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Article

On the q-Analogues of Srivastava's Triple Hypergeometric Functions

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Abstract: We find Euler integral formulas, summation and reduction formulas for q-analogues of Srivastava's three triple hypergeometric functions. The proofs use q-analogues of Picard's integral formula for the first Appell function, a summation formula for the first Appell function based on the Bayley–Daum formula, and a general triple series reduction formula of Karlsson. Many of the formulas are purely formal, since it is difficult to find convergence regions for these functions of several complex variables. We use the Ward q-addition to describe the known convergence regions of q-Appell and q-Lauricella functions.

Keywords: Euler integral formula; summation and reduction formula; q-integral

1. Introduction

The history of the subject of this article started in 1964 when Srivastava [1] investigated the domain of convergence for the first two functions in this article (for $q = 1$). He then obtained an Euler integral formula for $\mathrm{H_A}$ and the predecessor of formula (58). The proofs used the unique single integral formula for the first Appell function, and the Gauss summation formula and Kummer's first formula, respectively. Three years later [2], Srivastava defined the third function $\mathrm{H_C}$. In the year 1974, Karlsson [3], based on Horn, gave a complete explanation of the convergence regions for these triple functions. The main emphasis of this paper, though, is on rather concrete computations in the spirit of [4]. A thorough introduction to the formal logarithmic notation that we use is made in [5]. This paper is organized as follows. In section 2 we introduce the notation and define the q-Appell functions, the q-Lauricella functions and q-analogues of Srivastava's three triple hypergeometric functions. The convergence regions for some of the q-Appell functions and some of the q-Lauricella functions are given in terms of a certain Ward q-addition. In the third section we state the convergence regions for $\mathrm{H_A}$, $\mathrm{H_B}$ and $\mathrm{H_C}$. In Section 4 we use q-analogues of Euler integral formulas for the two first q-Appell functions from [4] to find Euler integral formulas for q-analogues of Srivastava's triple hypergeometric functions. Finally, we use a summation formula for Φ_1 to find a reduction formula for the q-analogue of $\mathrm{H_A}$. In Section 5 we use a general reduction formula of Karlsson to find two new reduction formulas for q-analogues of Srivastava's triple hypergeometric functions.

2. Definitions

We start by defining the umbral notation [6–8], a mixture of Heine 1846 and Gasper–Rahman [9].

Definition 1. The power function is defined by $q^a \equiv e^{a log(q)}$. We always assume that $0 < |q| < 1$. Let $\delta > 0$ be an arbitrary small number. We will use the following branch of the logarithm: $-\pi + \delta < \mathrm{Im}\,(\log q) \leq \pi + \delta$. This defines a simply connected space in the complex plane.

The variables $a, b, c, \ldots \in \mathbb{C}$ denote certain parameters. The variables i, j, k, l, m and n will denote natural numbers except for certain cases where it will be clear from the context that i will denote the imaginary unit. The q-analogues ($q \in \mathbb{C} \setminus \{1\}$) of a complex number a is defined by:

$$\{a\}_q \equiv \frac{1 - q^a}{1 - q} \tag{1}$$

The q-shifted factorial is given by

$$\langle a; q\rangle_n \equiv \prod_{m=0}^{n-1} (1 - q^{a+m});\ (a; q)_n \equiv \prod_{m=0}^{n-1} (1 - aq^m) \tag{2}$$

Since products of q-shifted factorials occur so often, to simplify them we shall frequently use the more compact notation

$$\langle a_1, \ldots, a_m; q\rangle_n \equiv \prod_{j=1}^{m} \langle a_j; q\rangle_n \tag{3}$$

The operator

$$\sim : \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{\pi i}{\log q} \tag{4}$$

By (4) it follows that

$$\widetilde{\langle a; q\rangle_n} = \prod_{m=0}^{n-1} (1 + q^{a+m}) \tag{5}$$

Assume that $(m, l) = 1$, *i.e.*, m and l relatively prime. The operator

$$\widetilde{\tfrac{m}{l}} : \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{2\pi i m}{l \log q} \tag{6}$$

We will also need another generalization of the tilde operator.

$$ {}_{k}\langle\widetilde{a};q\rangle_n \equiv \prod_{m=0}^{n-1}(\sum_{i=0}^{k-1} q^{i(a+m)}) \tag{7}$$

A q-analogue of a notation by MacRobert and Srivastava, which is equivalent to a product of nk q-shifted factorials.

$$\langle\lambda;q\rangle_{kn} \equiv \langle\triangle(q;k;\lambda);q\rangle_n \equiv \prod_{m=0}^{k-1}\langle\frac{\lambda+m}{k};q\rangle_n \times_k \langle\frac{\widetilde{\lambda+m}}{k};q\rangle_n \tag{8}$$

Furthermore,

$$(a;q)_\infty = \prod_{m=0}^{\infty}(1-aq^m),\ 0<|q|<1 \tag{9}$$

$$(a;q)_\alpha \equiv \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty},\ a \neq q^{-m-\alpha}, m=0,1,\ldots \tag{10}$$

The q-integral is the inverse of the q-derivative

$$\int_0^a f(t,q)\,d_q(t) \equiv a(1-q)\sum_{n=0}^{\infty} f(aq^n,q)q^n,\ 0<|q|<1,\ a\in\mathbb{R} \tag{11}$$

The q-gamma function is given by

$$\Gamma_q(z) \equiv \frac{\langle 1;q\rangle_\infty}{\langle z;q\rangle_\infty}(1-q)^{1-z},\ 0<|q|<1 \tag{12}$$

To save space, the following notation for quotients of Γ_q functions will often be used.

$$\Gamma_q\left[\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_r \end{array}\right] \equiv \frac{\Gamma_q(a_1)\ldots\Gamma_q(a_p)}{\Gamma_q(b_1)\ldots\Gamma_q(b_r)} \tag{13}$$

The following notation is often used when we have long exponents.

$$\mathrm{QE}(x) \equiv q^x \tag{14}$$

The following notation will sometimes be used:

$$\widehat{a} \equiv a \vee \tilde{a} \vee \widetilde{{}^{\frac{m}{n}}a} \vee_k \widetilde{a} \vee \triangle(q;l;\lambda) \tag{15}$$

