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Article

Pro-Lie Groups: A Survey with Open Problems

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Abstract: A topological group is called a pro-Lie group if it is isomorphic to a closed subgroup of a product of finite-dimensional real Lie groups. This class of groups is closed under the formation of arbitrary products and closed subgroups and forms a complete category. It includes each finite-dimensional Lie group, each locally-compact group that has a compact quotient group modulo its identity component and, thus, in particular, each compact and each connected locally-compact group; it also includes all locally-compact Abelian groups. This paper provides an overview of the structure theory and the Lie theory of pro-Lie groups, including results more recent than those in the authors' reference book on pro-Lie groups. Significantly, it also includes a review of the recent insight that weakly-complete unital algebras provide a natural habitat for both pro-Lie algebras and pro-Lie groups, indeed for the exponential function that links the two. (A topological vector space is weakly complete if it is isomorphic to a power $\mathbb{R}^X$ of an arbitrary set of copies of $\mathbb{R}$. This class of real vector spaces is at the basis of the Lie theory of pro-Lie groups.) The article also lists 12 open questions connected to pro-Lie groups.

Keywords: pro-Lie group; pro-Lie algebra; Lie group; Lie algebra; topological group; locally-compact group; unital topological algebra; exponential function; weakly-complete vector space

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1. Introduction

In 1900, David Hilbert presented a seminal address to the International Congress of Mathematicians in Paris. In this address, he initiated a program by formulating 23 problems, which influenced a vast amount of research of the 20th century. The fifth of these problems asked whether every locally-Euclidean topological group admits a Lie group structure, see [1]. This motivated an enormous volume of work on locally-compact groups during the first half of the 20th century. It culminated in the work of Gleason, Iwasawa, Montgomery, Yamabe and Zippin, yielding a positive answer to Hilbert's fifth problem and exposing the structure of almost connected locally-compact groups. (Recall that a topological group G is called almost connected if the quotient group G/G_0, modulo the connected component G_0 of the identity, is compact. The class of almost connected groups includes all compact groups and all connected locally-compact groups.) The advances in the second half of the 20th century shed much light on the structure and representation theory of locally compact groups.

Notwithstanding this, it was recognized that the class of locally-compact groups is obviously deficient in that it is not even closed under the formation of arbitrary products. It was unclear which class of topological groups would most appropriately extend the class of finite-dimensional Lie groups and avoid this defect.

At the beginning of the 21st century, the authors of this survey introduced the class of pro-Lie groups [2–4]. This class contains all finite-dimensional Lie groups, all compact groups, all locally-compact Abelian groups and all almost connected locally-compact groups. It is characterized as the class of all closed subgroups of arbitrary products of finite-dimensional Lie groups. Obviously, it is closed under the formation of arbitrary products and even the passage to closed subgroups.

This notion of pro-Lie group differs from that used in the late 20th century, where a group G is called a pro-Lie group if it is locally compact and contains arbitrarily small compact normal subgroups N, such that G/N is a finite-dimensional Lie group.

In order to understand the structure of pro-Lie groups in our sense, we developed a highly infinite-dimensional Lie theory of considerable complexity (see [5] and subsequent publications [6–8]). This Lie theory assigns to each pro-Lie group G a pro-Lie algebra $\mathfrak{g}$ and an exponential function $\exp\colon \mathfrak{g} \to G$. This approach was exploited very successfully in [9] for compact groups and was found to be an eminently useful extension of the classical theory of real Lie groups of finite dimension.

The theory of an n-dimensional real Lie group is based on the fact that open subsets of $\mathbb{R}^n$ have a rich differentiable structure that is transported to the group, allowing a differentiable multiplication and inversion in the group. It has been an ongoing effort to replace $\mathbb{R}^n$ by more general, possibly infinite-dimensional, topological vector spaces supporting differentiable structures. The most advanced such theory is the theory of Lie groups on differentiable or smooth manifolds modeled on open subsets of locally-convex vector spaces and their real analysis as used by Helge Glöckner and Karl-Hermann Neeb (see [10–12]). One may justifiably ask how the theory of pro-Lie groups and the theory of infinite-dimensional differentiable Lie groups in their spirit are related. It was shown in [13] that a pro-Lie group is a smooth Lie group in their sense if and only if it is locally contractible.

The theory of pro-Lie groups has been described in detail in the 678-page book [5] in 2007 and in later papers. An endeavor to summarize that work would therefore be futile. Rather, in this survey, we highlight central results and explain some key open problems.

The purely theoretical foundation of, and motivation for, the theory of pro-Lie groups, however, must be complemented by an outlook to areas in which they emerge naturally and by necessity. Section 8 therefore deals with the appearance of pro-Lie groups in the so-called character theory of Hopf algebras, which received attention in recent publications [14].

2. The Topology of Pro-Lie Groups

It has become a standard result in the literature that every connected locally-compact Abelian group is homeomorphic to $\mathbb{R}^n \times K$ where K is a compact connected Abelian group (see, e.g., [9], Theorem 7.57). The non-Abelian version of this (see [15], p. 188f) says that a connected locally-compact group is homeomorphic to $\mathbb{R}^n \times K$, where K is a compact connected group. We shall see in this section that a connected pro-Lie group is homeomorphic to $\mathbb{R}^{\mathbf{m}} \times K$ where $\mathbf{m}$ is an arbitrary cardinal and K is a compact connected group. This is convincing evidence that connected pro-Lie groups represent a natural extension of connected locally-compact groups. We also see from this observation that the pro-Lie group $\mathbb{R}^I$, for an arbitrary set I, is a critical example of a pro-Lie group. We shall return to this theme repeatedly in this text.

We shall give now a complete description of the topological structure of an almost connected pro-Lie group [16]. We present it here because it is perhaps easily understood without too much background information. A complete proof of the result is far from elementary or short.

A compact connected group S is said to be semisimple if its algebraic commutator subgroup is the whole group. Let us preface the main result by the remark that we know very explicitly the structure of compact connected semisimple groups from [9], Theorem 9.19. It is also a fact that in such a group, every element is itself a commutator.

Likewise, the structure of a compact connected Abelian group A is well understood. Indeed, a compact Abelian group A is connected if and only if its Pontryagin dual is a torsion-free Abelian group (see [9], Corollary 8.5). This allows the determination of details of the structure of such a group, as is expounded in [9], Chapter 8.

We denote by $\mathbb{Z}(n)$ the n-element group $\mathbb{Z}/n\mathbb{Z}$.

With this notation and information at hand, one can appreciate the power of the following result:

Theorem 1. ([7], Corollary 8.9, p. 381) *An almost connected pro-Lie group G contains a compact connected semisimple subgroup S and a compact connected Abelian subgroup A, such that for suitable sets I and J, the topological group G is homeomorphic to the topological group* $\mathbb{R}^I \times S \times A \times \Delta$, *where:*

$$\Delta = \begin{cases} \mathbb{Z}(n), & \textit{if } G \textit{ has finitely many components}, \\ \mathbb{Z}(2)^J, & \textit{otherwise}. \end{cases}$$

This result allows several immediate corollaries, which are of interest for the topology of pro-Lie groups.

Corollary 1. *The space underlying an almost connected pro-Lie group is a Baire space.*

This follows from Theorem 1 and Oxtoby's results in [17].

