axioms

MDPI

Article

Non-Abelian Pseudocompact Groups

W. W. Comfort [1,†,*] and Dieter Remus [2,†,*]

Received: 29 September 2015; Accepted: 23 December 2015; Published: 12 January 2016
Academic Editor: Sidney A. Morris

1 Department of Mathematics, Wesleyan University, Middletown, CT 06457, USA
2 Institut für Mathematik, Universität Paderborn, Warburger Str. 100, Paderborn D-33095, Germany
* Correspondence: wcomfort@wesleyan.edu (W.W.C.); remus@math.uni-paderborn.de (D.R.);
Tel.: +1-860-767-0069 (W.W.C.); +49-5404-7199 (D.R.)
† These authors contributed equally to this work.

Abstract: Here are three recently-established theorems from the literature. (A) (2006) Every non-metrizable compact abelian group K has $2^{|K|}$-many proper dense pseudocompact subgroups. (B) (2003) Every non-metrizable compact abelian group K admits $2^{2^{|K|}}$-many strictly finer pseudocompact topological group refinements. (C) (2007) Every non-metrizable pseudocompact abelian group has a proper dense pseudocompact subgroup and a strictly finer pseudocompact topological group refinement. (Theorems (A), (B) and (C) become false if the non-metrizable hypothesis is omitted.) With a detailed view toward the relevant literature, the present authors ask: What happens to (A), (B), (C) and to similar known facts about pseudocompact abelian groups if the abelian hypothesis is omitted? Are the resulting statements true, false, true under certain natural additional hypotheses, *etc.*? Several new results responding in part to these questions are given, and several specific additional questions are posed.

Keywords: topological group; group topology; pseudocompact topological group; G_δ-dense subgroup; countably compact group; ω-bounded group; refinement of topology; extremal pseudocompact group

1. Introduction

Specific references to the literature concerning Theorems (A), (B) and (C) of the Abstract are given in 5.7(d), 8.2.2 and 4(l), respectively. Every metrizable pseudocompact group, abelian or not, is compact, hence admits neither a proper dense pseudocompact subgroup nor a proper pseudocompact group refinement (see 4(a)); thus, (A), (B) and (C) all become false when the non-metrizability hypothesis is omitted.

All hypothesized topological spaces and topological groups in this paper are assumed to be Tychonoff spaces.

1.1. Brief Outline of the Paper

As our Title and Abstract indicate, our goal in this survey is to describe the historical development of the theory of pseudocompact topological groups. Many of the results we cite, especially the older results, require an abelian hypothesis; some questions, definitions and results make sense and are correct without that hypothesis, however, and we emphasize these. Thus, this paper has two goals: (1) to provide an overview of the (by now substantial) literature on pseudocompact groups; and (2) to offer several new results about non-abelian pseudocompact groups.

As an aid to the reader and to avoid uncertainty, algebraic statements and results known to hold also for non-abelian groups carry the symbol *.

We proceed as follows. Subsection 1.2 establishes the notation and terminology, and Subsection 1.3 reviews early works.

Section 2 describes several criteria, some algebraic and some cardinality-related, which are necessary or sufficient that a group admits a pseudocompact group topology. With a focus on compact groups, Section 3 describes the availability of proper dense subgroups. Section 4 recounts the principal incremental steps in the literature which led finally to a positive solution to these questions: Does every non-metrizable pseudocompact abelian group admit a proper dense subgroup and a strictly finer pseudocompact group topology? Section 5 considers briefly several miscellaneous issues and questions which concern pseudocompact groups.

Several workers have noted that those compact groups which admit a continuous epimorphism onto a product of the form F^κ ($|F| > 1$, $\kappa > \omega$) or of the form $\Pi_{i\in I} K_i$ ($|K_i| > 1$, $|I| > \omega$) admit (sometimes large) families of dense subgroups with special properties. Section 6 describes several instances in the literature.

Sections 7 and 8 concern respectively free compact (abelian and non-abelian) groups over a Tychonoff space and new results concerning non-abelian pseudocompact groups.

Insofar as expository clarity permits, we use the symbol K to denote a topological group known or assumed to be compact; and we use the symbol G for other groups and topological groups.

1.2. Notation and Terminology

As to notation and terminology, we generally follow Engelking [1] and Hewitt and Ross [2]. Here we record some supplemental definitions, notation and conventions.

(a) Given a cardinal number $\alpha = \alpha_0 \geq \omega$, the cardinal $\beth_\omega(\alpha)$ is defined as follows: $\alpha_{n+1} := 2^{\alpha_n}$ for $n < \omega$, and $\beth_\omega(\alpha_0) := \Sigma_n \alpha_n = \sup_n \alpha_n$.

(b) *For topological groups G_0 and G_1, we write $G_0 \simeq G_1$ if some bijection from G_0 onto G_1 is simultaneously an algebraic isomorphism and a topological homeomorphism.

(c) *A topological group $G = (G, \mathcal{T})$ is totally bounded (alternatively, precompact) if for every non-empty $U \in \mathcal{T}$ there is finite $F \subseteq G$ such that $G = FU$. We denote by $\mathrm{TB}(G)$ the set of totally bounded group topologies on a group G.

(d) (Hewitt [3]) A space X is pseudocompact if each continuous function $f : X \to \mathbb{R}$ is bounded.

(e) A space is countably compact if each of its infinite subsets has an accumulation point (equivalently ([1]) if each countable open cover admits a finite subcover).

(f) A space is ω-bounded if each of its countably-infinite subsets has compact closure.

(g) A space X is a Baire space if every intersection of countably many dense open subsets of X is again dense in X.

(h) *A cardinal κ is admissible if there is a pseudocompact group of cardinality κ. And κ is λ-admissible if there is a pseudocompact group such that $|G| = \kappa$ and $wG = \lambda$.

(i) *Given a topological group G, we write

$\mathbb{P}(G) := \{H : |H| \text{ is a dense pseudocompact subgroup of } G\}$,

and, following [4], for a compact group K we write

$\mathfrak{m}(K) := \min\{|H| : H \in \mathbb{P}(K)\}$.

(j) *For a topological space $X = (X, \mathcal{T})$, we denote by PX, or by $(X, P\mathcal{T})$, the set X with the smallest topology in which each G_δ-subset of $(X, \mathcal{T})$ is open.

It is clear from the definitions of PK and $\mathfrak{m}(K)$ that $^*\mathfrak{m}(K) = d(PK)$; hence $^*\mathrm{cf}(\mathfrak{m}(K)) > \omega$, for each infinite compact group K.

(k) It is well known [2] (7.7) that a compact group K is totally disconnected (equivalently: zero-dimensional) if and only if each neighborhood of 1_K contains a compact open normal subgroup. In this paper we follow many workers and call such compact groups K profinite.

1.3. Basic Early Works

Here we offer a brief history of the principal concepts and objects we deal with in this paper.

(a) Hewitt [3] showed *inter alia* that a space X is pseudocompact if and only if it is G_δ-dense in its Stone–Čech compactification βX, hence in every (Tychonoff) space in which it is densely embedded. Later, Glicksberg [5] characterized pseudocompact spaces as those in which each locally finite family of open subsets is finite. For a detailed treatment and extrapolation of Hewitt's work, including the many other fruitful concepts introduced there, see [6].

(b) It is a fundamental theorem of Weil [7] that *the totally bounded groups are exactly the topological groups G which embed as a dense topological subgroup of a compact group. Further, this compactification of G, called the Weil completion of G and here denoted $\overline{G}$, is unique in the obvious sense.

(c) It is easy to see ([8] (1.1)) that *every pseudocompact group is totally bounded. Identifying those totally bounded groups which are pseudocompact, Comfort and Ross [8] (1.2, 4.1) showed that *for a totally bounded group G, these conditions are equivalent: (1) G is pseudocompact; (2) G is G_δ-dense in $\overline{G}$; and (3) $\overline{G} = \beta G$.

(d) From the equivalence (c) ((1)$\Leftrightarrow$(3)) and Mycielski's theorem [9] that every compact divisible group is connected, Wilcox [10] deduced a useful consequence: *every divisible pseudocompact group is connected (we note in passing, as remarked by Wilcox [10] (p. 579), that a connected pseudocompact abelian group need not be divisible).

(e) From (c) and the uniqueness aspect of Weil's theorem it follows that *a dense subgroup H of a pseudocompact group G is itself pseudocompact if and only if H is G_δ-dense in G – in which case necessarily $\overline{H} = \overline{G}$; further, as in [8] (1.4), *the product of any set of pseudocompact groups is again pseudocompact. Those two statements have been vastly generalized by subsequent workers. We give some examples. In (1) and (2), $G = \Pi_{i\in I} G_i$ with each G_i an arbitrary (not necessarily abelian or pseudocompact) topological group, $F_i \subseteq G_i$ and $F := \Pi_{i\in I} F_i$. (1) [11] *If F_i is functionally bounded in G_i in the sense that each continuous $f : G_i \to \mathbb{R}$ is bounded on F_i, then F is functionally bounded in G; (2) [12] *If $F_i \subseteq G_i$ is pseudocompact and either each F_i is a G_δ-set in G_i or each F_i is a retract of G_i, then F is pseudocompact; (3) [13] *If K is compact and X is G_δ-dense in K, then X is pseudocompact and $K = \beta X$; (4) [14] *If G is pseudocompact and X is dense in G, then X is C-embedded in $\mathrm{cl}_{PG}\, X$.

(f) The equivalences of (c) were established in [8] using earlier theorems of Kakutani and Kodaira [15], Halmos [16] (§64) and Ross and Stromberg [17]. A more direct approach, avoiding reference to those works, was given subsequently by de Vries [18]. See also Hušek [19] and Tkachenko [11,20] for alternative approaches.

