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Article

Generalized q-Stirling Numbers and Their Interpolation Functions

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Abstract: In this paper, we define the generating functions for the generalized q-Stirling numbers of the second kind. By applying Mellin transform to these functions, we construct interpolation functions of these numbers at negative integers. We also derive some identities and relations related to q-Bernoulli numbers and polynomials and q-Stirling numbers of the second kind.

Keywords: q-Bernoulli numbers and polynomials; generalized q-Stirling numbers of the second kind; q-zeta function

Classification: MSC 11B68, 11B39, 11B73

1. Introduction, Definitions and Notations

q-Stirling numbers of the second kind were first defined by Carlitz [1]. After Carlitz's paper, many combinatorial papers have centered around the q-analogue, the earliest by Milne [2]; (among others) also see [3–20].

In [16], Simsek studied the generating functions of the fermionic and deformic Stirling numbers. By applying the derivative operator $\frac{d^n}{dt^n} \mid_{t=0}$ to these functions, he constructed interpolation functions of these numbers at negative integers.

It is well known that the Stirling numbers of the second kind $S(n,k)$ are defined by means of the generating function [16–21]:

$$F_S(t,k) = \frac{(-1)^k}{k!}(1-e^t)^k = \sum_{n=0}^{\infty} S(n,k)\frac{t^n}{n!} \tag{1}$$

It is also well known that the usual Stirling numbers of the second kind $S^{(\alpha)}(n,k)$ are defined by means of the generating function [16–21]:

$$F_\alpha(t,k) = \frac{(-1)^k}{k!}e^{\alpha t}(1-e^t)^k = \sum_{n=0}^{\infty} S^{(\alpha)}(n,k)\frac{t^n}{n!} \tag{2}$$

and

$$F_\alpha^r(t,k) = \frac{(-1)^k}{k!}e^{\alpha t}(1-e^{rt})^k = \sum_{n=0}^{\infty} S^{(\frac{\alpha}{r})}(n,k)\frac{t^n}{n!} \tag{3}$$

Let $q \in \mathbb{C}$ with $\mid q \mid< 1$. Some well known results related to the q-integers are given by (see for detail [1–28]):

$$\begin{aligned} [n,q] &= [n] = \frac{1-q^n}{1-q} \\ [k+j] &= [k] + q^k[j] \\ [kj] &= [k][j,q^k] \end{aligned}$$

and

$$[n]! = [n][n-1]...[2][1],\ [0]! = 1 \text{ and } \begin{pmatrix} n \\ k \end{pmatrix}_q = \frac{[n]!}{[n-k]![k]!}$$

Note that $\lim_{q\to 1}[n] = n$, [1–28].

Generating functions of the q-Stirling numbers of the second kind were defined in [8]:

$$F_q(t) = q^{\begin{pmatrix} k \\ 2 \end{pmatrix}} \frac{1}{[k]!}\sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} (-1)^{k-j} q^{\begin{pmatrix} k-j \\ 2 \end{pmatrix}} \exp([j]t) \tag{4}$$

and

$$[x]^n = \sum_{k=0}^{n} \begin{pmatrix} m \\ n \end{pmatrix}_q [k]! q^{\begin{pmatrix} k \\ 2 \end{pmatrix}} S_2(n,k,q)$$

By the above equation, we have [1,8]

$$S_2(n,k,q) = q^{-\begin{pmatrix} k \\ 2 \end{pmatrix}} \frac{1}{[k]!}\sum_{j=0}^{k}(-1)^j q^{\begin{pmatrix} j \\ 2 \end{pmatrix}} \begin{pmatrix} k \\ j \end{pmatrix}_q [k-j]^n$$