Corollary 2. *Every almost connected pro-Lie group is homotopy equivalent to a compact group.*

Indeed, $\mathbb{R}^I$ is homotopy equivalent to a singleton. The algebraic topology of an almost connected pro-Lie group therefore is that of a compact group (*cf.* [9]).

Corollary 3. *An almost connected pro-Lie group is locally compact if and only if in* Theorem 1*, the set I is finite.*

At the root of Theorem 1 is the following main theorem generalizing the Theorem on p. 188ff in [15].

Theorem 2. ([7], Main Theorem 8.1, p. 379) *Every almost connected pro-Lie group G has a maximal compact subgroup M, and any other compact subgroup is conjugate to a subgroup of M. Moreover,*

(1) $G = G_0M$,
(2) $M_0 = G_0 \cap M$, *and*
(3) M_0 *is a maximal compact subgroup of* G_0.

We record that the results of Theorem 2 enter into a proof of Theorem 1 in an essential way. In the process of proving Theorem 1, one also establishes the following theorem, which is more concise than Theorem 1 if one assumes the structure theory of compact groups as presented in [9].

Theorem 3. ([7], Theorem 8.4) *Let G be an almost connected pro-Lie group, and M a maximal compact subgroup of G. Then, G contains a subspace E homeomorphic to $\mathbb{R}^I$, for a set I, such that* $\quad (m, e) \mapsto me : M \times E \to G \quad$ *is a homeomorphism. Thus, G is homeomorphic to $\mathbb{R}^I \times M$.*

3. Pro-Lie Groups as Projective Limits of Lie Groups

We have defined pro-Lie groups as closed subgroups of products of finite-dimensional real Lie groups. In fact, they can be equivalently defined as special closed subgroups of such products, namely projective limits of finite-dimensional Lie groups.

At first pass, this is surprising, and its proof requires some effort.

Definition 1. *A family $\{f_{jk} : G_j \to G_k | j, k \in J\}$ of morphisms of topological groups is called a projective system if J is a directed set satisfying the conditions:*

(a) for all $j \in J$, f_{jj} is the identity map of G_{jj}, and

(b) for $i, j, k \in J$ with $i \leq j \leq k$, we have $f_{ik} = f_{ij} \circ f_{jk}$.

Given a projective system of morphisms of topological groups, we define a closed subgroup G of the product $\prod_{j \in J} G_j$ to be the set $\{(g_i)_{i \in J} : g_j = f_{jk}(g_k)$ for all $j \leq k$ in $J\}$. This group G is called the projective limit (of the projective system), denoted by $\lim_{j \in J} G_j$.

When we say in the following that a subgroup N of a topological group G is a co-Lie subgroup, we mean that N is a normal closed subgroup, such that the factor group G/N is a Lie group.

Theorem 4. ([5,18], Theorem 3.39 on p. 161, [6,19]) *For a topological group G, the following conditions are equivalent.*

(1) *G is complete, and every identity neighborhood contains a co-Lie subgroup.*
(2) *G is a projective limit of Lie groups.*
(3) *G is a pro-Lie group.*

This theorem, by the way, explains the choice of the name "pro-Lie group." (See also [20].) There is a considerable literature on the projective limits of finite discrete groups, called profinite groups (see [21,22]). Furthermore, amongst the pro-Lie groups, there are the prodiscrete groups, namely projective limits of discrete groups or, equivalently, closed subgroups of products of discrete groups. There is not much literature on prodiscrete groups. We formulate the following open question

Question 1: Is there a satisfactory structure theory for non-discrete prodiscrete groups? More particularly, is there a satisfactory structure theory even for Abelian non-discrete prodiscrete groups?

We do know that a quotient group of a pro-Lie group modulo a closed subgroup is a pro-Lie group if the quotient group is complete (see [5], Theorem 4.1.(i), p.170). One may reasonably ask whether, for a pro-Lie group G and a closed normal subgroup N, we have sufficient conditions for G/N to be complete and therefore a pro-Lie group (cf. [23]).

Theorem 5. ([5], Theorem 4.28, p. 202) *Let G be a pro-Lie group and N a closed normal subgroup of G. Each of the following condition suffices for G/N to be a pro-Lie group:*

(i) *N is almost connected, and G/G_0 is complete.*
(ii) *N satisfies the first axiom of countability.*
(iii) *N is locally compact.*

The answer to following question is unknown:

Question 2: Let G be a pro-Lie group with identity component G_0. Is G/G_0 complete (and therefore, prodiscrete)?

Indeed, this is the case when G is Abelian; see Theorem 6(iii) below.

4. Weakly-Complete Vector Groups

In this section, we discuss the previously mentioned connected pro-Lie groups $\mathbb{R}^I$, for a set I. For infinite sets I, these are the simplest connected pro-Lie groups outside the class of locally-compact groups. However they will appear many times in the Lie theory we shall present. In particular, they feature in the structure theory of Abelian pro-Lie groups.

All vector spaces considered here are understood to be vector spaces over $\mathbb{R}$. For a vector space E, let $\mathrm{Hom}(E, \mathbb{R})$ denote the set of all linear functionals on E with the vector space structure and topology it inherits from $\mathbb{R}^E$, the vector space of all functions $E \to \mathbb{R}$ with the product topology.

Proposition 1. *For a topological vector space V, the following conditions are equivalent:*

(1) *There is a vector space E, such that the topological vector spaces $\mathrm{Hom}(E, \mathbb{R})$ and V are isomorphic as topological groups.*
(2) *There is some set I, such that V is isomorphic to $\mathbb{R}^I$ with the product topology.*

Definition 2. *A topological vector space V is called weakly complete if it satisfies the equivalent conditions of Proposition 1.*

Every weakly-complete vector space is a nuclear locally-convex space (see [24], p. 100, Corollary to Theorem 7.4, p. 103). The vector space E in Proposition 1 is obtained as the vector space of all continuous linear maps $V \to \mathbb{R}$. In fact, the category of all weakly-complete vector spaces and continuous linear maps between them is dual to the category of all vector spaces and linear maps between them in a fashion analogous to the Pontryagin duality between compact Abelian groups and discrete Abelian groups (see [9], Theorem 7.30, [5], Appendix 2: Weakly Complete Topological Vector Spaces, pp. 629–650).

Theorem 6. ([5], Theorem 5.20, p. 230) *For any Abelian pro-Lie group G, there is a weakly-complete vector subgroup V and a closed subgroup H, such that (in additive notation):*

(i) *$(v, h) \mapsto v + h : V \times H \to G$ is an isomorphism of topological groups.*
(ii) *H_0 is a compact connected Abelian group, and every compact subgroup of G is contained in H.*
(iii) *$H/H_0 \cong G/G_0$, and, thus, G/G_0 is prodiscrete.*
(iv) *If G_a and H_a are the arc components of the identity of G and H, respectively, then $G_a = V \oplus H_a$.*

We note that in the present circumstances, the positive answer to Question 2 expressed in Conclusion (iii) follows from the compactness of H_0 via Theorem 5.

We have seen that products of pro-Lie groups are pro-Lie groups and that closed subgroups of pro-Lie groups are pro-Lie groups. As a consequence, projective limits of pro-Lie groups are pro-Lie groups.

