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Article

Free Boolean Topological Groups

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Abstract: Known and new results on free Boolean topological groups are collected. An account of the properties that these groups share with free or free Abelian topological groups and properties specific to free Boolean groups is given. Special emphasis is placed on the application of set-theoretic methods to the study of Boolean topological groups.

Keywords: free Boolean topological group; free Boolean linear topological group; free topological group; free Abelian topological group; almost discrete space; Ramsey filter; extremally disconnected group

MSC classifications: 54H11; 22A05; 54G05; 03E75; 03E35; 03C25

1. Introduction

In the very early 1940s, Markov [1,2] introduced the free topological group $F(X)$ and the free Abelian topological group $A(X)$ on an arbitrary completely regular Hausdorff topological space X as a topological-algebraic counterpart of the abstract free and free Abelian groups on a set, respectively; he also proved the existence and uniqueness of these groups. During the next decade, Graev [3,4], Nakayama [5] and Kakutani [6] simplified the proofs of the main statements of Markov's theory of free topological groups, generalized Markov's construction and proved a number of important theorems on free topological groups. In particular, Graev generalized the notions of the free and the free Abelian topological group on a space X by identifying the identity element of the free group with an (arbitrary) point of X (the free topological group on X in the sense of Markov coincides with Graev's group on X plus an isolated point), described the topology of free topological groups on compact spaces and extended any continuous pseudometric on X to a continuous invariant pseudometric on $F(X)$ (and on $A(X)$) which is maximal among all such extensions [3].

This study stimulated Mal'tsev, who believed that the most appropriate place of the theory of abstract free groups was in the framework of the general theory of algebraic systems, to introduce general free topological algebraic systems. In 1957, he published the large paper [7], where the basics of the theory of free topological universal algebras were presented.

Yet another decade later, Morris initiated the study of free topological groups in the most general aspect. Namely, he introduced the notion of a variety of topological groups and a full variety of topological groups and studied the free objects of these varieties [8–10] (see also [11]). (A definition of a variety of topological groups (determined by a so-called varietal free topological group) was also proposed in 1951 by Higman [12]; however, it is Morris' definition that has proven viable and developed into a rich theory.) Varieties of topological groups and their free objects were also considered by Porst [13], Comfort and van Mill [14], Kopperman, Mislove, Morris, Nickolas, Pestov and Svetlichny [15], and other authors. Special mention should be made of Dikranjan and Tkachenko's detailed study of varieties of Abelian topological groups with properties related to compactness [16].

The varieties of topological groups in which free objects have been studied best are, naturally, the varieties of general and Abelian topological groups; free and free Abelian precompact groups have also been considered (see, e.g., [17]). However, there is yet another natural variety: Boolean topological groups. Free objects in this variety and its subvarieties have been investigated much less extensively, although they arise fairly often in various studies (especially in the set-theoretic context). The author is aware of only three published papers considering free Boolean topological groups from a general point of view: [18], where free Boolean topological groups on compact spaces were studied fairly thoroughly; [19], where the topology of the free Boolean topological group on a compact metric space was explicitly described; and [20], where the free Boolean topological groups on compact initial segments of ordinals were classified (see also [21]). The purpose of this paper is to draw attention to these very interesting groups and to give a general impression of them. We collect some (known and new) results on free Boolean topological groups, which describe both properties that these groups share with free or free Abelian topological groups and properties specific to free Boolean groups.

2. Preliminaries and a General Description of Free Boolean Topological Groups

All topological spaces and groups considered in this paper are assumed to be completely regular and Hausdorff.

The notation ω is used for the set of all nonnegative integers and $\mathbb{N}$ for the set of all positive integers. By $\mathbb{Z}_2$, we denote the group of order two. The cardinality of a set A is denoted by $|A|$ and the closure of a set A in an ambient topological space by $\overline{A}$. We denote the disjoint union of spaces X and Y by $X \oplus Y$.

By a zero-dimensional space, we mean a space X with $\operatorname{ind} X = 0$ and by a strongly zero-dimensional space a space X with $\dim X = 0$.

A Boolean group is a group in which all elements are of order two. Clearly, all Boolean groups are Abelian. Algebraically, all Boolean groups are free, because any Boolean group is a linear space over the field $\mathbb{F}_2 = \{0, 1\}$ and must have a basis (a maximal linearly independent set) by Zorn's lemma. This basis freely generates the given Boolean group. Moreover, any Boolean group (linear space) with basis X is isomorphic to the direct sum $\bigoplus^{|X|}\mathbb{Z}_2$ of $|X|$ copies of $\mathbb{Z}_2$, *i.e.*, the set of finitely supported maps $g\colon X \to \mathbb{Z}_2$ with pointwise addition (in the field $\mathbb{F}_2$). Of course, such an isomorphic representation depends on the choice of the basis.

A variety of topological groups is a class of topological groups closed with respect to taking topological subgroups, topological quotient groups and Cartesian products of groups with the product topology. Thus, the abstract groups $\widetilde{G}$ underlying the topological groups G in a variety V of topological groups (that is, all groups $G \in V$ without topology) form a usual variety $\widetilde{V}$ of groups. A variety V of topological groups is full if any topological group G for which $\widetilde{G} \in \widetilde{V}$ belongs to V. The notions of a variety and a full variety of topological groups were introduced by Morris in [8,9], who also proved the existence of the free group of any full variety on any completely regular Hausdorff space X.

Free objects of varieties of topological groups are characterized by the corresponding universality properties (we give a somewhat specific meaning to the word "universality," but we use this word only in this meaning here). Thus, the free topological group $F(X)$ on a space X admits the following description: X is topologically embedded in $F(X)$ and, for any continuous map f of X to a topological group G, there exists a continuous homomorphism $\hat{f}\colon F(X) \to G$ for which $f = \hat{f} \restriction X$. As an abstract group, $F(X)$ is the free group on the set X. The topology of $F(X)$ can be defined as the strongest group topology inducing the initial topology on X. On the other hand, the free topological group $F(X)$ is the abstract free group generated by the set X (which means that any map of the set X to any abstract group can be extended to a homomorphism of $F(X)$) endowed with the weakest topology with respect to which all homomorphic extensions of continuous maps from X to topological groups are continuous. The free Abelian topological group $A(X)$ on X, the free Boolean topological group $B(X)$ on X and free (free Abelian, free Boolean) precompact groups are defined similarly; instead of continuous maps to any topological groups, continuous maps to topological Abelian groups, topological Boolean groups and precompact (Abelian precompact, Boolean precompact) groups should be considered.

There is yet another family of interesting varieties of topological groups. Following Malykhin (see also [17]), we say that a topological group is linear if it has a base of neighborhoods of the identity element which consists of open subgroups. The classes of all linear groups, all Abelian linear groups and all Boolean linear groups are varieties of topological groups. These varieties are not full, but for any zero-dimensional space X, there exist free groups of all of these three varieties on X. Indeed, Morris proved that a free group of a variety of topological groups on a given space exists if this space can be embedded as a subspace in a group from this variety ([8], Theorem 2.6). Thus, it suffices to embed any zero-dimensional X in a Boolean linear topological group (which belongs to all of the three varieties under consideration). We do this below, but first we introduce more notation.

Whenever X algebraically generates a group G, we can set the length of the identity element to zero, define the length of any non-identity $g \in G$ with respect to X as the least (positive) integer n such that $g = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}$ for some $x_i \in X$ and $\varepsilon_i = \pm 1$, $i = 1, 2, \dots, n$, and denote the set of elements of length at most k by G_k for $k \in \omega$; then, $G = \bigcup G_k$. Thus, we use $F_k(X)$, $A_k(X)$ and $B_k(X)$ to denote the sets of words of length at most k in $F(X)$, $A(X)$ and $B(X)$, respectively.

Now, we can describe the promised embedding.

Lemma 1. (i) *For any space X with* $\operatorname{ind} X = 0$*, there exists a Hausdorff linear topological group $F'(X)$ such that $F'(X)$ is an algebraically free group on X, X is a closed subspace of $F'(X)$, and all sets $F_n(X)$ of words of length at most n are closed in $F'(X)$.*

(ii) *For any space X with* $\operatorname{ind} X = 0$*, there exists a Hausdorff Abelian linear topological group $A'(X)$ such that $A'(X)$ is an algebraically free Abelian group on X, X is a closed subspace of $A'(X)$, and all sets $A_n(X)$ of words of length at most n are closed in $A'(X)$.*

(iii) *For any space X with* $\operatorname{ind} X = 0$*, there exists a Hausdorff Boolean linear topological group $B'(X)$ such that $B'(X)$ is an algebraically free Boolean group on X, X is a closed subspace of $B'(X)$, and all sets $B_n(X)$ of words of length at most n are closed in $B'(X)$.*

Proof. Assertion (i) was proven in [22], Theorem 10.5. Let us prove (ii). Given a disjoint open cover γ of X, we set:

$$H(\gamma) = \Big\{\sum_{i=1}^{n}(x_i - y_i)\colon n \in \mathbb{N} \text{ and for each } i \le n, \text{ there exists an } U_i \in \gamma \text{ for which } x_i, y_i \in U_i\Big\};$$

this is a subgroup of the free Abelian group on X. We can assume that all words in $H(\gamma)$ are reduced (if x_i is canceled with y_j, then $U_i = U_j$, because $U_i \cap U_j \ni x_i = y_j$ and γ is disjoint, and we can replace $x_i - y_i + x_j - y_j$ by $x_j - y_i$). All such subgroups generate a group topology on the free Abelian group on X; we denote the free Abelian group with this topology by $A'(X)$ (we might as well take only finite covers).

The space X is indeed embedded in $A'(X)$: given any clopen neighborhood U of any point $x \in X$, we have $x + H(\{U, X \setminus U\}) \cap X = U$.