2. New Generating Functions for q-Stirling Numbers of the Second Kind

Here, by using the same method of Simsek [16,18], we construct interpolation functions for the generalized q-Stirling numbers of the second kind. We shall define new functions to interpolate the second kind q-Stirling numbers. We define q-version of Equations (1) and (2) functions. Generalized q-Stirling numbers of the second kind are defined by means of the following generating functions:

$$\begin{aligned} F_{q,\alpha}(t,k) &= \frac{1}{[k]!}\left(-\sum_{n=0}^{\infty} q^n \exp\left([n-\frac{\alpha}{k}]t\right)\right)^{-k} \\ &= \sum_{n=0}^{\infty} S^{(\alpha)}(n,k,q)\frac{t^n}{n!} \end{aligned} \tag{5}$$

or

$$\begin{aligned} F^{*}_{q,\alpha}(t,k) &= \frac{1}{[k]!}\sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j} q^k e^{[\alpha+j]t} \\ &= \sum_{n=0}^{\infty}\frac{1}{[k]!}\sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j} q^k \frac{[\alpha+j]^n t^n}{n!} \end{aligned} \tag{6}$$

and

$$F^{*}_{q,\alpha}(t) = \sum_{n=0}^{\infty} S^{(\alpha)}(n,k,q)\frac{t^n}{n!}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equations, we easily obtain that

$$\begin{aligned} S^{(\alpha)}(n,k,q) &= \frac{1}{[k]!}\sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j} q^k [\alpha+j]^n \\ &= \frac{(1-q)^{-n}}{[k]!}\sum_{j=0}^{k}\sum_{c=0}^{n}(-1)^{k+j+c}\binom{k}{j}\binom{n}{c} q^{k+(\alpha+j)c} \end{aligned}$$

Observe that when $q \to 1$, Equations (5) and (6) reduce to Equation (2). When $q \to 1$ in Equation (5), we have

$$\begin{aligned} \lim_{q\to 1} F_{q,\alpha}(t,k) &= \frac{1}{k!}\left(-\sum_{n=0}^{\infty} q^n \exp\left(nt-\frac{\alpha}{k}t\right)\right)^{-k} \\ \sum_{n=0}^{\infty} S^{(\alpha)}(n,k)\frac{t^n}{n!} &= \frac{(-1)^k}{k!}\left(e^t-1\right)^k = \frac{(-1)^k \exp(\alpha t)}{k!}\sum_{y=0}^{k}\binom{k}{y}(-1)^y \exp(yt) \\ &= \sum_{n=0}^{\infty}\left(\frac{(-1)^k \exp(\alpha t)}{k!}\sum_{y=0}^{k}\binom{k}{y}(-1)^y y^n\right)\frac{t^n}{n!} \end{aligned}$$

Here we use the binomial expansion and the fact that

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equations, we easily obtain that

$$S(n,k)=\frac{(-1)^k\exp(\alpha t)}{k!}\sum_{y=0}^{k}\begin{pmatrix}k\\y\end{pmatrix}(-1)^y y^n$$

We also see that

$$S^{(\alpha)}(n,k)=\frac{(-1)^k}{k!}\sum_{y=0}^{k}\begin{pmatrix}k\\y\end{pmatrix}(-1)^y(\alpha+y)^n$$

with recurrence relation in [21]

$$S^{(\alpha)}(n,k)=S^{(\alpha)}(n-1,k-1)+(k+\alpha)S^{(\alpha)}(n-1,k)$$

We also define the following generating function which is generalized Equation (6):

$$\begin{aligned}F^{*}_{q,\frac{\alpha}{r}}(t,k) &= \frac{1}{[k]!}\sum_{j=0}^{k}(-1)^{k+j}\begin{pmatrix}k\\j\end{pmatrix}q^k e^{[\alpha+rj]t} \\ &= \sum_{n=0}^{\infty}S^{(\frac{\alpha}{r})}(n,k,q)\frac{t^n}{n!}\end{aligned}\tag{7}$$

By Equation (7), we obtain

$$S^{(\frac{\alpha}{r})}(n,k,q)=\frac{1}{[k]!}\sum_{j=0}^{k}\sum_{d=0}^{n}(-1)^{k+j}\begin{pmatrix}k\\j\end{pmatrix}\begin{pmatrix}n\\d\end{pmatrix}q^{k+d\alpha}[\alpha]^{n-d}[rj]^d$$

