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Article

On T-Characterized Subgroups of Compact Abelian Groups

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Abstract: A sequence $\{u_n\}_{n\in\omega}$ in abstract additively-written Abelian group G is called a T-sequence if there is a Hausdorff group topology on G relative to which $\lim_n u_n = 0$. We say that a subgroup H of an infinite compact Abelian group X is T-characterized if there is a T-sequence $\mathbf{u} = \{u_n\}$ in the dual group of X, such that $H = \{x \in X : (u_n, x) \to 1\}$. We show that a closed subgroup H of X is T-characterized if and only if H is a G_δ-subgroup of X and the annihilator of H admits a Hausdorff minimally almost periodic group topology. All closed subgroups of an infinite compact Abelian group X are T-characterized if and only if X is metrizable and connected. We prove that every compact Abelian group X of infinite exponent has a T-characterized subgroup, which is not an F_σ-subgroup of X, that gives a negative answer to Problem 3.3 in Dikranjan and Gabriyelya (*Topol. Appl.* **2013**, *160*, 2427–2442).

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1. Introduction

Notation and preliminaries: Let X be an Abelian topological group. We denote by $\widehat{X}$ the group of all continuous characters on X, and $\widehat{X}$ endowed with the compact-open topology is denoted by $X^\wedge$. The homomorphism $\alpha_X : X \to X^{\wedge\wedge}$, $x \mapsto (\chi \mapsto (\chi, x))$, is called the canonical homomorphism. Denote by $\mathbf{n}(X) = \cap_{\chi\in\widehat{X}}\ker(\chi) = \ker(\alpha_X)$ the von Neumann radical of X. The group X is called minimally almost periodic ($MinAP$) if $\mathbf{n}(X) = X$, and X is called maximally almost periodic

(MAP) if $\mathbf{n}(X) = \{0\}$. Let H be a subgroup of X. The annihilator of H we denote by $H^{\perp}$, *i.e.*, $H^{\perp} = \{\chi \in X^{\wedge} : (\chi, h) = 1 \text{ for every } h \in H\}$.

Recall that an Abelian group G is of finite exponent or bounded if there exists a positive integer n, such that $ng = 0$ for every $g \in G$. The minimal integer n with this property is called the exponent of G and is denoted by $\exp(G)$. When G is not bounded, we write $\exp(G) = \infty$ and say that G is of infinite exponent or unbounded. The direct sum of ω copies of an Abelian group G we denote by $G^{(\omega)}$.

Let $\mathbf{u} = \{u_n\}_{n\in\omega}$ be a sequence in an Abelian group G. In general, no Hausdroff topology may exist in which $\mathbf{u}$ converges to zero. A very important question of whether there exists a Hausdorff group topology τ on G, such that $u_n \to 0$ in (G, τ), especially for the integers, has been studied by many authors; see Graev [1], Nienhuys [2], and others. Protasov and Zelenyuk [3] obtained a criterion that gives a complete answer to this question. Following [3], we say that a sequence $\mathbf{u} = \{u_n\}$ in an Abelian group G is a T-sequence if there is a Hausdorff group topology on G in which u_n converges to zero. The finest group topology with this property we denote by $\tau_{\mathbf{u}}$.

The counterpart of the above question for precompact group topologies on $\mathbb{Z}$ is studied by Raczkowski [4]. Following [5,6] and motivated by [4], we say that a sequence $\mathbf{u} = \{u_n\}$ is a TB-sequence in an Abelian group G if there is a precompact Hausdorff group topology on G in which u_n converges to zero. For a TB-sequence $\mathbf{u}$, we denote by $\tau_{b\mathbf{u}}$ the finest precompact group topology on G in which $\mathbf{u}$ converges to zero. Clearly, every TB-sequence is a T-sequence, but in general, the converse assertion does not hold.

While it is quite hard to check whether a given sequence is a T-sequence (see, for example, [3,7–10]), the case of TB-sequences is much simpler. Let X be an Abelian topological group and $\mathbf{u} = \{u_n\}$ be a sequence in its dual group $X^{\wedge}$. Following [11], set:

$$s_{\mathbf{u}}(X) = \{x \in X : \ (u_n, x) \to 1\}.$$

In [5], the following simple criterion to be a TB-sequence was obtained:

Fact 1 ([5]). *A sequence* $\mathbf{u}$ *in a (discrete) Abelian group* G *is a* TB*-sequence if and only if the subgroup* $s_{\mathbf{u}}(X)$ *of the (compact) dual* $X = G^{\wedge}$ *is dense.*

Motivated by Fact 1, Dikranjan *et al.* [11] introduced the following notion related to subgroups of the form $s_{\mathbf{u}}(X)$ of a compact Abelian group X:

Definition 2 ([11]). Let H be a subgroup of a compact Abelian group X and $\mathbf{u} = \{u_n\}$ be a sequence in $\widehat{X}$. If $H = s_{\mathbf{u}}(X)$, we say that $\mathbf{u}$ characterizes H and that H is characterized (by $\mathbf{u}$).

Note that for the torus $\mathbb{T}$, this notion was already defined in [12]. Characterized subgroups have been studied by many authors; see, for example, [11–16]. In particular, the main theorem of [15] (see also [14]) asserts that every countable subgroup of a compact metrizable Abelian group is characterized. It is natural to ask whether a closed subgroup of a compact Abelian group is characterized. The following easy criterion is given in [13]:

Fact 3 ([13]). *A closed subgroup* H *of a compact Abelian group* X *is characterized if and only if* H *is a* G_δ*-subgroup. In particular,* X/H *is metrizable, and the annihilator* $H^{\perp}$ *of* H *is countable.*

The next fact follows easily from Definition 2:

Fact 4 ([17], see also [13])**.** *Every characterized subgroup H of a compact Abelian group X is an $F_{\sigma\delta}$-subgroup of X, and hence, H is a Borel subset of X.*

Facts 3 and 4 inspired in [13] the study of the Borel hierarchy of characterized subgroups of compact Abelian groups. For a compact Abelian group X, denote by $\mathrm{Char}(X)$ (respectively, $\mathrm{SF}_\sigma(X)$, $\mathrm{SF}_{\sigma\delta}(X)$ and $\mathrm{SG}_\delta(X)$) the set of all characterized subgroups (respectively, F_σ-subgroups, $F_{\sigma\delta}$-subgroups and G_δ-subgroups) of X. The next fact is Theorem E in [13]:

Fact 5 ([13])**.** *For every infinite compact Abelian group X, the following inclusions hold:*

$$\mathrm{SG}_\delta(X) \subsetneqq \mathrm{Char}(X) \subsetneqq \mathrm{SF}_{\sigma\delta}(X) \quad \textit{and} \quad \mathrm{SF}_\sigma(X) \nsubseteq \mathrm{Char}(X).$$

If in addition X has finite exponent, then:

$$\mathrm{Char}(X) \subsetneqq \mathrm{SF}_\sigma(X). \tag{1}$$

The inclusion Equation (1) inspired the following question:

Question 6 (Problem 3.3 in [13])**.** *Does there exist a compact Abelian group X of infinite exponent all of whose characterized subgroups are F_σ-subsets of X?*

Main results: It is important to emphasize that there is no restriction on the sequence $\mathbf{u}$ in Definition 2. If a characterized subgroup H of a compact Abelian group X is dense, then, by Fact 1, a characterizing sequence is also a TB-sequence. However, if H is not dense, we cannot expect in general that a characterizing sequence of H is a T-sequence. Thus, it is natural to ask:

Question 7. *For which characterized subgroups of compact Abelian groups can one find characterizing sequences that are also T-sequences?*

This question is of independent interest, because every T-sequence $\mathbf{u}$ naturally defines the group topology $\tau_{\mathbf{u}}$ satisfying the following dual property:

Fact 8 ([18])**.** *Let H be a subgroup of an infinite compact Abelian group X characterized by a T-sequence $\mathbf{u}$. Then, $(\widehat{X}, \tau_{\mathbf{u}})^\wedge = H (= s_{\mathbf{u}}(X))$ and $\mathbf{n}(\widehat{X}, \tau_{\mathbf{u}}) = H^\perp$ algebraically.*

This motivates us to introduce the following notion:

Definition 9. *Let H be a subgroup of a compact Abelian group X. We say that H is a T-characterized subgroup of X if there exists a T-sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$ in $\widehat{X}$, such that $H = s_{\mathbf{u}}(X)$.*

Denote by $\mathrm{Char}_T(X)$ the set of all T-characterized subgroups of a compact Abelian group X. Clearly, $\mathrm{Char}_T(X) \subseteq \mathrm{Char}(X)$. Hence, if a T-characterized subgroup H of X is closed, it is a G_δ-subgroup of X by Fact 3. Note also that X is T-characterized by the zero sequence.

The main goal of the article is to obtain a complete description of closed T-characterized subgroups (see Theorem 10) and to study the Borel hierarchy of T-characterized subgroups (see Theorem 18) of compact Abelian groups. In particular, we obtain a complete answer to Question 7 for closed characterized subgroups and give a negative answer to Question 6.

