} :=$ range of T^*.*

Proof. Proof from the complex theory can easily be adapted. □

Theorem 1. *Let $\{f_k\}_{k=1}^m$ be a frame for V_H^L with frame operator S. Then*

1. *S is invertible and self-adjoint.*
2. *Every $f \in V_H^L$, can be represented as*

$$f = \sum_{k=1}^m \left\langle f | S^{-1} f_k \right\rangle f_k = \sum_{k=1}^m \langle f | f_k \rangle S^{-1} f_k.$$

3. *If $f \in V_H^L$, and has the representation $f = \sum_{k=1}^m c_k f_k$ for some scalar coefficients $\{c_k\}_{k=1}^m$, then*

$$\sum_{k=1}^m |c_k|^2 = \sum_{k=1}^m \left| \left\langle f | S^{-1} f_k \right\rangle \right|^2 + \sum_{k=1}^m \left| c_k - \left\langle f | S^{-1} f_k \right\rangle \right|^2.$$

Proof. (1) $S : V_H^L \longrightarrow V_H^L$, by $Sf = TT^*f = \sum_{k=1}^m \langle f | f_k \rangle f_k$, for all $f \in V_H^L$. Now

$$S^* = (TT^*)^* = (T^*)^* T^* = TT^* = S.$$

It follows that S is self-adjoint. We have ker $S = \{f : Sf = 0\}$. Let $f \in$ ker S, then $Sf = 0$. Therefore

$$0 = \langle Sf | f \rangle = \left\langle \sum_{k=1}^m \langle f | f_k \rangle f_k | f \right\rangle = \sum_{k=1}^m |\langle f | f_k \rangle|^2.$$

Thereby $\sum_{k=1}^{m} |\langle f|f_k\rangle|^2 = 0$. Since $\{f_k\}_{k=1}^{m}$ be a frame for V_H^L, by Definition (3),

$$A\|f\|^2 \leq \sum_{k=1}^{m} |\langle f|f_k\rangle|^2 \leq B\|f\|^2, \text{ for all } f \in V_H^L.$$

Hence $A\|f\|^2 \leq 0 \leq B\|f\|^2$, for all $f \in V_H^L$ and $A, B > 0$. So $\|f\|^2 = 0$. Therefore $f = 0$, which implies $N_S = 0$. Hence S is one to one. Since V_H^L is of finite dimension, from the Lemma (8) S is onto . Therefore S is invertible.

(2) If S is self-adjoint then S^{-1} is self-adjoint. For, Consider

$$(S^{-1})^* = (S^*)^{-1} = (S)^{-1} \text{ as } S \text{ is self adjoint.}$$

Thereby S^{-1} is self-adjoint. If $S : V_H^L \longrightarrow V_H^L$, is linear and bijection then S^{-1} is linear. For, Since S is onto, $S^{-1} : V_H^L \longrightarrow V_H^L$. Let $f, g \in V_H^L$ then there exists $k, h \in V_H^L$ such that $S^{-1}(f) = k$ and $S^{-1}(g) = h$. Thereby $f = S(k)$ and $g = S(h)$. Let $\alpha, \beta \in H$, then

$$\begin{aligned}
S^{-1}(\alpha f + \beta g) &= S^{-1}(\alpha S(k) + \beta S(h)) \\
&= S^{-1}(S(\alpha k + \beta h)) \text{ as } S \text{ is linear.} \\
&= \alpha k + \beta h \\
&= \alpha S^{-1}(f) + \beta S^{-1}(g)
\end{aligned}$$

Thereby for all $f, g \in V_H^L$ and $\alpha, \beta \in H$,

$$S^{-1}(\alpha f + \beta g) = \alpha S^{-1}(f) + \beta S^{-1}(g)$$

Hence S^{-1} is linear. Let $f \in V_H^L$, then

$$f = SS^{-1}f = TT^*S^{-1}f = \sum_{k=1}^{m} \left\langle S^{-1}f|f_k\right\rangle f_k = \sum_{k=1}^{m} \left\langle f|(S^{-1})^*f_k\right\rangle f_k = \sum_{k=1}^{m} \left\langle f|S^{-1}f_k\right\rangle f_k.$$