In Section 7 of [25], Banaszczyk introduced nuclear Abelian groups. Since all Abelian Lie groups are nuclear and the class of nuclear groups is closed under projective limits, all Abelian pro-Lie groups are nuclear. (See also [26].)

In these circumstances, it is somewhat surprising that quotient groups of pro-Lie groups may fail to be pro-Lie groups. Indeed, as we shall see in the next Proposition 2, there is a quotient group of a very simple Abelian pro-Lie group, namely of $\mathbb{R}^{2^{\aleph_0}}$, which is an Abelian topological group that is not complete and, therefore, is not a pro-Lie group. However, this situation is not as concerning as it might first appear, because every quotient group of a pro-Lie group has a completion, and the completion is a pro-Lie group.

We consider the unit interval $\mathbb{I} = [0,1]$ as representative for the sets of continuum cardinality $2^{\aleph_0}$. Let $\delta_n \in \mathbb{Z}^{(\mathbb{N})}$ be defined by:

$$\delta_n = (e_{mn})_{m\in\mathbb{N}}, \quad e_{mn} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $B = \{\delta_n : n \in \mathbb{N}\}$ generates the free Abelian group $\mathbb{Z}^{(\mathbb{N})}$ algebraically.

Proposition 2. ([5], Proposition 5.2, p. 214) *The free Abelian group $\mathbb{Z}^{(\mathbb{N})}$ of countably-infinite rank has a non-discrete topology making it a prodiscrete group F, so that the following conditions are satisfied:*

(i) *F is a countable non-discrete non-metrizable pro-Lie group, which therefore is not a Baire space.*

(ii) *F can be considered as a closed subgroup of $V = \mathbb{R}^{\mathbb{I}}$ with dense $\mathbb{R}$-linear span in V, so that V/F is an incomplete group whose completion is a compact-connected and locally-connected, but not arcwise-connected group.*

(iii) *Every compact subset of F is contained in a finite rank subgroup.*

The structure theory results we discussed permit us to derive results on the duality of Abelian pro-Lie groups. Recall that this class contains the class of all locally-compact Abelian groups properly and is contained in the class of all Abelian nuclear groups.

For any Abelian topological group G, we let $\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$ denote its dual with the compact open topology, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (see e.g., [9], Chapter 7). There is a natural morphism of Abelian groups $\eta_G \colon G \to \widehat{\widehat{G}}$ given by $\eta_G(g)(\chi) = \chi(g)$, which may or may not be continuous; information regarding this issue is to be found for instance in [9], notably in Theorem 7.7. We shall call an Abelian topological group semireflexive if $\eta_G \colon G \to \widehat{\widehat{G}}$ is bijective and reflexive if η_G is an isomorphism of topological groups; in the latter case, G is also said to have duality (see [9], Definition 7.8).

There is an example of a non-discrete, but prodiscrete Abelian torsion group due to Banaszczyk (see [25], p. 159, Example 17.11), which is semireflexive, but not reflexive (see also [5], Chapter 14, p. 595, Example 14.15; attention: in Line 2 of the text of this example, read $N_\alpha = \mathbb{Z}(2)^{(\{\nu\colon \nu\geq\alpha\})}$, not $\nu < \alpha$). Therefore, we know that the category of Abelian pro-Lie groups is not self-dual under Pontryagin duality.

Theorem 7. ([5] Theorem 5.36, p. 239) *Every almost connected Abelian pro-Lie group is reflexive, and its character group is a direct sum of the additive topological group of a real vector space endowed with the finest locally-convex topology and a discrete Abelian group. Pontryagin duality establishes a contravariant functorial bijection between the categories of almost connected Abelian pro-Lie groups and the full subcategory of the category of topological Abelian groups containing all direct sums of vector groups with the finest locally-convex topology and discrete Abelian groups.*

By this theorem, duality applies to almost connected Abelian Lie groups. In particular, we recall that weakly-complete topological vector spaces have a good Pontryagin duality. By Theorem 7 above, the issue of duality of Abelian pro-Lie groups is reduced to groups with a compact identity component. Amongst this class there are all prodiscrete groups. In particular, nothing is known about the duality of prodiscrete Abelian groups. As all Abelian pro-Lie groups are nuclear, whatever is known on the duality of nuclear Abelian groups applies to pro-Lie groups.

Question 3: Which Abelian pro-Lie groups are reflexive?

Question 4: Which Abelian pro-Lie groups are strongly reflexive in the sense that all of their subgroups and Hausdorff quotient groups are reflexive?

5. The Open Mapping Theorem

We have just dealt with the question of whether a quotient group of a pro-Lie group is a pro-Lie groups, and we have seen that the answer is negative sometimes. We now deal with the question when a surjective morphism of pro-Lie groups is a quotient morphism. Specifically, let $f\colon G \to H$ be a surjective morphism of pro-Lie groups. Does this imply that f is an open mapping? This question is answered negatively in any first course on topological groups by the example of $G = \mathbb{R}_d$, the additive group of reals with the discrete topology and $H = \mathbb{R}$ with its natural topology. The quest for sufficient conditions that would secure the openness is answered by any of the so-called "open mapping theorems" in the classical literature in functional analysis and in the theory of topological groups. These impose countability conditions on G and a Baire space hypothesis on H. The latter is provided by such properties as complete metrizability, local compactness or the pro-Lie group property. If σ-compactness is taken as a countability condition on G, then the Baire space property of H will force local compactness upon $G/\ker f$. The induced bijection $G/\ker f \to H$ is an isomorphism if and only if f is open. Therefore, settling the issue for bijective f cannot be much of a restriction, assuming that the properties envisaged for G are preserved by passing to quotients.

Whichever way the issue is looked at, the proof of an open mapping theorem for pro-Lie groups [27,28] is quite different from all other open mapping theorems we know.

Once more, we encounter the class of almost connected pro-Lie groups as that class on which our methods, notably a Lie theory, which we have yet to discuss, yields an expected result.

Theorem 8. ([5], 9.60, p.409f) [6,19]) *Let G and H be pro-Lie groups, and let $f\colon G \to H$ be a surjective continuous homomorphism. Then, f is an open mapping if G is almost connected. In particular, the natural bijective morphism $G/\ker f \to H$ is an isomorphism of topological groups.*

The last conclusion yields another instance of a quotient group of a pro-Lie group, which is again a pro-Lie group, giving us a sufficient condition not included in Theorem 5 above, namely G is almost connected, and N is the kernel of a morphism onto a pro-Lie group.

A further corollary of our open mapping theorem is the following form of the second isomorphism theorem for pro-Lie groups:

Corollary 4. *Assume that N and H are two closed almost connected subgroups of a topological group with N being normal, and assume that NH is a pro-Lie group. Then, $H/(N \cap H) \cong NH/N$. Moreover, the natural morphism $\mu\colon N \rtimes H \to NH$, $mu(n,h) = nh$ is a quotient morphism (where H is acting as an automorphism group on N via inner automorphisms.*

Noting that $\mathbb{Z}^I$ is a pro-Lie group, but is not Polish, unless I is countable (see [5], pp. 235, 236, notably Lemma 5.30), we ask:

Question 5: Is a surjective morphism $f\colon \mathbb{Z}^I \to K$ onto a compact group open for every set I?