Note that, if a compact Abelian group X is finite, then every T-sequence $\mathbf{u}$ in $\widehat{X}$ is eventually equal to zero. Hence, $s_{\mathbf{u}}(X) = X$. Thus, X is the unique T-characterized subgroup of X. Therefore, in what follows, we shall consider only infinite compact groups.

The following theorem describes all closed subgroups of compact Abelian groups that are T-characterized.

Theorem 10. *Let H be a proper closed subgroup of an infinite compact Abelian group X. Then, the following assertions are equivalent:*

(1) *H is a T-characterized subgroup of X;*
(2) *H is a G_δ-subgroup of X, and the countable group $H^\perp$ admits a Hausdorff MinAPgroup topology;*
(3) *H is a G_δ-subgroup of X and one of the following holds:*
    (a) $H^\perp$ has infinite exponent;
    (b) $H^\perp$ has finite exponent and contains a subgroup that is isomorphic to $\mathbb{Z}\left(\exp(H^\perp)\right)^{(\omega)}$.

Corollary 11. *Let X be an infinite compact metrizable Abelian group. Then, the trivial subgroup $H = \{0\}$ is T-characterized if and only if $\widehat{X}$ admits a Hausdorff MinAP group topology.*

As an immediate corollary of Fact 3 and Theorem 10, we obtain a complete answer to Question 7 for closed characterized subgroups.

Corollary 12. *A proper closed characterized subgroup H of an infinite compact Abelian group X is T-characterized if and only if $H^\perp$ admits a Hausdorff MinAP group topology.*

If H is an open proper subgroup of X, then $H^\perp$ is non-trivial and finite. Thus, every Hausdorff group topology on $H^\perp$ is discrete. Taking into account Fact 3, we obtain:

Corollary 13. *Every open proper subgroup H of an infinite compact Abelian group X is a characterized non-T-characterized subgroup of X.*

Nevertheless (see Example 1 below), there is a compact metrizable Abelian group X with a countable T-characterized subgroup H, such that its closure $\bar{H}$ is open. Thus, it may happen that the closure of a T-characterized subgroup is not T-characterized.

It is natural to ask for which compact Abelian groups all of their closed G_δ-subgroups are T-characterized. The next theorem gives a complete answer to this question.

Theorem 14. *Let X be an infinite compact Abelian group. The following assertions are equivalent:*

(1) *All closed G_δ-subgroups of X are T-characterized;*
(2) *X is connected.*

By Corollary 2.8 of [13], the trivial subgroup $H = \{0\}$ of a compact Abelian group X is a G_δ-subgroup if and only if X is metrizable. Therefore, we obtain:

Corollary 15. *All closed subgroups of an infinite compact Abelian group X are T-characterized if and only if X is metrizable and connected.*

Theorems 10 and 14 are proven in Section 2.

In the next theorem, we give a negative answer to Question 6:

Theorem 16. *Every compact Abelian group of infinite exponent has a dense T-characterized subgroup, which is not an F_σ-subgroup.*

As a corollary of the inclusion Equation (1) and Theorem 16, we obtain:

Corollary 17. *For an infinite compact Abelian group X, the following assertions are equivalent:*

(i) *X has finite exponent;*
(ii) *every characterized subgroup of X is an F_σ-subgroup;*
(iii) *every T-characterized subgroup of X is an F_σ-subgroup.*

Therefore, $\mathrm{Char}(X) \subseteq \mathrm{SF}_\sigma(X)$ *if and only if X has finite exponent.*

In the next theorem, we summarize the obtained results about the Borel hierarchy of T-characterized subgroups of compact Abelian groups.

Theorem 18. *Let X be an infinite compact Abelian group X. Then:*

(1) $\mathrm{Char}_T(X) \subsetneqq \mathrm{SF}_{\sigma\delta}(X)$*;*
(2) $\mathrm{SG}_\delta(X) \bigcap \mathrm{Char}_T(X) \subsetneqq \mathrm{Char}_T(X)$*;*
(3) $\mathrm{SG}_\delta(X) \subseteq \mathrm{Char}_T(X)$ *if and only if X is connected;*
(4) $\mathrm{Char}_T(X) \bigcap \mathrm{SF}_\sigma(X) \subsetneqq \mathrm{SF}_\sigma(X)$*;*
(5) $\mathrm{Char}_T(X) \subseteq \mathrm{SF}_\sigma(X)$ *if and only if X has finite exponent.*

We prove Theorems 16 and 18 in Section 3.

The notions of $\mathfrak{g}$-closed and $\mathfrak{g}$-dense subgroups of a compact Abelian group X were defined in [11]. In the last section of the paper, in analogy to these notions, we define $\mathfrak{g}_T$-closed and $\mathfrak{g}_T$-dense subgroups of X. In particular, we show that every $\mathfrak{g}_T$-dense subgroup of a compact Abelian group X is dense if and only if X is connected (see Theorem 37).

2. The Proofs of Theorems 10 and 14

The subgroup of a group G generated by a subset A we denote by $\langle A \rangle$.

Recall that a subgroup H of an Abelian topological group X is called dually closed in X if for every $x \in X \setminus H$, there exists a character $\chi \in H^\perp$, such that $(\chi, x) \neq 1$. H is called dually embedded in X if every character of H can be extended to a character of X. Every open subgroup of X is dually closed and dually embedded in X by Lemma 3 of [19].

The next notion generalizes the notion of the maximal extension in the class of all compact Abelian groups introduced in [20].

Definition 19. Let $\mathcal{G}$ be an arbitrary class of topological groups. Let $(G,\tau) \in \mathcal{G}$ and H be a subgroup of G. The group (G,τ) is called a maximal extension of $(H,\tau|_H)$ in the class $\mathcal{G}$ if $\sigma \le \tau$ for every group topology on G, such that $\sigma|_H = \tau|_H$ and $(G,\sigma) \in \mathcal{G}$.

Clearly, the maximal extension is unique if it exists. Note that in Definition 19, we do not assume that $(H,\tau|_H)$ belongs to the class $\mathcal{G}$.

If H is a subgroup of an Abelian group G and $\mathbf{u}$ is a T-sequence (respectively, a TB-sequence) in H, we denote by $\tau_{\mathbf{u}}(H)$ (respectively, $\tau_{b\mathbf{u}}(H)$) the finest (respectively, precompact) group topology on H generated by $\mathbf{u}$. We use the following easy corollary of the definition of T-sequences.

Lemma 20. *For a sequence $\mathbf{u}$ in an Abelian group G, the following assertions are equivalent:*

(1) *$\mathbf{u}$ is a T-sequence in G;*
(2) *$\mathbf{u}$ is a T-sequence in every subgroup of G containing $\langle \mathbf{u} \rangle$;*
(3) *$\mathbf{u}$ is a T-sequence in $\langle \mathbf{u} \rangle$.*

In this case, $\langle \mathbf{u} \rangle$ is open in $\tau_{\mathbf{u}}$ (and hence, $\langle \mathbf{u} \rangle$ is dually closed and dually embedded in $(G,\tau_{\mathbf{u}})$), and $(G,\tau_{\mathbf{u}})$ is the maximal extension of $(\langle \mathbf{u} \rangle, \tau_{\mathbf{u}}(\langle \mathbf{u} \rangle))$ in the class **TAG** *of all Abelian topological groups.*

Proof. Evidently, (1) implies (2) and (2) implies (3). Let $\mathbf{u}$ be a T-sequence in $\langle \mathbf{u} \rangle$. Let τ be the topology on G whose base is all translationsof $\tau_{\mathbf{u}}(\langle \mathbf{u} \rangle)$-open sets. Clearly, $\mathbf{u}$ converges to zero in τ. Thus, $\mathbf{u}$ is a T-sequence in G. Therefore, (3) implies (1).

Let us prove the last assertion. By the definition of $\tau_{\mathbf{u}}$, we have also $\tau \le \tau_{\mathbf{u}}$, and hence, $\tau|_{\langle \mathbf{u} \rangle} = \tau_{\mathbf{u}}(\langle \mathbf{u} \rangle) \le \tau_{\mathbf{u}}|_{\langle \mathbf{u} \rangle}$. Thus, $\langle \mathbf{u} \rangle$ is open in $\tau_{\mathbf{u}}$, and hence, it is dually closed and dually embedded in $(G,\tau_{\mathbf{u}})$ by [19] (Lemma 3.3). On the other hand, $\tau_{\mathbf{u}}|_{\langle \mathbf{u} \rangle} \le \tau_{\mathbf{u}}(\langle \mathbf{u} \rangle) = \tau|_{\langle \mathbf{u} \rangle}$ by the definition of $\tau_{\mathbf{u}}(\langle \mathbf{u} \rangle)$. Therefore, $\tau_{\mathbf{u}}$ is an extension of $\tau_{\mathbf{u}}(\langle \mathbf{u} \rangle)$. Now, clearly, $\tau = \tau_{\mathbf{u}}$, and $(G,\tau_{\mathbf{u}})$ is the maximal extension of $(\langle \mathbf{u} \rangle, \tau_{\mathbf{u}}(\langle \mathbf{u} \rangle))$ in the class **TAG**. $\square$

For TB-sequences, we have the following:

Lemma 21. *For a sequence $\mathbf{u}$ in an Abelian group G, the following assertions are equivalent:*

(1) *$\mathbf{u}$ is a TB-sequence in G;*
(2) *$\mathbf{u}$ is a TB-sequence in every subgroup of G containing $\langle \mathbf{u} \rangle$;*
(3) *$\mathbf{u}$ is a TB-sequence in $\langle \mathbf{u} \rangle$.*

In this case, the subgroup $\langle \mathbf{u} \rangle$ is dually closed and dually embedded in $(G,\tau_{b\mathbf{u}})$, and $(G,\tau_{b\mathbf{u}})$ is the maximal extension of $(\langle \mathbf{u} \rangle, \tau_{b\mathbf{u}}(\langle \mathbf{u} \rangle))$ in the class of all precompact Abelian groups.