(g) *Many of the results cited above have been extended and generalized into the context of locally pseudocompact groups; see, for example, [21,22] and the references given there.

2. Pseudocompactifiability Criteria: Elementary Constraints

(a) Every pseudocompact space is a Baire space [1] (3.10.F(e)), so in particular *every pseudocompact group is a Baire space (alternatively one may argue as in [4] (2.4(b)): a G_δ-dense subspace of a Baire space is itself a Baire space, so a pseudocompact group G, being G_δ-dense in the compact space $\overline{G}$, is necessarily a Baire space).

(b) Using (a), several workers (e.g., [4,23], [24] (2.5), [25]) made elementary cardinality observations like these, valid for infinite pseudocompact groups G. (1) $^*|G| \geq \mathfrak{c}$; (2) $^*d(PG) \geq \mathfrak{c}$; (3) $^*\mathrm{cf}(d(PG)) > \omega$; (4) *if $|G|$ is a strong limit cardinal, then $\mathrm{cf}(|G|) > \omega$; (5) if G is abelian, then either $r_0(G) \geq \mathfrak{c}$ or G is torsion; (6) if G is a torsion abelian group, then G is of bounded order.

Concerning (b): Van Douwen [23], arguing in a more general context, proved $|X| \geq \mathfrak{c}$ and other inequalities of cardinality type for every infinite pseudocompact space X with no isolated point.

(c) We remark in passing that the relation $\omega = \mathrm{cf}(|G|)$ does occur for some pseudocompact abelian groups in some models of ZFC . For example, if $\mathfrak{c} = \aleph_1 < \aleph_\omega < 2^{\mathfrak{c}}$, then, as noted below in 3(e), the group $K = \mathbb{T}^{\mathfrak{c}}$ contains a proper dense countably compact subgroup H with $|H| = \mathfrak{c}$, and then any group G such that $H \subseteq G \subseteq K$, say with $|G| = \aleph_\omega$, is necessarily pseudocompact by 1.3(c) ((2)⇒(1)) (with $K = \overline{H} = \overline{G}$).

(d) The remarks in (b) are useful, but they are largely negative in flavor. Here are some simple examples. (1) *There is no countably infinite pseudocompact group; (2) *A compact group K such that $wK = \beth_\omega(\alpha)$, satisfies $dK = \beth_\omega(\alpha)$; (3) *If [CH] fails, no infinite pseudocompact group satisfies $|G| = \aleph_1$.

3. Dense Subgroups: Scattered Results

Some topological groups do, and some do not, have proper dense subgroups. Here we cite some representative results from the literature.

(a) *The relations $dK \leq wK < 2^{wK} = |K|$, valid for every infinite compact group K ([26] (28.58(c))), make it clear that each such K admits a (proper) dense subset D with $|D| < |K|$, which then in turn generates a proper dense subgroup of the same cardinality. For emphasis: *every infinite compact group K admits a proper dense subgroup G with $|G| < |K|$. Similarly it follows easily, as in [4] (2.2(b)), that *for K a compact group with $w(K) = \alpha \geq \omega$, one has $\mathfrak{m}(K) \leq (\log(\alpha))^\omega$.

(b) [4] (2.7(a)) *Infinite compact groups K, K' with $w(K) = w(K')$ satisfy $\mathfrak{m}(K) = \mathfrak{m}(K')$. Hence *$\mathfrak{m}(K)$ is determined fully by $w(K)$ and is not affected by algebraic properties of the group K. Following [4], we define $\mathfrak{m}(\alpha) := \mathfrak{m}(K)$ for (arbitrary) compact K with $w(K) = \alpha$. Note: for $\alpha \geq \omega$, the cardinal $\mathfrak{m}(\{0,1\}^\alpha)$, which is $\mathfrak{m}(\alpha)$, is denoted $\Delta(\alpha, \omega)$ in [27].

(c) ([27]) If the Singular Cardinals Hypothesis is assumed (that is: $\kappa^\lambda \leq 2^\lambda \cdot \kappa^+$ for all infinite κ, λ), then $\mathfrak{m}(\alpha) = (\log(\alpha))^\omega$.

(d) Every infinite pseudocompact group*, and every infinite connected abelian group, has a proper dense subgroup [28] (4.1, 4.2).

(e) For compact groups K with $w(K)$ of the form $w(K) = 2^\alpha$, it was shown by Itzkowitz [29] (the abelian case) and by Wilcox [30] in general that *K contains a (necessarily proper) dense pseudocompact subgroup H such that $|H| \leq \alpha^\omega \leq 2^\alpha < 2^{2^\alpha} = |K|$. It was noted later [31] that *H may be chosen countably compact.

(f) Negating the tempting conjecture that parallel results might hold for locally compact groups, Rajagopalan and Soundrarajan [32] show that for each infinite cardinal κ there is on the group $\mathbb{T}^\kappa$ a locally compact group topology which admits no proper dense subgroup. In the same vein, there are many infinite totally bounded abelian groups which admit no proper dense subgroup [28].

(g) In fact, given an abelian group G, the topology induced on G by $\mathrm{Hom}(G, \mathbb{T})$ is a totally bounded group topology [33] (1.5) in which every subgroup is closed [31] (2.1).

4. Extremal Phenomena

We adopt terminology introduced tentatively and partially in [34] (5.1) and finally fully formalized in [35] (4.1): *a pseudocompact group $(G, \mathcal{T})$ is *r*-extremal (resp., *s*-extremal) if no pseudocompact group topology on G strictly contains $\mathcal{T}$ (resp., $(G, \mathcal{T})$ admits no proper dense pseudocompact subgroup). Note: the letters *r* and *s* here are intended to invoke the words refinement and subgroup, respectively.

(a) Since a pseudocompact normal space is countably compact ([1] (3.10.21), [6] (3.D.2)) and a countably compact metric space is compact [1] (4.1.17), we have, as noted frequently in the literature ([36] (4.5(a)), [37] (3.1), [34] (2.4, 3.6)): *every pseudocompact group G with $w(G) \leq \omega$ is both *r*- and *s*-extremal. This explains the occurrence of the hypothesis "$w(G) > \omega$" (equivalently: "G is non-metrizable") in many of the theorems cited below.

It was conjectured in [34] (5.1ff.) that no non-metrizable pseudocompact abelian group is r- or s-extremal (see also Question 2.B.1 in [38]). In the earliest days of investigation, the non-abelian case seemed totally inaccessible; but some fragmentary non-abelian results have emerged serendipitously by now (see below). Concerning the abelian question, the reader interested not in preliminary or incremental stages but only in the dénouement may safely ignore (b)–(k) below and skip directly to (l). For a more leisurely treatment of the historical development of this theorem, and for the statement of several related unsolved problems, see [39].

(b) A non-metrizable compact abelian group is not r-extremal [37] (3.4).

(c) A non-metrizable compact totally disconnected abelian group is neither r- nor s-extremal [36] (4.3, 4.4).

(d) A non-metrizable compact abelian group is neither r- nor s-extremal; indeed, the witnessing dense subgroup may be chosen ω-bounded [34] (3.4).

(e) *A non-metrizable compact connected group is not r-extremal [40] (6.7).

(f) A non-metrizable zero-dimensional pseudocompact abelian group is neither r- nor s-extremal [34] (7.3).

(g) A pseudocompact abelian group G such that $|G| > \mathfrak{c}$ or $\omega_1 \leq w(G) \leq \mathfrak{c}$ is not s-extremal [41] (1.3).

(h) A pseudocompact abelian group G such that $r_0(G) > \mathfrak{c}$ or $\omega_1 \leq w(G) \leq \mathfrak{c}$ is not r-extremal [42] (5.10).

(i) A pseudocompact connected non-divisible abelian group is neither s-extremal [25] (7.1) nor r-extremal [43] (6.1), [42] (4.5(b)).

(j) A pseudocompact abelian group G with a closed G_δ-subgroup H (1) is r-extremal if H is r-extremal, and (2) is s-extremal if H is s-extremal ([42] (2.1)).

(k) If H is a closed pseudocompact subgroup of a pseudocompact abelian group G, then (1) G is not r-extremal if G/H is not r-extremal, and (2) G is not s-extremal if G/H is not s-extremal [43] (4.5), [42] (5.3).

(l) (See (C) of the Abstract.) Fully familiar with the sources cited in (b)–(k), and drawing on some of the arguments cited there, Comfort and van Mill showed [44,45] that no non-metrizable abelian pseudocompact group is r-extremal or s-extremal.

As suggested above and as our title indicates, we are interested in the present paper primarily in comparable and parallel results concerning non-abelian pseudocompact groups. See in this connection especially Sections 6 and 8.

4.1. Extremality Questions

As indicated in 4(l), the following two questions have been answered affirmatively [44,45] in the context of abelian groups. However, they remain unsettled in the general (possibly non-abelian) case.

Problem 4.1.1. (a) **Is every non-metrizable pseudocompact group not r-extremal?*

(b) **Is every non-metrizable pseudocompact group not s-extremal?*

(c) **Are those properties (r- extremal, s-extremal) equivalent?*

5. Related Concepts

It was natural that workers thinking about the issues raised in Section 4 might be drawn simultaneously to different but related questions. Here, with no pretense to completeness, we mention some of these.