The results in Theorem 8 and in Chapter 5 of [5] do not answer this question. If I is countable, then $\mathbb{Z}^I$ is a Polish group, and an open mapping theorem applies in this case and gives an affirmative answer. The open mapping theorem for Pro-Lie groups does not apply, since $\mathbb{Z}^I$ is never almost connected for I nonempty.

Added in Proof: Saak Gabriyelyan and the second author have recently announced a positive answer to Question 5.

6. Lie Theory

We started our discussion with a presentation of some remarkable structure theorems on almost connected pro-Lie groups. It is not surprising that the proofs of such results require some powerful tools. The crucial tool for a structure theory of pro-Lie groups is their Lie theory. It is a challenge to explain what we mean by "Lie theory." Minimally, one wants to attach to each pro-Lie group G a Lie algebra $\mathfrak{L}(G)$ with characteristics making it suitable for an exploitation of its topological algebraic structure for the topological group structure of the pro-Lie group G. Guided by our knowledge of classical Lie group theory, we shall link the group G with its Lie algebra $\mathfrak{L}(G)$ by an exponential function $\exp\colon \mathfrak{L}(G) \to G$, which mediates between Lie algebra theoretical properties of $\mathfrak{L}(G)$ and group theoretical properties of G.

As a start, for each $X \in \mathfrak{L}(G)$, the function $t \mapsto \exp(t.X) : \mathbb{R} \to G$ is a morphism of topological groups. Tradition calls a morphism $f\colon \mathbb{R} \to G$ of topological groups a one-parameter subgroup of G (admittedly, a misnomer). In classical Lie theory, every one-parameter subgroup is obtained via the exponential function in this fashion. In other words, however one defines a Lie algebra and an exponential function, it must establish a bijection β from the elements of the Lie algebra $\mathfrak{L}(G)$ to the set $\mathrm{Hom}(\mathbb{R}, G)$ of one-parameter subgroups of G, so that the following diagram commutes:

$$\begin{array}{ccc}
\mathfrak{L}(G) & \xrightarrow{\exp} & G \\
{\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_G} \\
\mathrm{Hom}(\mathbb{R}, G) & \xrightarrow[f\mapsto f(1)]{} & G
\end{array}$$

Therefore, if we are given a topological group G, as a first step, we may think of $\mathfrak{L}(G)$ as $\mathrm{Hom}(\mathbb{R}, G)$ with a scalar multiplication, such that $r.X(s) = X(sr)$ for $X \in \mathfrak{L}(G)$ and $r, s \in \mathbb{R}$. If G has an additional structure, such as that of a pro-Lie group, we will obtain the additional structure on $\mathfrak{L}(G)$. Therefore, if G is a closed subgroup of a product $P = \prod_{i\in I} G_i$ of finite-dimensional Lie groups, then an element $X \in \mathfrak{L}(P)$ may be identified with an element $(X_i)_{i\in I}$ of $\prod_{i\in I} \mathfrak{L}(G_i)$, and an element X of this kind is in $\mathfrak{L}(G)$ simply if $X(\mathbb{R}) \subseteq G$. As G_i is a finite-dimensional Lie group, each $\mathfrak{L}(G_i)$ has the structure of a Lie algebra, $\mathfrak{L}(P)$ is both a weakly-complete topological vector space and a topological Lie algebra. Since G is a closed subgroup of P, it is elementary that $\mathfrak{L}(G)$ is closed in $\mathfrak{L}(P)$, both topologically and in the sense of Lie algebras.

Definition 3. *A pro-Lie algebra* $\mathfrak{g}$ *is a topological real Lie algebra isomorphic to a closed subalgebra of a product of finite-dimensional Lie algebras.*

Clearly, every pro-Lie algebra has a weakly-complete topological vector space as the underlying topological vector space. In complete analogy to Theorem 4, we have the following characterization [29]:

Theorem 9. ([5], p. 138ff) *For a topological Lie algebra* $\mathfrak{g}$*, the following conditions are equivalent:*

(1) $\mathfrak{g}$ *is complete, and every neighborhood of zero contains a closed ideal* $\mathfrak{i}$*, such that the Lie algebra* $\mathfrak{g}/\mathfrak{i}$ *is finite-dimensional.*
(2) $\mathfrak{g}$ *is a projective limit of finite-dimensional Lie algebras.*
(3) $\mathfrak{g}$ *is a pro-Lie algebra.*

One notes that our procedure of identifying $\mathfrak{L}(G) = \mathrm{Hom}(\mathbb{R}, G)$ for a pro-Lie group G with the structure of a pro-Lie algebra yields an exponential function $\exp\colon \mathfrak{L}(G) \to G$ by $\exp X = X(1)$ for $X\colon \mathbb{R} \to G$ in $\mathfrak{L}(G)$. The implementation of this setup is secured in [5], summarized in the following:

Theorem 10. *To each pro-Lie group, there is uniquely and functorially assigned a pro-Lie algebra* $\mathfrak{L}(G)$ *together with an exponential function* $\exp_G\colon \mathfrak{L}(G) \to G$*, such that every one-parameter subgroup of* G *is of the form* $t \mapsto \exp t.X : \mathfrak{L}(G) \to G$ *with a unique element* $X \in \mathfrak{L}(G)$ *and that the subgroup* $\langle \exp \mathfrak{L}(G)\rangle$ *generated by all one parameter subgroups is dense in the identity component* G_0 *of the identity in* G*.*

To each pro-Lie algebra $\mathfrak{g}$ *there is a uniquely- and functorially-assigned connected pro-Lie group* $\Gamma(\mathfrak{g})$ *with Lie algebra* $\mathfrak{g}$*, and for each pro-Lie group* G *with Lie algebra* $\mathfrak{L}(G)$ *permitting an isomorphism* $f\colon \mathfrak{g} \to \mathfrak{L}(G)$ *of pro-Lie algebras, there is a unique isomorphism of pro-Lie groups* $f_\Gamma : \Gamma(\mathfrak{g}) \to G$*, such that the following diagram is commutative with* $\exp$ *denoting the exponential function of* $\Gamma(\mathfrak{g})$*:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\exp} & \Gamma(\mathfrak{g}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_\Gamma} \\ \mathfrak{L}(G) & \xrightarrow[\exp_G]{} & G \end{array}$$

A pro-Lie group G *is prodiscrete if and only if it is totally disconnected if and only if* $\mathfrak{L}(G) = \{0\}$*. Further,* $\Gamma(\mathfrak{g})$ *is always simply connected.*

Sophus Lie's third theorem applies and is perfectly coded into the existence of the functor Γ. The exactness properties of the functor $\mathfrak{L}$ are well understood (see [5], Theorem 4.20, p. 188). Structural results, such as we discussed at the beginning of our survey, are all based on a thorough application of the Lie theory of pro-Lie groups. Since $\mathfrak{L}(G) = \mathfrak{L}(G_0)$ from the very definition of $\mathfrak{L}(G)$, in the strictest sense it applies to connected pro-Lie groups, we saw that the essential facts reach out to almost connected pro-Lie groups.

Of course, since every locally-compact group has an open subgroup that is an almost connected pro-Lie group, this Lie theory applies to all locally-compact groups.