Thereby for every $f \in V_H^L$,

$$f = \sum_{k=1}^{m} \left\langle f|S^{-1}f_k\right\rangle f_k. \tag{15}$$

Similarly we have

$$f = S^{-1}Sf = S^{-1}TT^*f = S^{-1}\left(\sum_{k=1}^{m} \langle f|f_k\rangle f_k\right) = \sum_{k=1}^{m} S^{-1}(\langle f|f_k\rangle f_k) = \sum_{k=1}^{m} \langle f|f_k\rangle S^{-1}f_k.$$

Thereby for every $f \in V_H^L$,

$$f = \sum_{k=1}^{m} \langle f|f_k\rangle S^{-1}f_k. \tag{16}$$

From (15) and (16), for every $f \in V_H^L$,

$$f = \sum_{k=1}^{m} \left\langle f|S^{-1}f_k\right\rangle f_k = \sum_{k=1}^{m} \langle f|f_k\rangle S^{-1}f_k.$$

(3) Let $f \in V_H^L$, from Corollary (1)

$$f = \sum_{k=1}^{m} c_k f_k, \text{ for some } c_k \in H.$$

From the part (1),

$$f = \sum_{k=1}^{m} c_k f_k = \sum_{k=1}^{m} \left\langle f | S^{-1} f_k \right\rangle f_k. \tag{17}$$

Hence

$$\sum_{k=1}^{m} \left(c_k - \left\langle f | S^{-1} f_k \right\rangle \right) f_k = 0. \tag{18}$$

Thereby $\sum_{k=1}^{m} d_k f_k = 0$, for some $d_k = \left(c_k - \left\langle f | S^{-1} f_k \right\rangle \right) \in H$. From (7), $T : H^m \longrightarrow V_H^L$, is defined by $T \{d_k\}_{k=1}^{m} = \sum_{k=1}^{m} d_k f_k$, $d_k \in H$. We have $N_T = \left\{ \{d_k\}_{k=1}^{m} \mid T \{d_k\}_{k=1}^{m} = 0 \right\}$, therefore

$$\{d_k\}_{k=1}^{m} = \{c_k\}_{k=1}^{m} - \left\{ \left\langle f | S^{-1} f_k \right\rangle \right\}_{k=1}^{m} \in N_T. \tag{19}$$

From Lemma (10), $N_T = R_{T^*}^{\perp}$, then

$$\{c_k\}_{k=1}^{m} - \left\{ \left\langle f | S^{-1} f_k \right\rangle \right\}_{k=1}^{m} \in R_{T^*}^{\perp}. \tag{20}$$

From (7) and (8) we have $T^* : V_H^L \longrightarrow H^m$, $T^* f = \{\langle f | f_k \rangle\}_{k=1}^{m}$, and $S : V_H^L \longrightarrow V_H^L$, $Sf = TT^* f = \sum_{k=1}^{m} \langle f | f_k \rangle f_k$. Hence $T^*(S^{-1}f) = \left\{ \langle S^{-1} f | f_k \rangle \right\}_{k=1}^{m}$. Therefore

$$\left\{ \left\langle S^{-1} f | f_k \right\rangle \right\}_{k=1}^{m} \in R_{T^*}.$$

Since S^{-1} is self adjoint,

$$\left\{ \left\langle S^{-1} f | f_k \right\rangle \right\}_{k=1}^{m} = \left\{ \left\langle f | S^{-1} f_k \right\rangle \right\}_{k=1}^{m}.$$

Hence

$$\left\{ \left\langle f | S^{-1} f_k \right\rangle \right\}_{k=1}^{m} = \left\{ \left\langle S^{-1} f | f_k \right\rangle \right\}_{k=1}^{m} \in R_{T^*}. \tag{21}$$

Now we can write,

$$\{c_k\}_{k=1}^{m} = \{c_k\}_{k=1}^{m} - \left\{ \left\langle f | S^{-1} f_k \right\rangle \right\}_{k=1}^{m} + \left\{ \left\langle f | S^{-1} f_k \right\rangle \right\}_{k=1}^{m}. \tag{22}$$