5.1. Refinements of Maximal Weight

When a pseudocompact group G admits a proper pseudocompact refinement, can that be chosen of maximal weight (that is, of weight $2^{|G|}$)? Comfort and Remus [40] (5.5) responded positively for many (non-metrizable) compact abelian groups K, including for example those which are connected, or torsion, or which satisfy $\mathrm{cf}(w(K)) > \omega$. Later Comfort and Galindo [46] gave a positive answer for all non-metrizable compact abelian groups G [46] (5.1), also for non-metrizable pseudocompact abelian groups G which are torsion-free with $wG \leq |G| = |G|^{\omega}$ [46] (5.3) or (assuming [GCH]) which are torsion-free [46] (5.4(b)). Indeed ([46] (5.2)), in the compact abelian case K with $w(K) = \alpha > \omega$, there are $2^{2^{2^{\alpha}}}$-many pseudocompact group refinements of weight 2^{α}.

5.2. The Poset of Pseudocompact Refinements

Given a pseudocompact group $(G, \mathcal{T})$ with $w(G, \mathcal{T}) = \alpha$, let $Ps(G, \mathcal{T})$ (respectively, $CPs(G, \mathcal{T})$) be the partially-ordered set of group topologies $\mathcal{U}$ on G such that $\mathcal{U} \supseteq \mathcal{T}$ and $\mathcal{U}$ is pseudocompact (respectively, $\mathcal{U}$ is pseudocompact and connected). For each cardinal number γ set

$Ps_{\gamma} = \{\mathcal{U} \in Ps(G, \mathcal{T}) : w(G, \mathcal{U}) = \gamma\}$ and

$CPs_{\gamma} = \{\mathcal{U} \in CPs(G, \mathcal{T}) : w(G, \mathcal{U}) = \gamma\}$.

From [47] (3.11) it follows that each $\mathcal{U} \in CPs_{\gamma}(G, \mathcal{T})$ satisfies $\alpha \leq w(G, \mathcal{U}) \leq 2^{|G|}$.

We have shown [40] (6.6):

Theorem 5.2.1. **Let $K = (K, \mathcal{T})$ be a compact, connected group such that $w(K) = \alpha > \omega$, and let A be the connected component of the center of K. Then:*

(a) *if $\omega < \beta < \alpha$, then K admits a pseudocompact group topology $\mathcal{U}$ such that $\mathcal{U} \supseteq \mathcal{T}$, $\mathcal{U} \neq \mathcal{T}$, and $w(K, \mathcal{U}) = \alpha + 2^{2^{\beta}}$; and*

(b) *if $w(A) = \alpha$ or $\mathrm{cf}(\alpha) > \omega$, then K admits a pseudocompact group topology $\mathcal{U}$ with $\mathcal{U} \supseteq \mathcal{T}$, $\mathcal{U} \neq \mathcal{T}$, and $w(K, \mathcal{U}) = 2^{2^{\alpha}}$.*

As usual the (a) width, the (b) height and the (c) depth of a partially ordered set P are defined to be the supremum of the cardinality of those subsets of P which are respectively (a) an anti-chain, (b) well ordered, and (c) anti-well ordered. If there is an anti-chain $A \subseteq P$ such that $|A|$=width(P), then we say that width(P) is assumed, and similarly for height(P) and depth(P).

Comfort and Remus [47] (6.7) proved the following

Theorem 5.2.2. **Let $(K, \mathcal{T})$ be a compact, connected group, such that $w(K, \mathcal{T}) = \alpha$ with $\mathrm{cf}(\alpha) > \omega$, and let $\alpha \leq \gamma \leq 2^{|K|}$. Define $\bar{\gamma} = \min\{\gamma^{+}, 2^{|K|}\}$. Then:*

(a) $|Ps(K, \mathcal{T})| = |CPs(K, \mathcal{T})| = 2^{2^{|K|}}$;

(b) $|Ps_{\gamma}(K, \mathcal{T})| = |CPs_{\gamma}(K, \mathcal{T})| = 2^{\gamma \cdot |K|}$;

(c) *$width(Ps_{\gamma}(K, \mathcal{T})) = width(CPs_{\gamma}(K, \mathcal{T})) = 2^{\gamma \cdot |K|}$, and these widths are assumed;*

(d) *$height(Ps_{\gamma}(K, \mathcal{T})) = height(CPs_{\gamma}(K, \mathcal{T})) = \bar{\gamma}$, and these heights are assumed; and*

(e) *$depth(Ps_{\gamma}(K, \mathcal{T})) = depth(CPs_{\gamma}(K, \mathcal{T})) = \gamma$, and these depths are assumed.*

In the proof of Theorem 5.2.2 the main tools are Theorem 5.2.1 and the following theorem ([47] (6.4)).

Theorem 5.2.3. **Let $(K, \mathcal{T}_1)$ be a totally bounded topological group such that $w(K, \mathcal{T}_1) = \alpha_1 > \omega$ and the Weil completion is connected. Then every totally bounded group topology $\mathcal{T}_0$ on K such that $\mathcal{T}_0 \subseteq \mathcal{T}_1$ and $w(K, \mathcal{T}_0) = \alpha_0 < \alpha_1$ satisfies $|\,[\mathcal{T}_0, \mathcal{T}_1]\,| = 2^{\alpha_1}$.*

In the absence of the connectivity hypothesis, we proved this result [48] (2.6(a)).

Theorem 5.2.4. **Let G be a group, and let $\mathcal{T}_i \in \mathrm{TB}(G)$ $(i = 0, 1)$ with $w(G, \mathcal{T}_i) = \alpha_i \geq \omega$. If $\alpha_0 < \alpha_1$ and $\mathcal{T}_0 \subseteq \mathcal{T}_1$, then $|\,[\mathcal{T}_0, \mathcal{T}_1]\,| \geq \alpha_1$.*

Corollary 5.2.5. **Let $(G, \mathcal{T})$ be a pseudocompact group with $w(G, \mathcal{T}) = \alpha > |G|$. Then there are at least α-many pseudocompact group topologies on G which are coarser than $\mathcal{T}$.*

Proof. Using the technique of the proof of [49] (2.9), we obtain $\mathcal{U} \in \mathrm{TB}(G)$ with $w(G, \mathcal{U}) \leq |G| < \alpha = w(G, \mathcal{T})$ such that $\mathcal{U} \subseteq \mathcal{T}$. Then $|[\mathcal{U}, \mathcal{T}]| \geq \alpha$ by Theorem 5.2.4, and $(G, \mathcal{V})$ is pseudocompact for each $\mathcal{V} \in [\mathcal{U}, \mathcal{T}]$ (since $(G, \mathcal{T})$ is pseudocompact). □

In [47] (6.9), one finds

Problem 5.2.6. **Let G be a group, and let $\mathcal{T}_i \in \mathrm{TB}(G)$ $(i = 0, 1)$ with $w(G, \mathcal{T}_i) = \alpha_i \geq \omega$. If $\alpha_0 < \alpha_1$ and $\mathcal{T}_0 \subseteq \mathcal{T}_1$, must $|[\mathcal{T}_0, \mathcal{T}_1]| = 2^{\alpha_1}$?*

We add the following

Problem 5.2.7. **Let $(K, \mathcal{T})$ be a non-metrizable compact, connected group.*
(a) *Does $\mathcal{T}$ admit a proper (connected) pseudocompact refinement of maximal weight $2^{|K|}$?*
(b) *Are there $2^{2^{|K|}}$-many (connected) pseudocompact group topologies on K which are finer than $\mathcal{T}$?*

We note in passing in connection with Problem 5.2.7, as remarked in [47] (pp. 277–278) and in contrast with [47] (6.11), that a pseudocompact refinement of a connected (abelian) pseudocompact group need not itself be connected. We note also that when the non-metrizability hypothesis is omitted in Problem 5.2.7, the resulting Questions (a) and (b) have negative answers. See in this connection 4(a) above.

5.3. Totally Dense Subgroups

As usual, a subgroup D of a topological group G is totally dense in G if $D \cap H$ is dense in H for every closed normal subgroup H of G. Several workers have turned attention to the question of the existence of totally dense pseudocompact subgroups of a given (usually compact) group. Since this topic is a bit removed from our central focus here, for details in this direction we simply refer the reader to the relevant papers known to us: [50,51], [36] (5.3), [4] (5.8), [52–55]. We note explicitly that, building upon and extending results from her thesis [56], Giordano Bruno and Dikranjan [57] characterized those compact abelian groups with a proper totally dense pseudocompact subgroup as those with no closed torsion G_δ-subgroup.

5.4. Concerning the Group Topologies $\sup\mathrm{TB}(G)$ *and* $\sup Ps(G, \mathcal{T})$

It is easily seen that *the supremum of any nonempty set of totally bounded group topologies on a fixed group G is another such topology. In particular, then *each group G which admits a totally bounded group topology admits the largest such topology. As noted [33] (1.6, 1.7) above, for abelian groups G this is the topology induced on G by $\mathrm{Hom}(G, \mathbb{T})$. It is shown in [31] (2.2) that when G is infinite abelian, that topology on G is never pseudocompact; that is obvious now, in view of the result cited above in 4(l).

Concerning that supremum, we record here a conjecture of Comfort and van Mill [58].

Conjecture 5.4.1. *Let G be an abelian group which admits a pseudocompact group topology. Then the supremum of the pseudocompact group topologies on G coincides with the largest totally bounded group topology on G (that is, the topology induced on G by $\mathrm{Hom}(G, \mathbb{T})$).*

Conjecture 5.4.1 was established in [58] for abelian groups G which satisfy any of these (overlapping) conditions: (1) G is torsion; (2) $|G| \leq 2^{\mathfrak{c}}$; (3) $r_0(G) = |G| = |G|^\omega$; (4) $|G|$ is a strong limit cardinal with $r_0(G) = |G|$; (5) some pseudocompact group topology $\mathcal{T}$ on G satisfies $w(G, \mathcal{T}) \leq \mathfrak{c}$; (6) G admits a compact group topology. However, the conjecture remains unsettled in full generality.