Proof. Evidently, (1) implies (2) and (2) implies (3). Let $\mathbf{u}$ be a TB-sequence in $\langle \mathbf{u} \rangle$. Then, $(\langle \mathbf{u} \rangle, \tau_{b\mathbf{u}}(\langle \mathbf{u} \rangle))^\wedge$ separates the points of $\langle \mathbf{u} \rangle$. Let τ be the topology on G whose base is all translations of $\tau_{b\mathbf{u}}(\langle \mathbf{u} \rangle)$-open sets. Then, $(\langle \mathbf{u} \rangle, \tau_{b\mathbf{u}}(\langle \mathbf{u} \rangle))$ is an open subgroup of (G,τ). It is easy to see that $(G,\tau)^\wedge$ separates the points of G. Since $\mathbf{u}$ converges to zero in τ, it also converges to zero in τ^+, where τ^+ is the Bohr topology of (G,τ). Thus, $\mathbf{u}$ is a TB-sequence in G. Therefore, (3) implies (1).

The last assertion follows from Proposition 1.8 and Lemma 3.6 in [20]. $\square$

For a sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$ of characters of a compact Abelian group X, set:

$$K_{\mathbf{u}} = \bigcap_{n\in\omega} \ker(u_n).$$

The following assertions is proven in [13]:

Fact 22 (Lemma 2.2(i) of [13]). *For every sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$ of characters of a compact Abelian group X, the subgroup $K_{\mathbf{u}}$ is a closed G_δ-subgroup of X and $K_{\mathbf{u}} = \langle \mathbf{u}\rangle^{\perp}$.*

The next two lemmas are natural analogues of Lemmas 2.2(ii) and 2.6 of [13].

Lemma 23. *Let X be a compact Abelian group and $\mathbf{u} = \{u_n\}_{n\in\omega}$ be a T-sequence in $\widehat{X}$. Then, $s_{\mathbf{u}}(X)/K_{\mathbf{u}}$ is a T-characterized subgroup of $X/K_{\mathbf{u}}$.*

Proof. Set $H := s_{\mathbf{u}}(X)$ and $K := K_{\mathbf{u}}$. Let $q : X \to X/K$ be the quotient map. Then, the adjoint homomorphism $q^\wedge$ is an isomorphism from $(X/K)^\wedge$ onto $K^\perp$ in $X^\wedge$. For every $n \in \omega$, define the character $\widetilde{u}_n$ of X/K as follows: $(\widetilde{u}_n, q(x)) = (u_n, x)$ ($\widetilde{u}_n$ is well-defined, since $K \subseteq \ker(u_n)$). Then, $\widetilde{\mathbf{u}} = \{\widetilde{u}_n\}_{n\in\omega}$ is a sequence of characters of X/K, such that $q^\wedge(\widetilde{u}_n) = u_n$. Since $\mathbf{u} \subset K^\perp$, $\mathbf{u}$ is a T-sequence in $K^\perp$ by Lemma 20. Hence, $\widetilde{\mathbf{u}}$ is a T-sequence in $(X/K)^\wedge$ because $q^\wedge$ is an isomorphism.

We claim that $H/K = s_{\widetilde{\mathbf{u}}}(X/K)$. Indeed, for every $h + K \in H/K$, by definition, we have $(\widetilde{u}_n, h + K) = (u_n, h) \to 1$. Thus, $H/K \subseteq s_{\widetilde{\mathbf{u}}}(X/K)$. If $x + K \in s_{\widetilde{\mathbf{u}}}(X/K)$, then $(\widetilde{u}_n, x + K) = (u_n, x) \to 1$. This yields $x \in H$. Thus, $x + K \in H/K$. $\Box$

Let $\mathbf{u} = \{u_n\}_{n\in\omega}$ be a T-sequence in an Abelian group G. For every natural number m, set $\mathbf{u}_m = \{u_n\}_{n\geq m}$. Clearly, $\mathbf{u}_m$ is a T-sequence in G, $\tau_{\mathbf{u}} = \tau_{\mathbf{u}_m}$ and $s_{\mathbf{u}}(X) = s_{\mathbf{u}_m}(X)$ for every natural number m.

Lemma 24. *Let K be a closed subgroup of a compact Abelian group X and $q : X \to X/K$ be the quotient map. Then, $\widetilde{H}$ is a T-characterized subgroup of X/K if and only if $q^{-1}(\widetilde{H})$ is a T-characterized subgroup of X.*

Proof. Let $\widetilde{H}$ be a T-characterized subgroup of X/K, and let a T-sequence $\widetilde{\mathbf{u}} = \{\widetilde{u}_n\}_{n\in\omega}$-characterized $\widetilde{H}$. Set $H := q^{-1}(\widetilde{H})$. We have to show that H is a T-characterized subgroup of X.

Note that the adjoint homomorphism $q^\wedge$ is an isomorphism from $(X/K)^\wedge$ onto $K^\perp$ in $X^\wedge$. Set $\mathbf{u} = \{u_n\}_{n\in\omega}$, where $u_n = q^\wedge(\widetilde{u}_n)$. Since $q^\wedge$ is injective, $\mathbf{u}$ is a T-sequence in $K^\perp$. By Lemma 20, $\mathbf{u}$ is a T-sequence in $\widehat{X}$. Therefore, it is enough to show that $H = s_{\mathbf{u}}(X)$. This follows from the following chain of equivalences. By definition, $x \in s_{\mathbf{u}}(X)$ if and only if:

$$(u_n, x) \to 1 \ \Leftrightarrow\ (\widetilde{u}_n, q(x)) \to 1 \ \Leftrightarrow\ q(x) \in \widetilde{H} = H/K \ \Leftrightarrow\ x \in H.$$

The last equivalence is due to the inclusion $K \subseteq H$.

Conversely, let $H := q^{-1}(\widetilde{H})$ be a T-characterized subgroup of X and a T-sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$-characterized H. Proposition 2.5 of [13] implies that we can find $m \in \mathbb{N}$, such that $K \subseteq K_{\mathbf{u}_m}$. Therefore, taking into account that $H = s_{\mathbf{u}}(X) = s_{\mathbf{u}_m}(X)$ for every natural number m, without loss of generality, we can assume that $K \subseteq K_{\mathbf{u}}$. By Lemma 23, $H/K_{\mathbf{u}}$ is a T-characterized subgroup of $X/K_{\mathbf{u}}$. Denote by q_u the quotient homomorphism from X/K onto $X/K_{\mathbf{u}}$. Then, $\widetilde{H} = q_u^{-1}(H/K_{\mathbf{u}})$ is T-characterized in X/K by the previous paragraph of the proof. $\square$

The next theorem is an analogue of Theorem B of [13], and it reduces the study of T-characterized subgroups of compact Abelian groups to the study of T-characterized ones of compact Abelian metrizable groups:

Theorem 25. *A subgroup H of a compact Abelian group X is T-characterized if and only if H contains a closed G_δ-subgroup K of X, such that H/K is a T-characterized subgroup of the compact metrizable group X/K.*

Proof. Let H be T-characterized in X by a T-sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$ in $\widehat{X}$. Set $K := K_{\mathbf{u}}$. Since K is a closed G_δ-subgroup of X by Fact 22, X/K is metrizable. By Lemma 23, H/K is a T-characterized subgroup of X/K.