By using Pb. 189 in [24], we can write

$$S(n,k)=\frac{1}{k!}\left(k^n-\begin{pmatrix}k\\1\end{pmatrix}(k-1)^n+\begin{pmatrix}k\\2\end{pmatrix}(k-2)^n-...+(-1)^k 0^n\right),n\geq 1\tag{8}$$

We give the q-version of the above equation as follows

$$S(n,k,q)=\frac{1}{[k]!}\left([k]^n-\begin{pmatrix}k\\1\end{pmatrix}_q[k-1]^n+\begin{pmatrix}k\\2\end{pmatrix}_q[k-2]^n\_...+[-1]^k[0]^n\right)$$

3. Some Special Zeta Functions

Throughout this section, let $s\in\mathbb{C}$ with $Res>1$. By using the same method of Simsek [16,18], we construct interpolation functions for the generalized q-Stirling numbers of the second kind. By applying the Mellin transform to Equation (4), we have

$$\begin{aligned}&\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}F_q(-t)dt \\ =\ & q^{\begin{pmatrix}k\\2\end{pmatrix}}\frac{1}{[k]!}\sum_{j=0}^{k}\begin{pmatrix}k\\j\end{pmatrix}(-1)^{k-j}q^{\begin{pmatrix}k-j\\2\end{pmatrix}}\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\exp(-[j]t)dt \\ =\ & q^{\begin{pmatrix}k\\2\end{pmatrix}}\frac{1}{[k]!}\sum_{j=0}^{k}\begin{pmatrix}k\\j\end{pmatrix}(-1)^{k-j}q^{\begin{pmatrix}k-j\\2\end{pmatrix}}\frac{1}{[j]^{s-1}}\end{aligned}$$

Thus we define the following zeta function:

Definition 1 *Let $s \in \mathbb{C}$ and $k \in \mathbb{Z}^+$, the set of positive integers.*

$$z(s,k) = q^{\binom{k}{2}} \frac{1}{[k]!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} q^{\binom{k-j}{2}} \frac{1}{[j]^{s-1}}$$

By substituting $s = 1-n$ into above definition, we have

$$z_q(1-n,k) = q^{\binom{k}{2}} \frac{1}{[k]!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} q^{\binom{k-j}{2}} [j]^n$$

Using the above relation, we arrive at the following result:

Theorem 1 *Let n and k be positive integers. Then*

$$z_q(1-n,k) = S(n,k,q)$$

By applying the Mellin transform to the Equation (2), we have

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} F_\alpha(-t,k)dt = \frac{(-1)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{(j+\alpha)^{s-1}}$$

So we have the following definition:

Definition 2 *Let $s \in \mathbb{C}$ and $k \in \mathbb{Z}^+$.*

$$z^\alpha(s,k) = \frac{(-1)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{(j+\alpha)^{s-1}}$$

Remark 1 *If $\alpha = 0$ above, then we have*

$$z^0(s,k) = \frac{(-1)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{j^{s-1}}$$

For $s = 1-n$, $n \in \mathbb{Z}^+$ above equation, we obtain

$$\begin{aligned} z^\alpha(1-n,k) &= \frac{(-1)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{(j+\alpha)^{-n}} \\ &= S^{(\alpha)}(n,k) \end{aligned}$$

Therefore, we arrive at the following result:

Corollary 1 *Let $n \in \mathbb{Z}^+$. Then we have*

$$z^\alpha(1-n,k) = S^{(\alpha)}(n,k)$$

Remark 2

$$\begin{aligned}(e^t-1)^k &= \sum_{j=0}^{k}\begin{pmatrix}k\\ j\end{pmatrix}(-1)^j e^{jt}\\ &= \sum_{n=0}^{\infty}\left(k^n\begin{pmatrix}k\\ 0\end{pmatrix}-\begin{pmatrix}k\\ j\end{pmatrix}(k-1)^n+...+\begin{pmatrix}k\\ k\end{pmatrix}(-1)^k(k-k)^n\right)\frac{t^n}{n!}\end{aligned}$$