While neither the present authors nor the authors of [58] attempted to find the optimal non-abelian version of the theorems and conjecture just given, we note that the most naive non-abelian analogue, namely that the supremum of all pseudocompact group topologies on a (possibly non-abelian) group G which admits such a topology coincides with the largest totally bounded group topology, fails dramatically, even in the metrizable case. The following result is taken from [47].

Theorem 5.4.2. **Let K be a compact, connected Lie group with trivial center, and let $\mathcal{T}$ be the usual product topology on K^ω. Then $\mathcal{T}$ is the only pseudocompact group topology on K^ω [47] (7.4(a)), but $\mathcal{T}$ admits $2^{2^{\mathfrak{c}}}$-many totally bounded finer group topologies [47] (7.4(b)).*

5.5. Additional Extremality Theorems

The techniques used in the papers cited in Section 4 were adapted and extended by Giordano Bruno [56,59] to achieve parallel extremality results for pseudocompact abelian groups G which are even α-pseudocompact in the sense that G meets every non-empty intersection of α-many open subsets of $\overline{G} = \beta G$.

Prior to the appearance of [44,45], researchers in Udine, Italy, considered conditions weaker than metrizability which suffice to guarantee that a pseudocompact abelian group G is both r- and s-extremal [56,57,60,61]. Here is a sample result.

Theorem 5.5.1. *If some closed G_δ-subgroup N of G admits a dense pseudocompact subgroup H such that $r_0(N/H) \geq \mathfrak{c}$, then G itself has such a subgroup (hence is neither r- nor s-extremal).*

5.6. Closed Subgroups of Pseudocompact Groups

Since every subgroup of a totally bounded group is totally bounded and every closed subspace of a countably compact space is countably compact, it is reasonable (though perhaps naive) to inquire whether every closed subgroup of a pseudocompact group is necessarily pseudocompact. To the authors' knowledge, this question was first addressed in [31] (2.4), where a straightforward abelian counterexample is offered. Later this fact was noted (see [62] (2.1), [63] (2.9)):

Theorem 5.6.1. **Every totally bounded group H embeds as a closed subgroup of a pseudocompact group G. If H is abelian, G may be chosen as abelian.*

The construction of [62] (2.1) shows, though the authors did not record the fact explicitly, that, *when H as in Theorem 5.6.1 is non-metrizable (that is, when $w(H) > \omega$), one may choose G so that $w(G) = w(H)$.

In the abelian case, the correct locally bounded analogue of Theorem 5.6.1 has been recorded by Ursul [64]:

Theorem 5.6.2. *Every locally bounded abelian group is a closed subgroup of a locally pseudocompact group.*

More recently, Leiderman, Morris and Tkachenko [65] have focused on closed embeddings into pseudocompact groups of small density character. For example, they have shown this.

Theorem 5.6.3. **Every totally bounded group H such that $w(H) \leq \mathfrak{c}$ embeds as a closed subgroup of a separable pseudocompact group.*

Since there are many totally bounded non-separable pseudocompact groups of weight $\mathfrak{c}$, for example the ω-bounded group $H := \{x \in \mathbb{T}^{\mathfrak{c}} : |\{\eta < \mathfrak{c} : x_\eta \neq 1_{\mathbb{T}}\}| \leq \omega\}$, it follows from Theorem 5.6.3, as is remarked in [65], that a closed subgroup of a separable pseudocompact abelian group can be non-separable.

5.7. Miscellaneous Investigations

(a) *Dikranjan and Shakhmatov [66] (3.7) extended an important result of Zel′manov [67] (the compact case) to prove: every pseudocompact torsion group is locally finite.

(b) *The same authors investigated the following problem: Which infinite groups admit a pseudocompact group topology? We restrict attention here to non-abelian groups. A variety $\mathcal{V}$ of groups is said to be precompact if each $\mathcal{V}$-free group admits a precompact (totally bounded) group topology. An example is the variety of all groups. Here, we quote verbatim from [66] (1.3) *"a free group F in a variety $\mathcal{V}$ admits a non-discrete pseudocompact group topology if and only if $\mathcal{V}$ is precompact and $|F|$ is admissible". For further results see [66] (Chapter 5).

(c) Dikranjan [68] proved (among other interesting theorems concerning pseudocompact abelian groups) this statement: *Let F be a free group in a variety, and let $\alpha \geq \omega$. If $|F|$ is α-admissible, then the poset of all pseudocompact group topologies of weight α on F contains a copy of the power set of α.

(d) (See (A) of the Abstract.) *Comfort, Raczkowski and Trigos-Arrieta noted [69] (3.1) that in a compact group, every proper, G_δ-dense subgroup (that is, every proper dense pseudocompact subgroup) is non-measurable (in the sense of Haar). They showed that every infinite abelian group K of uncountable weight has $2^{|K|}$-many dense pseudocompact subgroups of cardinality $|K|$ [69] (3.2); hence such K admits $2^{|K|}$-many dense non-measurable subgroups of cardinality $|K|$ [69] (3.4). In the same vein, Itzkowitz [70] (2.1) showed that every non-metrizable product-like group, defined as in Section 6.1, satisfies $|\mathbb{P}(K)| = 2^{|K|}$ (the witnessing elements of $\mathbb{P}(K)$ being necessarily non-measurable). For further related results see [70].

Problem 5.7.1. *Let K be a non-metrizable compact group. Does $|\mathbb{P}(K)| = 2^{|K|}$ hold?*

By [71] (3.4) a strongly complete group is a profinite group in which every finite index subgroup is open. An infinite group is almost perfect if $|G/G'|$ is finite for the algebraic commutator subgroup G' of G. Hernández, Hofmann and Morris [71] (3.5) proved: an infinite group in which every subgroup is measurable is a strongly complete almost perfect group. More recently Brian and Mislove [72] showed that it is consistent with ZFC that every infinite compact group has a non-measurable subgroup.

Problem 5.7.2. *Does every infinite compact group K have $2^{|K|}$-many non-measurable subgroups (of cardinality $|K|$)?*

6. Epimorphisms onto Products

Comfort and Robertson [37] (3.2(b)) showed that each non-metrizable compact abelian group K maps by a continuous epimorphism onto a group of the form $M^{(\omega^+)}$ with M a compact subgroup of $\mathbb{T}$. From this they determined [37] (3.4) that such K is not r-extremal.

6.1. Product-Like Groups

According to a definition of Itzkowitz and Shakhmatov ([70,73–75]) *a compact group K with $\kappa = w(K)$ is product-like if there is a continuous epimorphism $h : K \twoheadrightarrow \Pi_{\xi<\kappa} M_\xi$ with each M_ξ a non-trivial (compact) metrizable group. A similar class was introduced by Varopoulos [76] (§3): a compact group K is called a $\prod$-group if $K \simeq \Pi_{i\in I} M_i$, where all M_i are (compact) metrizable groups. Not every $\prod$-group K is product-like: take for example for K an algebraically simple compact group. In [76] it is proved that if K is a connected, compact group with center Z, then K/Z is a $\prod$-group. Comfort and Robertson [4] (4.2) showed that (for non-trivial K) the group $K/Z(K)$ is a product of compact, connected, non-abelian Lie groups which are algebraically simple.

It is known that every non-metrizable compact group K which is either abelian or connected is product-like (see for example [40] (5.4) or [74] (1.11) for the abelian case, [40] (proof of 6.5) or [73] for the connected case). This allowed the authors of [70,73–75] to conclude that for every such group K the set $\Omega(K)$ of dense ω-bounded subgroups of K satisfies $|\Omega(K)| \geq |K|$. They asked whether that inequality may be improved to $|\Omega(K)| = 2^{|K|}$; the question was answered affirmatively in [77] for those K which in addition satisfy $w(K) = (w(K))^\omega$.

In this connection this question, raised in [77], appears still to be unsettled:

Problem 6.1.1. **Does $|\Omega(K)| = 2^{|K|}$ hold for every non-metrizable compact group K? What if K is product-like? What if* $\mathrm{cf}(wK) > \omega$*?*

*Itzkowitz [70] (p. 23) cites from [74,75] the statement that all non-metrizable compact groups which are connected or abelian are product-like. In the proof for connected groups ([75] (5)) the authors used [4] ((4.2), (4.3)). The paper [40] was cited in [74] (2.5), but not in [75]. The authors of [70,74,75] inadvertently failed to note that already in [40] (proof of (6.5)) the following more detailed result had been obtained in a more direct way (see also [40] (5.4) for a relevant result in the abelian context):

Lemma 6.1.2. **Let K be an infinite compact, connected group, and let A be the connected component of the center of K. Then:*

(a) *if $w(A) = w(K)$, then there is a continuous epimorphism $h : K \twoheadrightarrow \mathbb{T}^{w(K)}$; and*

(b) *if $w(A) < w(K)$, then there is a continuous epimorphism $h : K \twoheadrightarrow \Pi_{i \in I}\,(H_i/C_i)$ with $|I| = w(K)$.*

In the statement of Lemma 6.1.2(b), each H_i is a compact, simply connected, simple Lie group with finite center C_i. By a result of van der Waerden [78] all groups H_i/C_i are algebraically simple.

Lemma 6.1.2 is fundamental for the following theorem, which is a main tool in the proof of Theorem 5.2.1.