From (20)–(22) and Lemma (9),

$$\sum_{k=1}^{m} |c_k|^2 = \sum_{k=1}^{m} \left| \left\langle f | S^{-1} f_k \right\rangle \right|^2 + \sum_{k=1}^{m} \left| c_k - \left\langle f | S^{-1} f_k \right\rangle \right|^2. \tag{23}$$

□

Theorem (1) is one of the most important results about frames, and

$$f = \sum_{k=1}^{m} \left\langle f | S^{-1} f_k \right\rangle f_k = \sum_{k=1}^{m} \langle f | f_k \rangle S^{-1} f_k.$$

is called the *frame decomposition*. If $\{f_k\}_{k=1}^{m}$ is a frame but not a basis, there exists non-zero sequences $\{h_k\}_{k=1}^{m}$ such that $\sum_{k=1}^{m} h_k f_k = 0$. Thereby $f \in V_H^L$ can be written as

$$f = \sum_{k=1}^{m} \left\langle f | S^{-1} f_k \right\rangle f_k + \sum_{k=1}^{m} h_k f_k = \sum_{k=1}^{m} \left(\left\langle f | S^{-1} f_k \right\rangle + h_k \right) f_k$$

showing that f has many representations as superpositions of the frame elements.

Corollary 2. *Assume that* $\{f_k\}_{k=1}^{m}$ *is a basis for* V_H^L. *Then there exists a unique family* $\{g_k\}_{k=1}^{m}$ *in* V_H^L *such that*

$$f = \sum_{k=1}^{m} \langle f | g_k \rangle f_k, \text{ for all } f \in V_H^L. \tag{24}$$

In terms of the frame operator, $\{g_k\}_{k=1}^{m} = \left\{ S^{-1} f_k \right\}_{k=1}^{m}$. *Furthermore* $\langle f_j | g_k \rangle = \delta_{j,k}$.

Proof. Let $f \in V_H^L$, from the Theorem (1),

$$f = \sum_{k=1}^{m} \left\langle f | S^{-1} f_k \right\rangle f_k. \tag{25}$$

Now take $\{g_k\}_{k=1}^{m} = \left\{ S^{-1} f_k \right\}_{k=1}^{m}$, in (25) then,

$$f = \sum_{k=1}^{m} \langle f | g_k \rangle f_k. \tag{26}$$

Hence there exists a family $\{g_k\}_{k=1}^{m}$ in V_H^L such that

$$f = \sum_{k=1}^{m} \langle f | g_k \rangle f_k, \text{ for all } f \in V_H^L. \tag{27}$$

Uniqueness: Assume that there is another family $\{h_k\}_{k=1}^{m}$ in V_H^L such that

$$f = \sum_{k=1}^{m} \langle f | h_k \rangle f_k, \text{ for all } f \in V_H^L.$$

Then $\sum_{k=1}^{m} \langle f | g_k \rangle f_k = \sum_{k=1}^{m} \langle f | h_k \rangle f_k.$

$\implies \sum_{k=1}^{m} (\langle f | g_k \rangle - \langle f | h_k \rangle) f_k = 0.$

$\implies \sum_{k=1}^{m} \langle f | g_k - h_k \rangle f_k = 0.$

$\implies \langle f | g_k - h_k \rangle = 0, \text{ for all } k = 1, 2, \cdots, m, \text{ as } \{f_k\}_{k=1}^{m} \text{ is a basis for } V_H^L.$

$\implies \langle f | g_k - h_k \rangle = 0, \text{ for all } k = 1, 2, \cdots, m, \text{ for all } f \in V_H^L.$

$\implies g_k - h_k = 0, \text{ for all } k = 1, 2, \cdots, m.$

$\implies g_k = h_k, \text{ for all } k = 1, 2, \cdots, m.$ Hence there exists a unique family $\{g_k\}_{k=1}^{m}$ in V_H^L such that

$$f = \sum_{k=1}^{m} \langle f | g_k \rangle f_k, \text{ for all } f \in V_H^L. \tag{28}$$