Let us show that $A_n(X)$ is closed in $A'(X)$ for any $n \in \omega$. Take any reduced word $g = \varepsilon_1 x_1 + \varepsilon_2 x_2 + \dots + \varepsilon_k x_k$ with $k > n$, where $\varepsilon_i = \pm 1$ and $x_i \in X$ for $i \le k$. Let U_i be clopen neighborhoods of x_i such that U_i and U_j are disjoint if $x_j \ne x_i$ and coincide if $x_j = x_j$. We set:

$$\gamma = \Big\{U_1, \dots, U_k, X \setminus \bigcup_{i \le k} U_i\Big\}.$$

Take any reduced word $h = \sum_{i=1}^{m}(y_i - z_i)$ in $H(\gamma)$ and consider $g + h$. If, for some $i \le m$, both y_i and $-z_i$ are canceled in $g + h$ with some x_j and x_l, then, first, $x_j = x_l$ (because any different letters in g are separated by the cover γ, while y_i and z_i must belong to the same element of this cover), and secondly, $\varepsilon_j = -\varepsilon_l$ (because y_i and z_i occur in h with opposite signs). Hence, $\varepsilon_j x_j = -\varepsilon_l x_l$, which contradicts g being reduced. Thus, among any two letters y_i and $-z_i$ in h, only one can be canceled in $g + h$, so that $g + h$ cannot be shorter than g. In other words, $g + H(\gamma) \cap A'_n(X) = \varnothing$.

The proof that X is closed in $A'(X)$ is similar: given any $g \notin X$, we construct precisely the same γ as above (if $g \notin -X$) or set $\gamma = \{X\}$ (if $g \in -X$) and show that $g + H(\gamma)$ must contain at least one negative letter.

The Hausdorffness of $A'(X)$ is equivalent to the closedness of $A_0(X)$.

The proof of (iii) is similar. $\square$

This lemma and Morris' theorem cited above ([8], Theorem 2.6) immediately imply the following theorem.

Theorem 2. *For any space X with $\operatorname{ind} X = 0$, the free, free Abelian and free Boolean linear topological groups $F^{\mathrm{lin}}(X)$, $A^{\mathrm{lin}}(X)$ and $B^{\mathrm{lin}}(X)$ are defined. They are Hausdorff and contain X as a closed subspace, and all sets $F_n(X)$, $A_n(X)$ and $B_n(X)$ are closed in the respective groups.*

By definition, the free linear groups of a zero-dimensional space X have the strongest linear group topologies inducing the topology of X, that is, any continuous map from X to a linear topological group (Abelian linear topological group, Boolean linear topological group) extends to a continuous homomorphism from $F^{\mathrm{lin}}(X)$ ($A^{\mathrm{lin}}(X)$, $B^{\mathrm{lin}}(X)$) to this group.

Let X be a space, and let X_n, $n \in \omega$, be its subspaces such that $X = \bigcup X_n$. Suppose that any $Y \subset X$ is open in X if and only if each $Y \cap X_n$ is open in X_n (replacing "open" by "closed," we obtain an equivalent condition). Then, X is said to have the inductive limit topology (with respect to the decomposition $X = \bigcup X_n$). When talking about inductive limit topologies on $F(X)$, $A(X)$ and $B(X)$, we always mean the decompositions $F(X) = \bigcup F_k(X)$, $A(X) = \bigcup A_k(X)$ and $B(X) = \bigcup B_k(X)$ and assume the sets $F_k(X)$, $A_k(X)$ and $B_k(X)$ to be endowed with the topology induced by the respective free topological groups.

For any space X, the free Abelian topological group $A(X)$ is the quotient topological group of $F(X)$ by the commutator subgroup, and the free Boolean topological group $B(X)$ is the quotient of $A(X)$ by the subgroup of squares $A(2X)$ (which is generated by all words of the form $2x$, $x \in X$) (the universality of free objects in varieties of topological groups implies that the corresponding homomorphisms are continuous and open). Thus, $B(X)$ is the image of $A(X)$ (and of $F(X)$) under a continuous open homomorphism.

The topology of free groups can be described explicitly. The first descriptions were given for free topological groups on compact spaces and free Abelian topological groups by Graev [3,4]; Tkachenko [23,24] and Pestov [25] gave explicit descriptions of the topology of general free topological groups. There are also descriptions due to the author (see, e.g., [26,27]). Mal'tsev proposed a universal approach to describing the topology of free topological algebras, which is not quite constructive, but looks very promising [7]. All descriptions of the topology of free and free Abelian topological groups of which the author is aware are given in [22]. The descriptions of the free topological group topology are very complex (except in a few special cases); the topologies of free Abelian and Boolean topological groups look much simpler. Thanks to the fact that $B(X) = A(X)/A(2X)$, the descriptions of the free Abelian topological group topology given in [22] immediately imply the following descriptions of the free topology of $B(X)$.

I For each $n \in \mathbb{N}$, we fix an arbitrary entourage $W_n \in \mathcal{U}$ of the diagonal of $X \times X$ in the universal uniformity of X and set:

$$\widetilde{W} = \{W_n\}_{n\in\mathbb{N}},$$
$$U(W_n) = \{x + y\colon (x, y) \in W_n\},$$
$$U(\widetilde{W}) = \bigcup_{n\in\mathbb{N}} (U(W_1) + U(W_2) + \cdots + U(W_n)).$$

The sets $U(\widetilde{W})$, where $\widetilde{W}$ ranges over all sequences of uniform entourages of the diagonal, form a neighborhood base at zero for the topology of the free Boolean topological group $B(X)$.

II For each $n \in \mathbb{N}$, we fix an arbitrary normal (or merely open) cover γ_n of the space X and set:

$$\Gamma = \{\gamma_n\}_{n\in\mathbb{N}},$$
$$U(\gamma_n) = \{x + y \colon (x, y) \in U \in \gamma_n\},$$
$$U(\Gamma) = \bigcup_{n\in\mathbb{N}} (U(\gamma_1) + U(\gamma_2) + \cdots + U(\gamma_n)).$$

The sets $U(\Gamma)$, where Γ ranges over all sequences of normal (or arbitrary open) covers, form a neighborhood base at zero for the topology of $B(X)$.

III For an arbitrary continuous pseudometric d on X, we set:

$$U(d) = \Big\{x_1 + y_1 + x_2 + y_2 + \cdots + x_n + y_n \colon n \in \mathbb{N},\ x_i, y_i \in X,\ \sum_{i=1}^{n} d(x_i, y_i) < 1\Big\}.$$

The sets $U(d)$, where d ranges over all continuous pseudometrics on X, form a neighborhood base at zero for the topology of $B(X)$.

It follows directly from the second description that the base of neighborhoods of zero in $B^{\mathrm{lin}}(X)$ (for zero-dimensional X) is formed by the subgroups:

$$\langle U(\gamma)\rangle = \Big\{\sum_{i=1}^{n}(x_i + y_i) \colon n \in \mathbb{N},\ (x_i, y_i) \in U_i \in \gamma \text{ for } i \le n\Big\}$$

generated by the sets $U(\gamma)$ with γ ranging over all normal covers of X. By definition, any normal cover of a strongly zero-dimensional space has a disjoint open refinement. Therefore, for X with $\dim X = 0$, the covers γ can be assumed to be disjoint, and for disjoint γ, we have:

$$\langle U(\gamma)\rangle = \Big\{\sum_{i=1}^{n}(x_i + y_i) \colon n \in \mathbb{N},\ (x_i, y_i) \in U_i \in \gamma \text{ for } i \le n, \text{ the word } \sum_{i=1}^{n}(x_i + y_i) \text{ is reduced}\Big\}$$

(see the proof of Lemma 1). A similar description is valid for the Abelian groups $A^{\mathrm{lin}}(X)$ (the pluses must be replaced by minuses). This leads to the following statement.

Proposition 3. *For any strongly zero-dimensional space X and any $n \in \omega$, the topology induced on $A_n(X)$ (on $B_n(X)$) by $A^{\mathrm{lin}}(X)$ (by $B^{\mathrm{lin}}(X)$) coincides with that induced by $A(X)$ (by $B(X)$).*

Proof. We can assume without loss of generality that n is even. Given any neighborhood U of zero in $A(X)$ (in $B(X)$), it suffices to take a sequence $\Gamma = \{\gamma_k\}_{k\in\mathbb{N}}$ of disjoint covers such that $\frac{n}{2} \cdot U(\Gamma) \subset U$ and note that $\langle U(\gamma_1)\rangle \cap A_n(X) \subset \frac{n}{2} \cdot U(\gamma_1) \subset U$. □

Graev's procedure for extending any continuous pseudometric d on X to a maximal invariant pseudometric $\hat{d}$ on $F(X)$ is easy to adapt to the Boolean case. Following Graev, we first consider free topological groups in the sense of Graev, in which the identity element is identified with a point of the generating space and the universality property is slightly different: only continuous maps of the generating space to topological groups G that take the distinguished point to the identity elements of G must extend to continuous homomorphisms [3]. Graev showed that the free topological and Abelian topological groups $F_G(X)$ and $A_G(X)$ in the sense of Graev are unique (up to topological isomorphism) and do not depend on the choice of the distinguished point; moreover, the free topological group in the sense of Markov is nothing but the Graev free topological group on the same space to which an isolated point is added (and identified with the identity element).

The extension of a continuous pseudometric d on X to a maximal invariant continuous pseudometric $\hat{d}$ on the Graev free Boolean topological group $B_G(X)$ is defined by setting:

$$\hat{d}(g,h) = \inf\Big\{\sum_{i=1}^{n} d(x_i,y_i)\colon n\in\mathbb{N},\ x_i,y_i\in X, g=\sum_{i=1}^{n} x_i,\ h=\sum_{i=1}^{n} y_i\Big\}$$

for any $g,h\in B_G(X)$. The infimum is taken over all representations of g and h as (reducible) words of equal lengths. The corresponding Graev seminorm $\|\cdot\|_d$ (defined by $\|g\|_d=\hat{d}(g,0)$ for $g\in B_G(X)$, where 0 is the zero element of $B_G(X)$) is given by:

$$\|g\|_d = \inf\Big\{\sum_{i=1}^{n} d(x_i,y_i)\colon g=\sum_{i=1}^{n}(x_i+y_i),\ x_i,y_i\in X\Big\}.$$

The infimum is attained at a word representing g which may contain one zero (if the length of g is odd) and is otherwise reduced. Indeed, if the sum representing g contains terms of the form $x+z$ and $z+y$, then these terms can be replaced by one term $x+y$; the sum $\sum_{i=1}^{n} d(x_i,y_i)$ does not increase under such a change thanks to the triangle inequality.