Definition 2. Generalizing Heine's series we shall define a q-hypergeometric series by

$$\begin{aligned} &{}_{p+p'}\phi_{r+r'}\left[\begin{array}{c} \hat{a}_1,\ldots,\hat{a}_p \\ \hat{b}_1,\ldots,\hat{b}_r \end{array}|q;z||\begin{array}{c}\prod_i f_i(k) \\ \prod_j g_j(k)\end{array}\right] \equiv \\ &\sum_{k=0}^{\infty}\frac{\langle\hat{a}_1;q\rangle_k\ldots\langle\hat{a}_p;q\rangle_k}{\langle 1,\hat{b}_1;q\rangle_k\ldots\langle\hat{b}_r;q\rangle_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+r+r'-p-p'} z^k\frac{\prod_i f_i(k)}{\prod_j g_j(k)} \end{aligned} \tag{16}$$

We assume that the $f_i(k)$ and $g_j(k)$ contain p' and r' factors of the form $\langle\widehat{a(k)};q\rangle_k$ or $(s(k);q)_k$ respectively.

Definition 3. The notation $\sum_{\vec{m}}$ denotes a multiple summation with the indices $m_1, \ldots, m_n$ running over all non-negative integer values. In this connection we put $m \equiv \sum_{j=1}^{n} m_j$.

The first definition is a q-analogue of ([10][(24), p.38]), in the spirit of Srivastava. The second definition is a q-analogue of ([10][(24), p.38]), with the restraint ([10][(29), p.38]), due to Karlsson. It will be clear from the context which of the definitions we use. The vectors

$$(a), (b), (g_i), (h_i), (a'), (b'), (g_i'), (h_i')$$

have dimensions

$$A, B, G_i, H_i, A', B', G_i', H_i'$$

Let

$$1 + B + B' + H_i + H_i' - A - A' - G_i - G_i' \geq 0, i = 1, \ldots, n$$

Then the generalized q–Kampé de Fériet function is defined by

$$\begin{gathered}
\Phi^{A+A':G_1+G_1';\ldots;G_n+G_n'}_{B+B':H_1+H_1';\ldots;H_n+H_n'}\left[\begin{matrix}(\hat{a}) : (\hat{g_1}); \ldots ; (\hat{g_n}) \\ (\hat{b}) : (\hat{h_1}); \ldots ; (\hat{h_n})\end{matrix} |\vec{q}; \vec{x}|| \begin{matrix}(a') : (g_1'); \ldots ; (g_n') \\ (b') : (h_1'); \ldots ; (h_n')\end{matrix}\right] \equiv \\
\sum_{\vec{m}} \frac{\langle(\hat{a}); q_0\rangle_m (a')(q_0, m) \prod_{j=1}^{n} (\langle(\hat{g_j}); q_j\rangle_{m_j} ((g_j')(q_j, m_j) x_j^{m_j})}{\langle(\hat{b}); q_0\rangle_m (b')(q_0, m) \prod_{j=1}^{n} (\langle(\hat{h_j}); q_j\rangle_{m_j} (h_j')(q_j, m_j)\langle 1; q_j\rangle_{m_j})} \times \\
(-1)^{\sum_{j=1}^{n} m_j(1+H_j+H_j'-G_j-G_j'+B+B'-A-A')} \times \\
\text{QE}\left((B + B' - A - A')\binom{m}{2}, q_0\right) \prod_{j=1}^{n} \text{QE}\left((1 + H_j + H_j' - G_j - G_j')\binom{m_j}{2}, q_j\right)
\end{gathered} \tag{17}$$

It is assumed that there are no zero factors in the denominator. We assume that $(a')(q_0, m), (g_j')(q_j, m_j), (b')(q_0, m), (h_j')(q_j, m_j)$ contain factors of the form $\langle a(\hat{k}); q\rangle_k$, $(s; q)_k$, $(s(k); q)_k$ or QE $(f(\vec{m}))$.

For double (triple) q-hypergeometric series, we use $m, n(, p)$ as default values. This notation is used in the last theorem of Section 4.

Definition 4. The vectors

$$(a), (b), (g_i), (h_i), (a'), (b'), (g_i'), (h_i')$$

have dimensions

$$A, B, G, H, A', B', G', H'$$

Let

$$1 + B + B' + H + H' - A - A' - G - G' \geq 0$$

Then the generalized q–Kampé de Fériet function is defined by

$$\Phi^{A+A':G+G'}_{B+B':H+H'}\left[\begin{array}{c}(\hat{a}):(\hat{g_1});\ldots;(\hat{g_n})\\(\hat{b}):(\hat{h_1});\ldots;(\hat{h_n})\end{array}|\vec{q};\vec{x}||\begin{array}{c}(a'):(g_1');\ldots;(g_n')\\(b'):(h_1');\ldots;(h_n')\end{array}\right]\equiv$$
$$\sum_{\vec{m}}\frac{\langle(\hat{a});q_0\rangle_m(a')(q_0,m)\prod_{j=1}^n(\langle(\hat{g_j});q_j\rangle_{m_j}((g_j')(q_j,m_j)x_j^{m_j})}{\langle(\hat{b});q_0\rangle_m(b')(q_0,m)\prod_{j=1}^n(\langle(\hat{h_j});q_j\rangle_{m_j}(h_j')(q_j,m_j)\langle 1;q_j\rangle_{m_j})}\times$$
$$(-1)^{\sum_{j=1}^n m_j(1+H+H'-G-G'+B+B'-A-A')}\times$$
$$\text{QE}\left((B+B'-A-A')\binom{m}{2},q_0\right)\prod_{j=1}^n\text{QE}\left((1+H+H'-G-G')\binom{m_j}{2},q_j\right) \quad (18)$$

where

$$\hat{a}\equiv a\vee\tilde{a}\vee\widetilde{\tfrac{m}{n}a}\vee_k\widetilde{a}\vee\triangle(q;l;\lambda) \quad (19)$$

It is assumed that there are no zero factors in the denominator. We assume that $(a')(q_0,m),(g_j')(q_j,m_j),(b')(q_0,m),(h_j')(q_j,m_j)$ contain factors of the form $\langle a(\hat{k});q\rangle_k$, $(s;q)_k$, $(s(k);q)_k$ or QE $(f(\vec{m}))$. In the rest of the paper, we write $\langle a;q\rangle_m$ instead of $\langle\hat{a};q\rangle_m$.