Theorem 6.1.3 ([40] (6.5)). **Let K be a compact, connected group such that $w(K) = \alpha > \omega$, and let A be the connected component of the center of K. Then:*

(a) *if $\omega < \beta < \alpha$, then there are a compact group F with $|F| > 1$ and a continuous epimorphism from K onto F^β; and*

(b) *if $w(A) = \alpha$ or $\mathrm{cf}(\alpha) > \omega$, then there are a compact group F with $|F| > 1$ and a continuous epimorphism from K onto F^α.*

The proof of Theorem 5.2.1 depends crucially on the following result.

Theorem 6.1.4 ([40] (5.2)). **Let $\kappa > \omega$, let $K = (K, \mathcal{T})$ be a compact group, and let $h : K \twoheadrightarrow F^\kappa$ be a continuous epimorphism with F a compact group, $|F| > 1$. Then K admits a pseudocompact group topology $\mathcal{U}$ such that $\mathcal{U} \supseteq \mathcal{T}, \mathcal{U} \neq \mathcal{T}$, and $w(K, \mathcal{U}) = w(K, \mathcal{T}) + 2^{2^\kappa}$.*

Remarks 6.1.5. (a) By no means is every compact group product-like. It is shown in [34] (4.10(d)) that *for every cardinal $\kappa \geq \omega$ there is a compact (non-abelian) group K with $wK = \kappa$ such that no homomorphism $h : K \twoheadrightarrow H_0 \times H_1$ with $|H_i| > 1$ is surjective.

(b) It is shown in [46] (6.2) that when $\alpha := \beth_\omega(\alpha_0)$ ($\alpha_0 \geq \omega$), then with $K := \Pi_{n<\omega}\,(\mathbb{Z}(p_n))^{\alpha_n}$ we have: every continuous surjective epimorphism $h : K \twoheadrightarrow F^\kappa$ with $|F| > 1$ satisfies $\kappa < \alpha = w(K)$ (indeed $2^{2^\kappa} < 2^{2^\alpha}$). Seeking a non-abelian result with a similar flavor, we have formulated (but not proved) several reasonable conjectures. Of these, Conjecture 6.1.6 below seems particularly attractive and accessible. Here we say as usual that a topological group K is topologically simple if the only closed normal subgroups of K are $\{1_K\}$ and K itself, and we recall this characterization of compact topologically simple groups, due to Yu [79] (1.8).

Every compact topologically simple group is either a finite simple group or a compact, algebraically simple, connected Lie group.

In particular, then, each such group K is metrizable, *i.e.*, satisfies $wK \leq \omega$.

We recall also, for example from [80] (7.3.11), that for K as hypothesized below, every closed normal subgroup N of K (in particular the subgroup $N = \ker(h)$), has the form $N = \Pi_{n \in I}\, K_n^{\beta_n} \times \{1_{\omega \setminus I}\}$ for suitable $I \subseteq \omega$ and for suitable cardinals $\beta_n \leq \alpha_n$.

Conjecture 6.1.6. **Let $(\alpha_n)_{n<\omega}$ be a strictly increasing sequence of infinite cardinals with each $\mathrm{cf}(\alpha_n) > \omega$, and set $\alpha := \sup_n \alpha_n = \Sigma_n\, \alpha_n$. Let $\{K_n : n < \omega\}$ be a sequence of pairwise non-isomorphic, non-abelian, topologically simple compact groups, and set $K := \Pi_n\, K_n^{\alpha_n}$. Then:*

(a) *every continuous epimorphism $h : K \twoheadrightarrow F^\kappa$ with $|F| > 1$ satisfies $\kappa < \alpha = wK$; and*

(b) *if $\alpha_{n+1} = 2^{\alpha_n}$ for all n, so that $\alpha = \beth_\omega(\alpha_0)$, then $\kappa^\beta < wK$ for all $\beta < \alpha$.*

Of course (b) follows from (a), since for such κ and β there is $n < \omega$ such that $\kappa < \alpha_n$ and $\beta < \alpha_n$, and then $\kappa^\beta \leq \alpha_n^{\alpha_n} = 2^{\alpha_n} = \alpha_{n+1} < \alpha$.

7. Concerning Free Compact Topological Groups

7.1. Characterizations of FX and FAX

Here we follow generally the conventions of Hofmann and Morris [81] (Chapter 11). See also [2] (8.8) for a less extensive, more constructive approach to free topological groups.

(a) For every space X there is a compact group FX, the free compact group on X, such that

(1) $X \subseteq \langle X \rangle \subseteq FX$ with X closed in $\langle X \rangle$ and $\langle X \rangle$ dense in FX;

(2) algebraically, $\langle X \rangle$ is the free group on the set X; and

(3) for every continuous $f : X \to K$ with K a compact group there is a (unique) continuous homomorphism $\overline{f} : FX \to K$ such that $\overline{f}|X = f$.

The free compact abelian group FAX has analogous properties, with $\overline{f} \supseteq f : X \to K$ with K a compact abelian group.

(b) The role of 1_{FX} is played in FX by the empty word. In contrast, some workers prefer to work with pointed spaces (X, p), then with the identification $p \to 1 \cdot p = 1_{FX} \in \langle X \rangle \subseteq FX$.

The theorem cited in (a) is rooted in the work of Markov [82,83] and Graev [84,85] concerning free topological groups. Alternate latter-day constructions abound, some achieved independently of [82–85] and some based on those works, some with algebraic emphasis [86,87], [2] (8.8,8.9), [24] (2.3–2.5), some topological [88–90], some functorial or categorical [91]. See [92] (§4) for a comprehensive introduction to the groups FX and FAX, and see [62] for generalizations to "free $\mathbb{P}$-spaces" for some other classes $\mathbb{P}$.

The reader will note that in our present convention, the "free compact group FX" is a compact group which is not algebraically the free group on X. In the sources about free topological groups cited above, that is reversed: "the free topological group over a space X" is itself not compact, it is algebraically the free group on X. Note that FX is the Bohr compactification (see [93] (Chapter 5.4)) of the free topological group over the space X.

7.2. Basic Properties of FX and FAX

(a) We list four basic facts about the free groups FX and FAX.

(i) ([92] (4.2.2)) *For each space X the free compact group FX is naturally isomorphic to the free compact group $F\beta X$, where βX is the Stone–Čech compactification of X.

(ii) *FX is connected if and only if X is connected; similarly for FAX.

(iii) ([81] (1.4), [92] (4.2.4)) *With X given and with FX in hand, the group FAX may be "realized" in concrete form by the rule $FAX = FX/\overline{(FX)'}$, with $(FX)'$ denoting the commutator subgroup of FX;

(iv) ([92] (4.2.1(i), [81] (11.6)) *For X compact and infinite one has $w(FX) = w(FAX) = (wX)^\omega$; hence $\mathrm{cf}(w(FX)) = \mathrm{cf}(w(FAX)) > \omega$.

It follows from (ii), (iii) and (iv) that the free compact groups FX and FAX for X compact and connected satisfy the conditions of Theorem 5.2.1(b) of Section 5. By applying Theorem 8.2.3 of Section 8 below we have this result.

Theorem 7.2.1. **Let $FX = (FX, \mathcal{T})$ be the free compact group over the compact space X with $|X| > 1$ and $w(FX) = w(FAX) = \alpha > \omega$. Then with $\kappa := 2^{2^{2^\alpha}}$ there are κ-many pseudocompact group topologies $\mathcal{U}_\eta$ ($\eta < \kappa$) on FX such that $\mathcal{U}_\eta \supseteq \mathcal{T}$, $\mathcal{U}_\eta \neq \mathcal{T}$, and $w(FX, \mathcal{U}_\eta) = 2^{|FX|}$.*

8. New Results, Non-Abelian Emphasis

8.1. Three Preliminary Lemmas

It is shown in [46] (5.1) that every non-metrizable compact abelian group K admits a pseudocompact group refinement of maximal weight (that is, of weight $2^{|K|}$). With a view toward generalizing that statement and its corollaries from [46] into the non-abelian context, we begin this section (making no claim for novelty in either case) with two simple lemmas.

Lemma 8.1.1. **Let H be a closed normal subgroup of an infinite totally bounded group G. Then $wG = wH + w(G/H)$.*

Proof. The result is well known (see for example [40] (6.1)) when G, hence also H, is compact. Denoting as in 1.3(b) by $\overline{G}$ the Weil completion of (an arbitrary) totally bounded group G, we have in the present case $\overline{G/H} = \overline{G}/\overline{H}$ (see also in this connection [93] (5.4.3) and [94] (2.6)), and hence

$$wG = w\overline{G} = w\overline{H} + w(\overline{G}/\overline{H}) = wH + w(G/H). \quad \square$$

Lemma 8.1.2. [47] (3.11) **Let $\mathcal{S}$ and $\mathcal{U}$ be totally bounded group topologies on a group G such that $\mathcal{S} \supseteq \mathcal{U}$. Then $w(G, \mathcal{S}) \geq w(G, \mathcal{U})$.*

Proof. The continuous map $\mathrm{id} : (G, \mathcal{S}) \twoheadrightarrow (G, \mathcal{U})$ extends continuously to $\overline{\mathrm{id}} : \overline{(G, \mathcal{S})} \twoheadrightarrow \overline{(G, \mathcal{U})}$, and a continuous surjection between compact spaces cannot raise weight [1] (3.1.22), so we have

$$w(G, \mathcal{S}) = w\overline{(G, \mathcal{S})} \geq w\overline{(G, \mathcal{U})} = w(G, \mathcal{U}). \quad \square$$

Now for an arbitrary totally bounded group $(G, \mathcal{T})$ we denote by $\mathcal{M}(\mathcal{T})$ the set of totally bounded group topologies on G which contain $\mathcal{T}$. Further, given a closed normal subgroup H of G, we denote by ϕ the usual quotient map from G onto G/H and by $\mathcal{T}_q$ the quotient topology on G/H. We note that for a group topology $\mathcal{S}$ on G/H, the initial topology $\phi^{-1}(\mathcal{S})$ induced on G by ϕ and $\mathcal{S}$ is a group topology which is not in general a Hausdorff topology.