For the usual (Markov's) free Boolean topological group $B(X)$, which is the same as $B_G(X\oplus\{0\})$ (where 0 is an isolated point identified with zero), the Graev metric depends on the distances from the points of X to the isolated point (they can be set to 1 for all $x\in X$). The corresponding seminorm $\|\cdot\|_d$ on the subgroup $B_{\text{even}}(X)$ of $B(X)$ consisting of words of even length does not change. The subgroup $B_{\text{even}}(X)$ is open and closed in $B(X)$, because this is the kernel of the continuous homomorphism $\hat{f}\colon B(X)\to\{0,1\}$ extending the constant continuous map $f\colon X\to\{0,1\}$ taking all $x\in X$ to 1. Thus, in fact, it does not matter how to extend $\|\cdot\|_d$ to $B(X)\setminus B_{\text{even}}(X)$; for convenience, we set:

$$\|g\|_d = \begin{cases} \min\Big\{\sum_{i=1}^{n} d(x_i,y_i)\colon g=\sum_{i=1}^{n}(x_i+y_i),\ x_i,y_i\in X, \\ \qquad\qquad \text{the word } \sum_{i=1}^{n}(x_i+y_i) \text{ is reduced}\Big\} & \text{if } g\in B_{\text{even}}(X), \\ 1 & \text{if } g\in B(X)\setminus B_{\text{even}}(X). \end{cases}$$

All open balls (as well as all open balls of any fixed radius not exceeding one) in all seminorms $\|\cdot\|_d$ for d ranging over all continuous pseudometrics on X form a base of open neighborhoods of zero in $B(X)$.

Topological spaces X and Y are said to be M-equivalent (A-equivalent) if their free (free Abelian) topological groups are topologically isomorphic. We shall say that X and Y are B-equivalent if $B(X)$ and $B(Y)$ are topologically isomorphic.

Given $X\supset Y$, we use $B(Y|X)$ to denote the topological subgroup of $B(X)$ generated by Y.

A special role in the theory of topological groups and in set-theoretic topology is played by Boolean topological groups generated by almost discrete spaces, that is, spaces having only one non-isolated point. With each free filter $\mathcal{F}$ on any set X, we associate the almost discrete space $X_{\mathcal{F}}=X\cup\{*\}$ ($*$ is a point not belonging to X); all points of X are isolated, and the neighborhoods of $*$ are $\{*\}\cup A$, $A\in\mathcal{F}$. For a space with infinitely many isolated points, there is no difference between the canonical definition of the groups $F(X)$, $A(X)$ and $B(X)$ and Graev's generalizations $F_G(X)$, $A_G(X)$ and $B_G(X)$. Indeed, Graev showed that $F_G(X)$ and $A_G(X)$ are unique (up to topological isomorphism) and do not depend on the choice of the distinguished point. Graev's argument, which uses only the universality property, carries over word for word to free Boolean topological groups. Thus, when dealing with spaces $X_{\mathcal{F}}$ associated with filters, we can identify $B(X_{\mathcal{F}})$ with $B_G(X_{\mathcal{F}})$ and assume that the only non-isolated point of $X_{\mathcal{F}}$ is the zero of $B(X_{\mathcal{F}})$; the descriptions of the neighborhoods of zero and the Graev seminorm are altered accordingly. To understand how they change, take the new (but in fact, the same) space $\widetilde{X}_{\mathcal{F}} = X_{\mathcal{F}} \cup \{0\}$, where 0 is one more isolated point, represent $B(X_{\mathcal{F}})$ as the Graev free Boolean topological group $B_G(\widetilde{X}_{\mathcal{F}})$ with distinguished point (zero of $B_G(\widetilde{X}_{\mathcal{F}})$) 0, and consider the topological isomorphism $g \mapsto g + 0$ between this group and the similar group with distinguished point (zero) $*$.

For example, since any open cover of $X_{\mathcal{F}}$ can be assumed to consist of a neighborhood of $*$ and singletons, Description II reads as follows in this case: For each $n \in \mathbb{N}$, we fix an arbitrary neighborhood V_n of $*$, that is, $A_n \cup \{*\}$, where $A_n \in \mathcal{F}$, and set:

$$W = \{V_n\}_{n\in\mathbb{N}},$$
$$U(V_n) = \{x + *\colon x \in V_n\} = \{x\colon x \in A_n\} \qquad (* \text{ is zero}),$$
$$U(W) = \bigcup_{n\in\mathbb{N}} (U(V_1) + U(V_2) + \cdots + U(V_n)) = \bigcup_{n\in\mathbb{N}} \{x_1 + \cdots + x_n\colon x_i \in A_i \text{ for } i \le n\}.$$

The sets $U(W)$, where the W range over all sequences of neighborhoods of $*$, form a neighborhood base at zero for the topology of $B(X_{\mathcal{F}})$. Strictly speaking, to obtain a full analogy with Description II of the Markov free group topology, we should set:

$$U(W) = \bigcup_{n\in\mathbb{N}} (2U(V_1) + 2U(V_2) + \cdots + 2U(V_n)) = \bigcup_{n\in\mathbb{N}} \{x_1 + y_1 \cdots + x_n + y_n\colon x_i, y_i \in V_i \text{ for } i \le n\},$$

but this would not affect the topology: the former $U(W)$ equals the latter for a sequence of smaller neighborhoods, say $V'_n = \bigcap_{i\le 2n} V_i$ (remember that some of the x_i and y_i in the expression for $U(W)$ may equal $*$, that is, vanish).

Similarly, the base neighborhoods of zero in Description III take the form:

$$U(d) = \Big\{x_1 + x_2 + \cdots + x_n\colon n \in \mathbb{N},\ x_1, \ldots, x_n \in X,\ \sum_{i=1}^{n} d(x_i, *) < 1\Big\},$$

where d ranges over continuous pseudometrics on $X_{\mathcal{F}}$ (again, we should set $U(d) = \Big\{x_1 + y_1 + x_2 + y_2 + \cdots + x_n + y_n\colon n \in \mathbb{N},\ x_i, y_i \in X \cup \{*\},\ \sum_{i=1}^{n} d(x_i, y_i) < 1\Big\}$, but this would not make any difference).

It is also easy to see that the isomorphism between $B_G(\widetilde{X}_{\mathcal{F}})$ (with distinguished point $*$) and $B(X_{\mathcal{F}})$ does not essentially affect the sets of words of length at most n; in particular, they remain closed, and $B_G(\widetilde{X}_{\mathcal{F}})$ is the inductive limit of these sets with the induced topology if and only if $B(X_{\mathcal{F}})$ has the inductive limit topology. In what follows, by $B(X_{\mathcal{F}})$, we shall usually mean the Graev free Boolean topological group with zero $*$.

Thus, $B(X_{\mathcal{F}})$ is naturally identified with the group $[X]^{<\omega}$ of all finite subsets of X under the operation $\triangle$ of symmetric difference ($A \triangle B = (A \setminus B) \cup (B \setminus A)$). The point $*$, which is the zero element of

$B(X_{\mathcal{F}})$, is identified with the empty set $\emptyset$, which belongs to $[X]^{<\omega}$ as the zero element. In the context of free Boolean groups on almost discrete spaces, we also identify each $x \in X$ with the one-point set $\{x\} \in [X]^{<\omega}$.

Sets of the form $[X]^{<\omega}$ often arise in set-theoretic topology and in forcing. The role of X is often played by ω, and the filter $\mathcal{F}$ is usually an ultrafilter with certain properties.

We assume all filters $\mathcal{F}$ on ω to be free, *i.e.*, to contain the Fréchet filter (of all cofinite sets).

A filter $\mathcal{F}$ on ω is said to be a P-filter if, for any family of $A_i \in \mathcal{F}$, $i \in \omega$, the filter $\mathcal{F}$ contains a pseudo-intersection of this family, *i.e.*, a set $A \subset \omega$ such that $|A \setminus A_i| < \omega$ for all $i \in \omega$. For ultrafilters, this property is equivalent to being a P-point, or weakly selective, ultrafilter. A filter $\mathcal{F}$ on ω is said to be a Ramsey filter if, for any family of $A_i \in \mathcal{F}$, $i \in \omega$, the filter $\mathcal{F}$ contains a diagonal of this family, *i.e.*, a set $D \subset \omega$ such that, whenever $i, j \in D$ and $i < j$, we have $j \in A_i$. Ultrafilters with this property are known as Ramsey, or selective, ultrafilters.

We use the standard notation $[\omega]^{<\omega}$ for the set of all finite subsets of ω and $\omega^{<\omega}$ for the set of all finite sequences of elements of ω. Given $s, t \in [\omega]^{<\omega}$, $s \sqsubset t$ means that s is an initial segment of t, *i.e.*, $s \subset t$ and all elements of $t \setminus s$ are greater than all elements of s. For $s \in [\omega]^{<\omega} \setminus \{\emptyset\}$, by $\max s$, we mean the greatest element of s in the ordering of ω. We also set $\max \emptyset = -1$.

3. A Comparison of Free, Free Abelian and Free Boolean Topological Groups

3.1. Similarity

There are a number of known properties of free and free Abelian topological groups that automatically carry over to free Boolean topological groups simply because they are preserved by taking topological quotient groups or, more generally, by continuous maps. Thus, if $F(X)$ (and $A(X)$) is separable, Lindelöf, ccc, and so on, then so is $B(X)$. It is also quite obvious that X is discrete if and only if so are $F(X)$, $A(X)$ and $B(X)$.

Let X be a space, and let Y be its subspace. The topological subgroup $B(Y|X)$ of $B(X)$ generated by Y is not always the free Boolean topological group on Y (the induced topology of $B(Y|X)$ may be weaker). Looking at Description I of the free group topology on $B(X)$, we see that X and Y equipped with the universal uniformities $\mathcal{U}_X$ and $\mathcal{U}_Y$ are uniform subspaces of $B(X)$ and $B(Y)$ with their group uniformities $\mathcal{W}_{B(X)}$ and $\mathcal{W}_{B(Y)}$ (generated by entourages of the form $W(U) = \{(g, h) \colon h \in g + U\}$, where U ranges over all neighborhoods of zero in the corresponding group), which completely determine the topologies of $B(X)$ and $B(Y)$. Thus, if the topology of $B(Y|X)$ coincides with that of $B(Y)$, then, like in the case of free and free Abelian topological groups [28,29], $(Y, \mathcal{U}_Y)$ must be a uniform subspace of $(X, \mathcal{U}_X)$, which means that any bounded continuous pseudometric on Y can be extended to a continuous pseudometric on X (in this case, Y is said to be P-embedded in X [30]). The converse has been proven to be true for free Abelian (presented in [28] with an incomplete proof and completely proven in [29]) and even free [27] (see also [22], where a minor misprint in the condition 3° on p. 186 of [27] is corrected) topological groups. Since $B(X)$ and $B(Y)$ are the topological quotients of $A(X)$ and $A(Y)$ by the subgroups of squares $A(2X)$ and $A(2Y)$ and $A(2Y) = A(2X) \cap A(Y)$, we immediately obtain the following theorem.