Definition 5. Let a and b be any elements with commutative multiplication. Then the NWA q-addition is given by

$$(a\oplus_q b)^n\equiv\sum_{k=0}^n\binom{n}{k}_q a^kb^{n-k},\ n=0,1,2,\ldots \quad (20)$$

In many cases, see [11,12], the convergence condition can be stated $(x\oplus_q y)^n<1$.

Definition 6. The statement

$$(x\oplus_q y)^n<1,\ n>N_0,\ n\in\mathbb{N} \quad (21)$$

is denoted by $x\oplus_q y<1$.

And similarly for a finite number of letters.

Definition 7. The four q-Appell functions [11] are given by

$$\Phi_1(a;b,b';c|q;x_1,x_2)\equiv\sum_{m_1,m_2=0}^\infty\frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}}x_1^{m_1}x_2^{m_2}$$
$$\max(|x_1|,|x_2|)<1 \quad (22)$$

$$\Phi_2(a;b,b';c,c'|q;x_1,x_2)\equiv\sum_{m_1,m_2=0}^\infty\frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}}x_1^{m_1}x_2^{m_2},$$
$$|x_1|\oplus_q|x_2|<1 \quad (23)$$

$$\Phi_3(a,a';b,b';c|q;x_1,x_2)\equiv\sum_{m_1,m_2=0}^\infty\frac{\langle a;q\rangle_{m_1}\langle a';q\rangle_{m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}}x_1^{m_1}x_2^{m_2},$$
$$\max(|x_1|,|x_2|)<1 \quad (24)$$

$$\Phi_4(a;b;c,c'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_1+m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1}\langle c';q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}, \quad |\sqrt{x_1}| \oplus_q |\sqrt{x_2}| < 1 \tag{25}$$

Two q-Lauricella functions [12] are given by

$$\Phi_{\mathrm{A}}^{(n)}(a,\vec{b};\vec{c}|q;\vec{x}) \equiv \sum_{\vec{m}} \frac{\langle a;q\rangle_m \langle \vec{b};q\rangle_{\vec{m}} \vec{x}^{\vec{m}}}{\langle \vec{c},\vec{1};q\rangle_{\vec{m}}}, \ |x_1| \oplus_q \ldots \oplus_q |x_n| < 1 \tag{26}$$

$$\Phi_{\mathrm{D}}^{(n)}(a,b_1,\ldots,b_n;c|q;x_1,\ldots,x_n) \equiv \sum_{\vec{m}} \frac{\langle a;q\rangle_m \prod_{j=1}^n \langle b_j;q\rangle_{m_j} x_j^{m_j}}{\langle c;q\rangle_m \prod_{j=1}^n \langle 1;q\rangle_{m_j}}, \ \max(|x|,\ldots,|x_n|) < 1 \tag{27}$$

The q-analogues of the Srivastava triple hypergeometric functions are (the convergence regions will be given in the next session):

$$\mathrm{H_A}(a,b_1,b_2;c_1,c_2|q;x_1,x_2,x_3) \equiv \sum_{m,n,p=0}^{\infty} \frac{\langle a;q\rangle_{m+p}\langle b_1;q\rangle_{m+n}\langle b_2;q\rangle_{n+p}}{\langle 1,c_1;q\rangle_m \langle 1;q\rangle_n \langle 1;q\rangle_p \langle c_2;q\rangle_{n+p}} x_1^m x_2^n x_3^p \tag{28}$$

$$\mathrm{H_B}(a,b_1,b_2;c_1,c_2,c_3|q;x_1,x_2,x_3) \equiv \sum_{m,n,p=0}^{\infty} \frac{\langle a;q\rangle_{m+p}\langle b_1;q\rangle_{m+n}\langle b_2;q\rangle_{n+p}}{\langle 1,c_1;q\rangle_m \langle 1,c_2;q\rangle_n \langle 1,c_3;q\rangle_p} x_1^m x_2^n x_3^p \tag{29}$$

$$\mathrm{H_C}(a,b_1,b_2;c|q;x_1,x_2,x_3) \equiv \sum_{m,n,p=0}^{\infty} \frac{\langle a;q\rangle_{m+p}\langle b_1;q\rangle_{m+n}\langle b_2;q\rangle_{n+p}}{\langle 1;q\rangle_m \langle 1;q\rangle_n \langle 1;q\rangle_p \langle c;q\rangle_{m+n+p}} x_1^m x_2^n x_3^p \tag{30}$$

In the whole paper, the notation $\cong$ denotes a formal equality. We will use the following formulas in the proofs:

$$_2\phi_1(a,b;c|q;z) = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 t^{b-1} \frac{(qt;q)_{c-b-1}}{(zt;q)_a} \, d_q(t) \tag{31}$$

$$\mathrm{B}_q(x,y) = \int_0^1 t^{x-1} \frac{(tq;q)_\infty}{(tq^y;q)_\infty} \, d_q(t), \ y \neq 0,-1,-2,\ldots \tag{32}$$

Proof. Put $a = 0$ in (31). □

Theorem 2.1. *[13], ([14][p.78]), a q-analogue of Picard [15]:*

$$\frac{\Gamma_q(a)\Gamma_q(c-a)}{\Gamma_q(c)} \Phi_1(a;b,b';c|q;x_1,x_2) \cong \int_0^1 t^{a-1} \frac{(qt;q)_{c-a-1}}{(x_1 t;q)_b (x_2 t;q)_{b'}} \, d_q(t) \tag{33}$$

3. Convergence Regions

We adhere to the terminology in Horn [16], Exton ([17][p. 68]) and the notation in Srivastava ([1][p. 98–99]). The positive quantities $\{r_i\}_{i=1}^{3}$ are called the associated radii of convergence for the triple series

$$\sum_{m,n,p} A_{m,n,p} x_1^m x_2^n x_3^p \tag{34}$$

if (34) is absolutely convergent when

$$|x_i| < r_i, i = 1, 2, 3$$

and divergent when

$$|x_i| > r_i, i = 1, 2, 3$$

The q-analogue $\{n\}_q$ is an interpolation between the integer n and 1. Since we will mainly be concerned with the case $\lim_{n\to\infty}$ in this section, the following formula will be of interest to us:

$$\frac{\{2n\}_q}{\{n\}_q} = 1 + q^n \tag{35}$$

Put

$$f_1(m, n, p) \equiv \frac{A_{m+1,n,p}}{A_{m,n,p}},\ f_2(m, n, p) \equiv \frac{A_{m,n+1,p}}{A_{m,n,p}},\ f_3(m, n, p) \equiv \frac{A_{m,n,p+1}}{A_{m,n,p}} \tag{36}$$

and

$$\Psi_i(m, n, p) \equiv \lim_{\epsilon\to\infty} f_i(\epsilon m, \epsilon n, \epsilon p),\ i = 1, 2, 3 \tag{37}$$

