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Article

On the Content Bound for Real Quadratic Field Extensions

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Abstract: Let K be a finite extension of $\mathbb{Q}$ and let $S = \{\nu\}$ denote the collection of normalized absolute values on K. Let V_K^+ denote the additive group of adeles over K and let $\mathbf{c} : V_K^+ \to \mathbb{R}_{\geq 0}$ denote the content map defined as $\mathbf{c}(\{a_\nu\}) = \prod_{\nu \in S} \nu(a_\nu)$ for $\{a_\nu\} \in V_K^+$. A classical result of J. W. S. Cassels states that there is a constant $c > 0$ depending only on the field K with the following property: if $\{a_\nu\} \in V_K^+$ with $\mathbf{c}(\{a_\nu\}) > c$, then there exists a non-zero element $b \in K$ for which $\nu(b) \leq \nu(a_\nu)$, $\forall \nu \in S$. Let c_K be the greatest lower bound of the set of all c that satisfy this property. In the case that K is a real quadratic extension there is a known upper bound for c_K due to S. Lang. The purpose of this paper is to construct a new upper bound for c_K in the case that K has class number one. We compare our new bound with Lang's bound for various real quadratic extensions and find that our new bound is better than Lang's in many instances.

Keywords: adele group; content map; real quadratic extension

Classification: MSC 11R56, 11R11

1. Introduction

Let K be a finite extension of $\mathbb{Q}$ and let $S = \{\nu\}$ denote the collection of normalized absolute values on K. Let V_K^+ denote the additive group of adeles over K and let K^+ denote the additive group of K viewed as a subgroup of V_K^+. Let $\mathbf{c} : V_K^+ \to \mathbb{R}_{\geq 0}$ denote the *content map* defined as $\mathbf{c}(\{a_\nu\}) = \prod_{\nu \in S} \nu(a_\nu)$ for $\{a_\nu\} \in V_K^+$. We have the following classical result due to J. W. S. Cassels [1](Lemma, p. 66).

Proposition 1.1 *(J. W. S. Cassels) There is a constant $c > 0$ depending only on the field K with the following property: Let $\{a_\nu\} \in V_K^+$ be an adele for which $\mathbf{c}(\{a_\nu\}) > c$. Then there exists a non-zero element $b \in K^+ \subseteq V_K^+$ for which $\nu(b) \leq \nu(a_\nu)$, $\forall \nu \in S$.*

Let $\{c\}$ denote the set of all positive constants for which Proposition 1.1 holds. Then $\{c\}$ is a non-empty set of real numbers that is bounded below by 0. Thus $\inf(\{c\})$ exists. We define $c_K = \inf(\{c\})$ to be the *content bound for* K. In the case that K is a real quadratic field extension there is a known upper bound for c_K due to S. Lang [2](Chapter V, §1, Theorem 0).

Proposition 1.2 *(S. Lang) Let d be a positive square-free integer and let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic extension.*

(i) If $d \equiv 1 \pmod 4$, then $c_K \leq (2 + 2\sqrt{d})^2$,
(ii) if $d \equiv 2, 3 \pmod 4$, then $c_K \leq 16d$.

In this paper we construct a new upper bound for c_K in the case that K is a real quadratic extension with class number one. We prove the following proposition.

Proposition 1.3 *Let K be a real quadratic extension with class number one. Let f be a fundamental unit of K with $f > 1$. Then $c_K \leq f$.*

It is of interest to compare our new bound with Lang's bound for various extensions with class number one. For example, if $K = \mathbb{Q}(\sqrt{86})$, then the fundamental unit $f = 10405 + 1122\sqrt{86} > 20810$. Since $16 \cdot 86 = 1376 < 20810$, in this case Lang's bound is better. On the other hand, if $K = \mathbb{Q}(\sqrt{93})$, then the fundamental unit $f = \frac{29+3\sqrt{93}}{2} < 29$. Since $(2 + 2\sqrt{93})^2 > 453$, the new bound of Proposition 1.3 is better. Overall, our new bound is better than Lang's in many instances.

For the convenience of the reader we begin with a review of some preliminary material (§2, §3.) In §4 we prove the formula for our new bound and in §5 we compare our new bound on c_K with Lang's bound for some real quadratic extensions K.

2. Absolute Values

Let K be a finite extension of $\mathbb{Q}$ with ring of integers R. An *absolute value on K* is a function $\eta : K \to \mathbb{R}_{\geq 0}$ that satisfies

(i) $\eta(x) = 0$ if and only if $x = 0$,
(ii) $\eta(xy) = \eta(x)\eta(y), \forall x, y \in K$,
(iii) there exists a constant M so that $\eta(1 + x) \leq M$ whenever $\eta(x) \leq 1$.