Theorem 4. *Let X be a space, and let Y be its subspace. The topological subgroup of the free Boolean groups $B(X)$ generated by Y is the free topological group $B(Y)$ if and only if each bounded continuous pseudometric on Y can be extended to a continuous pseudometric on X.*

Any space X is closed in its free Boolean topological group $B(X)$ (see, e.g., ([9], Theorems 2.1 and 2.2)), as well as in $F(X)$ and $A(X)$ [1,2]. Moreover, all $F_n(X)$, $A_n(X)$ and $B_n(X)$ (the sets of words of length at most n) are closed in their respective groups, as well. The most elegant proof of this fact was first proposed by Arkhangel'skii in the unavailable book [31] (for $F(X)$, but the argument works for $A(X)$ and $B(X)$ without any changes): Note that all $F_n(\beta X) \subset F(\beta X)$ are compact, since these are the continuous images of $(X \oplus \{e\} \oplus X^{-1})^n$ under the natural multiplication maps $i_n\colon (x_1^{\varepsilon_1}, \dots, x_n^{\varepsilon_n}) \mapsto x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ (here, e denotes the identity element of $F(X)$, $\varepsilon_i = \pm 1$, and the word $x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ may be reducible, *i.e.*, have length shorter than n). Therefore, the $F_n(\beta X)$ are closed in $F(\beta X)$, and hence, the sets $F_n(X) = F_n(\beta X) \cap F(X|\beta X)$ are closed in $F(X|\beta X)$. It follows that these sets are also closed in $F(X)$, which is the same group as $F(X|\beta X)$, but has stronger topology.

The topological structure of a free group becomes much clearer when this group has the inductive limit topology (or, equivalently, when the inductive limit topology is a group topology). The problem of describing all spaces for which $F(X)$ (or $A(X)$) possesses this property has proven extremely difficult (and is still unsolved). Apparently, the problem was first stated explicitly by Pestov and Tkachenko in 1985 [32], but it was tackled as early as in 1948 by Graev [3], who proved that the free topological group of a compact space has the inductive limit topology. Then, Mack, Morris and Ordman [33] proved the same for k_ω-spaces. The strongest (to the author's knowledge) result in this direction was obtained by Tkachenko [34], who proved that if X is a P-space or a C_ω-space (the latter means X is the inductive limit of an increasing sequence $\{X_n\}$ of its closed subsets such that all finite powers of each X_n are countably compact and strictly collection-wise normal), then $F(X)$ has the inductive limit topology. All of these sufficient conditions are also valid for $A(X)$ and $B(X)$ by virtue of the following simple observation.

Proposition 5. *Suppose that $X = \bigcup_{n\in\mathbb{N}} X_n$, $Y = \bigcup_{n\in\mathbb{N}} Y_n$, X is the inductive limit of its subspaces X_n, $n \in \mathbb{N}$, and $f\colon X \to Y$ is a quotient map such that $f(X_n) = Y_n$ for each $n \in \mathbb{N}$. Then, Y is the inductive limit of its subspaces Y_n.*

Proof. Let $U \subset Y$ be such that all $U_n = U \cap Y_n$ are open in Y_n. Consider $V = f^{-1}(U)$ and $V_n = f^{-1}(U_n) \cap X_n$ for $n \in \omega$. Each V_n is open in X_n, because the restriction of f to X_n is continuous and $f(X_n) = Y_n$. On the other hand,

$$V_n = f^{-1}(U \cap Y_n) \cap X_n = (f^{-1}(U) \cap f^{-1}(Y_n)) \cap X_n = V \cap X_n;$$

therefore, V is open in X. Since the map f is quotient, it follows that $U = f(V)$ is an open set. □

For X of the form $\omega_{\mathcal{F}}$ (where $\mathcal{F}$ is a filter on ω), not only the sufficient conditions mentioned above, but also a necessary and sufficient condition for $F(X)$ and $A(X)$ to have the inductive limit topology is known. This condition is also valid for $B(X)$.

Theorem 6. *Given a filter $\mathcal{F}$ on ω, $B(\omega_{\mathcal{F}})$ has the inductive limit topology if and only if $\mathcal{F}$ is a P-filter.*

Proof. This theorem is true for free and free Abelian topological groups [35]. Therefore, by Proposition 5, $B(\omega_{\mathcal{F}})$ has the inductive limit topology for any P-filter. It remains to prove that if $B(\omega_{\mathcal{F}})$ is the inductive limit of the $B_n(\omega_{\mathcal{F}})$, then $\mathcal{F}$ is a P-filter.

Thus, suppose that $B(\omega_{\mathcal{F}})$ is the inductive limit of the $B_n(\omega_{\mathcal{F}})$, but $\mathcal{F}$ is not a P-filter, that is, there exists a decreasing sequence of $A_n \in \mathcal{F}$, $n \in \omega$, such that, for any $A \in \mathcal{F}$, there is an i for which the intersection $A \cap A_i$ is infinite. As usual, we assume that the zero element of $B(\omega_{\mathcal{F}})$ is the non-isolated point $*$ of $\omega_{\mathcal{F}}$.

Without loss of generality, we can assume that $A_0 = \omega$ and all sets $A_n \setminus A_{n+1}$ are infinite. We enumerate these sets as:

$$A_n \setminus A_{n+1} = \{x_{ni} \colon i \in \omega\}$$

and put:

$$D_n = \{x_{nm} + x_{i_1 j_1} + x_{i_2 j_2} + \cdots + x_{i_n j_n} \colon n < i_1 < i_2 < \cdots < i_n < j_1 < j_2 < \cdots < j_n < m\}$$

for all $n \in \omega$. Let us show that each D_n is a closed discrete subset of $B(\omega_{\mathcal{F}})$. Fix n and consider $X = \{*\} \cup \{x_{ni} \colon i \in \omega\}$ and the retraction $r \colon \omega_{\mathcal{F}} \to X$ that takes $\omega_{\mathcal{F}} \setminus X$ to $\{*\}$. Clearly, X is discrete, and the map r is continuous. Let $\hat{r} \colon B(\omega_{\mathcal{F}}) \to B(X)$ be the homomorphic extension of r; then, $\hat{r}$ continuously maps $B(\omega_{\mathcal{F}})$ onto the discrete group $B(X)$. For any $g \in B(X)$, the set $\hat{r}^{-1}(g) \cap D_n$ is finite: if $\hat{r}^{-1}(g) \cap D_n$ is nonempty, then we have $g = \hat{r}(x_{nm_0} + x_{i_{0_1} j_{0_1}} + x_{i_{0_2} j_{0_2}} + \cdots + x_{i_{0_n} j_{0_n}})$ for some $m_0, i_{0_k}, j_{0_k} \in \omega$ such that $n < i_{0_1} < i_{0_2} < \cdots < i_{0_n} < j_{0_1} < j_{0_2} < \cdots < j_{0_n} < m_0$, whence $g = x_{nm_0}$ and:

$$\hat{r}^{-1}(g) \cap D_n = \{x_{nm_0} + x_{i_1 j_1} + x_{i_2 j_2} + \cdots + x_{i_n j_n} \colon \\ n < i_1 < i_2 < \cdots < i_m < j_1 < j_2 < \cdots < j_n < m_0\}.$$

Since the sets $\hat{r}^{-1}(g)$, $g \in B(X)$, form an open cover of $B(\omega_{\mathcal{F}})$, it follows that D_n is a closed discrete subspace of $B(\omega_{\mathcal{F}})$.

The length of each word in D_n equals $n + 1$. Therefore, $D = \bigcup_n D_n$ is closed in the inductive limit topology. It remains to show that $*$ (the zero of $B(\omega_{\mathcal{F}})$) belongs to the closure of D in the free group topology, *i.e.*, that $U(d) \cap D \neq \emptyset$ for any continuous pseudometric d on $\omega_{\mathcal{F}}$ (see Description III of the topology of $B(\omega_{\mathcal{F}})$).

Take an arbitrary (continuous) pseudometric d on $\omega_{\mathcal{F}}$. In $\omega_{\mathcal{F}}$, the ball $B_d(*, \frac{1}{2})$ of radius $\frac{1}{2}$ centered at $*$ with respect to d is a neighborhood of $*$; that is, the punctured ball (with $*$ removed) belongs to $\mathcal{F}$. By assumption, there is an $n \in \omega$ for which the set $M = \{i \in \omega : d(*, x_{ni}) < \frac{1}{2})\}$ is infinite. Since $B_d(*, \frac{1}{2n}) \cap A_{n+1}$ is a punctured neighborhood of $*$ and hence belongs to $\mathcal{F}$, it follows by assumption that the sets $J_i = \{j \in \omega : d(*, x_{ij}) < \frac{1}{2n}\}$ are infinite for infinitely many i. Choose $i_1 < i_2 < \cdots < i_n$ greater than n so that all J_{i_k} are infinite, then choose $j_k \in J_{i_k}$, $k \leq n$, so that $i_n < j_1 < \cdots < j_n$, and take $m \in M$ such that $m > j_n$. We have $g = x_{nm} + x_{i_1 j_1} + x_{i_2 j_2} + \cdots + x_{i_n j_n} \in D_n$. We also have $g \in U(d)$, because:

$$d(*, x_{nm}) + \sum_{k=1}^{n} d(*, x_{i_k j_k}) < \frac{1}{2} + n\frac{1}{2n} = 1.$$

Therefore, $g \in D_n \cap U(d)$. $\square$

In [36] Tkachuk proved that the free Abelian topological group of a disjoint union of two spaces X and Y is topologically isomorphic to the direct sum $A(X) \bigoplus A(Y) = A(X) \times A(Y)$. His argument carries over to varieties of Abelian topological groups closed under direct sums (or, in topological terminology, σ-products with respect to the zero elements of factors) with the box topology. We denote such sums by $\sigma\square$.