The convergence region, a q-deformed hypersurface in $\mathbb{R}^3$, has the parametric representation

$$r_i = |\Psi_i(m, n, p)|^{-1}, i = 1, 2, 3 \tag{38}$$

Consider the function $\mathrm{H_A}$. We have

$$f_1(m, n, p) = \frac{\langle a + m + p, b_1 + m + n; q\rangle_1}{\langle c_1 + m, 1 + m; q\rangle_1},\ \Psi_1(m, n, p) = \frac{\{m + n\}_q\{m + p\}_q}{\{m\}_q^2} \tag{39}$$

$$f_2(m, n, p) = \frac{\langle b_1 + m + n, b_2 + n + p; q\rangle_1}{\langle 1 + n, c_2 + n + p; q\rangle_1},\ \Psi_2(m, n, p) = \frac{\{m + n\}_q}{\{n\}_q} \tag{40}$$

$$f_3(m, n, p) = \frac{\langle a + m + p, b_2 + n + p; q\rangle_1}{\langle 1 + p, c_2 + n + p; q\rangle_1},\ \Psi_3(m, n, p) = \frac{\{m + p\}_q}{\{p\}_q} \tag{41}$$

The convergence border has the Cartesian equation

$$r_1 = \frac{\{m\}_q^2}{\{m + n\}_q\{m + p\}_q},\ r_2 = \frac{\{n\}_q}{\{m + n\}_q},\ r_3 = \frac{\{p\}_q}{\{m + p\}_q} \tag{42}$$

By formula (35), combined with the hypothesis $m = n = p$, this border curve can be approximated by the hypersurface $|x_1| = |x_2x_3|,\ |x_1| < 1$. For $q = 1$, the convergence region is bounded by the curve

$$r + s + t = 1 + st \tag{43}$$

where

$$|x_1| < r, \ |x_2| < s, \ |x_3| < t \tag{44}$$

Consider the function $\mathrm{H_B}$. We have

$$f_1(m,n,p) = \frac{\langle a+m+p, b_1+m+n; q\rangle_1}{\langle 1+m, c_1+m; q\rangle_1}, \ \Psi_1(m,n,p) = \frac{\{m+n\}_q\{m+p\}_q}{\{m\}_q^2} \tag{45}$$

$$f_2(m,n,p) = \frac{\langle b_1+m+n, b_2+n+p; q\rangle_1}{\langle 1+n, c_2+n; q\rangle_1}, \ \Psi_2(m,n,p) = \frac{\{m+n\}_q\{n+p\}_q}{\{n\}_q^2} \tag{46}$$

$$f_3(m,n,p) = \frac{\langle a+m+p, b_2+n+p; q\rangle_1}{\langle 1+p, c_3+p; q\rangle_1}, \ \Psi_3(m,n,p) = \frac{\{m+p\}_q\{n+p\}_q}{\{p\}_q^2} \tag{47}$$

The convergence border has the Cartesian equation

$$r_1 = \frac{\{m\}_q^2}{\{m+n\}_q\{m+p\}_q}, \ r_2 = \frac{\{n\}_q^2}{\{m+n\}_q\{n+p\}_q}, \ r_3 = \frac{\{p\}_q^2}{\{m+p\}_q\{n+p\}_q} \tag{48}$$

For $q = 1$, the convergence region is bounded by the curve

$$r + s + t + 2\sqrt{rst} = 1 \tag{49}$$

where

$$|x_1| < r, \ |x_2| < s, \ |x_3| < t \tag{50}$$

Consider the function $\mathrm{H_C}$. We have

$$f_1(m,n,p) = \frac{\langle a+m+p, b_1+m+n; q\rangle_1}{\langle 1+m, c+m+n+p; q\rangle_1}, \ \Psi_1(m,n,p) = \frac{\{m+n\}_q\{m+p\}_q}{\{m\}_q\{m+n+p\}_q} \tag{51}$$

$$f_2(m,n,p) = \frac{\langle b_1+m+n, b_2+n+p; q\rangle_1}{\langle 1+n, c+m+n+p; q\rangle_1}, \ \Psi_2(m,n,p) = \frac{\{m+n\}_q\{n+p\}_q}{\{n\}_q\{m+n+p\}_q} \tag{52}$$

$$f_3(m,n,p) = \frac{\langle a+m+p, b_2+n+p; q\rangle_1}{\langle 1+p, c+m+n+p; q\rangle_1}, \ \Psi_3(m,n,p) = \frac{\{m+p\}_q\{n+p\}_q}{\{p\}_q\{m+n+p\}_q} \tag{53}$$

The convergence border has the Cartesian equation

$$r_1 = \frac{\{m\}_q\{m+n+p\}_q}{\{m+n\}_q\{m+p\}_q}, \ r_2 = \frac{\{n\}_q\{m+n+p\}_q}{\{m+n\}_q\{n+p\}_q}, \ r_3 = \frac{\{p\}_q\{m+n+p\}_q}{\{m+p\}_q\{n+p\}_q} \tag{54}$$

For $q = 1$, the convergence region is bounded by the curve

$$r < 1, \ s < 1, \ t < 1, \ \text{where } |x_1| < r, \ |x_2| < s, \ |x_3| < t \tag{55}$$

4. q-Integral Formulas

We first collect the formulas that are necessary for the proofs in this section.