In classical Lie theory, Lie algebras are more directly amenable to structural analysis than Lie groups, as they are purely algebraic. While pro-Lie algebras are both topological and algebraic, they are nevertheless more easily analyzed than pro-Lie groups, as well.

Let us look as some characteristic features of pro-Lie algebras.

Definition 4. *A pro-Lie algebra $\mathfrak{g}$ is called:*

(i) reductive if every closed ideal is algebraically and topologically a direct summand,
(ii) prosolvable if every finite-dimensional quotient algebra is solvable,
(iii) pronilpotent if every finite-dimensional quotient algebra is nilpotent.

The center of $\mathfrak{g}$ is denoted $\mathfrak{z}(\mathfrak{g})$. A pro-Lie algebra is called semisimple if it is reductive and satisfies $\mathfrak{z}(\mathfrak{g}) = \{0\}$.

Theorem 11. ([5] Theorem 7.27, p. 283) *A reductive pro-Lie algebra $\mathfrak{g}$ is a product of finite-dimensional ideals, each of which is either simple or else is isomorphic to $\mathbb{R}$.*

The algebraic commutator algebra $[\mathfrak{g}, \mathfrak{g}]$ is closed and a product of simple ideals. Furthermore,

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}],$$

algebraically and topologically.

If a maximal prosolvable (respectively, pronilpotent) ideal of a pro-Lie algebra $\mathfrak{g}$ exists, it is called the (solvable) radical $\mathfrak{r}(\mathfrak{g})$ (respectively, the nilradical $\mathfrak{n}(\mathfrak{g})$). If there is a smallest closed ideal $\mathfrak{i}$ of $\mathfrak{g}$, such that $\mathfrak{g}/\mathfrak{i}$ is reductive, then we call it the co-reductive radical $\mathfrak{n}_{\text{cored}}(g)$ of $\mathfrak{g}$.

Theorem 12. ([5], Chapter 7) *Every pro-Lie algebra has a radical, a nilradical and a co-reductive radical, and:*

$$\mathfrak{n}_{\text{cored}}(g) \subseteq \mathfrak{n}(\mathfrak{g}) \subseteq \mathfrak{r}(\mathfrak{g}).$$

Moreover,

$$\mathfrak{n}_{\text{cored}}(\mathfrak{g}) = \overline{[\mathfrak{g}, \mathfrak{g}]} \cap \mathfrak{r}(\mathfrak{g}) = \overline{[\mathfrak{g}, \mathfrak{r}(\mathfrak{g})]}.$$

There is a closed semisimple subalgebra $\mathfrak{s}$, such that $\mathfrak{g} = \mathfrak{r}(\mathfrak{g}) \oplus \mathfrak{s}$, where $(r, s) \mapsto r + s : \mathfrak{r}(\mathfrak{g}) \times \mathfrak{s} \to \mathfrak{g}$ is an isomorphism of weakly-complete topological vector spaces (Levi–Mal'cev decomposition). Moreover,

$$\overline{[\mathfrak{g}, \mathfrak{g}]} = \mathfrak{n}_{\text{cored}}(\mathfrak{g}) \oplus \mathfrak{s}.$$

All of this fine structure can be translated to the group level with due circumspection and ensuing complications. One can get an idea of this translation from the process of Lie's third theorem. Among other things, Theorem 10 yields for each of the pro-Lie algebras $\mathfrak{g}$ a pro-Lie group $\Gamma(\mathfrak{g})$ with an exponential function $\exp\colon \mathfrak{g} \to \Gamma(\mathfrak{g})$.

If $\mathfrak{g}$ is pronilpotent, then $\exp$ is a homeomorphism. In fact, the Baker–Campbell–Hausdorff series is summable on the weakly-complete topological vector space $\mathfrak{g}$ yielding a binary operation $\star$, so that for $\Gamma(\mathfrak{g})$, we may take $(\mathfrak{g}, \star)$ and for $\exp$ the identity map. This applies, in particular, to the co-reductive radical $\mathfrak{n}_{\text{cored}}(\mathfrak{g})$, for which $\mathfrak{g}/\mathfrak{n}_{\text{cored}}(\mathfrak{g})$ is reductive.

For reductive $\mathfrak{g}$, however, the product structure of $\mathfrak{g}$ expressed in Theorem 10 carries over to a clean product structure:

$$\Gamma(\mathfrak{g}) \cong \mathbb{R}^I \times \prod_{j \in J} S_j$$

for a family of simply-connected simple real Lie groups S_j, producing, in fact, a simply-connected reductive group $\Gamma(\mathfrak{g})$.

These observations show again how far connected pro-Lie groups reach outside the domain of locally-compact connected groups while their structure remains close to that which is familiar from finite-dimensional Lie groups due to a fairly lucid topological-algebraic structure of pro-Lie algebras. We note that for every connected pro-Lie group G with $\mathfrak{L}(G) \cong \mathfrak{g}$, one has a morphism $f : \Gamma(\mathfrak{g}) \to G$ with a prodiscrete kernel and a dense image, such that the following diagram is commutative:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\exp_{\Gamma(\mathfrak{g})}"] \arrow[d, "\cong"'] & \Gamma(\mathfrak{g}) \arrow[d, "f"] \\
\mathfrak{L}(G) \arrow[r, "\exp_G"'] & G
\end{tikzcd}$$

In [5], it is demonstrated that this tool allows a structural analysis of G.

For instance, the existence of the various radicals of a pro-Lie algebra has its correspondence in respective radicals in any connected pro-Lie group. For example, every connected pro-Lie group G has a unique largest normal connected solvable subgroup $\mathbf{R}(G)$, called its (solvable) radical. If G is any topological group whose identity component G_0 is a pro-Lie group, then one writes $\mathbf{R}(G) = \mathbf{R}(G_0)$ and also calls $\mathbf{R}(G)$ the radical of G.

The volume of additional details of the theory of pro-Lie algebras and connected pro-Lie groups presented in [5] is immense. It cannot be expected that a survey such as this can do complete justice to it.

7. Later Developments

In this section, we report on some developments in the theory of pro-Lie groups since the appearance of [5]. Of course, we have already included some important material, which appeared subsequent to [5], namely [7,13].

The article [7] contributed the insight that some essential structure theorems on connected pro-Lie groups could be formulated so as to include almost connected pro-Lie groups. This provided a common generalization of the structure theories of connected pro-Lie groups and compact groups. This generalization is both significant and satisfying. In this survey, this was illustrated by Theorem 1 and its corollaries and Theorems 2 and 3.

Hofmann-Morris [7] contains another interesting result, which we think has yet to be exploited in the literature.

Theorem 13. *Let G be an arbitrary topological group whose identity component G_0 is a pro-Lie group. Then, there is a closed subgroup G_1 whose identity component is the radical $\mathbf{R}(G)$, such that the following conditions hold:*

(i) *$G = G_0G_1$ and $G_0 \cap G_1 = \mathbf{R}(G)$.*
(ii) *The factor group $G/\mathbf{R}(G)$ is the semidirect product of the connected normal subgroup $G_0/\mathbf{R}(G)$ and the totally disconnected closed subgroup $G_1/\mathbf{R}(G)$.*
(iii) *In particular,*
$$\frac{G}{G_0} \cong \frac{G/\mathbf{R}(G)}{G_0/\mathbf{R}(G)} = \frac{G/\mathbf{R}(G_0)}{G_0/\mathbf{R}(G_0)} \cong \frac{G_1}{(G_1)_0}$$
with a prosolvable pro-Lie group $(G_1)_0$.