Proposition 7. *For any family $\{X_\alpha\colon \alpha \in A\}$ of spaces,*

$$A\Big(\bigoplus_{\alpha\in A} X_\alpha\Big) \cong \sigma\square_{\alpha\in A} A(X_\alpha) \qquad \textit{and} \qquad B\Big(\bigoplus_{\alpha\in A} X_\alpha\Big) \cong \sigma\square_{\alpha\in A} B(X_\alpha).$$

If all X_α are zero-dimensional, then:

$$A^{\mathrm{lin}}\Big(\bigoplus_{\alpha\in A} X_\alpha\Big) \cong \sigma\square_{\alpha\in A} A^{\mathrm{lin}}(X_\alpha) \qquad \textit{and} \qquad B^{\mathrm{lin}}\Big(\bigoplus_{\alpha\in A} X_\alpha\Big) \cong \sigma\square_{\alpha\in A} B^{\mathrm{lin}}(X_\alpha).$$

Proof. Let T stand for A, B, A^{lim} or B^{lim}, and let 0_α denote the zero element of $T(X_\alpha)$. For each $\alpha \in A$, we set $X'_\alpha = \sigma\square_{\beta\in A} Y_\beta$, where $Y_\alpha = X_\alpha$ and $Y_\beta = \{0_\beta\}$ for $\beta \neq \alpha$. Every X'_α is embedded in the group $T'_\alpha(X_\alpha)$ defined accordingly as a product of $T(X_\alpha)$ and zeros. Clearly, the union $\bigcup_{\alpha\in A} X'_\alpha$ algebraically generates $\sigma\square_{\alpha\in A} T(X_\alpha)$ and is homeomorphic to $\bigoplus_{\alpha\in A} X_\alpha$. It remains to show that the homomorphic extension of any continuous map of this union to any topological group from the corresponding variety is continuous. Let $f\colon \bigcup_{\alpha\in A} X'_\alpha \to G$ be such a map. For each $\alpha \in A$, the homomorphic extension $\hat{f}_\alpha\colon T'_\alpha(X_\alpha) \to G$ of the restriction of f to X'_α is continuous. We define $\hat{f}\colon \sigma\square_{\alpha\in A} T(X_\alpha) \to G$ by setting $\hat{f}\big((g_\alpha)_{\alpha\in A}\big) = \sum_{\alpha\in A} \hat{f}_\alpha(g_\alpha)$ for each $(g_\alpha)_{\alpha\in A} \in \sigma\square_{\alpha\in A} T(X_\alpha)$; the sum is defined, because any element of $\sigma\square_{\alpha\in A} T(X_\alpha)$ has only finitely many nonzero components. Let us show that $\hat{f}$ is continuous. It suffices to check continuity at the zero element of $\sigma\square_{\alpha\in A} T(X_\alpha)$. Take any neighborhood U of zero in G. Its preimages V_α under the component maps $\hat{f}_\alpha$ are open neighborhoods of zero in $T'_\alpha(X_\alpha)$. The product $\sigma\square_\alpha V_\alpha$ is the preimage of U under $\hat{f}$, and it is open in the box topology. $\square$

The free Boolean topological group of a non-discrete space is never metrizable (as well as the free and free Abelian topological groups). Indeed, if $B(X)$ is metrizable and X is non-discrete, then X contains a convergent sequence S with limit point $*$, and $B(S) = B(S|X)$ (see Theorem 4); thus, it suffices to show that $B(S)$ is non-metrizable. Suppose that it is metrizable. Then, the topology of $B(S)$ is generated by a continuous norm $\|\cdot\|$. For all pairs of positive integers n and $m \leq n$, choose different $s_{n_m} \in S$ so that $\|s_{n_m} + *\| < \frac{1}{n^2}$. Clearly, the set:

$$D = \{(s_{n_1} + *) + (s_{n_2} + *) + \ldots (s_{n_n} + *)\colon n \geq 0\}$$

has finite intersection with each $B_k(S)$; hence, it must be discrete, because $B(S)$ has the inductive limit topology. On the other hand, D is a sequence convergent to zero, since:

$$\|(s_{n_1} + *) + (s_{n_2} + *) + \ldots (s_{n_n} + *)\| \leq \sum_{i=1}^{n} (s_{n_i} + *) < n \cdot \frac{1}{n^2} = \frac{1}{n}.$$

The list of properties shared by free, free Abelian and free Boolean topological groups that can be proven without much effort is very long. Many of these properties are proven for Boolean groups by analogy, but sometimes, their proofs are drastically simplified. We conclude our brief excursion by one such examples. The proof of the following theorem for free topological groups given in [27] is extremely complicated (it is based on a more general construction). The proof given in [22] is much shorter, but still very cumbersome. In the Boolean case, the proof becomes almost trivial.

Theorem 8. *If* $\dim X = 0$*, then* $\operatorname{ind} B(X) = 0$.

Proof. Any continuous pseudometric d on X is majorized by a non-Archimedean pseudometric ρ (a pseudometric ρ is said to be non-Archimedean if $\rho(x, z) \le \max\{\rho(x, y), \rho(y, z)\}$ for any $x, y, z \in X$) taking only values of the form $\frac{1}{2^n}$. To see this, it suffices to consider the elements $V_0, V_1, \dots$ of the universal uniformity on X which are determined by decreasing disjoint open refinements $\gamma_0, \gamma_1, \dots$ of the covers of X by balls of radii $\frac{1}{2^1}, \frac{1}{2^2}, \dots$ with respect to d and apply the construction in the proof of Theorem 8.1.10 of [37] (see also [38]). Since the covers γ_n determining the entourages V_n are disjoint and each γ_{i+1} is a refinement of γ_i, it follows that the function f in this construction has the property $f(x, z) \le \max\{f(x, y), f(y, z)\}$, and therefore, the pseudometric ρ constructed there from f is non-Archimedean and takes the values $\frac{1}{2^n}$. Clearly, it majorizes d.

Each value $\|g\|_\rho$, $g \in B(X)$, of the Graev extension $\|\cdot\|_\rho$ of ρ is either one or a finite sum of values of d (recall that the minimum in the expression for $\|g\|_\rho$ is attained at an irreducible representation of g). Hence, $\|\cdot\|_\rho$ takes only rational values, and the balls with irrational radii centered at zero in this norm are open and closed. They form a base of neighborhoods of zero, and their translates form a base of the entire topology on $B(X)$. □

3.2. Difference

Pestov gave an example of a space X for which $F(X)$ is not homeomorphic to $A(X)$ [39]. Spaces for which $A(X)$ is not homeomorphic to $B(X)$ exist, too.

Proposition 9. *The free Abelian topological group of any connected space has infinitely many connected components. The free Boolean topological group of any connected space has two connected components.*

Proof. Consider a connected space X. The connected component of zero in $A(X)$ is the subgroup $A^{\mathrm{c}}(X)$ consisting of all words $\sum_{i=1}^n x_i^{\varepsilon_i}$ with $\sum_{i=1}^n \varepsilon_i = 0$ (see ([17], Lemma 7.10.2)). Clearly, all words in this subgroup are of even length, and the canonical homomorphism $A(X) \to B(X)$ takes $A^{\mathrm{c}}(X)$ to the subgroup $B^{\mathrm{c}}(X)$ of $B(X)$ consisting of all words of even length. Since the canonical homomorphism is continuous and open, the subgroup $B^{\mathrm{c}}(X)$ is connected and open (and hence, closed), and it has index two in $B(X)$. Thus, $B(X)$ has two connected components, while $A(X)$ has infinitely many connected components, because $A(X)/A^{\mathrm{c}}(X) \cong \mathbb{Z}$ (see ([17], Lemma 7.10.2)). □

There is a fundamental difference in the very topological-algebraic nature of free, free Abelian and free Boolean groups. Thus, nontrivial free and free Abelian groups admit no compact group topologies (see [40]); this follows from the well-known algebraic description of infinite compact Abelian groups ([41], Theorem 25.25). On the other hand, for any infinite cardinal κ, the direct sum $\bigoplus_{2^\kappa} \mathbb{Z}_2$ of 2^κ copies of $\mathbb{Z}_2$ (that is, the free Boolean group of rank 2^κ) is algebraically isomorphic to the

Cartesian product $(\mathbb{Z}_2)^\kappa$ ([42], Lemma 4.5) and, therefore, admits compact group topologies (e.g., the product topology).

The free and free Abelian groups are never finite, while the free Boolean group of any finite set is finite.

The free and free Abelian topological groups of any completely regular Hausdorff topological space X contain all finite powers X^n of X as closed subspaces. Thus, each X^n is homeomorphic to the closed subset $\{x_1 \dots x_n \colon x_i \in X \text{ for } i = 1 \leq n\}$ of $F(X)$ [43] and to the closed subset $\{x_1 + 2x_2 + \dots + nx_n \colon x_i \in X \text{ for } i = 1 \leq n\}$ of $A(X)$ [44]. (Arkhangel'skii announced the result for $F(X)$ in [43] and proved it in [31] by considering the Stone–Čech compactification of X and its free topological group; details can be found in Theorem 7.1.13 of [17]. Unfortunately, the book [31], which is a rotaprint edition of a lecture course, is (and always was) virtually unavailable, even in Russia. Thus, the result was rediscovered by Joiner [45] and the idea of proof by Morris [9] (see also [46]). In fact, both Arkhangel'skii and Joiner proved a stronger statement; namely, they gave the same complete description of the topological structure of all $F_n(X)$, although obtained by different methods (Arkhangel'skii proof is much shorter).)

However, the situation with free Boolean topological groups is much more complicated. For example, consider extremally disconnected free topological groups.

Extremally disconnected groups are discussed in the next section. Here, we only mention that non-discrete $F(X)$ and $A(X)$ are never extremally disconnected, while $B(X)$ may be non-discrete and extremally disconnected under certain set-theoretic assumptions (e.g., under CH), even for countable X of the form $\omega_{\mathcal{F}}$, and that any hereditarily normal, in particular, countable, extremally disconnected space, is hereditarily extremally disconnected (this is shown in the next section). It follows that if X is a non-discrete countable space for which $B(X)$ is extremally disconnected, then $B(X)$ does not contain X^2 as a subspace. Indeed, otherwise, X^2 is extremally disconnected (and non-discrete), and the existence of such spaces is prohibited by the following simple observation; it must be known, although the author failed to find a reference.

Proposition 10. *If $X \times X$ is extremally disconnected, then X is discrete.*

This immediately follows from Frolík's general theorem that the fixed-point set of any surjective self-homeomorphism of an extremally disconnected space is clopen [47]: it suffices to consider the self-homeomorphism of $X \times X$ defined by $(x, y) \mapsto (y, x)$. (Frolík proved this theorem for compact extremally disconnected spaces and not necessarily surjective self-homeomorphisms; in the surjective case, the theorem is extended to non-compact spaces by considering their Stone–Čech compactifications, which are always extremally disconnected for extremally disconnected spaces (this and other fundamental properties of extremally disconnected spaces can be found in the book [48]).)