Since $f = \sum_{k=1}^{m} \langle f | g_k \rangle f_k$, for all $f \in V_H^L$, for fixed $f_j \in V_H^L$,

$$f_j = \sum_{k=1}^{m} \langle f_j | g_k \rangle f_k. \tag{29}$$

Since $\{f_k\}_{k=1}^m$ is a basis for V_H^L, $\{f_k\}_{k=1}^m$ is linearly independent. Therefore in (29), $\langle f_j | g_k \rangle = \delta_{j,k}$. Otherwise ($\langle f_j | g_k \rangle \neq \delta_{j,k}$.), $\{f_k\}_{k=1}^m$ becomes linearly dependent. Hence,

$$\langle f_j | g_k \rangle = \delta_{j,k}. \tag{30}$$

□

We can give a perceptive clarification of why frames are important in signal transmission. Let us say we want to transmit a signal f that belonging to a left quaternion Hilbert space from a transmitter T to a receiver R. Suppose that both T and R have the knowledge of frame $\{f_k\}_{k=1}^m$ for V_H^L. Let T transmits the frame coefficients $\{\langle f | S^{-1} f_k \rangle\}_{k=1}^m$. Using the received numbers, the receiver R can reconstruct the signal f using the frame decomposition. If R receives a perturbed (noisy) signal, $\{\langle f | S^{-1} f_k \rangle + c_k\}_{k=1}^m$ of the correct frame coefficients, using the received coefficients, R will reconstruct the transmitted signal as

$$\sum_{k=1}^m \left(\left\langle f | S^{-1} f_k \right\rangle + c_k \right) f_k \;=\; \sum_{k=1}^m \left\langle f | S^{-1} f_k \right\rangle f_k + \sum_{k=1}^m c_k f_k = f + \sum_{k=1}^m c_k f_k$$

this differs from the correct signal f by the noise $\sum_{k=1}^m c_k f_k$. Minimizing this noise for various signals with different types of noises has been a hot topic in signal processing. Since the frame $\{f_k\}_{k=1}^m$ is an over complete set, it is possible that some part the noise contribution may sum to zero. At the same time, if $\{f_k\}_{k=1}^m$ is an orthonormal basis this scenario is never a possibility. In that case

$$\| \sum_{k=1}^m c_k f_k \|^2 = \sum_{k=1}^m |c_k|^2,$$

so each noise contribution will make the reconstruction worse.

Definition 8. *For $0 < p < \infty$,*

$$\ell^p = \left\{ x = \{x_n\} \subset H \mid \sum_n |x_n|^p < \infty \right\}.$$

If $p \geq 1$, $\|x\|_p = \left(\sum_n |x_n|^p \right)^{\frac{1}{p}}$ defines a norm in ℓ^p. In fact ℓ^p is a complete metric space with respect to this norm.

We have already seen that, for $f \in V_H^L$, the frame coefficients $\{\langle f | S^{-1} f_k \rangle\}_{k=1}^m$ have minimal ℓ^2 norm among all sequences $\{c_k\}_{k=1}^m$ for which $f = \sum_{k=1}^m c_k f_k$. In the next theorem, let us see that the existence of coefficients minimizing the ℓ^1 norm.

Theorem 2. *Let $\{f_k\}_{k=1}^m$ be a frame for a finite-dimensional left quaternion Hilbert space V_H^L. Given $f \in V_H^L$, there exist coefficients $\{d_k\}_{k=1}^m \in H^m$ such that $f = \sum_{k=1}^m d_k f_k$, and*

$$\sum_{k=1}^m |d_k| = \inf \left\{ \sum_{k=1}^m |c_k| : f = \sum_{k=1}^m c_k f_k \right\}. \tag{31}$$

Proof. Fix $f \in V_H^L$. It is clear that we can choose a set of coefficients $\{c_k\}_{k=1}^m$, $c_k \in H$ such that $f = \sum_{k=1}^m c_k f_k$. Let $r := \sum_{k=1}^m |c_k|$. Since we want to minimize the ℓ^1 norm of the coefficients, we can now restrict our search for a minimizer to sequences $\{d_k\}_{k=1}^m$ belonging to the compact set

$$M := \left\{ \{d_k\}_{k=1}^m \in H^m \; : \; |d_k| \le r, \; k = 1, 2, ..., m. \right\} \tag{32}$$