The significance of Theorem 13 emerges even when it is specialized to the case that G is a pro-Lie group. As was emphasized by formulating Question 2, we do not know whether the component factor group G/G_0 is complete and, therefore, is a prodiscrete group. Theorem 13 reduces the problem to the case that the identity component of G is prosolvable. For instance, we obtain a positive answer to Question 2 if we know that the radical $\mathbf{R}(G)$ is locally compact or first countable (see Theorem 5 above).

In the process of extending the structure theory of pro-Lie groups from connected ones to almost connected ones, G. Michael, A.A. has proven the following structure theorem guaranteeing a local splitting, provided the nilradical is not too big on the Lie algebra side.

Theorem 14. ([8,30]) *Assume that G is an almost connected pro-Lie group G whose Lie algebra $\mathfrak{g}$ has a finite-dimensional co-reductive radical $\mathfrak{n}_{\text{cored}}(\mathfrak{g})$. Then, there are arbitrarily small closed normal subgroups N, such that there exists a simply-connected Lie group L_N and a morphism $\alpha\colon L_N \to G$, such that the morphism:*
$$(n, x) \mapsto n\alpha(x) : N \times L_N \to G$$
is open and has a discrete kernel. In particular, G and $N \times L_N$ are locally isomorphic.

Let us recall Iwasawa's local splitting theorem for locally-compact groups as it is reported in [9], Theorem 10.89.

Theorem 15. *Let G be a locally-compact group. Then, there exists an open almost connected subgroup A, such that for each identity neighborhood U, there is:*

— *a compact normal subgroup N of A contained in U,*
— *a simply-connected Lie group L, and*
— *an open and continuous surjective morphism $\phi\colon N \times L \to A$ with a discrete kernel, such that $\phi(n, 1) = n$ for all $n \in N$.*

The way to compare the two preceding theorems is to look at the Lie algebra $\mathfrak{g}$ of G in Theorem 15. We notice that $\mathfrak{g} = \mathfrak{L}(N) \oplus \mathfrak{l}$, $\mathfrak{l} = \mathfrak{L}(L)$. As the Lie algebra of a compact group N of the first direct summand is of the form $\mathfrak{L}(N) = \mathfrak{c} \oplus \mathfrak{s}_1$ with a central ideal $\mathfrak{c}$ and a compact semisimple ideal $\mathfrak{s}_1$, the finite-dimensional Lie algebra $\mathfrak{l}$ has a Levi–Mal'cev decomposition $\mathfrak{r} \oplus \mathfrak{s}_2$ with its radical $\mathfrak{r}$ and a finite-dimensional semisimple subalgebra $\mathfrak{s}_2$, so that we have:
$$\mathfrak{g} = (\mathfrak{c} \oplus \mathfrak{s}_1) \oplus (\mathfrak{r} +_{\text{sdir}} \mathfrak{s}_2) = (\mathfrak{r} \oplus \mathfrak{c}) +_{\text{sdir}} (\mathfrak{s}_1 + \mathfrak{s}_2).$$

We observe that the radical $\mathfrak{r}(\mathfrak{g})$ is $\mathfrak{r} \oplus \mathfrak{c}$ and that the co-reductive radical $\mathfrak{n}_{\text{cored}}(\mathfrak{g})$ of $\mathfrak{g}$ are contained in the finite-dimensional subalgebra $\mathfrak{r}$ and are, therefore, finite-dimensional. The hypothesis that was imposed in Theorem 14, namely that the Lie algebra dimension of the co-reductive radical is finite, thus emerges as a necessary condition in the more classical Iwasawa local decomposition theorem of locally-compact groups. Additional comments on the local decomposition of pro-Lie groups may be found in [31].

8. A Natural Occurrence of Pro-Lie Groups and Pro-Lie Algebras

We emphasized that weakly-complete topological vector spaces play an essential role in the theory of pro-Lie groups and pro-Lie algebras. Now, we record that they are crucial in describing a mathematical environment where they occur naturally; this was pointed out recently in [14]. Each of the categories of real vector spaces and of their dual category of weakly-complete topological vector spaces (see Section 3, Definition 3.2ff) in fact has a tensor product (see [14], Appendix C, notably C4, and [32]), making each of them a commutative monoidal category (see, e.g., [9], Appendix 3, Definition A3.62 ff). Let us denote by $\mathcal{V}$ the category of (real) vector spaces E, $F\dots$, and of linear maps, equipped with the usual tensor product $E \otimes F$, and let us call $\mathcal{W}$ the category of weakly-complete vector spaces V, $W, \dots$, *etc*., with continuous linear maps, equipped with the (completed) tensor product $V\widetilde{\otimes}W$. Then, a monoid in $\mathcal{W}$ (see [9] Appendix 3, the discussion preceding Definition A3.64) is a weakly-complete topological (associative) algebra with identity, specifically, a morphism $m\colon A\widetilde{\otimes}A \to A$ plus a morphism $\mathbb{R} \to A$ representing the identity (see also [32]). Its dual $E \stackrel{\text{def}}{=} A' = \text{Hom}(A, \mathbb{R})$ then is an (associative unital) coalgebra $m'\colon E \to E \otimes E$ with a coidentity (augmentation) $u\colon E \to \mathbb{R}$. The theory of coalgebras (see [33]) culminates in one theorem holding without any further hypotheses:

Theorem 16. (The fundamental theorem of coalgebras [33], 4.12, p. 742) *Each associative unitary coalgebra is the directed union of its finite-dimensional unitary sub-coalgebras.*

By duality this implies at once Rafael Dahmen's fundamental theorem of weakly-complete algebras.

Theorem 17. (Fundamental theorem of weakly-complete algebras [14]) *For every weakly-complete associative unital algebra A, there is a projective system of surjective linear morphisms $f_{jk}\colon A_k \to A_j$, $j \le k$, $j, k \in J$ of finite-dimensional associative unital algebras and a natural isomorphism $\phi_A\colon A \to \lim_{j\in J} A_j$ onto the projective limit of this system.*

Conversely, by definition, every projective limit of finite-dimensional real unital associative algebras is a weakly-complete associative unital algebra.

Every associative algebra A becomes a Lie algebra A_{Lie} when it is equipped with the commutator bracket $(x, y) \mapsto [x, y] \stackrel{\text{def}}{=} xy - yx$. Each of the Lie algebras $(A_j)_{\text{Lie}}$ with $j \in J$ is finite-dimensional. From Theorem 17 and the Definition 3 plus Theorem 9, we thus have:

Corollary 5. *For every weakly-complete associative unital algebra A, the Lie algebra A_{Lie} is a pro-Lie algebra, and $\phi_A\colon A_{\text{Lie}} \to \lim_{j\in J}(A_j)_{\text{Lie}}$ is an isomorphism of pro-Lie algebras.*

Each of the morphisms f_{jk} maps the group A_k^{-1} of units, that is invertible elements, into the group of units A_j^{-1}, and so, we obtain a natural isomorphism:

$$\phi_A|A^{-1} : A^{-1} \to \lim_{j\in J} A_j^{-1}.$$

Now, every A_j^{-1} is a (linear) Lie group (see [5], Chapter 5, 5.1–5.32) with exponential function $\exp_{A_j} : (A_j)_{\mathrm{Lie}} \to A_j^{-1}$. By Theorem 4, as a consequence, we have Dahmen's corollary.