Conversely, let H contain a closed G_δ-subgroup K of X, such that H/K is a T-characterized subgroup of the compact metrizable group X/K. Then, H is a T-characterized subgroup of X by Lemma 24. $\square$

As was noticed in [21] before Definition 2.33, for every T-sequence $\mathbf{u}$ in an infinite Abelian group G, the subgroup $\langle\mathbf{u}\rangle$ is open in $(G,\tau_{\mathbf{u}})$ (see also Lemma 20), and hence, by Lemmas 1.4 and 2.2 of [22], the following sequences are exact:

$$\begin{aligned} &0 \to (\langle\mathbf{u}\rangle,\tau_{\mathbf{u}}) \to (G,\tau_{\mathbf{u}}) \to G/\langle\mathbf{u}\rangle \to 0, \\ &0 \to (G/\langle\mathbf{u}\rangle)^\wedge \to (G,\tau_{\mathbf{u}})^\wedge \to (\langle\mathbf{u}\rangle,\tau_{\mathbf{u}}|_{\langle\mathbf{u}\rangle})^\wedge \to 0, \end{aligned} \tag{2}$$

where $(G/\langle\mathbf{u}\rangle)^\wedge \cong \langle\mathbf{u}\rangle^\perp$ is a compact subgroup of $(G,\tau_{\mathbf{u}})^\wedge$ and $(\langle\mathbf{u}\rangle,\tau_{\mathbf{u}})^\wedge \cong (G,\tau_{\mathbf{u}})^\wedge/\langle\mathbf{u}\rangle^\perp$.

Let $\mathbf{u} = \{u_n\}_{n\in\omega}$ be a T-sequence in an Abelian group G. It is known [10] that $\tau_{\mathbf{u}}$ is sequential, and hence, $(G,\tau_{\mathbf{u}})$ is a k-space. Therefore, the natural homomorphism $\alpha := \alpha_{(G,\tau_{\mathbf{u}})} : (G,\tau_{\mathbf{u}}) \to (G,\tau_{\mathbf{u}})^{\wedge\wedge}$ is continuous by [23] (5.12). Let us recall that $(G,\tau_{\mathbf{u}})$ is MinAP if and only if $(G,\tau_{\mathbf{u}}) = \ker(\alpha)$.

To prove Theorem 10, we need the following:

Fact 26 ([16])**.** *For each T-sequence $\mathbf{u}$ in a countably infinite Abelian group G, the group $(G,\tau_{\mathbf{u}})^\wedge$ is Polish.*

Now, we are in a position to prove Theorem 10.

Proof of Theorem 10. (1) $\Rightarrow$ (2) Let H be a proper closed T-characterized subgroup of X and a T-sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$-characterized$H$. Since H is also characterized, it is a G_δ-subgroup of X by Fact 3. We have to show that $H^\perp$ admits a MinAP group topology.

Our idea of the proof is the following. Set $G := \widehat{X}$. By Fact 8, $H^\perp$ is the von Neumann radical of $(G,\tau_{\mathbf{u}})$. Now, assume that we found another T-sequence $\mathbf{v}$ that characterizes H and such that $\langle\mathbf{v}\rangle = H^\perp$ (maybe $\mathbf{v} = \mathbf{u}$). By Fact 8, we have $\mathbf{n}(G,\tau_{\mathbf{v}}) = H^\perp = \langle\mathbf{v}\rangle$. Lemma 20 implies that the subgroup $(\langle\mathbf{v}\rangle,\tau_{\mathbf{v}}|_{\langle\mathbf{v}\rangle})$ of $(G,\tau_{\mathbf{v}})$ is open, and hence, it is dually closed and dually embedded in $(G,\tau_{\mathbf{v}})$.

Hence, $\mathbf{n}(\langle\mathbf{v}\rangle, \tau_{\mathbf{v}}|_{\langle\mathbf{v}\rangle}) = \mathbf{n}(G, \tau_{\mathbf{v}})(= \langle\mathbf{v}\rangle)$ by Lemma 4 of [16]. Therefore, $(\langle\mathbf{v}\rangle, \tau_{\mathbf{v}}|_{\langle\mathbf{v}\rangle})$ is MinAP. Thus, $H^{\perp} = \langle\mathbf{v}\rangle$ admits a MinAP group topology, as desired.

We find such a T-sequence $\mathbf{v}$ in four steps (in fact, we show that $\mathbf{v}$ has the form $\mathbf{u}_m$ for some $m \in \mathbb{N}$).

Step 1. Let $q : X \to X/K_{\mathbf{u}}$ be the quotient map. For every $n \in \omega$, define the character $\widetilde{u}_n$ of $X/K_{\mathbf{u}}$ by the equality $u_n = \widetilde{u}_n \circ q$ (this is possible since $K_{\mathbf{u}} \subseteq \ker(u_n)$). As was shown in the proof of Lemma 23, the sequence $\widetilde{\mathbf{u}} = \{\widetilde{u}_n\}_{n\in\omega}$ is a T-sequence, which characterizes $H/K_{\mathbf{u}}$ in $X/K_{\mathbf{u}}$. Set $\widetilde{X} := X/K_{\mathbf{u}}$ and $\widetilde{H} := H/K_{\mathbf{u}}$. Therefore, $\widetilde{H} = s_{\widetilde{\mathbf{u}}}(\widetilde{X})$. By [24] (5.34 and 24.11) and since $K_{\mathbf{u}} \subseteq H$, we have:

$$H^{\perp} \cong (X/H)^{\wedge} \cong \left(\widetilde{X}/\widetilde{H}\right)^{\wedge} \cong \widetilde{H}^{\perp}. \tag{3}$$

By Fact 3, $\widetilde{X}$ is metrizable. Hence, $\widetilde{H}$ is also compact and metrizable, and $\widetilde{G} := \widehat{\widetilde{X}}$ is a countable Abelian group by [24] (24.15). Since H is a proper closed subgroup of X, Equation (3) implies that $\widetilde{G}$ is non-zero.

We claim that $\widetilde{G}$ is countably infinite. Indeed, suppose for a contradiction that $\widetilde{G}$ is finite. Then, $X/K_{\mathbf{u}} = \widetilde{X}$ is also finite. Now, Fact 22 implies that $\langle\mathbf{u}\rangle$ is a finite subgroup of G. Since $\mathbf{u}$ is a T-sequence, $\mathbf{u}$ must be eventually equal to zero. Hence, $H = s_{\mathbf{u}}(X) = X$ is not a proper subgroup of X, a contradiction.

Step 2. We claim that there is a natural number m, such that the group $(\langle\widetilde{\mathbf{u}}_m\rangle, \tau_{\widetilde{\mathbf{u}}}|_{\langle\widetilde{\mathbf{u}}_m\rangle}) = (\langle\widetilde{\mathbf{u}}_m\rangle, \tau_{\widetilde{\mathbf{u}}_m}|_{\langle\widetilde{\mathbf{u}}_m\rangle})$ is MinAP.

Indeed, since $\widetilde{G}$ is countably infinite, we can apply Fact 8. Therefore, $\widetilde{H} = (\widetilde{G}, \tau_{\widetilde{\mathbf{u}}})^{\wedge}$ algebraically. Since $\widetilde{H}$ and $(\widetilde{G}, \tau_{\widetilde{\mathbf{u}}})^{\wedge}$ are Polish groups (see Fact 26), $\widetilde{H}$ and $(\widetilde{G}, \tau_{\widetilde{\mathbf{u}}})^{\wedge}$ are topologically isomorphic by the uniqueness of the Polish group topology. Hence $(\widetilde{G}, \tau_{\widetilde{\mathbf{u}}})^{\wedge\wedge} = \widetilde{H}^{\wedge}$ is discrete. As was noticed before the proof, the natural homomorphism $\widetilde{\alpha} : (\widetilde{G}, \tau_{\widetilde{\mathbf{u}}}) \to (\widetilde{G}, \tau_{\widetilde{\mathbf{u}}})^{\wedge\wedge}$ is continuous. Since $(\widetilde{G}, \tau_{\widetilde{\mathbf{u}}})^{\wedge\wedge}$ is discrete, we obtain that the von Neumann radical $\ker(\widetilde{\alpha})$ of $(\widetilde{G}, \tau_{\widetilde{\mathbf{u}}})$ is open in $\tau_{\widetilde{\mathbf{u}}}$. Therefore, there exists a natural number m, such that $\widetilde{u}_n \in \ker(\widetilde{\alpha})$ for every $n \geq m$. Hence, $\langle\widetilde{\mathbf{u}}_m\rangle \subseteq \ker(\widetilde{\alpha})$. Lemma 20 implies that the subgroup $\langle\widetilde{\mathbf{u}}_m\rangle$ is open in $(\widetilde{G}, \tau_{\widetilde{\mathbf{u}}})$, and hence, it is dually closed and dually embedded in $(\widetilde{G}, \tau_{\widetilde{\mathbf{u}}})$. Now, Lemma 4 of [16] yields $\langle\widetilde{\mathbf{u}}_m\rangle = \ker(\widetilde{\alpha})$, and $(\langle\widetilde{\mathbf{u}}_m\rangle, \tau_{\widetilde{\mathbf{u}}}|_{\langle\widetilde{\mathbf{u}}_m\rangle})$ is MinAP.