Theorem 4.1. *[4] A q-analogue of a well-known double integral representation of the first Appell function by Feldheim 1943 and Koschmieder 1947.*

$$\Phi_1(a;b,b';c|q;x,y)\Gamma_q\left[\begin{array}{c} b,d-b,b',d'-b' \\ d,d' \end{array}\right] \cong \sum_{m,n=0}^{\infty} \frac{\langle a;q\rangle_{m+n}\langle d;q\rangle_m \langle d';q\rangle_n}{\langle 1;q\rangle_m \langle 1;q\rangle_n \langle c;q\rangle_{m+n}} \\ x^m y^n \int_{s=0}^{1}\int_{t=0}^{1} \frac{s^{b-1+m}t^{b'-1+n}}{(q^{d-b}s;q)_{1-d+b}(q^{d'-b'}t;q)_{1-d'+b'}}\, d_q(s)\, d_q(t) \tag{56}$$

Theorem 4.2. *[4] A q-analogue of a well-known double integral representation of the second Appell function.*

$$\int_{u=0}^{1}\int_{v=0}^{1} \frac{u^{b-1}v^{b'-1}}{(q^{c-b}u;q)_{1-c+b}(q^{c'-b'}v;q)_{1-c'+b'}} \sum_{m=0}^{\infty} \frac{\langle a;q\rangle_m}{\langle 1;q\rangle_m} \frac{y^m v^m}{(ux;q)_{a+m}}\, d_q(u)\, d_q(v) \\ \cong \Phi_2(a;b,b';c,c'|q;x,y)\Gamma_q\left[\begin{array}{c} c-b,b,c'-b',b' \\ c,c' \end{array}\right] \tag{57}$$

Theorem 4.3. *[5] A summation formula for the first q-Appell function.*

$$\Phi_1(a;b,b';1+a+b-b'|q;q^{1-b'},-q^{1-b'}) \\ = \Gamma_q\left[\begin{array}{c} 1+a+b-b',1-b',1+\frac{a}{2},1+\frac{a}{2}+\beta,1-b'+\beta \\ 1+a,1+b-b',1+\frac{a}{2}-b',1+\frac{a}{2}-b'+\beta,1+\beta \end{array}\right] \tag{58}$$

where $|q^{1-b'}| < 1$ *and* $\beta \equiv \frac{\log(-1)}{\log q}$.

We now come to the new formulas, which all follow the same pattern: hypergeometric formula implies q-hypergeometric formula implies q-Appell function formula implies triple q-hypergeometric formula. The q-integral formulas are all separated, and all these formulas are purely formal. The difficult thing is not to prove the formulas, but to find them with the help of old literature. At first sight, there is no connection to the Schlosser–Gustavsson theory.

Theorem 4.4. *A q-integral representation of the first q-Srivastava function. A q-analogue of ([1][p. 100]), ([18][p. 2754 (2.1)]).*

$$\mathrm{H_A}(a,b_1,b_2;c_1,c_2|q;x_1,x_2,x_3) \\ \cong \Gamma_q\left[\begin{array}{c} c_2 \\ b_2,c_2-b_2 \end{array}\right] \int_{u=0}^{1} u^{b_2-1}\frac{(qu;q)_{c_2-b_2-1}}{(x_2u;q)_{b_1}(x_3u;q)_a} \\ {}_4\phi_3\left[\begin{array}{c} a,b_1,\infty,\infty \\ c_1 \end{array} |q;x_1|| \begin{array}{c} \cdot \\ (x_2uq^{b_1};q)_k,(x_3uq^a;q)_k \end{array}\right] d_q(u) \tag{59}$$

Proof. Use formula (33).

$$\text{LHS} = \sum_{m=0}^{\infty} \frac{\langle a, b_1; q\rangle_m x_1^m}{\langle 1, c_1; q\rangle_m} \Phi_1(b_2; b_1+m, a+m; c_2|q; x_2, x_3) \stackrel{\text{by}(33)}{=} \sum_{m=0}^{\infty} \frac{\langle a, b_1; q\rangle_m x_1^m}{\langle 1, c_1; q\rangle_m} \Gamma_q \begin{bmatrix} c_2 \\ b_2, c_2-b_2 \end{bmatrix} \int_0^1 u^{b_2-1} \frac{(qu;q)_{c_2-b_2-1}}{(x_2u;q)_{b_1+m}(x_3u;q)_{a+m}} d_q(u) = \text{RHS} \tag{60}$$

□

Theorem 4.5. *A q-integral representation of the third q-Srivastava function.*

$$\begin{aligned} &\mathrm{H_C}(a, b_1, b_2; c|q; x_1, x_2, x_3) \\ &\cong \Gamma_q \begin{bmatrix} c \\ b_2, c-b_2 \end{bmatrix} \int_{u=0}^1 u^{b_2-1} \frac{(qu;q)_{c-b_2-1}}{(x_2u;q)_{b_1}(x_3u;q)_a} \\ &{}_4\phi_3 \begin{bmatrix} a, b_1, \infty & & (uq^{c-b_2};q)_k \\ c-b_2 & |q; x_1|| & (x_2uq^{b_1};q)_k, (x_3uq^a;q)_k \end{bmatrix} d_q(u) \end{aligned} \tag{61}$$

Proof. Use formula (33) again.

$$\text{LHS} = \sum_{m=0}^{\infty} \frac{\langle a, b_1; q\rangle_m x_1^m}{\langle 1, c; q\rangle_m} \Phi_1(b_2; b_1+m, a+m; c+m|q; x_2, x_3) \stackrel{\text{by}(33)}{=} \sum_{m=0}^{\infty} \frac{\langle a, b_1; q\rangle_m x_1^m}{\langle 1, c; q\rangle_m} \Gamma_q \begin{bmatrix} c+m \\ b_2, c+m-b_2 \end{bmatrix} \int_0^1 u^{b_2-1} \frac{(qu;q)_{c+m-b_2-1}}{(x_2u;q)_{b_1+m}(x_3u;q)_{a+m}} d_q(u) = \text{RHS} \tag{62}$$

□

Theorem 4.6. *A q-analogue of ([18][p. 2759 (4.1)]). A q-integral representation of the third q-Srivastava function.*

$$\begin{aligned} &\mathrm{H_C}(a, b_1, b_2; c|q; x_1, x_2, x_3) \\ &\cong \Gamma_q \begin{bmatrix} c \\ a, c-a \end{bmatrix} \int_{u=0}^1 u^{a-1} \frac{(qu;q)_{c-a-1}}{(x_1u;q)_{b_1}(x_3u;q)_{b_2}} \\ &{}_4\phi_3 \begin{bmatrix} b_1, b_2, \infty & & (uq^{c-a};q)_k \\ c-a & |q; x_2|| & (x_1uq^{b_1};q)_k, (x_3uq^{b_2};q)_k \end{bmatrix} d_q(u) \end{aligned} \tag{63}$$

Proof. Similar to above, permute a, b_2 and x_1, x_2 respectively. □

The next three formulas, one for each of $\mathrm{H_{A_q}}$, $\mathrm{H_{B_q}}$ and $\mathrm{H_{C_q}}$, respectively, seem to be new even for the case $q = 1$.