Lemma 8.1.3. **Let $K = (K, \mathcal{T})$ be a compact group with a closed normal subgroup H. Then:*
(a) $\mathcal{S} \in \mathcal{M}(\mathcal{T}_q) \Rightarrow \mathcal{T} \vee \phi^{-1}(\mathcal{S}) \in \mathcal{M}(\mathcal{T})$;
(b) *the map $\mathcal{M}(\mathcal{T}_q) \rightarrow \mathcal{M}(\mathcal{T})$ given in* (a) *is injective; and*
(c) *if $\mathcal{S} \in \mathcal{M}(\mathcal{T}_q)$ is pseudocompact, then $\mathcal{T} \vee \phi^{-1}(\mathcal{S})$ is pseudocompact.*

Proof. With only notational changes, the proof follows the argument of our work with Szambien [95] (3.5). $\square$

8.2. Refinements of Large Weight (the Non-abelian Case)

The following theorem is a generalization of [40] (6.2).

Theorem 8.2.1. **Let $K = (K, \mathcal{T})$ be a compact group such that $w(K) = \alpha > \omega$. Let K' be the closure in $(K, \mathcal{T})$ of the commutator subgroup of K. If $w(K/K', \mathcal{T}_q) = \beta > \omega$, then there is a pseudocompact group topology $\mathcal{U}$ on K such that $\mathcal{U} \supseteq \mathcal{T}, \mathcal{U} \neq \mathcal{T}$, and $w(K, \mathcal{U}) = \alpha + 2^{2^{\beta}}$.*

Proof. From [46] (5.1) applied to the compact group $(K/K', \mathcal{T}_q)$ we have: there is a pseudocompact group topology $\mathcal{S} \in \mathcal{M}(\mathcal{T}_q)$ such that $\mathcal{S} \neq \mathcal{T}_q$ and $w(K/K', \mathcal{S}) = 2^{2^{\beta}}$. The topology $\mathcal{U} := \mathcal{T} \vee \phi^{-1}(\mathcal{S})$ on K is pseudocompact by Lemma 8.1.3(c).

Lemma 8.1.1 implies $w(K, \mathcal{U}) = w(K', \mathcal{U}_0) + w(K/K', \mathcal{S})$, because $\mathcal{S}$ is the quotient topology of $\mathcal{U}$; here $\mathcal{U}_0$ denotes the topology induced by $\mathcal{U}$ on K'. This topology coincides with the topology induced by $\mathcal{T}$ on K', so $w(K, \mathcal{U}) \leq \alpha + 2^{2^{\beta}}$.

For the reverse equality we note that $w(K, \mathcal{U}) \geq w(K/K', \mathcal{S}) = 2^{2^{\beta}}$. Lemma 8.1.2 gives $w(K, \mathcal{U}) \geq \alpha$. Hence $w(K, \mathcal{U}) \geq \alpha + 2^{2^{\beta}}$. $\square$

In preparation for Theorem 8.2.3 we recall this result from [46] (5.2) (see (B) of the Abstract).

Lemma 8.2.2. *For every compact abelian group $K = (K, \mathcal{T})$ with $wK = \alpha > \omega$ there are $2^{2^{|K|}}$-many pseudocompact group topologies $\mathcal{U}$ on K such that $\mathcal{U} \supseteq \mathcal{T}, \mathcal{U} \neq \mathcal{T}$, and $w(K, \mathcal{U}) = 2^{|K|}$.*

Theorem 8.2.3. **Let $K = (K, \mathcal{T})$ be a compact group such that $w(K, \mathcal{T}) = w(K/K', \mathcal{T}_q) = \alpha > \omega$. Then with $\kappa := 2^{2^{2^{\alpha}}}$ there are κ-many pseudocompact group topologies $\mathcal{U}_\eta$ $(\eta < \kappa)$ on K such that $\mathcal{U}_\eta \supseteq \mathcal{T}$, $\mathcal{U}_\eta \neq \mathcal{T}$, and $w(K, \mathcal{U}_\eta) = 2^{|K|}$.*

Proof. In view of Lemma 8.1.1 and Lemma 8.1.3, it is enough to know that there are κ-many pseudocompact group topologies $\mathcal{S}_\eta$ ($\eta < \kappa$) on the abelian group K/K' such that $\mathcal{S}_\eta \supseteq \mathcal{T}_q$, $\mathcal{S}_\eta \neq \mathcal{T}_q$, and $w(K/K', \mathcal{S}_\eta) = 2^{|K/K'|} = 2^{|K|}$. This is given by Lemma 8.2.2 (with K there replaced by K/K' here). □

Remark 8.2.4. We adopt this definition from [81] (9.92). Given a group G, the set F of all elements whose conjugacy class is finite is called the FC-center of G. If $F = G$, then G is an FC-group. Hofmann and Morris [81] (9.99) proved this theorem:

*Let K be a compact group. Then the following three statements are equivalent:

(i) K is an FC-group;

(ii) $K/Z(K)$ is finite; and

(iii) the commutator subgroup of K is finite.

From the implication (i) ⇒ (iii) it is clear that every FC-group $K = (K, \mathcal{T})$ with $w(K) = \alpha > \omega$ satisfies the condition $w(K) = w(K/K') = \alpha$, so from Theorem 8.2.3 we have this consequence for such K: with $\kappa := 2^{2^{\alpha}}$ there are κ-many pseudocompact group topologies $\mathcal{U}_\eta$ ($\eta < \kappa$) on K such that $\mathcal{U}_\eta \supseteq \mathcal{T}, \mathcal{U}_\eta \neq \mathcal{T}$, and $w(K, \mathcal{U}_\eta) = 2^{|K|}$.

Theorem 8.2.5. **Let $K = (K, \mathcal{T})$ be a compact group, such that $w(K) = \alpha > \omega$, and let $Z_0(K)$ be the connected component of the center $Z(K)$ of K. If $w(Z_0(K)) = \beta > \omega$, then there is a pseudocompact group topology $\mathcal{U}$ on K such that $\mathcal{U} \supseteq \mathcal{T}, \mathcal{U} \neq \mathcal{T}$, and $w(K, \mathcal{U}) \geq \alpha + 2^{2^{\beta}}$.*

Proof. Let K' be the closure of the commutator subgroup of K. By [81] (9.23(iii)) the connected component C_0 of K/K' is topologically isomorphic to $Z_0(K)/H$, where H is the intersection of $Z_0(K)$ and K'. The group H is totally disconnected, since $Z(K) \cap K'$ is totally disconnected by [81] (9.23(i)). Thus [96] (3.2) implies $w(Z_0(K)) = w(C_0)$. Hence $\beta \leq w(K/K')$. Now apply Theorem 8.2.1 to complete the proof. □

Let $\mathcal{R}$ denote a set of representatives for the isomorphism classes of the class of all compact simple groups. It was noted by Hofmann and Morris [97] (p. 412) that $\mathcal{R}$ is an infinite countable set. In [97] (2.2), a compact group is called strictly reductive if it is isomorphic to a Cartesian product of compact algebraically simple groups. For a compact group K and $S \in \mathcal{R}$, the smallest closed subgroup K_S of K containing all closed normal subgroups isomorphic to S is called in [97] the S-socle of K.

Theorem 8.2.6 ([97] (2.3)). **Let K be a strictly reductive compact group, and let $(K_S)_{S\in\mathcal{R}}$ denote the sequence of S-socles of K. Then there is a sequence of cardinals $(J(K,S))_{S\in\mathcal{R}}$ such that $K \simeq \Pi_{S\in\mathcal{R}} K_S$, with $K_S \simeq S^{J(K,S)}$ for each $S \in \mathcal{R}$.*

Theorem 8.2.7. **Let $(K, \mathcal{T})$ be a strictly reductive compact group with $w(K, \mathcal{T}) = \alpha > \omega$. Then:*

(a) if $\omega < \beta < \alpha$, then K admits a pseudocompact group topology $\mathcal{U}$ such that $\mathcal{U} \supseteq \mathcal{T}, \mathcal{U} \neq \mathcal{T}$, and $w(K, \mathcal{U}) = \alpha + 2^{2^{\beta}}$; and

(b) if cf(α)$> \omega$, then K admits a pseudocompact group topology $\mathcal{U}$ with $\mathcal{U} \supseteq \mathcal{T}$, $\mathcal{U} \neq \mathcal{T}$, and $w(K, \mathcal{U}) = 2^{2^{\alpha}}$.