Step 3. Set $\mathbf{v} = \{v_n\}_{n\in\omega}$, where $v_n = u_{n+m}$ for every $n \in \omega$. Clearly, $\mathbf{v}$ is a T-sequence in G characterizing H, $\tau_{\mathbf{u}} = \tau_{\mathbf{v}}$ and $K_{\mathbf{u}} \subseteq K_{\mathbf{v}}$. Let $t : X \to X/K_{\mathbf{v}}$ and $r : X/K_{\mathbf{u}} \to X/K_{\mathbf{v}}$ be the quotient maps. Analogously to Step 1 and the proof of Lemma 23, the sequence $\widetilde{\mathbf{v}} = \{\widetilde{v}_n\}_{n\in\omega}$ is a T-sequence in $\widehat{X/K_{\mathbf{v}}}$, which characterizes $H/K_{\mathbf{v}}$ in $X/K_{\mathbf{v}}$, where $v_n = \widetilde{v}_n \circ t$. Since $t = r \circ q$, we have:

$$v_n = \widetilde{v}_n \circ t = t^{\wedge}(\widetilde{v}_n) = q^{\wedge}\left(r^{\wedge}(\widetilde{v}_n)\right),$$

where $t^{\wedge}$, $r^{\wedge}$ and $q^{\wedge}$ are the adjoint homomorphisms to t, r and q, respectively.

Since $q^{\wedge}$ and $r^{\wedge}$ are embeddings, we have $r^{\wedge}(\widetilde{v}_n) = \widetilde{u}_{n+m}$. In particular, $\langle\mathbf{v}\rangle \cong \langle\widetilde{\mathbf{v}}\rangle \cong \langle\widetilde{\mathbf{u}}_m\rangle$ and :

$$(\langle\widetilde{\mathbf{u}}_m\rangle, \tau_{\widetilde{\mathbf{u}}}|_{\langle\widetilde{\mathbf{u}}_m\rangle}) = (\langle\widetilde{\mathbf{u}}_m\rangle, \tau_{\widetilde{\mathbf{u}}_m}|_{\langle\widetilde{\mathbf{u}}_m\rangle}) \cong (\langle\widetilde{\mathbf{v}}\rangle, \tau_{\widetilde{\mathbf{v}}}|_{\langle\widetilde{\mathbf{v}}\rangle}) \cong (\langle\mathbf{v}\rangle, \tau_{\mathbf{v}}|_{\langle\mathbf{v}\rangle}).$$

By Step 2, $(\langle\widetilde{\mathbf{u}}_m\rangle, \tau_{\widetilde{\mathbf{u}}_m}|_{\langle\widetilde{\mathbf{u}}_m\rangle})$ is MinAP. Hence, $(\langle\mathbf{v}\rangle, \tau_{\mathbf{v}}|_{\langle\mathbf{v}\rangle})$ is MinAP, as well.

Step 4. By the second exact sequence in Equation (2) applying to $\mathbf{v}$, Fact 8, and since $(\langle\mathbf{v}\rangle, \tau_{\mathbf{v}}|_{\langle\mathbf{v}\rangle})$ is MinAP (by Step 3), we have $H = s_{\mathbf{v}}(X) = (G, \tau_{\mathbf{v}})^{\wedge} = (G/\langle\mathbf{v}\rangle)^{\wedge} = \langle\mathbf{v}\rangle^{\perp}$ algebraically. Thus, $H^{\perp} = \langle\mathbf{v}\rangle$, and hence, $H^{\perp}$ admits a MinAP group topology generated by the T-sequence $\mathbf{v}$.

(2) $\Rightarrow$ (1): Since H is a G_δ-subgroup of X, H is closed by [13] (Proposition 2.4) and X/H is metrizable (due to the well-known fact that a compact group of countable pseudo-character is metrizable). Hence, $H^\perp = (X/H)^\wedge$ is countable. Since $H^\perp$ admits a MinAP group topology, $H^\perp$ must be countably infinite. By Theorem 3.8 of [9], $H^\perp$ admits a MinAP group topology generated by a T-sequence $\widetilde{\mathbf{u}} = \{\widetilde{u}_n\}_{n\in\omega}$. By Fact 8, this means that $s_{\widetilde{\mathbf{u}}}(X/H) = \{0\}$. Let $q : X \to X/H$ be the quotient map. Set $u_n = \widetilde{u}_n \circ q = q^\wedge(\widetilde{u}_n)$. Since $q^\wedge$ is injective, $\mathbf{u}$ is a T-sequence in $\widehat{X}$ by Lemma 20. We have to show that $H = s_{\mathbf{u}}(X)$. By definition, $x \in s_{\mathbf{u}}(X)$ if and only if:

$$(u_n, x) = (\widetilde{u}_n, q(x)) \to 1 \Leftrightarrow q(x) \in s_{\widetilde{\mathbf{u}}}(X/H) \Leftrightarrow q(x) = 0 \Leftrightarrow x \in H.$$

(2)$\Leftrightarrow$(3) follows from Theorem 3.8 of [9]. The theorem is proven. □

Proof of Theorem 14. (1) $\Rightarrow$ (2): Suppose for a contradiction that X is not connected. Then, by [24] (24.25), the dual group $G = X^\wedge$ has a non-zero element g of finite order. Then, the subgroup $H := \langle g\rangle^\perp$ of X has finite index. Hence, H is an open subgroup of X. Thus, H is not T-characterized by Corollary 13. This contradiction shows that X must be connected.

(2) $\Rightarrow$ (1): Let H be a proper G_δ-subgroup of X. Then, H is closed by [13] (Proposition 2.4), and X/H is connected and non-zero. Hence, $H^\perp \cong (X/H)^\wedge$ is countably infinite and torsion free by [24] (24.25). Thus, $H^\perp$ has infinite exponent. Therefore, by Theorem 10, H is T-characterized. □

The next proposition is a simple corollary of Theorem B in [13].

Proposition 27. *The closure $\bar{H}$ of a characterized (in particular, T-characterized) subgroup H of a compact Abelian group X is a characterized subgroup of X.*

Proof. By Theorem B of [13], H contains a compact G_δ-subgroup K of X. Then, $\bar{H}$ is also a G_δ-subgroup of X. Thus, $\bar{H}$ is a characterized subgroup of X by Theorem B of [13]. □

In general, we cannot assert that the closure $\bar{H}$ of a T-characterized subgroup H of a compact Abelian group X is also T-characterized, as the next example shows.

Example 1. Let $X = \mathbb{Z}(2) \times \mathbb{T}$ and $G = \widehat{X} = \mathbb{Z}(2) \times \mathbb{Z}$. It is known (see the end of $(\mathbf{1})$ in [7]) that there is a T-sequence $\mathbf{u}$ in G, such that the von Neumann radical $\mathbf{n}(G, \tau_{\mathbf{u}})$ of $(G, \tau_{\mathbf{u}})$ is $\mathbb{Z}(2) \times \{0\}$, the subgroup $H := s_{\mathbf{u}}(X)$ is countable and $\bar{H} = \{0\} \times \mathbb{T}$. Therefore, the closure $\bar{H}$ of the countable T-characterized subgroup H of X is open. Thus, $\bar{H}$ is not T-characterized by Corollary 13.

We do not know the answers to the following questions:

Problem 28. *Let H be a characterized subgroup of a compact Abelian group X, such that its closure $\bar{H}$ is T-characterized. Is H a T-characterized subgroup of X?*

Problem 29. *Does there exists a metrizable Abelian compact group that has a countable non-T-characterized subgroup?*

3. The Proofs of Theorems 16 and 18

Recall that a Borel subgroup H of a Polish group X is called polishable if there exists a Polish group topology τ on H, such that the inclusion map $i : (H, \tau) \to X$ is continuous. Let H be a T-characterized subgroup of a compact metrizable Abelian group X by a T-sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$. Then, by [16] (Theorem 1), H is polishable by the metric:

$$\rho(x, y) = d(x, y) + \sup\{|(u_n, x) - (u_n, y)|,\ n \in \omega\}, \tag{4}$$

where d is the initial metric on X. Clearly, the topology generated by the metric ρ on H is finer than the induced one from X.

To prove Theorem 16 we need the following three lemmas.

For a real number x, we write $[x]$ for the integral part of x and $\|x\|$ for the distance from x to the nearest integer. We also use the following inequality proven in [25]:

$$\pi|\varphi| \le |1 - e^{2\pi i\varphi}| \le 2\pi|\varphi|, \quad \varphi \in \left[-\frac{1}{2}, \frac{1}{2}\right). \tag{5}$$

Lemma 30. *Let $\{a_n\}_{n\in\omega} \subset \mathbb{N}$ be such that $a_n \to \infty$ and $a_n \ge 2, n \in \omega$. Set $u_n = \prod_{k\le n} a_n$ for every $n \in \omega$. Then, $\mathbf{u} = \{u_n\}_{n\in\omega}$ is a T-sequence in $X = \mathbb{T}$, and the T-characterized subgroup $H = s_{\mathbf{u}}(\mathbb{T})$ of $\mathbb{T}$ is a dense non-F_σ-subset of $\mathbb{T}$.*

Proof. We consider the circle group $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$ and write it additively. Therefore, $d(0, x) = \|x\|$ for every $x \in \mathbb{T}$. Recall that every $x \in \mathbb{T}$ has the unique representation in the form:

$$x = \sum_{n=0}^{\infty} \frac{c_n}{u_n}, \tag{6}$$

where $0 \le c_n < a_n$ and $c_n \ne a_n - 1$ for infinitely many indices n.