Theorem 4.7.

$$\begin{aligned} &\mathrm{H_A}(a, b_1, b_2; c_1, c_2|q; x_1, x_2, x_3) \\ &\cong \Gamma_q \begin{bmatrix} d, d' \\ b_1, d-b_1, a, d'-a \end{bmatrix} \int_{s=0}^1 \int_{t=0}^1 \sum_{m=0}^{\infty} \frac{\langle b_1-d+1, a-d'+1; q\rangle_m x_1^m}{\langle 1, c_1; q\rangle_m} \times \\ &\times \mathrm{QE}\left(-2\binom{m}{2} - m(2+b_1-d+a-d')\right) \sum_{n,p=0}^{\infty} \frac{\langle b_2; q\rangle_{n+p} \langle d; q\rangle_n \langle d'; q\rangle_p}{\langle 1; q\rangle_n \langle 1; q\rangle_p \langle c_2; q\rangle_{n+p}} \times \\ &\times x_2^n x_3^p \frac{s^{b_1-1+m+n} t^{a+m-1+p}}{(q^{d-b_1-m}s;q)_{1-d+b_1+m}(q^{d'-a-m}t;q)_{1-d'+a+m}} d_q(s)\, d_q(t) \end{aligned} \tag{64}$$

Proof. Use formula (56).

$$
\begin{aligned}
\text{LHS} &= \sum_{m=0}^{\infty} \frac{\langle a, b_1; q\rangle_m x_1^m}{\langle 1, c_1; q\rangle_m} \Gamma_q \left[\begin{matrix} d, d' \\ b_1+m, d-b_1-m, a+m, d'-a-m \end{matrix} \right] \times \\
&\times \sum_{n,p=0}^{\infty} \frac{\langle b_2; q\rangle_{n+p} \langle d; q\rangle_n \langle d'; q\rangle_p}{\langle 1; q\rangle_n \langle 1; q\rangle_p \langle c_2; q\rangle_{n+p}} \times \\
&\times x_2^n x_3^p \int_{s=0}^{1} \int_{t=0}^{1} \frac{s^{b_1-1+m+n} t^{a+m-1+p}}{(q^{d-b_1-m}s; q)_{1-d+b_1+m} (q^{d'-a-m}t; q)_{1-d'+a+m}} \, d_q(s)\, d_q(t) \\
&= \sum_{m=0}^{\infty} \frac{\langle d-b_1-m, d'-a-m; q\rangle_m x_1^m}{\langle 1, c_1; q\rangle_m} \Gamma_q \left[\begin{matrix} d, d' \\ b_1, d-b_1, a, d'-a \end{matrix} \right] \sum_{n,p=0}^{\infty} \frac{\langle b_2; q\rangle_{n+p} \langle d; q\rangle_n \langle d'; q\rangle_p}{\langle 1; q\rangle_n \langle 1; q\rangle_p \langle c_2; q\rangle_{n+p}} \times \\
&\times x_2^n x_3^p \int_{s=0}^{1} \int_{t=0}^{1} \frac{s^{b_1-1+m+n} t^{a+m-1+p}}{(q^{d-b_1-m}s; q)_{1-d+b_1+m} (q^{d'-a-m}t; q)_{1-d'+a+m}} \, d_q(s)\, d_q(t) = \text{RHS}
\end{aligned}
\tag{65}
$$

□

Theorem 4.8.

$$
\begin{aligned}
&\mathrm{H_C}(a, b_1, b_2; c|q; x_1, x_2, x_3) \\
&\cong \Gamma_q \left[\begin{matrix} d, d' \\ b_1, d-b_1, a, d'-a \end{matrix} \right] \int_{s=0}^{1} \int_{t=0}^{1} \sum_{m=0}^{\infty} \frac{\langle b_1-d+1, a-d'+1; q\rangle_m x_1^m}{\langle 1, c; q\rangle_m} \times \\
&\times \mathrm{QE}\left(-2\binom{m}{2} - m(2+b_1-d+a-d') \right) \sum_{n,p=0}^{\infty} \frac{\langle b_2; q\rangle_{n+p} \langle d; q\rangle_n \langle d'; q\rangle_p}{\langle 1; q\rangle_n \langle 1; q\rangle_p \langle c+m; q\rangle_{n+p}} \times \\
&\times x_2^n x_3^p \frac{s^{b_1-1+m+n} t^{a+m-1+p}}{(q^{d-b_1-m}s; q)_{1-d+b_1+m} (q^{d'-a-m}t; q)_{1-d'+a+m}} \, d_q(s)\, d_q(t)
\end{aligned}
\tag{66}
$$

Proof. Use formula (56) again.

$$
\begin{aligned}
\text{LHS} &= \sum_{m=0}^{\infty} \frac{\langle a, b_1; q\rangle_m x_1^m}{\langle 1, c; q\rangle_m} \Gamma_q \left[\begin{matrix} d, d' \\ b_1+m, d-b_1-m, a+m, d'-a-m \end{matrix} \right] \times \\
&\times \sum_{n,p=0}^{\infty} \frac{\langle b_2; q\rangle_{n+p} \langle d; q\rangle_n \langle d'; q\rangle_p}{\langle 1; q\rangle_n \langle 1; q\rangle_p \langle c+m; q\rangle_{n+p}} \times \\
&\times x_2^n x_3^p \int_{s=0}^{1} \int_{t=0}^{1} \frac{s^{b_1-1+m+n} t^{a+m-1+p}}{(q^{d-b_1-m}s; q)_{1-d+b_1+m} (q^{d'-a-m}t; q)_{1-d'+a+m}} \, d_q(s)\, d_q(t) \\
&= \sum_{m=0}^{\infty} \frac{\langle d-b_1-m, d'-a-m; q\rangle_m x_1^m}{\langle 1, c; q\rangle_m} \Gamma_q \left[\begin{matrix} d, d' \\ b_1, d-b_1, a, d'-a \end{matrix} \right] \sum_{n,p=0}^{\infty} x_2^n x_3^p \times \\
&\times \frac{\langle b_2; q\rangle_{n+p} \langle d; q\rangle_n \langle d'; q\rangle_p}{\langle 1; q\rangle_n \langle 1; q\rangle_p \langle c+m; q\rangle_{n+p}} \int_{s=0}^{1} \int_{t=0}^{1} \frac{s^{b_1-1+m+n} t^{a+m-1+p}}{(q^{d-b_1-m}s; q)_{1-d+b_1+m} (q^{d'-a-m}t; q)_{1-d'+a+m}} \, d_q(s)\, d_q(t) \\
&= \text{RHS}
\end{aligned}
\tag{67}
$$