Now,

$$\left\{ \{d_k\}_{k=1}^m \in M \,|\, f = \sum_{k=1}^m d_k f_k \right\} \quad \text{is compact.} \tag{33}$$

Define a function

$$\varphi : H^m \longrightarrow \mathbb{R}, \; \varphi \{d_k\}_{k=1}^m := \sum_{k=1}^m |d_k|. \tag{34}$$

We can prove φ is continuous by similar proof of Proposition (2). From (32) and Lemma (2),

$$\sum_{k=1}^m |d_k| = \inf \left\{ \sum_{k=1}^m |c_k| : f = \sum_{k=1}^m c_k f_k \right\}. \tag{35}$$

Hence for given $f \in V_H^L$, there exist coefficients $\{d_k\}_{k=1}^m \in H^m$ such that $f = \sum_{k=1}^m d_k f_k$, and

$$\sum_{k=1}^m |d_k| = \inf \left\{ \sum_{k=1}^m |c_k| : f = \sum_{k=1}^m c_k f_k \right\}. \tag{36}$$

□

Let $W = $ *left span* $\{f_k\}_{k=1}^m$, then in view of Proposition (2), the set of vectors $\{f_k\}_{k=1}^m$ is a frame for W. If $W \ne V_H^L$, then using the frame decomposition of the frame $\{f_k\}_{k=1}^m$ one can obtain useful expression for the orthogonal projection onto the subspace W.

Theorem 3. *Let $\{f_k\}_{k=1}^m$ be a frame for a subspace W of the left quaternion Hilbert space V_H^L. Then the orthogonal projection of V_H^L onto W is given by*

$$Pf = \sum_{k=1}^m \left\langle f | S^{-1} f_k \right\rangle f_k. \tag{37}$$

Proof. Define an operator P from V_H^L to W by

$$P : V_H^L \longrightarrow W \; \text{by} \; Pf = \sum_{k=1}^m \left\langle f | S^{-1} f_k \right\rangle f_k, \text{ for all } f \in V_H^L. \tag{38}$$

First let us prove that P is onto. For, let $f_1 \in W$ and $S : W \longrightarrow W$ be a frame operator in W. Since $\{f_k\}_{k=1}^m$ be a frame for the subspace W, we have

$$f_1 = \sum_{k=1}^m \left\langle f_1 | S^{-1} f_k \right\rangle f_k$$

But $W \subseteq V_H^L$, thereby $f_1 \in V_H^L$. Since f_1 is arbitrary, for given $f \in W$, there exists $g \in V_H^L$ such that $Pg = f$. Thereby P is onto. Now we want to prove that P is an orthogonal projection. Hence our claims are

(i) $Pf = f$, for $f \in W$.

(ii) $Pf = 0$, for $f \in W^{\perp}$.

For,

(i)   The mapping $P : V_H^L \xrightarrow{onto} W$ is given by

$$Pf = \sum_{k=1}^{m} \left\langle f | S^{-1} f_k \right\rangle f_k. \tag{39}$$

Since $\{f_k\}_{k=1}^{m}$ be a frame for a subspace W of the left quaternion Hilbert space V_H^L, from (7) the frame operator S is given by

$$S : W \longrightarrow W, \ \text{ by } Sf = \sum_{k=1}^{m} \langle f | f_k \rangle f_k, \text{ for all } f \in W. \tag{40}$$