It is known [26] (see also (12) in the proof of Lemma 1 of [25]) that x with representation Equation (6) belongs to H if and only if:

$$\lim_{n\to\infty} \frac{c_n}{a_n} (\mathrm{mod}\ 1) = 0. \tag{7}$$

Hence, H is a dense subgroup of $\mathbb{T}$. Thus, $\mathbf{u}$ is even a TB-sequence in $\mathbb{Z}$ by Fact 1.

We have to show that H is not an F_σ-subset of $\mathbb{T}$. Suppose for a contradiction that H is an F_σ-subset of $\mathbb{T}$. Then, $H = \cup_{n\in\mathbb{N}} F_n$, where F_n is a compact subset of $\mathbb{T}$ for every $n \in \mathbb{N}$. Since H is a subgroup of $\mathbb{T}$, without loss of generality, we can assume that $F_n - F_n \subseteq F_{n+1}$. Since all F_n are closed in (H, ρ), as well, the Baire theorem implies that there are $0 < \varepsilon < 0.1$ and $m \in \mathbb{N}$, such that $F_m \supseteq \{x : \rho(0, x) \le \varepsilon\}$.

Fix arbitrarily $l > 0$, such that $\frac{2}{u_{l-1}} < \frac{\varepsilon}{20}$. For every natural number $k > l$, set:

$$x_k := \sum_{n=l}^{k} \frac{1}{u_n} \cdot \left[\frac{(a_n - 1)\varepsilon}{20}\right].$$

Then, for every $k > l$, we have:

$$x_k = \sum_{n=l}^{k} \frac{1}{u_n} \cdot \left[\frac{(a_n - 1)\varepsilon}{20}\right] < \sum_{n=l}^{k} \frac{1}{u_{n-1}} \cdot \frac{\varepsilon}{20} < \frac{1}{u_{l-1}} \sum_{n=0}^{k-l} \frac{1}{2^n} < \frac{2}{u_{l-1}} < \frac{\varepsilon}{20} < \frac{1}{2}.$$

This inequality and Equation (5) imply that:

$$d(0, x_k) = \|x_k\| = x_k < \frac{\varepsilon}{20}, \quad \text{for every } k > l. \tag{8}$$

For every $s \in \omega$ and every natural number $k > l$, we estimate $|1 - (u_s, x_k)|$ as follows.
Case 1. Let $s < k$. Set $q = \max\{s+1, l\}$. By the definition of x_k, we have:

$$\begin{aligned} 2\pi\left[(u_s \cdot x_k) \pmod 1\right] &= 2\pi \left[u_s \sum_{n=l}^{k} \frac{1}{u_n} \cdot \left[\frac{(a_n - 1)\varepsilon}{20}\right] \pmod 1 \right] < 2\pi \sum_{n=q}^{k} \frac{u_s}{u_n} \cdot \frac{(a_n-1)\varepsilon}{20} \\ &< \frac{\pi\varepsilon}{10}\left(1 + \frac{1}{a_{s+1}} + \frac{1}{a_{s+1}a_{s+2}} + \frac{1}{a_{s+1}a_{s+2}a_{s+3}} + \dots\right) \\ &< \frac{\pi\varepsilon}{10}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots\right) = \frac{\pi\varepsilon}{10} \cdot 2 < \frac{2\varepsilon}{3} < \frac{1}{2}. \end{aligned}$$

This inequality and Equation (5) imply:

$$|1 - (u_s, x_k)| = |1 - \exp\{2\pi i \cdot [(u_s \cdot x_k) \pmod 1)]\}| < \frac{2\varepsilon}{3}. \tag{9}$$

Case 2. Let $s \geq k$. By the definition of x_k, we have:

$$|1 - (u_s, x_k)| = 0. \tag{10}$$

In particular, Equation (10) implies that $x_k \in H$ for every $k > l$.
Now, for every $k > l$, Equations (4) and (8)–(10) imply:

$$\rho(0, x_k) < \frac{\varepsilon}{20} + \frac{2\varepsilon}{3} < \varepsilon.$$

Thus, $x_k \in F_m$ for every natural number $k > l$. Clearly,

$$x_k \to x := \sum_{n=l}^{\infty} \frac{1}{u_n} \cdot \left[\frac{(a_n-1)\varepsilon}{20}\right] \quad \text{in } \mathbb{T}.$$

Since F_m is a compact subset of $\mathbb{T}$, we have $x \in F_m$. Hence, $x \in H$. On the other hand, we have:

$$\lim_{n\to\infty} \frac{1}{a_n} \cdot \left[\frac{(a_n-1)\varepsilon}{20}\right] \pmod 1 = \frac{\varepsilon}{20} \neq 0.$$

Therefore, Equation (7) implies that $x \notin H$. This contradiction shows that $H = s_{\mathbf{u}}(\mathbb{T})$ is not an F_σ-subset of $\mathbb{T}$. $\Box$

For a prime number p, the group $\mathbb{Z}(p^\infty)$ is regarded as the collection of fractions $m/p^n \in [0, 1)$. Let Δ_p be the compact group of p-adic integers. It is well known that $\widehat{\Delta_p} = \mathbb{Z}(p^\infty)$.

Lemma 31. *Let $X = \Delta_p$. For an increasing sequence of natural numbers $0 < n_0 < n_1 < \dots$, such that $n_{k+1} - n_k \to \infty$, set:*

$$u_k = \frac{1}{p^{n_k+1}} \in \mathbb{Z}(p^\infty).$$

Then, the sequence $\mathbf{u} = \{u_k\}_{k\in\omega}$ is a T-sequence in $\mathbb{Z}(p^\infty)$, and the T-characterized subgroup $H = s_{\mathbf{u}}(\Delta_p)$ is a dense non-F_σ-subset of Δ_p.

Proof. Let $\omega = (a_n)_{n\in\omega} \in \Delta_p$, where $0 \leq a_n < p$ for every $n \in \omega$. Recall that, for every $k \in \omega$, [24] (25.2) implies:

$$(u_k, \omega) = \exp\left\{ \frac{2\pi i}{p^{n_k+1}} \left(a_0 + pa_1 + \cdots + p^{n_k} a_{n_k}\right) \right\}. \tag{11}$$

Further, by [24] (10.4), if $\omega \neq 0$, then $d(0,\omega) = 2^{-n}$, where n is the minimal index, such that $a_n \neq 0$.

Following [27] (2.2), for every $\omega = (a_n) \in \Delta_p$ and every natural number $k > 1$, set:

$$m_k = m_k(\omega) = \max\{j_k, n_{k-1}\},$$

where:

$$j_k = n_k \text{ if } 0 < a_{n_k} < p - 1,$$

and otherwise:

$$j_k = \min\{j : \text{ either } a_s = 0 \text{ for } j < s \leq n_k, \text{or } a_s = p-1 \text{ for } j < s \leq n_k\}.$$

In [27] (2.2), it is shown that:

$$\omega \in s_{\mathbf{u}}(\Delta_p) \text{ if and only if } n_k - m_k \to \infty. \tag{12}$$

Therefore, $H := s_{\mathbf{u}}(\Delta_p)$ contains the identity $\mathbf{1} = (1, 0, 0, \dots)$ of Δ_p. By [24] (Remark 10.6), $\langle \mathbf{1} \rangle$ is dense in Δ_p. Hence, H is dense in Δ_p, as well. Now, Fact 1 implies that $\mathbf{u}$ is a T-sequence in $\mathbb{Z}(p^\infty)$.

We have to show that H is not an F_σ-subset of Δ_p. Suppose for a contradiction that $H = \cup_{n\in\mathbb{N}} F_n$ is an F_σ-subset of Δ_p, where F_n is a compact subset of Δ_p for every $n \in \mathbb{N}$. Since H is a subgroup of Δ_p, without loss of generality, we can assume that $F_n - F_n \subseteq F_{n+1}$. Since all F_n are closed in (H, ρ), as well, the Baire theorem implies that there are $0 < \varepsilon < 0.1$ and $m \in \mathbb{N}$, such that $F_m \supseteq \{x : \rho(0, x) \leq \varepsilon\}$.

Fix a natural number s, such that $\frac{1}{2^s} < \frac{\varepsilon}{20}$. Choose a natural number $l > s$, such that, for every natural number $w \geq l$, we have:

$$n_{w+1} - n_w > s. \tag{13}$$

For every $r \in \mathbb{N}$, set:

$$\omega_r := (a_n^r), \text{ where } a_n^r = \begin{cases} 1, \text{ if } n = n_{l+i} - s \text{ for some } 1 \leq i \leq r, \\ 0, \text{ otherwise.} \end{cases}$$

Then, for every $r \in \mathbb{N}$, Equation (13) implies that ω_r is well defined and:

$$d(0, \omega_r) = \frac{1}{2^{n_{l+1}-s}} < \frac{1}{2^{n_l}} \leq \frac{1}{2^l} < \frac{1}{2^s} < \frac{\varepsilon}{20}. \tag{14}$$

Note that:

$$1 + p + \cdots + p^k = \frac{p^{k+1} - 1}{p - 1} < p^{k+1}. \tag{15}$$

For every $k \in \omega$ and every $r \in \mathbb{N}$, we estimate $|1 - (u_k, \omega_r)|$ as follows.