Proof. By Theorem 8.2.6 there is a sequence of cardinals $(J(K,S))_{S\in\mathcal{R}}$ such that $K \simeq \Pi_{S\in\mathcal{R}} K_S$, $K_S \simeq S^{J(K,S)}$. Hofmann and Morris [97](2.7) showed $w(K, \mathcal{T}) = \max\{\omega, \sup\{J(K,S) : S \in \mathcal{R}\}\}$. Then Theorem 6.1.4 completes the proof. □

Corollary 8.2.8. **Let $(K, \mathcal{T})$ be a strictly reductive compact group with $w(K, \mathcal{T}) = \alpha > \omega$. Then K admits a pseudocompact group topology $\mathcal{U}$ with $\mathcal{U} \supseteq \mathcal{T}, \mathcal{U} \neq \mathcal{T}$.*

Proof. If $\alpha > \omega^+$, use Theorem 8.2.7(a). If $\alpha = \omega^+$, use Theorem 8.2.7(b). □

Corollary 8.2.9. **Let $(K, \mathcal{T})$ be a strictly reductive compact group with $w(K, \mathcal{T}) = \alpha$. If cf$(\alpha) > \omega$, then K admits a pseudocompact group topology $\mathcal{U}$ with $\mathcal{U} \supseteq \mathcal{T}, \mathcal{U} \neq \mathcal{T}$, and $|[\mathcal{T}, \mathcal{U}]| \geq 2^{|K|}$.*

Proof. Use Theorem 5.2.4 and Theorem 8.2.7(b). □

Remus considered in [98] uncountable powers of a non-abelian, compact, topologically simple group.

Theorem 8.2.10 ([98] (3.40))**.** **Let F be a non-abelian, compact, topologically simple group, and let $\alpha > \omega$. Let $K = F^\alpha$ endowed with the product topology $\mathcal{T}$. Then:*

(a) *there is a pseudocompact group topology $\mathcal{U}$ on K with $w(K, \mathcal{U}) = 2^{|K|}$ and $\mathcal{T} \subset \mathcal{U}$ such that there is an order-isomorphism f of the power set $\mathcal{P}(2^{|K|})$, ordered by the inclusion, onto a subset of $[\mathcal{T}, \mathcal{U}]$. The compact Weil completion of $(K, \mathcal{U})$ is topologically isomorphic to $F^{2^{2^\alpha}}$; and*

(b) *for each cardinal γ such that $\alpha \le \gamma \le 2^{|K|}$ and for every $M \in \mathcal{P}(2^{|K|})$ with $|M| = \gamma$, one has $w(K, f(M)) = \gamma$.*

From the proof of [98] (3.36) this statement follows: let $(K, \mathcal{T})$ be a profinite group with $w(K, \mathcal{T}) = \alpha$. If $\mathrm{cf}(\alpha) > \omega$, there is a subnormal series $H_1 \subset H_2 \ldots \subset H_k = K$ of open subgroups of K such that there is a pseudocompact group topology $\mathcal{U}$ on H_1, finer than the topology induced by $\mathcal{T}$ on H_1, such that $w(H_1, \mathcal{U}) = 2^{|G|}$. By [98] ((3.36)(b)) there is a linear totally bounded group topology $\mathcal{V}$ of weight $2^{|G|}$ on K which is finer than $\mathcal{T}$ (this is constructed using the topology $\mathcal{U}$ and the subnormal series). It remains open if $\mathcal{V}$ can be chosen pseudocompact. It is natural to pose the following

Problem 8.2.11. **Let $(K, \mathcal{T})$ be a profinite group of uncountable weight.*

(a) *Does $\mathcal{T}$ admit a proper pseudocompact refinement of maximal weight $2^{|K|}$?*

(b) *Are there $2^{2^{|K|}}$-many pseudocompact group topologies on K which are finer than $\mathcal{T}$?*

Acknowledgments: With many thanks, the authors acknowledge insightful, knowledgeable reports received from two referees. Collectively they have helped us enhance both the scope of the paper and its style of presentation, and with great clarity, they helped us identify some factual errors which were present in the earliest version.

Author Contributions: W. W. Comfort and Dieter Remus contributed equally to this work.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Engelking, R. *General Topology*; Heldermann Verlag: Berlin, Germany, 1989.
2. Hewitt, E.; Ross, K.A. *Abstract Harmonic Analysis I*; Springer-Verlag: Heidelberg, Germany, 1963.
3. Hewitt, E. Rings of real-valued continuous functions I. *Trans. Am. Math. Soc.* **1948**, *64*, 45–99.
4. Comfort, W.W.; Robertson, L.C. Cardinality constraints for pseudocompact and for totally dense subgroups of compact Abelian groups. *Pac. J. Math.* **1985**, *119*, 265–285.
5. Glicksberg, I. The representation of functionals by integrals. *Duke Math. J.* **1952**, *19*, 253–261.
6. Gillman, L.; Jerison, M. *Rings of Continuous Functions*; D. Van Nostrand Co.: New York, NY, USA, 1960.
7. Weil, A. *Sur les Espaces à Structure Uniforme et sur la Topologie Générale*; Hermann & Cie: Paris, France, 1938.
8. Comfort, W.W.; Ross, K.A. Pseudocompactness and uniform continuity in topological groups. *Pac. J. Math.* **1966**, *16*, 483–496.
9. Mycielski, J. Some properties of connected compact groups. *Colloq. Math.* **1958**, *5*, 162–166.
10. Wilcox, H.J. Dense subgroups of compact groups. *Proc. Am. Math. Soc.* **1971**, *28*, 578–580.
11. Tkachenko, M.G. Compactness type properties in topological groups. *Czechoslov. Math. J.* **1988**, *38*, 324–341.
12. Uspenskii, V.V. Topological groups and Dugundji compacta. *Math. USSA Sb.* **1990**, *67*, 555–580.
13. Hernández, S.; Sanchis, M. G_δ-open functionally bounded subsets in topological groups. *Topol. Appl.* **1993**, *53*, 289–299.
14. Arhangel′skiĭ, A.V. On a theorem of W. W. Comfort and K. A. Ross. *Comment. Math. Univ. Carol.* **1999**, *40*, 133–151.
15. Kakutani, S.; Kodaira, K. Über das Haarsche Maß in der lokal bikompakten Gruppe. *Proc. Imp. Acad. Tokyo* **1944**, *20*, 444–450.
16. Halmos, P.R. *Measure Theory*; Springer-Verlag: New York, NY, USA, 1974.

17. Ross, K.A.; Stromberg, K.R. Baire sets and Baire measures. *Ark. Matemat.* **1967**, *6*, 151–160.
18. De Vries, J. Pseudocompactess and the Stone-Čech compactification for topological groups. *Nieuw Arch. Wisk.* **1975**, *23*, 35–48.
19. Hušek, M. Productivity of properties of topological groups. *Topol. Appl.* **1994**, *44*, 189–196.
20. Tkachenko, M.G. Boundedness and pseudocompactness in topological groups. *Math. Notes* **1987**, *41*, 299–231.
21. Comfort, W.W.; Trigos-Arrieta, F.J. Locally pseudocompact topological groups. *Topol. Appl.* **1995**, *62*, 263–280.
22. Dikranjan, D.; Lukács, G. On zero-dimensionality and the connected component of locally pseudocompact groups. *Proc. Am. Math. Soc.* **2011**, *139*, 2995–3008.
23. Van Douwen, E.K. The weight of a pseudocompact (homogeneous) space whose cardinality has countable cofinality. *Proc. Am. Math. Soc.* **1980**, *80*, 678–682.
24. Comfort, W.W. Topological Groups. In *Handbook of Set-theoretic Topology*; Kunen, K., Vaughan, J.E., Eds.; North-Holland: Amsterdam, the Netherlands, 1984; pp. 1143–1263.
25. Comfort, W.W.; van Mill, J. Concerning connected, pseudocompact Abelian groups. *Topol. Appl.* **1989**, *33*, 21–45.
26. Hewitt, E.; Ross, K.A. *Abstract Harmonic Analysis II*; Springer-Verlag: Heidelberg, Germany, 1970.
27. Cater, F.S.; Erdős, P.; Galvin, F. On the density of λ-box products. *Gen. Topol. Appl.* **1978**, *9*, 307–312.
28. Comfort, W.W.; van Mill, J. Some topological groups with, and some without, proper dense subgroups. *Topol. Appl.* **1991**, *41*, 3–15.
29. Itzkowitz, G.L. Extensions of Haar measure for compact connected Abelian groups. *Bull. Am. Math. Soc.* **1965**, *71*, 152–156.
30. Wilcox, H.J. Pseudocompact groups. *Pac. J. Math.* **1966**, *19*, 365–379.
31. Comfort, W.W.; Saks, V. Countably compact groups and finest totally bounded topologies. *Pac. J. Math.* **1973**, *49*, 33–44.
32. Rajagopalan, M.; Subrahmanian, H. Dense subgroups of locally compact groups. *Colloq. Math.* **1976**, *35*, 289–292.
33. Comfort, W.W.; Ross, K.A. Topologies induced by groups of characters. *Fundam. Math.* **1964**, *55*, 283–291.
34. Comfort, W.W.; Robertson, L.C. Extremal phenomena in certain classes of totally bounded groups. *Diss. Math.* **1988**, *272*, 1–48.
35. Comfort, W.W. Tampering with pseudocompact groups. *Topol. Proc.* **2004**, *28*, 401–424.
36. Comfort, W.W.; Soundararajan, T. Pseudocompact group topologies and totally dense subgroups. *Pac. J. Math.* **1982**, *100*, 61–84.
37. Comfort, W.W.; Robertson, L.C. Proper pseudocompact extensions of compact Abelian group topologies. *Proc. Am. Math. Soc.* **1982**, *86*, 173–178.
38. Comfort, W.W. Problems on topological groups and other homogeneous spaces. In *Open Problems in Topology*; van Mill, J., Reed, G.M., Eds.; North-Holland Publishing Company: Amsterdam, the Netherlands, 1990; pp. 313–347.
39. Comfort, W.W. Pseudocompact groups: Progress and problems. *Topol. Appl.* **2008**, *155*, 172–179.
40. Comfort, W.W.; Remus, D. Pseudocompact refinements of compact group topologies. *Math. Z.* **1994**, *215*, 337–346.
41. Comfort, W.W.; Gladdines, H.; van Mill, J. Proper pseudocompact subgroups of pseudocompact Abelian groups. *Ann. N. Y. Acad. Sci.* **1994**, *728*, 237–247.
42. Comfort, W.W.; Galindo, J. Extremal pseudocompact topological groups. *J. Pure Appl. Algebra* **2005**, *197*, 59–81.
43. Galindo, J. Dense pseudocompact subgroups and finer pseudocompact group topologies. *Sci. Math. Jpn.* **2002**, *55*, 627–640.
44. Comfort, W.W.; van Mill, J. Extremal pseudocompact abelian groups are compact metrizable. *Proc. Am. Math. Soc.* **2007**, *135*, 4039–4044.
45. Comfort, W.W.; van Mill, J. Extremal pseudocompact abelian groups: A unifed treatment. *Comment. Math. Univ. Carol.* **2013**, *54*, 197–227.
46. Comfort, W.W.; Galindo, J. Pseudocompact topological group refinements of maximal weight. *Proc. Am. Math. Soc.* **2003**, *131*, 1311–1320.