Corollary 6. ([14], Proposition 5.4) *The group A^{-1} of units of every weakly-complete associative unital algebra A is a pro-Lie group with Lie algebra A_{Lie} and exponential function:*

$$\exp_A : A_{\mathrm{Lie}} \to A^{-1}, \quad \exp(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots .$$

What we have exposed here is the basis of a theory that applies to group objects in the commutative monoidal category $\mathcal{W}$ as defined in [9], Definition A3.64(ii). These objects are commonly called Hopf algebras, and so, we shall fix the following definition.

Definition 5. *A weakly-complete Hopf algebra is a group object in $\mathcal{W}$ according to Definition A3.64(ii) of [9].*

In particular, the multiplicative structure of a weakly-complete Hopf algebra A is a weakly-complete associative unital algebra. It also has a comultiplication $c\colon A \to A\widetilde{\otimes}A$ linked with the multiplication $m\colon A\widetilde{\otimes}A \to A$ through several commutative diagrams, for which we refer to [9], A3.63(ii), and which express the fact that c is indeed a morphism of algebras.

Definition 6. *Let A be a weakly-complete Hopf algebra with a comultiplication c. An element $x \in A$ is called group-like if it satisfies $c(x) = x\widetilde{\otimes}x$ and $u(x) = 1$, and it is called primitive if it satisfies $c(x) = x\widetilde{\otimes}1 + 1\widetilde{\otimes}x$. The set of group-like (respectively, primitive) elements will be denoted by $G(A)$ (respectively, $P(A)$).*

One shows the following fact:

Theorem 18. *$G(A)$ is a closed subgroup of A^{-1}, the group of units of the underlying algebra, and $P(A)$ is a closed Lie subalgebra of A_{Lie}.*

The link to the previous remarks is provided by the following theorem:

Theorem 19. *Let A be a weakly-complete Hopf algebra. Then, the set of group-like elements $G(A)$ is a pro-Lie group, and the set of primitive elements $P(A)$ is a pro-Lie algebra and is the Lie algebra of $G(A)$ with the exponential function:*

$$\exp_A |P(A) : P(A) \to G(A)$$

with the restriction of the exponential function of Corollary 5.

A proof is based on the fact that for the algebra morphism c, we have $\exp_A \circ c = c \circ (\exp_A \widetilde{\otimes} \exp_A)$. If x is primitive, then $c(x) = (x\widetilde{\otimes}1) + (1\widetilde{\otimes}x)$, and thus, $c(\exp x) = \exp(c(x)) = \exp((x\widetilde{\otimes}1) + (1\widetilde{\otimes}x)) = (\exp x\widetilde{\otimes}1)(1\widetilde{\otimes}\exp x) = \exp x\widetilde{\otimes}\exp x$. Therefore, $\exp$ maps $P(A)$ into $G(A)$. To see the converse, let $t \mapsto \exp t{\cdot}x\colon \mathbb{R} \to G(A)$ be a one-parameter subgroup. Then, $\exp t{\cdot}x$ is group-like for all t, *i.e.*,

$$\begin{aligned}
\exp t{\cdot}c(x) &= \exp c(t{\cdot}x) = c(\exp t{\cdot}x) = (\exp t{\cdot}x)\widetilde{\otimes}(\exp t{\cdot}x)\\
&= \big((\exp t{\cdot}x)\widetilde{\otimes}1\big)\big(1\widetilde{\otimes}(\exp t{\cdot}x)\big)\\
&= \exp(t{\cdot}x\widetilde{\otimes}1)\exp(1\widetilde{\otimes}t{\cdot}x) = \exp((t{\cdot}x\widetilde{\otimes}1) + (1\widetilde{\otimes}t{\cdot}x))\\
&= \exp t{\cdot}(x\widetilde{\otimes}1 + 1\widetilde{\otimes}x), \quad \text{for all } t \in \mathbb{R},
\end{aligned}$$

and this implies:

$$c(x) = x\widetilde{\otimes}1 + 1\widetilde{\otimes}x$$

which means $x \in P(A)$.

If the weakly-complete Hopf algebra A arises as the dual of an (abstract) Hopf algebra H (*i.e.*, a group object in $\mathcal{V}$), then the members of $G(A)$ are multiplicative linear functionals on H, the so-called characters of H. Thus, Theorem 19 may be interpreted as saying that the character group of a Hopf algebra is a pro-Lie group (see also Theorem 5.6 in [14]).

The simplest example is $A = \mathbb{R}[[x]]$, the algebra of formal power series in one variable. As a vector space, A is isomorphic to $\mathbb{R}^{\{1,x,x^2,\ldots\}}$, and this is weakly complete. Then, $A\widetilde{\otimes}A$ is isomorphic to $\mathbb{R}[[y,z]]$, the formal power series algebra in two commuting variables y and z. The algebra morphism $\mathbb{R}[[x]] \to \mathbb{R}[[y,z]]$ generated by $x \mapsto y+z$ gives a comultiplication $c\colon A \to A\widetilde{\otimes}A$ making A into a weakly-complete Hopf algebra. The multiplicative subgroup $G(A) = \{\exp t{\cdot}x : t \in \mathbb{R}\}$ is a Lie group isomorphic to $(\mathbb{R},+)$, and $P(A) \cong \mathbb{R}{\cdot}x$ is (trivially) a Lie algebra mapped by $\exp$ onto $G(A)$.

Question 6: (i) Is there a more elaborate duality theory of real Hopf algebras and weakly-complete Hopf algebras in which these facts on pro-Lie group and pro-Lie algebra theory play a role?
(ii) Does the existence of weakly-complete enveloping algebras of weakly-complete Lie algebras secure for each pro-Lie algebra L, an associative weakly-complete Hopf Algebra $U(L)$, such that L is isomorphic to a closed Lie subalgebra of $P(U(L))$?

9. Further Open Questions

In this last section, we record a few additional questions on pro-Lie groups that do not fit naturally with the material previously discussed in this paper, but that are of some significance to pro-Lie group theory. In so doing, we rely on definitions and concepts defined and discussed in [5].

Question 7: Is an Abelian prodiscrete compactly-generated group without nondegenerate compact subgroups a discrete group?

For a a definition and discussion of compactly-generated groups, see [5], Definition 5.6, pp. 218ff. For a discussion of Abelian compactly generated pro-Lie groups, see [5], Theorem 5.32, p. 236.

Question 8: Is a compactly-generated Abelian prodiscrete compact-free group a finitely-generated-free Abelian group?

Note that by compact free we mean the group has no nontrivial compact subgroups.

See [5], Theorem 5.32, the Compact Generation Theorem for Abelian Pro-Lie Groups, p. 236.

In the proof of the structure of reductive pro-Lie algebras in Theorem 11 ([5], Theorem 7.27), one uses the lemma that in every finite-dimensional real semisimple Lie algebra every element is a sum of at most two Lie brackets. For brackets in semisimple Lie algebras, see [5], Appendix 3, p. 651ff.