Case 1. Let $k \leq l$. By Equations (11) and (13) and the definition of ω_r, we have:

$$|1 - (u_k, \omega_r)| = 0. \tag{16}$$

Case 2. Let $l < k \le l + r$. Then, Equation (15) yields:

$$\frac{2\pi}{p^{n_k+1}} \left|p^{n_{l+1}-s} + \cdots + p^{n_k - s}\right| < \frac{2\pi}{p^{n_k+1}} \cdot p^{n_k - s+1} = \frac{2\pi}{p^s} \le \frac{2\pi}{2^s} < \frac{\varepsilon}{2} < \frac{1}{2}.$$

This inequality and the inequality Equations (5) and (11) imply:

$$|1 - (u_k, \omega_r)| = \left|1 - \exp\left\{\frac{2\pi i}{p^{n_k+1}}\left(p^{n_{l+1}-s} + \cdots + p^{n_k-s}\right)\right\}\right| < \frac{\varepsilon}{2}. \tag{17}$$

Case 3. Let $l + r < k$. By Equation (15), we have:

$$\begin{aligned}\frac{2\pi}{p^{n_k+1}} \left|p^{n_{l+1}-s} + \cdots + p^{n_{l+r}-s}\right| &< \frac{2\pi}{p^{n_k+1}} \cdot p^{n_{l+r}-s+1} \\ &< \frac{2\pi}{p^{n_k+1}} \cdot p^{n_k-s+1} = \frac{2\pi}{p^s} \le \frac{2\pi}{2^s} < \frac{\varepsilon}{2}.\end{aligned}$$

These inequalities, Equations (5) and (11) immediately yield:

$$|1 - (u_k, \omega_r)| = \left|1 - \exp\left\{\frac{2\pi i}{p^{n_k+1}}\left(p^{n_{l+1}-s} + \cdots + p^{n_{l+r}-s}\right)\right\}\right| < \frac{\varepsilon}{2}, \tag{18}$$

and:

$$|1 - (u_k, \omega_r)| < \frac{2\pi}{p^{n_k+1}} \cdot p^{n_{l+r}-s+1} \to 0, \quad \text{as } k \to \infty. \tag{19}$$

Therefore, Equation (19) implies that $\omega_r \in H$ for every $r \in \mathbb{N}$.

For every $r \in \mathbb{N}$, by Equations (4), (14) and (16)–(18), we have:

$$\rho(0, \omega_r) = d(0, \omega_r) + \sup\left\{|1 - (u_k, \omega_r)|,\ k \in \omega\right\} < \frac{\varepsilon}{20} + \frac{\varepsilon}{2} < \varepsilon.$$

Thus, $\omega_r \in F_m$ for every $r \in \mathbb{N}$. Evidently,

$$\omega_r \to \widetilde{\omega} = (\widetilde{a}_n) \text{ in } \Delta_p, \text{ where } \widetilde{a}_n = \begin{cases} 1, \text{ if } n = n_{l+i} - s \text{ for some } i \in \mathbb{N}, \\ 0, \text{ otherwise.}\end{cases}$$

Since F_m is a compact subset of Δ_p, we have $\widetilde{\omega} \in F_m$. Hence, $\widetilde{\omega} \in H$. On the other hand, it is clear that $m_k(\widetilde{\omega}) = n_k - s$ for every $k \ge l + 1$. Thus, for every $k \ge l + 1$, $n_k - m_k(\widetilde{\omega}) = s \not\to \infty$. Now, Equation (12) implies that $\widetilde{\omega} \notin H$. This contradiction shows that H is not an F_σ-subset of Δ_p. $\square$

Lemma 32. *Let $X = \prod_{n\in\omega} \mathbb{Z}(b_n)$, where $1 < b_0 < b_1 < \ldots$ and $G := \widehat{X} = \bigoplus_{n\in\omega} \mathbb{Z}(b_n)$. Set $\mathbf{u} = \{u_n\}_{n\in\omega}$, where $u_n = 1 \in \mathbb{Z}(b_n)^\wedge \subset G$ for every $n \in \omega$. Then, $\mathbf{u}$ is a T-sequence in G, and the T-characterized subgroup $H = s_{\mathbf{u}}(X)$ is a dense non-F_σ-subset of X.*

Proof. Set $H := s_{\mathbf{u}}(X)$. In [27] (2.3), it is shown that:

$$\omega = (a_n) \in s_{\mathbf{u}}(X) \text{ if and only if } \left\|\frac{a_n}{b_n}\right\| \to 0. \tag{20}$$

Therefore, $\bigoplus_{n\in\omega} \mathbb{Z}(b_n) \subseteq H$. Thus, H is dense in X. Now, Fact 1 implies that $\mathbf{u}$ is a T-sequence in G.

We have to show that H is not an F_σ-subset of X. Suppose for a contradiction that $H = \cup_{n\in\mathbb{N}} F_n$ is an F_σ-subset of X, where F_n is a compact subset of X for every $n \in \mathbb{N}$. Since H is a subgroup of X, without loss of generality, we can assume that $F_n - F_n \subseteq F_{n+1}$. Since all F_n are closed in (H, ρ), as well, the Baire theorem yields that there are $0 < \varepsilon < 0.1$ and $m \in \mathbb{N}$, such that $F_m \supseteq \{\omega \in X : \rho(0, \omega) \leq \varepsilon\}$.

Note that $d(0, \omega) = 2^{-l}$, where $0 \neq \omega = (a_n)_{n\in\omega} \in X$ and l is the minimal index, such that $a_l \neq 0$. Choose l, such that $2^{-l} < \varepsilon/3$. For every natural number $k > l$, set:

$$\omega_k := (a_n^k), \text{ where } a_n^k = \begin{cases} \left[\dfrac{\varepsilon b_n}{20}\right], \text{ for every } n \text{ such that } l \leq n \leq k, \\ 0, \text{ if either } 1 \leq n < l \text{ or } k < n. \end{cases}$$

Since $(u_n, \omega_k) = 1$ for every $n > k$, we obtain that $\omega_k \in H$ for every $k > l$. For every $n \in \omega$, we have:

$$2\pi \cdot \frac{1}{b_n}\left[\frac{\varepsilon b_n}{20}\right] < \frac{2\pi\varepsilon}{20} < \varepsilon < \frac{1}{2}.$$

This inequality and the inequality Equations (4) and (5) imply:

$$\begin{aligned} \rho(0, \omega_k) &= d(0, \omega_k) + \sup\{|1 - (u_n, \omega_k)|\,,\ n \in \omega\} \\ &\leq \frac{1}{2^l} + \max\left\{\left|1 - \exp\left\{2\pi i \frac{1}{b_n}\left[\frac{\varepsilon b_n}{20}\right]\right\}\right|,\ l \leq n \leq k\right\} \\ &\leq \frac{\varepsilon}{3} + 2\pi \cdot \max\left\{\frac{1}{b_n}\left[\frac{\varepsilon b_n}{20}\right],\ l \leq n \leq k\right\} < \frac{\varepsilon}{3} + \frac{2\pi\varepsilon}{20} < \varepsilon. \end{aligned}$$

Thus, $\omega_k \in F_m$ for every natural number $k > l$. Evidently,

$$\omega_k \to \widetilde{\omega} = (\widetilde{a}_n)_{n\in\omega} \text{ in } X, \text{ where } \widetilde{a}_n = \begin{cases} 0, \text{ if } 0 \leq n < l, \\ \left[\dfrac{\varepsilon b_n}{20}\right], \text{ if } l \leq n. \end{cases}$$

Since F_m is a compact subset of X, we have $\widetilde{\omega} \in F_m$. Hence, $\widetilde{\omega} \in H$. On the other hand, since $b_n \to \infty$, we have:

$$\lim_{n\to\infty}\left\|\frac{\widetilde{a}_n}{b_n}\right\| = \lim_{n\to\infty}\frac{1}{b_n}\left[\frac{\varepsilon b_n}{20}\right] = \frac{\varepsilon}{20} \neq 0.$$

Thus, $\widetilde{\omega} \notin H$ by Equation (20). This contradiction shows that H is not an F_σ-subset of X. $\square$

Now, we are in a position to prove Theorems 16 and 18.

Proof of Theorem 16. Let X be a compact Abelian group of infinite exponent. Then, $G := \widehat{X}$ also has infinite exponent. It is well-known that G contains a countably-infinite subgroup S of one of the following form:

(a) $S \cong \mathbb{Z}$;
(b) $S \cong \mathbb{Z}(p^\infty)$;
(c) $S \cong \bigoplus_{n\in\omega}\mathbb{Z}(b_n)$, where $1 < b_0 < b_1 < \dots$.