47. Comfort, W.W.; Remus, D. Long chains of topological group topologies—A continuation. *Topol. Appl.* **1997**, *75*, 51–79.
48. Comfort, W.W.; Remus, D. Intervals of totally bounded group topologies. *Ann. N. Y. Acad. Sci.* **1996**, *806*, 121–129.
49. Remus, D. Minimal and precompact group topologies on free groups. *J. Pure Appl. Algebra* **1990**, *70*, 147–157.
50. Dikranjan, D.N.; Shakhmatov, D.B. Compact-like totally dense subgroups of compact groups. *Proc. Am. Math. Soc.* **1992**, *144*, 1119–1129.
51. Grant, D.L. Topological Groups Which Satisfy an Open Mapping Theorem. *Pac. J. Math.* **1997**, *68*, 411–423.
52. Comfort, W.W.; Dikranjan, D. On the poset of totally dense subgroups of compact groups. *Topol. Proc.* **1999**, *24*, 103–127.
53. Comfort, W.W.; Dikranjan, D. Essential density and total density in topological groups. *J. Group Theory* **2002**, *5*, 325–350.
54. Comfort, W.W.; Dikranjan, D. The density nucleus of a topological group. *Topol. Proc.* **2014**, *44*, 325–356.
55. Comfort, W.W.; Grant, D.L. Cardinal invariants, pseudocompactness and minimality: Some recent advances in the topological theory of topological groups. *Topol. Proc.* **1981**, *6*, 227–265.
56. Giordano Bruno, A. Extremal Pseudocompact Groups. Ph.D. Thesis, Università di Udine, Udine, Italy, 2004.
57. Dikranjan, D.; Giordano Bruno, A. Pseudocompact totally dense subgroups. *Proc. Am. Math. Soc.* **2008**, *136*, 1093–1103.
58. Comfort, W.W.; van Mill, J. On the supremum of the pseudocompact group topologies. *Topol. Appl.* **2008**, *155*, 213–224.
59. Giordano Bruno, A. Extremal α-pseudocompact abelian groups. *Forum Math.* **2009**, *21*, 639–659.
60. Dikranjan, D.; Giordano Bruno, A.; Milan, C. Weakly metrizable pseudocompact groups. *Appl. Gen. Topol.* **2006**, *7*, 1–39.
61. Giordano Bruno, A. Dense minimal subgroups of compact abelian groups. *Topol. Appl.* **2008**, *155*, 1919–1928.
62. Comfort, W.W.; van Mill, J. On the existence of free topological groups. *Topol. Appl.* **1988**, *29*, 245–269.
63. Tkachenko, M.G. Pseudocompact topological groups and their properties. *Sib. Math. J.* **1989**, *30*, 120–128.
64. Ursul, M.I. Imbeddings of locally precompact groups in locally pseudocompact ones. *Izv. Akad. Nauk Mold. SSR* **1989**, *3*, 54–56.
65. Leiderman, A.; Morris, S.A.; Tkachenko, M.G. Density character of subgroups of topological groups. *Trans. Am. Math. Soc.* **2015**, doi:10.1090/tran/6832.
66. Dikranjan, D.N.; Shakhmatov, D.B. Algebraic structure of pseudocompact groups. *Mem. Am. Math. Soc.* **1998**, *633*, 1–83.
67. Zel'manov, E.I. On periodic compact groups. *Israel J. Math.* **1992**, *77*, 83–95.
68. Dikranjan, D. Chains of pseudocompact group topologies. *J. Pure Appl. Algebra* **1998**, *124*, 65–100.
69. Comfort, W.W.; Raczkowski, S.U.; Trigos-Arrieta, F.J. Making group topologies with, and without, convergent sequences. *Appl. Gen. Topol.* **2006**, *7*, 109–124.
70. Itzkowitz, G.L. Cardinal numbers asociated with dense pseudocompact, countably compact, and ω-bounded subgroups. *Topol. Appl.* **1998**, *84*, 21–32.
71. Hernández, S.; Hofmann, K.H.; Morris, S.A. Nonmeasurable subgroups of compact groups. *J. Group Theory* **2016**, *19*, 179–189.
72. Brian, W.R.; Mislove, M.W. Every compact group can have a non-measurable subgroup. 2015, arXiv:1503.01385.
73. Itzkowitz, G.; Shakhmatov, D. Large families of dense pseudocompact subgroups of compact groups. *Fundam. Math.* **1995**, *147*, 197–212.
74. Itzkowitz, G.L.; Shakhmatov, D. Dense countably compact subgroups of compact groups. *Math. Jpn.* **1997**, *45*, 497–501.
75. Itzkowitz, G.L.; Shakhmatov, D. Haar non-measurable partitions of compact groups. *Tsukuba J. Math.* **1997**, *21*, 251–262.
76. Varopoulos, N.T. A theorem on the continuity of homomorphisms of locally compact groups. *Math. Proc. Camb. Philos. Soc.* **1964**, *60*, 449–463.

77. Comfort, W.W.; van Mill, J. How many ω-bounded subgroups? *Topol. Appl.* **1997**, *77*, 105–113.
78. Van der Waerden, B.L. Stetigkeitssätze für halbeinfache Liesche Gruppen. *Math. Z.* **1933**, *36*, 780–786.
79. Yu, Y.K. Topologically complete semisimple groups. *Proc. Lond. Math. Soc.* **1976**, *33*, 515–534.
80. Dikranjan, D.N.; Prodanov, I.R.; Stoyanov, L.N. *Topological Groups (Characters, Dualities and Minimal Group Topologies)*; Marcel Dekker, Inc.: New York, NY, USA, 1990.
81. Hofmann, K.H.; Morris, S.A. *The Structure of Compact Groups; A Primer for Students, A Handbook for Experts*, 3rd ed.; Walter De Gruyter Inc.: Berlin, Germany, 2013.
82. Markov, A.A. On free topological groups. *Dokl. Akad. Nauk SSSR* **1941**, *31*, 299–301.
83. Markov, A.A. *On Free Topological Groups, Topology and Topological Algebra, Translations Series 1*; American Mathematical Society: Providence, RI, USA, 1962; volume 8, pp. 195–272.
84. Graev, M.I. Free Topological Groups. *Izv. Akad. Nauk SSSR Ser. Mat.* **1948**, *12*, 279–323.
85. Graev, M.L. On free products of topological groups. *Izv. Akad. Nauk SSSR Ser. Mat.* **1950**, *14*, 343–354.
86. Smith-Thomas, B.V. Free topological groups. *Topol. Appl.* **1974**, *4*, 51–72.
87. Smith-Thomas, B.V. Categories of topological groups. *Quaest. Math.* **1977**, *2*, 355–377.
88. Kakutani, S. Free topological groups and infinite direct product topological groups. *Proc. Imp. Acad. Tokyo* **1944**, *20*, 595–598.
89. Nakayama, T. Note on free topological groups. *Proc. Imp. Acad. Tokyo* **1943**, *19*, 471–475.
90. Samuel, P. On universal mappings and free topological groups. *Bull. Am. Math. Soc.* **1948**, *54*, 591–598.
91. Morris, S.A. Free Abelian topological groups. In Proceedings of the 1983 University of Toledo Ohio Conference on Categorical Topology, Toledo, OH, USA, 1–5 August 1983.
92. Comfort, W.W.; Hofmann, K.-H.; Remus, D. Topological groups and semigroups. In *Recent Progress in General Topology*; Hušek, M., van Mill, J., Eds.; Elsevier Science Publishers: Amsterdam, the Netherlands, 1992; pp. 57–144.
93. Heyer, H. *Dualität lokalkompakter Gruppen*; Springer-Verlag: Heidelberg, Germany, 1970.
94. Poguntke, D. Zwei Klassen lokalkompakter maximal fastperiodischer Gruppen. *Monatsh. Math.* **1976**, *81*, 15–40.
95. Comfort, W.W.; Remus, D.; Szambien, H. Extending ring topologies. *J. Algebra* **2000**, *232*, 21–47.
96. Hofmann, K.H.; Morris, S.A. Weight and $\mathfrak{c}$. *J. Pure Appl. Algebra* **1990**, *68*, 181–194.
97. Hofmann, K.H.; Morris, S.A. A structure theorem on compact groups. *Math. Proc. Camb. Philos. Soc.* **2001**, *130*, 409–426.
98. Remus, D. *Anzahlbestimmungen von gewissen präkompakten bzw. nicht-präkompakten hausdorffschen Gruppentopologien*; Habilitationsschrift Universität Hannover: Hannover, Germany, 1995.