By using Equation (8), we have

$$(e^t-1)^k = k!\sum_{n=0}^{\infty} z^0(1-n,k)\frac{t^n}{n!}$$

By applying the Mellin transform to Equations (6) and (7), we define the following functions, respectively:

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}F^*_{q,\alpha}(-t,k)dt = z_q^{\alpha}(s,k)$$
$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}F^*_{q,\frac{\alpha}{r}}(-t,k)dt = z_q^{\frac{\alpha}{r}}(s,k)$$

The above functions interpolate the numbers $S^{(\alpha)}(n,k,q)$ and $S^{(\frac{\alpha}{r})}(n,k,q)$ at negative integers, respectively.

4. Relations between Bernoulli Numbers of Order k and Stirling Numbers of the Second Kind

Let

$$F_B(t) = \frac{t^k}{(e^t-1)^k} = \sum_{n=0}^{\infty} B_n^{(k)}\frac{t^n}{n!} \tag{9}$$

where the coefficients $B_n^{(k)}$ are called Bernoulli numbers of order k [19,20,28]. By Equation (1), we have

$$F_S(t,k) = \frac{1}{k!}(e^t-1)^k \tag{10}$$

By using Equations (9) and (10), relation between $F_B(t)$ and $F_S(t)$ is given by

$$F_B(t)F_S(t) = \frac{t^k}{k!}$$

By using the above relation, we have

$$\sum_{n=0}^{\infty} B_n^{(k)}\frac{t^n}{n!}\sum_{n=0}^{\infty} S(n,k)\frac{t^n}{n!} = \frac{t^k}{k!}$$

By using Cauchy product above, we get

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n} B_j^{(k)} S(n-j,k)\frac{1}{k!}\frac{t^n}{(n-k)!} = \frac{t^k}{k!}$$

By comparing coefficients of t^k in both sides of the above equation, we arrive at the following theorem:

Theorem 2 *Let $n, k \in \mathbb{N}$. We have*

$$n!k!\sum_{j=0}^{n}\binom{n}{j}B_j^{(k)}S(n-j,k)=\begin{cases}1, \text{ if } n=k\\ 0, \text{ if } n\neq k\end{cases}$$

Remark 3 *The Barnes' type multiple Changhee q-Bernoulli polynomials are defined by means of the following generating function (see for details [28]):*

$$\begin{aligned} G_q^{(k)}(w,t \quad &| \quad w_1,w_2,...,w_k) \\ &= (-t)^k(\prod_{i=1}^{k}w_i)\sum_{n_1,n_2,...,n_k=0}^{\infty}q^{w+n_1w_1+n_2w_2+...+n_kw_k}e^{[w+n_1w_1+n_2w_2+...+n_kw_k]t} \\ &= \sum_{n=0}^{\infty}\frac{B_n^{(k)}(w:q\mid w_1,w_2,...,w_k)t^n}{n!} \quad (\mid t\mid<2\pi\ ) \end{aligned} \tag{11}$$

with as usual,

$$\sum_{n_1,n_2,...,n_k=0}^{\infty}=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}...\sum_{n_k=0}^{\infty}$$

It follows from Equation (11) that

$$\lim_{q\to1}G_q^{(k)}(w,t\mid w_1,w_2,...,w_k)=\frac{e^{tw}(tw_1)(tw_2)...(tw_k)}{(e^{tw_1}-1)(e^{tw_2}-1)...(e^{tw_k}-1)} \tag{12}$$

This gives the generating function of Barnes' type multiple Bernoulli numbers. Thus we get the following limit relationship:

$$\lim_{q\to1}B_n^{(k)}(w:q\mid w_1,w_2,...,w_k)=B_n^{(k)}(w\mid w_1,w_2,...,w_k)$$

This gives the Barnes' type multiple Bernoulli numbers as a limit when q approaches 1.