□

Theorem 4.9.

$$\begin{aligned}
&\mathrm{H_B}(a,b_1,b_2;c_1,c_2,c_3|q;x_1,x_2,x_3) \cong \Gamma_q \begin{bmatrix} c_2, c_3 \\ a, b_1, -b_1+c_2, -a+c_3 \end{bmatrix} \times \\
&\times \sum_{m=0}^{\infty} \frac{\langle b_1-c_2+1, a-c_3+1;q\rangle_m x_1^m}{\langle 1,c_1;q\rangle_m} \mathrm{QE}\left(-2\binom{m}{2} - m(2+b_1+a-c_2-c_3)\right) \times \\
&\times \int_{u=0}^{1}\int_{v=0}^{1} \frac{u^{b_1+m-1}v^{a+m-1}}{(q^{c_2-b_1-m}u;q)_{1-c_2+b_1+m}(q^{c_3-a-m}v;q)_{1-c_3+a+m}} \sum_{n=0}^{\infty} \frac{\langle b_2;q\rangle_n x_3^n v^n}{\langle 1;q\rangle_n (ux_2;q)_{b_2+n}}\, d_q(u)\, d_q(v)
\end{aligned} \tag{68}$$

Proof. Use formula (57).

$$\begin{aligned}
\mathrm{LHS} &= \sum_{m=0}^{\infty} \frac{\langle a,b_1;q\rangle_m x_1^m}{\langle 1,c_1;q\rangle_m} \Phi_2(b_2;b_1+m,a+m;c_2,c_3|q;x_2,x_3) \\
&= \sum_{m=0}^{\infty} \frac{\langle a,b_1;q\rangle_m x_1^m}{\langle 1,c_1;q\rangle_m} \Gamma_q \begin{bmatrix} c_2, c_3 \\ c_2-b_1-m, b_1+m, c_3-a-m, a+m \end{bmatrix} \times \\
&\times \int_{u=0}^{1}\int_{v=0}^{1} \frac{u^{b_1+m-1}v^{a+m-1}}{(q^{c_2-b_1-m}u;q)_{1-c_2+b_1+m}(q^{c_3-a-m}v;q)_{1-c_3+a+m}} \sum_{n=0}^{\infty} \frac{\langle b_2;q\rangle_n x_3^n v^n}{\langle 1;q\rangle_n (ux_2;q)_{b_2+n}}\, d_q(u)\, d_q(v) \\
&= \mathrm{RHS}
\end{aligned} \tag{69}$$

□

For the notation in the next formula, see Section 2.

Theorem 4.10. *A summation formula for the q-analogue of Srivastava's first triple hypergeometric function, which seems to be new even for the case $q = 1$.*

$$\begin{aligned}
&\mathrm{H_A}(a,b_1,b_2;c,1-a+b_1+b_2|q;x,q^{1-a},-q^{1-a}||q^{-mn-mp}) \\
&= \Gamma_q \begin{bmatrix} 1-a+b_1+b_2, 1-a, 1+\frac{b_2}{2}, 1+\frac{b_2}{2}+\beta, 1-a+\beta \\ 1+b_2, 1+b_1-a, 1+\frac{b_2}{2}-a, 1+\frac{b_2}{2}-a+\beta, 1+\beta \end{bmatrix} \times \\
&\times\ {}_4\phi_3(a,b_1,a-\tfrac{b_2}{2},a-\tfrac{b_2}{2}-\beta;c,a,a-\beta|q;xq^{-b_2}),\ \text{where } \beta \equiv \frac{\log(-1)}{\log q}
\end{aligned} \tag{70}$$

Proof. Use formula (58).

$$\begin{aligned}
\mathrm{LHS} &= \sum_{m=0}^{\infty} \frac{\langle a,b_1;q\rangle_m x^m}{\langle 1,c;q\rangle_m} \Phi_1(b_2;b_1+m,a+m;1-a+b_1+b_2|q;q^{1-a-m},-q^{1-a-m}) \\
&= \sum_{m=0}^{\infty} \frac{\langle a,b_1;q\rangle_m x^m}{\langle 1,c;q\rangle_m} \Gamma_q \begin{bmatrix} 1-a+b_1+b_2, 1-a-m, 1+\frac{b_2}{2}, 1+\frac{b_2}{2}+\beta, 1-a-m+\beta \\ 1+b_2, 1+b_1-a, 1+\frac{b_2}{2}-a-m, 1+\frac{b_2}{2}-a-m+\beta, 1+\beta \end{bmatrix} \\
&= \sum_{m=0}^{\infty} \frac{\langle a,b_1;q\rangle_m x^m}{\langle 1,c;q\rangle_m} \Phi_1(b_2;b_1+m,a+m;1-a+b_1+b_2|q;q^{1-a-m},-q^{1-a-m}) \\
&= \sum_{m=0}^{\infty} \frac{\langle a,b_1;q\rangle_m x^m}{\langle 1,c;q\rangle_m} \Gamma_q \begin{bmatrix} 1-a+b_1+b_2, 1-a, 1+\frac{b_2}{2}, 1+\frac{b_2}{2}+\beta, 1-a+\beta \\ 1+b_2, 1+b_1-a, 1+\frac{b_2}{2}-a, 1+\frac{b_2}{2}-a+\beta, 1+\beta \end{bmatrix} \times \\
&\times \frac{\langle 1+\frac{b_2}{2}-a-m, 1+\frac{b_2}{2}-a-m+\beta;q\rangle_m}{\langle 1-a-m, 1-a-m+\beta;q\rangle_m} = \mathrm{RHS}
\end{aligned} \tag{71}$$

□

5. A q-Analogue of a General Reduction Formula by Karlsson

We continue our treatment from the book [5], where a reduction formula for a q-Lauricella function was presented. We first give the general reduction formula, then we specify to triple functions.