Question 9: Is every element in an arbitrary real semisimple Lie algebra a bracket?

For the concept of transfinitely-solvable pro-Lie algebras and pro-Lie groups, see [5], Definition 7.32, pp. 285ff, respectively pp. 420ff. For the concept of transfinitely-nilpotent pro-Lie algebras and pro-Lie groups, see [5], pp. 296ff, respectively, pp. 443 ff.

Question 10: Is a transfinitely-nilpotent connected pro-Lie group pronilpotent?

In Question 10, such a group has to be prosolvable, since it is transfinitely solvable, and then, the Equivalence Theorem for Solvability of Connected Pro-Lie Groups 10.18 of [5] applies. The impediment to a proof is the failure of transfinite nilpotency to be preserved by passing to quotients. Free topological groups are free groups in the algebraic sense and, thus, are countably nilpotent; but, every topological group is a quotient of a free topological group and, thus, of a transfinitely-countably nilpotent topological group.

For the definition of an analytic subgroup, see [5], Definition 9.5 on p. 360.

Question 11: Let $\mathfrak{h}$ be a closed subalgebra of the Lie algebra $\mathfrak{g}$ of a connected pro-Lie group G. Let $A(\mathfrak{h})$ denote the analytic group generated by $\mathfrak{h}$. Is $\overline{A(\mathfrak{h})}/A(\mathfrak{h})$ Abelian?

(See [5], Theorem 9.32.)

Question 12: Is there a satisfactory theory of Polish pro-Lie groups (respectively, separable pro-Lie groups, or compactly-generated pro-Lie groups), notably in the connected case?

For information in [5] on the Abelian case, see pp. 235ff.

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Conflicts of Interest

The authors declare no conflict of interest.

References

1. Tao, T. *Hilbert's Fifth Problem and Related Topics*; Amer. Math. Soc.: Providence, RI, USA, 2014.

2. Hofmann, K.H.; Morris, S.A. On a category of topological groups suitable for a structure theory of locally compact groups. *Topology Proc.* **2001–2002**, *26*, 651–665.
3. Hofmann, K.H.; Morris, S.A. Projective limits of finite dimensional Lie groups. *Proc. London Math. Soc.* **2003**, *87*, 647–676.
4. Hofmann, K.H.; Morris, S.A. Lie theory and the structure of pro-Lie groups and Pro-Lie Algebras. *Topology Proceedings* **2004**, *28*, 541–567.
5. Hofmann, K.H.; Morris, S.A. *The Lie Theory of Connected Pro-Lie Groups*; European Mathematical Society: Zürich, Switzerland, 2007.
6. Hofmann, K.H.; Morris, S.A. On the Pro-Lie Group Theorem and the Closed Subgroup Theorem. *J. Lie Theory* **2008**, *18*, 383–390.
7. Hofmann, K.H.; Morris, S.A. The structure of almost connected pro-Lie groups. *J. Lie Theory* **2011**, *21*, 347–383.
8. Hofmann, K.H.; Morris, S.A. Local Splitting of Locally Compact Groups and Pro-Lie Groups. *J. Group Theory* **2011**, *14*, 931–935.
9. Hofmann, K.H.; Morris, S.A. *The Structure of Compact Groups*, 3rd ed., Revised and Augmented; Verlag Walter De Gruyter Berlin: Berlin, Germany, 2013.
10. Glöckner, H.; Neeb, K.-H. Infinite Dimensional Lie Groups, I, Basic Theory and Main Examples. 2015, in preparation.
11. Glöckner, H.; Neeb, K.-H. Infinite Dimensional Lie Groups, II, Geometry and Topology. 2015, in preparation.
12. Neeb, K.-H. Towards a Lie theory of locally convex group. *Japanese J. Math.* **2006**, *1*, 291–468.
13. Hofmann, K.H.; Neeb, K.-H. Pro-Lie groups which are infinite dimensional Lie groups. *Math. Proc. Cambridge Philos. Soc.* **2008**, *46*, 351–378.
14. Bogfjellmo, G.; Dahmen, R.; Schmeding, A. Character groups of Hopf algebras as infinite-dimensional Lie groups. arXiv:1501.05221v2 [math.GR], 2015.
15. Montgomery, D.; Zippin, L. *Topological Transformation Groups*; Interscience Publishers: New York, NY, USA, 1955.
16. Hofmann, K.H.; Morris, S.A. Contributions to the structure theory of connected pro-Lie groups. *Topology Proc.* **2008**, *33*, 225–237.
17. Oxtoby, J.C. Cartesian products of Baire spaces. *Fundamenta Math.* **1961**, *49*, 157–166.
18. Glöckner, H. Simplified Proofs for the Pro-Lie Group Theorem and the One-Parameter Subgroup Lifting Lemma. *J. Lie Theory* **2007**, *17*, 899–902.
19. Michael, G.A.A. On Inverse Limits of Finite-Dimensional Lie Groups. *J. Lie Theory* **2006**, *16*, 221–224.
20. Borel, A. Limites projectives de groupes de Lie. *C.R. Acad. Sci. Paris* **1950**, *230*, 1127–1128.
21. Ribes, L.; Zalesskii, P. *Profinite Groups*; Springer-Verlag Berlin: Heidelberg, Germany, 2000.
22. Wilson, J.S. *Profinite Group*; Oxford Science Publ.: Oxford, UK, 1998.
23. Roelcke, W.; Dierolf, S. *Uniform Structures on Topological Groups and their Quotients*; McGraw-Hill: New York, NY, USA, 1981.
24. Schaefer, H.; Wolff, M. *Topological Vector Spaces*; Springer: New York, NY, USA, 1999.

25. Banaszczyk, W. *Additive Subgroups of Topological Vector Spaces*; Springer-Verlag: Berlin, Germany, 1991.
26. Außenhöfer, L. The Lie Algebra of a Nuclear Group. *J. Lie Theory* **2003**, *13*, 263–270.
27. Hofmann, K.H.; Morris, S.A. An Open Mapping Theorem for pro-Lie Groups. *J. Austral. Math. Soc.* **2007**, *83*, 55–77.
28. Hofmann, K.H.; Morris, S.A. Open Mapping Theorems for Topological Groups. *Topology Proc.* **2007**, *31*, 533–551.
29. Hofmann, K.H.; Morris, S.A. Sophus Lie's Third Fundamental Theorem and the Adjoint Functor Theorem. *J. Group Theory* **2005**, *8*, 115–133.
30. Michael, G.A.A. The generalized Gluškov-Iwasawa local splitting theorem. *J. Math. Sci. Univ. Tokyo* **2010**, *17*, 123–133.
31. Hofmann, K.H.; Morris, S.A. Iwasaswa's local splitting theorem for Pro-Lie groups. *Forum Mathematicum* **2008**, *20*, 607–629.
32. Dahmen, R. Tensor Products of Weakly Complete Vector Spaces. *Diplomarbeit am FB. Math. der Technischen Universität Darmstad* **2007**, *230*, 66.
33. Michaelis, W. *Coassociative Coalgebras*; In *Handbook of Algebra*; Elsevier: Amsterdam, The Netherlands, 2003; pp. 587–788.