Thus, there exist (under CH) filters $\mathcal{F}$ on ω for which $(\omega_{\mathcal{F}})^2$ is not contained in $B(\omega_{\mathcal{F}})$ as a subspace. However, in the simplest case where $\mathcal{F}$ is the Fréchet filter (*i.e.*, $\omega_{\mathcal{F}}$ is a convergent sequence), $B(\omega_{\mathcal{F}})$ not merely contains $(\omega_{\mathcal{F}})^n$, but is topologically isomorphic to $B(\omega_{\mathcal{F}})^n$ for all n by virtue of Proposition 7 and the fact that a convergent sequence is B-equivalent to the disjoint union of two convergent sequences, which can be demonstrated as follows.

Any M-equivalent spaces are A-equivalent, and any A-equivalent spaces are B-equivalent, because $A(X)$ ($B(X)$) is the quotient of $F(X)$ ($A(X)$) by an algebraically determined subgroup not depending on X. Therefore, all known sufficient conditions for M- and A-equivalence (see, e.g., [3,4,49–51]) remain valid for B-equivalence. In particular, if X_0 is a space, K is a retract of X_0, X is the space obtained by adding an isolated point to X_0 and $Y = X_0/K \oplus K$, then X and Y are M-equivalent ([50], Theorem 2.4). This immediately implies the required B-equivalence of a convergent sequence S and the disjoint union $S \oplus S$ of two convergent sequences: it suffices to take $S \oplus S$ for X_0 and X and the two-point set of the two limit points in $S \oplus S$ for K.

However, there exist B-equivalent spaces, which are neither F- nor A-equivalent. Genze, Gul'ko and Khmyleva obtained necessary and sufficient conditions for infinite initial segments of ordinals to be F-, A- and B-equivalent [20] (see also [21]). It turned out that the criteria for F- and A-equivalence are the same, and the criterion for B-equivalence differs from them; see [20] for details.

Finally, the following theorem shows that there is also a fundamental difference between free groups of the varieties of Abelian and Boolean linear topological groups.

Theorem 11. *The free Boolean linear topological group of any strongly zero-dimensional pseudocompact space is precompact.*

Proof. Let X be a strongly zero-dimensional pseudocompact space. As mentioned in the preceding section, a base of neighborhoods of zero in $B^{\mathrm{lim}}(X)$ is formed by subgroups of the form:

$$\langle U(\gamma)\rangle = \Big\{\sum_{i=1}^{n}(x_i + y_i)\colon n \in \omega,\ (x_i, y_i) \in U_i \in \gamma \text{ for } i \le n\Big\},$$

where γ in a disjoint open cover of X; note that all such covers are finite. Clearly,

$$\langle U(\gamma)\rangle = \Big\{\sum_{i=1}^{2n} x_i\colon n \in \omega,\ |\{i \le 2n\colon x_i \in U\}| \text{ is even for each } U \in \gamma\Big\}.$$

Every such subgroup has finite index. Therefore, $B(X)$ is precompact. $\square$

This theorem is not true for Abelian groups; moreover, free Abelian linear groups are never precompact. Indeed, the group $A^{c}(X) = \Big\{\sum_{i=1}^{n} x_i^{\varepsilon_i}\colon n \in \mathbb{N},\ \sum_{i=1}^{n}\varepsilon_i = 1\Big\}$ considered above is always open, being the preimage of the isolated point zero under the homomorphism $A(X) \to \mathbb{Z}_2 = \{0, 1\}$, which extends the constant map $X \to \{0, 1\}$ taking everything to one. As already mentioned, $A^{c}(X)$ has infinite index in $A(X)$.

4. Extremally Disconnected Groups

There is an old problem of Arkhangel'skii on the existence in ZFC of a non-discrete Hausdorff extremally disconnected topological group; it was posed in 1967 [52] and has been extensively studied since then. The problem is still open even for countable groups, although several consistent examples have been constructed [53–58]. An impression of the state-of-the-art in this area can be gained from Zelenyuk's book [59] and the author's papers [60] and [61]. The most recent result (presented in [61]) asserts that, under additional set-theoretic assumptions, a countable extremally disconnected group cannot contain a sequence of open subgroups whose intersection has an empty interior; in other words, if there exists in ZFC a non-discrete countable extremally disconnected group, then there must exist such a group without open subgroups (note in this connection that any extremally disconnected space is strongly zero-dimensional, and any zero-dimensional free Boolean topological group contains a family of open subgroups with trivial intersection (see Theorem 2)). Here, we present a new observation closely related to free Boolean topological groups.

A space X is said to be extremally disconnected if the closure of each open set in this space is open or, equivalently, if any two disjoint open sets have disjoint closures. In particular, the space $X_{\mathcal{F}}$ associated with a filter $\mathcal{F}$ is extremally disconnected if and only if $\mathcal{F}$ is an ultrafilter. The most fundamental properties of extremally disconnected spaces can be found in the book [48]. Much useful information (especially in the topological-algebraic context) is contained in [62]. The central place in the theory of extremally disconnected topological groups is occupied by Boolean topological groups because of the following theorem of Malykhin.

Theorem 12 (Malykhin [54])**.** *Any extremally disconnected group contains an open (and therefore closed) Boolean subgroup.*

This theorem follows from Frolík's fixed-point theorem mentioned at the end of the preceding section. In [54], Malykhin reproved Frolík's theorem for the particular self-homeomorphism $g \mapsto g^{-1}$; its fixed-point set U is an open neighborhood of the identity element, and the subgroup generated by an open neighborhood V of the identity for which $V^2 \subset U$ is as required.

Thus, in the theory of extremally disconnected groups, only Boolean groups matter. As is known, the existence of a non-discrete extremally disconnected free Boolean topological group implies the existence of either measurable cardinals or Ramsey ultrafilters [60] (this is proven by reduction to the free Boolean topological group on a countable space with one non-isolated point); it is also known that the simultaneous nonexistence of measurable cardinals and Ramsey ultrafilters is consistent with ZFC (see [63]). The following two theorems have a stronger consequence.

Theorem 13. *Any hereditarily normal extremally disconnected space is hereditarily extremally disconnected.*

Proof. Let X be a hereditarily normal extremally disconnected space. We must prove that any $Y \subset X$ is extremally disconnected. We can assume that Y is closed in X, because, obviously, any dense subspace of an extremally disconnected space is extremally disconnected. We must show that the closures in Y of any disjoint sets U and V which are open in Y are disjoint. Note that such sets U and V are separated (in Y and, therefore, in X), that is, $\overline{U} \cap V = U \cap \overline{V} = \emptyset$. Since X is hereditarily normal, there exist disjoint open (in X) sets $U' \supset U$ and $V' \supset V$ ([37], Theorem 2.1.7). Their closures in X cannot intersect, because X is extremally disconnected; thus, the closures in Y of the smaller sets U and V do not intersect either. □

Theorem 14. *If G is a hereditarily normal extremally disconnected Boolean group, then any closed linearly independent subset of G contains at most one non-isolated point.*

Proof. Let $A \subset G$ be a closed linearly-independent subset of G. Suppose that $a \in A$ and $b \in A$ are distinct limit points of A. Take their disjoint closed neighborhoods $U \ni a$ and $V \ni b$. Since A is linearly independent and closed, it follows that $a + (V \cap A) \cap b + (U \cap A) = \{a + b\}$, and the sets $a + (V \cap A)$ and $b + (U \cap A)$ are closed. Therefore, the sets $a + ((V \setminus \{b\}) \cap A)$ and $b + ((U \setminus \{a\}) \cap A)$ are closed in the normal subspace $(a + (V \cap A) \cup b + (U \cap A)) \setminus \{a + b\}$ of G and, hence, can be separated by disjoint open neighborhoods in this subspace. These neighborhoods remain open in $a + (V \cap A) \cup b + (U \cap A)$; obviously, $a + b$ belongs to the closure of each of them, which contradicts the hereditary extremal disconnectedness of G. $\square$

Corollary 15. *If X is a non-discrete countable space for which $B(X)$ is extremally disconnected, then X is almost discrete.*

We shall see in the next section that, in fact, the space X in Corollary 15 must be associated with a Ramsey ultrafilter.

5. Free Boolean Groups on Filters on ω

We have already seen in the preceding sections that free Boolean groups on almost discrete countable spaces (associated with filters on ω) exhibit interesting behavior. Moreover, they are encountered more often than it may seem at first glance.

Consider any Boolean group $B(X)$ with countable basis X. As mentioned in Section 2, this group is (algebraically) isomorphic to the direct sum (or, in topological terminology, σ-product) $\bigoplus^{\aleph_0} \mathbb{Z}_2$ of countably many copies of $\mathbb{Z}_2$. There is a familiar natural topology on this σ-product, namely the usual product topology; let us denote it by τ_{prod}. This topology induces the topology of a convergent sequence on $X \oplus \{0\}$ (where 0 denotes the zero element of $B(X)$) and is metrizable; therefore, it never coincides with the topology τ_{free} of the free Boolean topological group on X. Moreover, τ_{prod} is contained in τ_{free} only when X is discrete or has the form $\omega_{\mathcal{F}}$ for some filter (recall that we assume all filters to be free, *i.e.*, contain the filter of cofinite sets, and identify the non-isolated points of the associated spaces with the zeros of their free Boolean groups). On the other hand, any countable space is zero-dimensional; therefore, any countable free Boolean topological group contains a sequence of subgroups with trivial intersection (see Theorem 2). In [61], the following lemma was proven.

Lemma 16 ([61]). *Let G be a countable non-discrete Boolean topological group that contains a family of open subgroups with trivial intersection. Then, there exists a basis of G such that the isomorphism $G \to \bigoplus^{\aleph_0} \mathbb{Z}_2$ taking this basis to the canonical basis of $\bigoplus^{\aleph_0} \mathbb{Z}_2$ is continuous with respect to the product topology on $\bigoplus^{\aleph_0} \mathbb{Z}_2 = \sigma(\mathbb{Z}_2)^{\aleph_0}$.*

This immediately implies the following assertion.