If $w=0$ and $w_1=w_2=w_k=1$ in Equation (12), we have

$$\lim_{q\to1}G_q^{(k)}(0,t\mid 1,1,...,1)=\frac{t^k}{(e^t-1)^k}=\sum_{n=0}^{\infty}B_n^{(k)}\frac{t^n}{n!}$$

Using Equation (12), we define

$$\begin{aligned} F_B^{(n)}(t \quad &| \quad w_1,w_2,...,w_k)=\frac{t^{kn}(w_1w_2...w_k)^n}{(e^{tw_1}-1)^n(e^{tw_2}-1)^n...(e^{tw_k}-1)^n} \\ &= \sum_{n=0}^{\infty}B_n^{(k,n)}(0\mid w_1,w_2,...,w_k)\frac{t^n}{n!} \end{aligned} \tag{13}$$

Observe that when $n=1$, $B_n^{(k,n)}(0\mid w_1,w_2,...,w_k)$ reduces to $B_n^{(k)}(0\mid w_1,w_2,...,w_k)$. We define

$$\begin{aligned} F_S^{(n)}(t \quad &| \quad w_1,w_2,...,w_k)=\left(\frac{1}{k!}\right)^n(e^{tw_1}-1)^n(e^{tw_2}-1)^n...(e^{tw_k}-1)^n \\ &= \sum_{n=0}^{\infty}Y^{(n)}(n,k\mid w_1,w_2,...,w_k)\frac{t^n}{n!} \end{aligned} \tag{14}$$

By Equations (13) and (14), we have

$$\left(\frac{k!}{w_1 w_2 ... w_k}\right)^n F_S^{(n)}(t \mid w_1, w_2, ..., w_k) F_B^{(n)}(t \mid w_1, w_2, ..., w_k) = t^{kn}$$

By using the above equation, we have

$$\left(\frac{k!}{w_1 w_2 ... w_k}\right)^n \sum_{m=0}^{\infty} Y^{(n)}(m, k \mid w_1, w_2, ..., w_k) \frac{t^m}{m!} \sum_{m=0}^{\infty} B_m^{(k,n)}(0 \mid w_1, w_2, ..., w_k) \frac{t^m}{m!} = t^{kn}$$

By applying the Cauchy product to the above, we arrive at the following theorem, which is the generalized form of Theorem 2:

Theorem 3 *Let* $n, k \in \mathbb{N}$. *We have*

$$m!\left(\frac{k!}{w_1 w_2 ... w_k}\right)^n \sum_{j=0}^{m} \binom{m}{j} B_n^{(k,n)}(0 \mid w_1, w_2, ..., w_k) Y^{(n)}(m-j, k \mid w_1, w_2, ..., w_k) = \begin{cases} 1, \text{ if } m = nk \\ 0, \text{ if } m \neq nk \end{cases}$$

Observe that

$$\begin{aligned} F_S^{(1)}(t \mid 1, 1, ..., 1) = \sum_{n=0}^{\infty} Y^{(1)}(n, k \mid 1, 1, ..., 1) \frac{t^n}{n!} \\ = \frac{1}{k!}(e^t - 1)^k \\ = \sum_{n=0}^{\infty} S(n, k) \frac{t^n}{n!} \end{aligned}$$

Thus we have

$$Y^{(1)}(n, k \mid 1, 1, ..., 1) = S(n, k)$$

5. Conclusions

q-Stirling numbers of the second kind arise in many different generating functions for various statistical partitions. The theory of q-Stirling numbers is enriched by combinatorial interpretations. By using these numbers, one can investigate orthogonality relations, recurrences, explicit expressions, and generating functions for the generalized (q-) Stirling numbers. Recently, many authors have generalized the Stirling numbers by differential operators. The Stirling numbers are related to Newton's interpolation, (q-) Lah numbers, exponential generating functions, q-calculus and related topics, combinatorial enumeration problems, Binomial coefficients and Bell numbers.

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