Theorem 5.1. *A q-analogue of ([19][3.1 p.202]).*

If $\{C_{m,n}\}_{m,n=0}^{\infty}$ is a sequence of bounded complex numbers then

$$\begin{aligned}&\sum_{m,n,p=0}^{\infty}\frac{C_{m,n+p}\langle b;q\rangle_{n+m}\langle b;q\rangle_{p+m}x_1^m x_2^n(-x_2)^p}{\langle 1;q\rangle_m\langle 1;q\rangle_n\langle 1;q\rangle_p}\\ &=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{C_{m,2k}\langle \widetilde{b+m};q\rangle_k\langle b;q\rangle_m\langle b;q\rangle_{m+k}x_1^m x_2^{2k}}{\langle 1;q\rangle_m\langle 1,\widetilde{1};q\rangle_k}\end{aligned}\tag{72}$$

Proof.

$$\begin{aligned}\text{LHS}=&\sum_{m,n,p=0}^{\infty}\frac{C_{m,n}\langle b;q\rangle_{n+m-p}\langle b;q\rangle_{p+m}x_1^m x_2^n(-1)^p}{\langle 1;q\rangle_m\langle 1;q\rangle_{n-p}\langle 1;q\rangle_p}\\ =&\sum_{m,n,p=0}^{\infty}\frac{C_{m,n}\langle b;q\rangle_m\langle b+m;q\rangle_p\langle b;q\rangle_{m+n}\langle -n;q\rangle_p x_1^m x_2^n(-1)^p q^{p(-b+1-m)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n\langle 1,-b+1-m-n;q\rangle_p}\\ =&\sum_{m=0}^{\infty}\sum_{t=0,t\text{ even}}^{\infty}\frac{C_{m,t}\langle b;q\rangle_m\langle b;q\rangle_{m+t}x_1^m x_2^t}{\langle 1;q\rangle_m\langle 1;q\rangle_t}\frac{\langle \widetilde{b+m};q\rangle_{\frac{t}{2}}\langle \frac{1}{2};q^2\rangle_{\frac{t}{2}}}{\langle b+m+\frac{t}{2};q\rangle_{\frac{t}{2}}}=\text{RHS}\end{aligned}\tag{73}$$

□

Corollary 5.2. *A reduction formula for the q-analogue of Srivastava's first triple hypergeometric function, a q-analogue of ([19][3.2 p.202]).*

$$\begin{aligned}&\mathrm{H_A}(a,a,b;c_1,c_2|q;x_1,x_2,-x_2)\\ &=\Phi_{1:1;5}^{1:2;6}\left[\begin{array}{c}a:a,\infty;\triangle(q;2;b),\infty\\ \infty:c_1;\widetilde{1},\triangle(q;2;c_2)\end{array}|q;x_1;x_2^2||\begin{array}{c}\langle \widetilde{a+m};q\rangle_n\\ .\end{array}\right]\end{aligned}\tag{74}$$

Proof.

$$\text{Put } C_{m,n}\equiv\frac{\langle b;q\rangle_n}{\langle c_1;q\rangle_m\langle c_2;q\rangle_n}\text{ in (73)}$$

$$\text{LHS}=\sum_{m,k=0}^{\infty}\frac{\langle \widetilde{a+m};q\rangle_k\langle b;q\rangle_{2k}\langle a;q\rangle_{m+k}\langle a;q\rangle_m x_1^m x_2^{2k}}{\langle c_2;q\rangle_{2k}\langle 1,c_1;q\rangle_m\langle 1,\widetilde{1};q\rangle_k}=\text{RHS}\tag{75}$$

□

Corollary 5.3. *A reduction formula for the q-analogue of Srivastava's third triple hypergeometric function, which seems to be new even for the case $q=1$.*

$$\begin{aligned}&\mathrm{H_C}(a,a,b;a|q;x_1,x_2,-x_2)\\ &=\Phi_{0:0;4}^{0:1;5}\left[\begin{array}{c}-:a;\triangle(q;2;b)\\ -:-;\widetilde{1},2\infty\end{array}|q;x_1;x_2^2||\begin{array}{c}\langle \widetilde{a+m};q\rangle_n\\ \langle a+m+n;q\rangle_n\end{array}\right]\end{aligned}\tag{76}$$

Proof.

$$\text{Put } C_{m,n} \equiv \frac{\langle b;q\rangle_n}{\langle a;q\rangle_{m+n}} \text{ in (73)}$$

$$\text{LHS} = \sum_{m,k=0}^{\infty} \frac{\langle \widetilde{a+m};q\rangle_k \langle b;q\rangle_{2k} \langle a;q\rangle_{m+k} \langle a;q\rangle_m x_1^m x_2^{2k}}{\langle a;q\rangle_{m+2k} \langle 1;q\rangle_m \langle 1,\widetilde{1};q\rangle_k} = \text{RHS} \tag{77}$$

□

6. Conclusions

We note that the convergence regions for some of the q-Appell functions are greater than for the case $q = 1$ as has been expounded in [11,12]. This is where the convergence discussions in Section 3 of the present paper started. This article is the second in a series of four articles treating roughly the same theme. In the first article [4], mainly based on Euler integral representations of q-Appell functions from two papers by Koschmieder, we found formal q-integral representations of Appell functions. This made the way for Section 4 and q-integral representations for triple functions. The third article in the series [20] contains convergence aspects and numerical function values for q-Horn functions, where a much increased convergence region is found. The fourth article in the series [21] contains convergence aspects for triple q-hypergeometric functions. In the last section we found a q-analogue of a general reduction formula for a triple sum, which is not to be confused with the reduction formulas for vector double sums in [22]. The summation formula for the first q-Appell function as well as two other q-analogues of a general reduction formula for a triple sum can be found in the book [5].

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