Fix such a subgroup S. Set $K = S^\perp$ and $Y = X/K \cong S_d^\wedge$, where S_d denotes the group S endowed with the discrete topology. Since S is countable, Y is metrizable. Hence, $\{0\}$ is a G_δ-subgroup of

Y. Thus, K is a G_δ-subgroup of X. Let $q : X \to Y$ be the quotient map. By Lemmas 30–32, the compact group Y has a dense T-characterized subgroup $\widetilde{H}$, which is not an F_σ-subset of Y. Lemma 24 implies that $H := q^{-1}(\widetilde{H})$ is a dense T-characterized subgroup of X. Since the continuous image of an F_σ-subset of a compact group is an F_σ-subset, as well, we obtain that H is not an F_σ-subset of X. Thus, the subgroup H of X is T-characterized, but it is not an F_σ-subset of X. The theorem is proven. □

Proof of Theorem 18. (1) Follows from Fact 5.

(2) By Lemma 3.6 in [13], every infinite compact Abelian group X contains a dense characterized subgroup H. By Fact 1, H is T-characterized. Since every G_δ-subgroup of X is closed in X by Proposition 2.4 of [13], H is not a G_δ-subgroup of X.

(3) Follows from Theorem 14 and the aforementioned Proposition 2.4 of [13].

(4) Follows from Fact 5.

(5) Follows from Corollary 17. □

It is trivial that $\mathrm{Char}_T(X) \subseteq \mathrm{Char}(X)$ for every compact Abelian group X. For the circle group $\mathbb{T}$, we have:

Proposition 33. $\mathrm{Char}_T(\mathbb{T}) = \mathrm{Char}(\mathbb{T})$.

Proof. We have to show only that $\mathrm{Char}(\mathbb{T}) \subseteq \mathrm{Char}_T(\mathbb{T})$. Let $H = s_{\mathbf{u}}(\mathbb{T}) \in \mathrm{Char}(\mathbb{T})$ for some sequence $\mathbf{u}$ in $\mathbb{Z}$.

If H is infinite, then H is dense in $\mathbb{T}$. Therefore, $\mathbf{u}$ is a T-sequence in $\mathbb{Z}$ by Fact 1. Thus, $H \in \mathrm{Char}_T(\mathbb{T})$.

If H is finite, then H is closed in $\mathbb{T}$. Clearly, $H^\perp$ has infinite exponent. Thus, $H \in \mathrm{Char}_T(\mathbb{T})$ by Theorem 10. □

Note that, if a compact Abelian group X satisfies the equality $\mathrm{Char}_T(X) = \mathrm{Char}(X)$, then X is connected by Fact 3 and Theorem 14. This fact and Proposition 33 justify the next problem:

Problem 34. *Does there exists a connected compact Abelian group X, such that $\mathrm{Char}_T(X) \neq \mathrm{Char}(X)$? Is it true that $\mathrm{Char}_T(X) = \mathrm{Char}(X)$ if and only if X is connected?*

For a compact Abelian group X, the set of all subgroups of X that are both $F_{\sigma\delta}$- and $G_{\delta\sigma}$-subsets of X we denote by $\mathrm{S}\Delta^0_3(X)$. To complete the study of the Borel hierarchy of (T-)characterized subgroups of X, we have to answer the next question.

Problem 35. *Describe compact Abelian groups X of infinite exponent for which $\mathrm{Char}(X) \subseteq \mathrm{S}\Delta^0_3(X)$. For which compact Abelian groups X of infinite exponent there exists a T-characterized subgroup H that does not belong to $\mathrm{S}\Delta^0_3(X)$?*

4. $\mathfrak{g}_T$-Closed and $\mathfrak{g}_T$-Dense Subgroups of Compact Abelian Groups

The following closure operator $\mathfrak{g}$ of the category of Abelian topological groups is defined in [11]. Let X be an Abelian topological group and H its arbitrary subgroup. The closure operator $\mathfrak{g} = \mathfrak{g}_X$ is defined as follows:

$$\mathfrak{g}_X(H) := \bigcap_{\mathbf{u}\in\widehat{X}^{\mathbb{N}}} \{s_{\mathbf{u}}(X) : \ H \leq s_{\mathbf{u}}(X)\},$$

and we say that H is $\mathfrak{g}$-closed if $H = \mathfrak{g}(H)$, and H is $\mathfrak{g}$-dense if $\mathfrak{g}(H) = X$.

The set of all T-sequences in the dual group $\widehat{X}$ of a compact Abelian group X we denote by $\mathcal{T}_s(\widehat{X})$. Clearly, $\mathcal{T}_s(\widehat{X}) \subsetneqq \widehat{X}^{\mathbb{N}}$. Let H be a subgroup of X. In analogy to the closure operator $\mathfrak{g}$, $\mathfrak{g}$-closure and $\mathfrak{g}$-density, the operator $\mathfrak{g}_T$ is defined as follows:

$$\mathfrak{g}_T(H) := \bigcap_{\mathbf{u}\in\mathcal{T}_s(\widehat{X})} \{s_{\mathbf{u}}(X) : \ H \leq s_{\mathbf{u}}(X)\},$$

and we say that H is $\mathfrak{g}_T$-closed if $H = \mathfrak{g}_T(H)$, and H is $\mathfrak{g}_T$-dense if $\mathfrak{g}_T(H) = X$.

In this section, we study some properties of $\mathfrak{g}_T$-closed and $\mathfrak{g}_T$-dense subgroups of a compact Abelian group X. Note that every $\mathfrak{g}$-dense subgroup of X is dense by Lemma 2.12 of [11], but for $\mathfrak{g}_T$-dense subgroups, the situation changes:

Proposition 36. *Let X be a compact Abelian group.*

(1) *If H is a $\mathfrak{g}_T$-dense subgroup of X, then the closure $\bar{H}$ of H is an open subgroup of X.*
(2) *Every open subgroup of a compact Abelian group X is $\mathfrak{g}_T$-dense.*

Proof. (1) Suppose for a contradiction that $\bar{H}$ is not open in X. Then, $X/\bar{H}$ is an infinite compact group. By Lemma 3.6 of [13], $X/\bar{H}$ has a proper dense characterized subgroup S. Fact 1 implies that S is a T-characterized subgroup of $X/\bar{H}$. Let $q : X \to X/\bar{H}$ be the quotient map. Then, Lemma 24 yields that $q^{-1}(S)$ is a T-characterized dense subgroup of X containing H. Since $q^{-1}(S) \neq X$, we obtain that H is not $\mathfrak{g}_T$-dense in X, a contradiction.

(2) Let H be an open subgroup of X. If $H = X$, the assertion is trivial. Assume that H is a proper subgroup (so X is disconnected). Let $\mathbf{u}$ be an arbitrary T-sequence, such that $H \subseteq s_{\mathbf{u}}(X)$. Since H is open, $s_{\mathbf{u}}(X)$ is open, as well. Now, Corollary 13 implies that $s_{\mathbf{u}}(X) = X$. Thus, H is $\mathfrak{g}_T$-dense in X. □

Proposition 36(2) shows that $\mathfrak{g}_T$-density may essentially differ from the usual $\mathfrak{g}$-density. In the next theorem, we characterize all compact Abelian groups for which all $\mathfrak{g}_T$-dense subgroups are also dense.

Theorem 37. *All $\mathfrak{g}_T$-dense subgroups of a compact Abelian group X are dense if and only if X is connected.*

Proof. Assume that all $\mathfrak{g}_T$-dense subgroup of X are dense. Proposition 36(2) implies that X has no open proper subgroups. Thus, X is connected by [24] (7.9).

Conversely, let X be connected and H be a $\mathfrak{g}_T$-dense subgroup of X. Proposition 36(1) implies that the closure $\bar{H}$ of H is an open subgroup of X. Since X is connected, we obtain that $\bar{H} = X$. Thus, H is dense in X. □

For $\mathfrak{g}_T$-closed subgroups, we have:

Proposition 38. *Let X be a compact Abelian group.*

(1) *Every proper open subgroup H of X is a $\mathfrak{g}$-closed non-$\mathfrak{g}_T$-closed subgroup.*
(2) *If every $\mathfrak{g}$-closed subgroup of X is $\mathfrak{g}_T$-closed, then X is connected.*

Proof. (1) The subgroup H is $\mathfrak{g}_T$-dense in X by Proposition 36. Therefore, H is not $\mathfrak{g}_T$-closed. On the other hand, H is $\mathfrak{g}$-closed in X by Theorem A of [13].

(2) Item (1) implies that X has no open subgroups. Thus, X is connected by [24] (7.9). $\square$

We do not know whether the converse in Proposition 38(2) holds true:

Problem 39. *Let a compact Abelian group X be connected. Is it true that every $\mathfrak{g}$-closed subgroup of X is also $\mathfrak{g}_T$-closed?*

Conflicts of Interest

The authors declare no conflict of interest.

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