From the Theorem (1), every $f \in W$, can be represented as

$$f = \sum_{k=1}^{m} \left\langle f | S^{-1} f_k \right\rangle f_k. \tag{41}$$

From (39) and (41),

$$Pf = f, \text{ for all } f \in W. \tag{42}$$

(ii)   Let $f \in W^{\perp}$. The mapping $P : V_H^L \xrightarrow{onto} W$ is given by

$$Pf = \sum_{k=1}^{m} \left\langle f | S^{-1} f_k \right\rangle f_k. \tag{43}$$

From the Theorem (1), the frame operator $S : W \longrightarrow W$ is bijective. Hence the range of S^{-1} is W. That is,

$$S^{-1} : W \longrightarrow W.$$

So $S^{-1} f_k = g$ for some $g \in W$. Hence (43) gives

$$\begin{aligned}
Pf &= \sum_{k=1}^{m} \left\langle f | S^{-1} f_k \right\rangle f_k \\
&= \sum_{k=1}^{m} \langle f | g \rangle f_k, \text{ for some } g \in W \\
&= 0 \text{ as } g \in W \text{ and } f \in W^{\perp}.
\end{aligned}$$

Therefore

$$Pf = 0 \text{ for all } f \in W^{\perp}. \tag{44}$$

From (42) and (44), P is an orthogonal projection.   □

Definition 9. *The numbers*

$$\left\langle f | S^{-1} f_k \right\rangle, \ k = 1, \cdots, m \tag{45}$$

are called frame coefficients. The frame $\left\{ S^{-1} f_k \right\}_{k=1}^{m}$ is called the canonical dual of $\{f_k\}_{k=1}^{m}$.

Author Contributions: M. Khokulan: The essence of this article is part of the M. Phil thesis of Khokulan. Initial proofs were done by him. K. Thirulogasanthar: Supervised the thesis, provided ideas to the proofs, corrected

all the proofs and wrote the article. S. Srisatkunarajah: Supervised the thesis, proof read the article and made corrections.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Duffin, R.J.; Schaeffer, A.C. A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.* **1952**, *72*, 341–366.
2. Daubechies, I.; Grossmann, A.; Meyer, Y. Painless nonorthogonal expansions. *J. Math. Phys.* **1986**, *27*, 1271–1283.
3. Daubechies, I. *Ten Lectures on Wavelets*; SIAM: Philadelphia, PA, USA, 1992.
4. Ali, S.T.; Antoine, J.-P.; Gazeau, J.-P. *Coherent States, Wavelets, and Their Generalization*, 2nd ed.; Springer: New York, NY, USA, 2014.
5. Christensen, O. *An Introduction to Frames and Riesz Bases*; Birkhauser Boston Inc.: New York, NY, USA, 2003.
6. Grochenig, K.H. *Foundations of Time-Frequency Analysis*; Birkhauser Boston Inc.: New York, NY, USA, 2000.
7. Cotfas, N.; Gazeau, J.-P. Finite tight frames and some applications. *J. Phys. A Math. Theor.* **2010**, *43*, 193001.
8. Cotfas, N.; Gazeau, J.-P.; Vourdas, A. Finite dimensional Hilbert spaces and frame quantization. *J. Phys. A Math. Theor.* **2011**, *44*, 17303.
9. Goyal, V.K.; Kovacevic, J.; Kelner, A.J. Quantized frame expansions with erasures. *Appl. Comp. Harm. Anal.* **2000**, *10*, 203–233.
10. Thirulogasanthar, K.; Bahsoun, W. Frames built on fractal sets. *J.Geom. Phys.* **2004**, *50*, 79–98.
11. Ali, S.T.; Antoine, J.-P.; Gazeau, J.-P. Continuous frames in Hilbert spaces. *Ann. Phys.* **1993**, *222*, 1–37.
12. Adler, S.L. *Quaternionic Quantum Mechanics and Quantum Fields*; Oxford University Press: New York, NY, USA, 1995.
13. Thirulogasanthar, K.; Ali, S.T. Regular subspace of a quaternionic Hilbert space from quaternionic Hermite polynomials and associated coherent states. *J. Math. Phys.* **2013**, *54*, 013506.
14. Thirulogasanthar, K.; Honnouvo, G.; Krzyzak, A. Coherent states and Hermite polynomials on quaternionic Hilbert spaces. *J. Phys. A Math. Theor.* **2010**, *43*, 385205.
15. Rolewicz, S. *Metric Linear Spaces*; D. Riedel Publishing Company: Dordrecht, The Netherlands, 1985.