Theorem 17. *Any countable Boolean topological group containing a family of open subgroups with trivial intersection (in particular, any free Boolean topological or linear topological group on a countable space) has either a discrete closed basis or a closed basis homeomorphic to the space $\omega_{\mathcal{F}}$ associated with a filter $\mathcal{F}$ on ω.*

Spaces of the form $\omega_{\mathcal{F}}$ are one of the rare examples where the free Boolean topological group is naturally embedded in the free and free Abelian topological groups as a closed subspace. The embedding of $B(\omega_{\mathcal{F}})$ into $A(\omega_{\mathcal{F}})$ is defined simply by $x_1 + x_2 + \cdots + x_n \mapsto x_1 + x_2 + \cdots + x_n$ (for the Graev free groups, which are the same as Markov ones for such spaces), and the embedding into $F(\omega_{\mathcal{F}})$ is $x_1 + x_2 + \cdots + x_n \mapsto x_1 x_2 \ldots x_n$, provided that $x_1 < x_2 < \cdots < x_n$. These embeddings take $B(\omega_{\mathcal{F}})$ to:

$$A = \{x_1 + x_2 + \cdots + x_n = (x_1 - *) + (x_2 - *) + \cdots + (x_n - *) \colon n \in \mathbb{N},\ x_i \in \omega\} \subset A(\omega_{\mathcal{F}})$$

and:

$$F = \{x_1 x_2 \ldots x_n = x_1 *^{-1} x_2 *^{-1} \ldots x_n *^{-1} \colon n \in \mathbb{N},\ x_i \in \omega,\ x_1 < x_2 < \cdots < x_n\} \subset F(\omega_{\mathcal{F}}).$$

The topologies induced on A and F by $A(\omega_{\mathcal{F}})$ and $F(\omega_{\mathcal{F}})$ are easy to describe; the restrictions of base neighborhoods of the zero (identity) element to these sets are determined by sequences of open covers of $\omega_{\mathcal{F}}$ (*i.e.*, of neighborhoods of the non-isolated point $*$) in the same manner as in Description II (see [22]). A straightforward verification shows that A, B and $B(\omega_{\mathcal{F}})$ are homeomorphic. The rigorous proof of this fact is rather tedious, and we omit it.

As mentioned in the Introduction, for any filter $\mathcal{F}$, the free Boolean group on $\omega_{\mathcal{F}}$ is simply $[\omega]^{<\omega}$. Any topology on $[\omega]^{<\omega}$ (as well as on any other set) is a partially ordered (by inclusion) family of subsets. Partial orderings of subsets of $[\omega]^{<\omega}$ have been extensively studied in forcing, and countable Boolean topological groups turn out to be closely related to them. In this section, we shall try to give an intuitive explanation of this relationship. The basic definitions and facts related to forcing can be found in Jech's book [64].

By a notion of forcing, we mean a partially ordered set (briefly, poset) $(\mathbb{P}, \leq)$. Elements of a notion of forcing are called conditions; given two conditions $p, q \in \mathbb{P}$, we say that p is stronger than q if $p \leq q$. A partially ordered set $(\mathbb{P}, \leq)$ is separative if, whenever $p \not\leq q$, there exists an $r \leq p$ which is incompatible with q. Thus, any topology is a generally non-separative notion of forcing, and the family of all regular open sets in a topology is a separative notion of forcing. Any separative forcing notion $(P, \leq)$ is isomorphic to a dense subset of a complete Boolean algebra. Indeed, consider the set $\mathbb{P} \downarrow p = \{q : q \leq p\}$ for each $p \in \mathbb{P}$. The family $\{X \subset \mathbb{P} : (\mathbb{P} \downarrow p) \subset X \text{ for every } p \in X\}$ generates a topology on $\mathbb{P}$. The complete Boolean algebra mentioned above is the algebra $\mathrm{RO}(\mathbb{P})$ of regular open sets in this topology.

Two notions of forcing $\mathbb{P}$ and $\mathbb{Q}$ are said to be forcing equivalent if the algebras $\mathrm{RO}(\mathbb{P})$ and $\mathrm{RO}(\mathbb{Q})$ are isomorphic or, equivalently, if $\mathbb{P}$ can be densely embedded in $\mathbb{Q}$ and *vice versa* (which means that $\mathbb{P}$ and $\mathbb{Q}$ give the same generic extensions).

Roughly speaking, given a countable transitive model $\mathbf{M}$ of set theory, the method of forcing extends $\mathbf{M}$ by adding a so-called generic subset (called also a generic filter) G of $\mathbb{P}$ not belonging to $\mathbf{M}$; the extended model, called a generic extension of $\mathbf{M}$, contains $\bigcup G$, which has certain desired properties ensured by the choice of $\mathbb{P}$ and G.

In the context of free Boolean groups on filters, most interesting are two well-known notions of forcing, Mathias forcing and Laver forcing relativized to (usual) filters on ω.

In Mathias forcing relative to a filter $\mathcal{F}$, the forcing poset, denoted $\mathbb{M}(\mathcal{F})$, is formed by pairs (s, A) consisting of a finite set $s \subset \omega$ and an (infinite) set $A \in \mathcal{F}$ such that every element of s is less than every element of A in the ordering of ω. A condition (t, B) is stronger than (s, A) ($(t, B) \le (s, A)$) if $s \sqsubset t$, $B \subset A$ and $t \setminus s \subset A$.

The poset in Laver forcing consists of subsets of the set $\omega^{<\omega}$ of ordered finite sequences in ω. However, it is more convenient for our purposes to consider its modification consisting of subsets of $[\omega]^{<\omega}$. Thus, we restrict the Laver forcing poset to the set $\omega^{\uparrow<\omega}$ of strictly increasing finite sequences in ω (this restricted poset is forcing equivalent to the original one) and note that the latter is naturally identified with $[\omega]^{<\omega}$. Below, we give the definition of the corresponding modification of Laver forcing.

The definition of Laver forcing uses the notion of a Laver tree. A Laver tree is a set p of finite subsets of ω such that:

(i) p is a tree (*i.e.*, if $t \in p$, then p contains any initial segment of t),
(ii) p has a stem, *i.e.*, a maximal node $s(p) \in p$, such that $s(p) \sqsubset t$ or $t \sqsubset s(p)$ for all $t \in p$ and
(iii) if $t \in p$ and $s(p) \sqsubset t$, then the set $\operatorname{succ}(t) = \{n \in \omega \colon n > \max t,\ t \cup \{n\} \in p\}$ is infinite.

In Laver forcing relative to $\mathcal{F}$, the poset, denoted $\mathbb{L}(\mathcal{F})$, is the set of Laver trees p such that $\operatorname{succ}(t) \in \mathcal{F}$ for any $t \in p$ with $s(p) \sqsubset t$, ordered by inclusion.

The Mathias and Laver forcings $\mathbb{M}(\mathcal{F})$ and $\mathbb{L}(\mathcal{F})$ have the special feature that they diagonalize the filter $\mathcal{F}$ (*i.e.*, add its pseudo-intersection). They determine two natural topologies on $[\omega]^{<\omega}$: the Mathias topology τ_M generated by the base:

$$\{[s, A] \colon s \in [\omega]^{<\omega},\ A \in \mathcal{F}\}, \qquad \text{where} \quad [s, A] = \{t \in [\omega]^{<\omega} \colon s \sqsubset t,\ t \setminus s \subset A\},$$

and the Laver topology τ_L generated by all sets $U \subset [\omega]^{<\omega}$ such that:

$$t \in U \implies \{n > \max t \colon t \cup \{n\} \in U\} \in \mathcal{F}.$$

It is easy to see that the Mathias topology is nothing but the topology of the free Boolean linear topological group on $\omega_{\mathcal{F}}$ (recall that linear groups are those with topology generated by subgroups): a base of neighborhoods of zero is formed by the sets $[\emptyset, A]$ with $A \in \mathcal{F}$, that is, by all subgroups generated by elements of $\mathcal{F}$.

The neighborhoods of zero in the Laver topology are not so easy to describe explicitly; their recursive definition immediately follows from that given above for general open sets (the only condition that must be added is $\emptyset \in U$). Thus, U is an open neighborhood of zero if, first, $\emptyset \in U$; by definition, U must also contain all $n \in A(\emptyset)$ for some $A(\emptyset) \in \mathcal{F}$ (moreover, U may contain no other elements of size one); for each of these n, there must exist an $A(n) \in \mathcal{F}$ such that $A(n) \cap \{0, 1, \dots, n\} = \emptyset$ and U contains all $\{n, m\}$ with $m \in A(n)$ (moreover, U may contain no other element of size two); for any such $\{n, m\}$ ($m > n$), there must exist an $A(\{n, m\}) \in \mathcal{F}$ such that $A(\{n, m\}) \cap \{0, 1, \dots, m\} = \emptyset$ and U contains all $\{n, m, l\}$ with $l \in A(\{n, m\})$, and so on. Thus, each neighborhood of zero is determined by a family $\{A(s) \colon s \in [\omega]^{<\omega}\}$ of elements of $\mathcal{F}$. Clearly, the topology τ_L is invariant with respect to translation by elements of $[\omega]^{<\omega}$; upon a little reflection, it becomes clear that τ_L is the maximal invariant topology on $[\omega]^{<\omega}$ in which the filter $\mathcal{F}$ converges to zero. (An invariant topology is a topology with respect to which the group operation is separately continuous; groups with an invariant topology are said to be semi-topological. The convergence of $\mathcal{F}$ to zero means that τ_L induces the initially given topology on $\omega_{\mathcal{F}}$.) Since the free group topology is invariant as well, it is weaker than τ_L.

The Mathias topology is, so to speak, the uniform version of the Laver topology: a neighborhood of zero in the Laver topology determined by a family $\{A(s) \in \mathcal{F}\colon s \in [\omega]^{<\omega}\}$ is open in the Mathias topology if and only if there exists a single $A \in \mathcal{F}$ such that $A(s) = A \setminus \{0, 1, \dots, \max s\}$ for each s. (In [65], the corresponding relationship between Mathias and Laver forcings was discussed from a purely set-theoretic point of view.) Hence, $\tau_M \subset \tau_L$.

The topology of the free Boolean topological group on $\omega_{\mathcal{F}}$ occupies an intermediate position between the Mathias and the Laver topology: it is not so uniform as the former, but more uniform than the latter. A neighborhood of zero is determined not by a single element of the filter (like in the Mathias topology), but by a family of elements of $\mathcal{F}$ assigned to $s \in [\omega]^{<\omega}$ (like in the Laver topology), but these elements depend only on the lengths of s.

The following theorem shows that the Laver topology is a group topology only for special filters. This theorem was proven in 2007 by Egbert Thümmel, who kindly communicated it, together with a complete proof, to the author. The symbols τ_{free} and τ_{indlim} in its statement denote the topology of the free topological group $B(\omega_{\mathcal{F}})$ and the inductive limit topology of $B(\omega_{\mathcal{F}})$, respectively.

Theorem 18 (Thümmel, 2007 [66])**.** *For any filter on ω, the following conditions are equivalent:*

(i) *$\mathcal{F}$ is Ramsey;*
(ii) $\tau_M = \tau_{\text{free}} = \tau_{\text{indlim}} = \tau_L$;
(iii) *τ_L is a group topology;*
(iv) *for any sequence of $A_i \in \mathcal{F}$, $i \in \omega$, the set $U = \{\emptyset\} \cup \bigcup_{i\in\omega}[i, A_i]$ is open in τ_{free}.*

This theorem is particularly interesting because its original (Thümmel's) proof uses an argument that is simple and still quite typical of the method of forcing. The proof given below only slightly differs from Thümmel's and uses this argument, as well.

Proof. First, note that $\tau_M \subset \tau_{\text{free}} \subset \tau_{\text{indlim}} \subset \tau_L$. Indeed, the first two inclusions are obvious, and the third one follows from Proposition 3 (or from the inclusion $\tau_{\text{free}} \subset \tau_L$ noted above) and the observation that τ_L is the inductive limit of its restrictions to $B_n(\omega_{\mathcal{F}})$.

Thus, to prove the implication (i) ⇒ (ii), it suffices to show that $\tau_M = \tau_L$ for any Ramsey filter. Let U be a neighborhood of $\emptyset$ in τ_L. For each $i \in \omega$, we set:

$$A_i = \bigcap\bigl\{\{n > \max s\colon s \cup \{n\} \in U\}\colon s \in U,\ \max s \le i\bigr\}.$$

Since the number of $s \in [\omega]^{<\omega}$ with $\max s \le i$ is finite, it follows that $A_i \in \mathcal{F}$. Take a diagonal $D \in \mathcal{F}$ for the family $\{A_i\colon i \in \omega\}$. We can assume that $D \subset A_0$. Clearly, $[\emptyset, D] \subset U$, whence $U \in \tau_M$.

The implication (ii) ⇒ (iii) is trivial.

Let us prove (iii) ⇒ (iv). Note that it follows from (iii) that $\tau_{\text{free}} = \tau_L$, because $\tau_{\text{free}} \subset \tau_L$ and τ_{free} is the strongest group topology inducing the initially given topology on $\omega_{\mathcal{F}}$. It remains to note that any set of the form $\{\emptyset\} \cup \bigcup_{i\in\omega}[i, A_i]$, where $A_i \in \mathcal{F}$, is open in τ_L.

We proceed to the last implication (iv) ⇒ (i). Take any family $\{A_i\colon i \in \omega\} \subset \mathcal{F}$ and consider the set U defined as in (iv). Since this is an open neighborhood of zero in the group topology τ_{free}, there exists an open neighborhood V of zero (in τ_{free}) such that $V + V \subset U$. The set $D = \{i \in \omega\colon i \in V\}$ belongs to $\mathcal{F}$ (because τ_{free} induces the initially given topology on $\omega_{\mathcal{F}}$) and is a diagonal of $\{A_i\colon i \in \omega\}$. □

This theorem is worth comparing to Judah and Shelah's proof that if $\mathcal{F}$ is a Ramsey ultrafilter, then $\mathbb{M}(\mathcal{F})$ is forcing equivalent to $\mathbb{L}(\mathcal{F})$ ([67], Theorem 1.20 (i)).

Thümmel also obtained the following remarkable result as a simple corollary of Theorem 18.

Theorem 19 (Thümmel, 2007 [66])**.** *Given a filter $\mathcal{F}$ on ω, the group $B(\omega_{\mathcal{F}})$ is extremally disconnected if and only if $\mathcal{F}$ is a Ramsey ultrafilter.*

Proof. The proof of the if part is essentially contained in Sirota's construction of a (consistent) example of an extremally disconnected group [56]. The proof of the only if part is based on the equivalence (iv) $\Leftrightarrow$ (i) of Theorem 18: for any family $\{A_i \colon i \in \omega\} \subset \mathcal{F}$, the set $\bigcup_{i\in\omega}[i, A_i]$ is open even in the Mathias topology, and its closure in τ_{free}, which must be open by virtue of extremal disconnectedness, is $\{\emptyset\} \cup \bigcup_{i\in\omega}[i, A_i]$. The assertion (iv) $\Leftrightarrow$ (i) implies that $\mathcal{F}$ is a Ramsey filter. It remains to apply Theorem 13 and recall that $\omega_{\mathcal{F}}$ is extremally disconnected if and only if $\mathcal{F}$ is an ultrafilter. □

Thümmel has never published these results, and Theorem 19 was rediscovered by Zelenyuk, who included it, among other impressive results, in his book [59] (see Theorem 5.1 in [59]).

Combining Theorem 19 with Corollary 15, we obtain yet another corollary.

Corollary 20. *The free Boolean group on a non-discrete countable space X is extremally disconnected if and only if X is an almost discrete space associated with a Ramsey ultrafilter.*

Free Boolean topological and free Boolean linear (that is, Mathias) topological groups on spaces associated with filters, as well as Boolean groups with other topologies determined by filters, are the main tool in the study of topological groups with extreme topological properties (see [59] and the references therein). However, free Boolean (linear) topological groups on filters arise also in more "conservative" domains. We conclude with mentioning an instance of this kind.

The most elegant (in the author's opinion) example of a countable non-metrizable Fréchet–Urysohn group was constructed by Nyikos in [68] under the relatively mild assumption $\mathfrak{p} = \mathfrak{b}$ (Hrušák and Ramos-García have recently proven that such an example cannot be constructed in ZFC [69]).

It is clear from general considerations that test spaces most convenient for studying convergence properties that can be defined pointwise (such as the Fréchet–Urysohn property and the related α_i-properties) are countable almost discrete spaces (that is, spaces of the form $\omega_{\mathcal{F}}$), and the most convenient test groups for studying such properties in topological groups are those generated by such spaces, simplest among which are free Boolean linear topological groups. Thus, it is quite natural that Nyikos' example is $B^{\text{lin}}(\omega_{\mathcal{F}})$ for a very cleverly constructed filter $\mathcal{F}$. In fact, he constructed it on $\omega \times \omega$ (which does not make any difference, of course) as the set of neighborhoods of the only non-isolated point in a Ψ-like space defined by using graphs of functions $\omega \to \omega$ from a special family. In the same paper, Nyikos proved many interesting convergence properties of groups $B^{\text{lin}}(\omega_{\mathcal{F}})$ for arbitrary filters $\mathcal{F}$ on ω. We do not give any more details here: the interested reader will gain much more benefit and pleasure from reading Nyikos' original paper.

6. A Few Open Problems

Free Boolean topological groups have not yet been extensively studied, and related unsolved problems are numerous. Some of the problems most interesting to the author are suggested below.

Problem 1. Describe those spaces X whose finite powers are embedded in the free Boolean topological groups $B(X)$. Is it true that if $\mathcal{F}$ is a free ultrafilter on ω, then $\omega_{\mathcal{F}} \times \omega_{\mathcal{F}}$ cannot be embedded in $B(\omega_{\mathcal{F}})$?

The following problem is open not only for free Boolean topological groups, but also for free and free Abelian ones.

Problem 2. Describe those spaces X for which $B(X)$ ($F(X)$, $A(X)$) is normal.

Of course, if $F(X)$ or $A(X)$ is normal, then so are all finite powers of X, because they are embedded in $F(X)$ and $A(X)$ as closed subspaces. However, in the Boolean case, even this has not been proven.

Problem 3. Does there exist a space X such that $B(X)$ is normal, but X^2 is not?

Similarly, for Boolean groups, the following problem becomes nontrivial.

Problem 4. Describe spaces X for which $B(X)$ is Lindelöf. Does there exist a space X such that $B(X)$ is Lindelöf, but X^2 is not?

Problem 5. Does there exist a space X for which $B(X)$ is normal (Lindelöf, ccc), but $F(X)$ or $A(X)$ is not?

Problem 6. Is it true that $B(X)$ is Weil complete for any Dieudonné complete space X?

A positive answer to this question in the case where X is a product of metrizable spaces would imply a positive answer in the general case. Indeed, any Dieudonné complete space X can be embedded in a product P of metric spaces as a closed subspace in such a way that every bounded continuous pseudometric on X can be extended to P. Therefore, by Theorem 4, $B(X)$ is a subgroup of $B(P)$; it is easy to see that $B(X)$ is closed in $B(P)$, and hence, $B(X)$ is Weil complete if so is $B(P)$. For free and free Abelian topological groups, Problem 6 has been completely solved: Tkachenko proved that if X is Dieudonné complete, then $A(X)$ is Weil complete [28]; Uspenskii proved the Weil completeness of $F(X)$ in the case where X is a product of metrizable spaces [29]; and the author extended Uspenskii's result to arbitrary Dieudonné complete spaces [27].

The following problem has been solved only for free Abelian topological groups [70].

Problem 7. Is it true that the free (Boolean) topological group of any stratifiable space is stratifiable?

The following two problems have been extensively studied and proven very difficult for free and free Abelian topological groups. Results related to the inductive limit topology were mentioned in Section 3.1, and results related to the natural multiplication maps being quotient can be found, e.g., in [25,34,71–73].

Problem 8. Describe spaces X for which $B(X)$ has the inductive limit topology.

Problem 9. Describe spaces X for which all (or some) of the natural addition maps $i_n\colon X \cup X^{-1} \to B(X)$ defined by $(x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, \dots, x_n^{\varepsilon_n}) \mapsto x_1 + x_2 + \dots + x_n$ for $n \in \mathbb{N}$, $x_i \in X$ and $\varepsilon_i = \pm 1$ $(i \le n)$ are quotient.

We conclude this short list of problems with a problem closely related to extremally disconnected groups.

Problem 10. Does there exist a (countable) non-discrete Boolean topological group in which all linearly independent sets are closed and discrete?

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Conflicts of Interest

The author declares no conflict of interest.

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