The *trivial absolute value* is defined as $\eta(0) = 0$ and $\eta(x) = 1$ for $x \neq 0$.
Two absolute values η_1 and η_2 on K are *equivalent* if there exist $r \in \mathbb{R}_{>0}$ so that $\eta_1(x) = (\eta_2(x))^r$, $\forall x \in K$. Thus, the absolute values on K can be partitioned into equivalence classes. It is well-known that up to equivalence the non-trivial absolute values on $\mathbb{Q}$ consist of

$$| \ |_\infty, | \ |_2, | \ |_3, | \ |_5, | \ |_7, \ldots .$$

Here $|\ |_\infty$ is the ordinary absolute value on $\mathbb{R}$ restricted to $\mathbb{Q}$, and for a rational prime p, $|\ |_p$ is the *p-adic absolute value* defined as $|0|_p = 0$ and

$$|x|_p = \frac{1}{p^m}$$

for $x = (r/s)p^m$, $(r,p) = (s,p) = 1$, $m \in \mathbb{Z}$.

Let η be an absolute value on K. Then η determines a topology on K where the basic open sets are of the form $U_{x,\epsilon}$, $x \in K$, $\epsilon > 0$, with

$$U_{x,\epsilon} = \{y \in K : \eta(x-y) < \epsilon\}$$

The topology thus described is the *η-topology on K*. Let K_η denote the completion of K with respect to the η-topology. In a natural way the absolute value η on K extends to a unique absolute value on K_η, which we also denote by η, cf. [3](Chapter XII, §2). In the case $K = \mathbb{Q}$, $\eta = |\ |_\infty$, the completion $\mathbb{Q}_{|\ |_\infty}$ is the set of real numbers $\mathbb{R}$. If $K = \mathbb{Q}$, $\eta = |\ |_p$, then the completion $\mathbb{Q}_{|\ |_p}$ is the field of p-adic rationals, $\mathbb{Q}_p$. If L is a finite extension of the completion K_η, then the absolute value η on K_η extends uniquely to an absolute value η^* on L and L is complete with respect to the η^*-topology [3](Chapter XII, Proposition 2.5).

If K is a finite extension of $\mathbb{Q}$ of degree N, then each absolute value on $\mathbb{Q}$ extends to a finite number $(\leq N)$ of absolute values η on K [1](Chapter II, Theorem, p. 57). To see how the ordinary absolute value $|\ |_\infty$ extends to K, let $K = \mathbb{Q}(\alpha)$ for some $\alpha \in \mathbb{C}$, and let $p(x) = \mathrm{irr}(\alpha; \mathbb{Q})$. Let $p(x) = \prod_{i=1}^{g} p_i(x)$ denote the factorization of $p(x)$ over $\mathbb{R}$ into irreducible polynomials. Note that $g \leq N$. For each i, $1 \leq i \leq g$, there exists an embedding $\lambda_i : K \to \mathbb{R}(\alpha_i)$, $\alpha \mapsto \alpha_i$, where α_i is a root of $p_i(x)$. One defines an absolute value η_i on K by setting

$$\eta_i(x) = |\lambda_i(x)|_\infty^*,\ \forall x \in K$$

where $|\ |_\infty^*$ is the unique extension of $|\ |_\infty$ to $\mathbb{R}(\alpha_j)$. The collection $\eta_1, \eta_2, \ldots, \eta_g$ is the set of extensions of $|\ |_\infty$ to K.

The p-adic absolute value $|\ |_p$ extends to K in the following manner. Let $(p) = P_1^{e_1} P_2^{e_2} \cdots P_g^{e_g}$ be the unique factorization of (p) into prime ideals P_i of R. Each P_i corresponds to an extension η_i of $|\ |_p$ to K as follows. Put $\eta_i(0) = 0$. For $r \in R$, $r \neq 0$, let t_r be the integer $t_r \geq 0$ for which $(r) \subseteq P_i^{t_r}$, $(r) \not\subseteq P_i^{t_r+1}$. Now let $x = r/s \in K$, $r \neq 0$, $s \neq 0$. One then puts

$$\eta_i(x) = \frac{1}{p^{(t_r - t_s)/e_i}}$$

The collection $\eta_1, \eta_2, \ldots, \eta_g$ is the set of extensions of $|\ |_p$ to K. Since $g \leq N$, there are at most N extensions.

The extensions η of $|\ |_\infty$ are the *Archimedean* absolute values on K. The extensions of η of $|\ |_p$ are the *non-Archimedean (or discrete)* absolute values on K. Absolute values η on K obtained as extensions constitute all of the absolute values on K (up to equivalence.)

If η_i is Archimedean and corresponds to a real embedding λ_i, then the *local degree* $d_{\eta_i} = [\mathbb{R}(\alpha_i) : \mathbb{R}] = 1$, and we define the *normalized absolute value* to be

$$\nu_i(x) = \eta_i(x), \forall x \in K$$

If η_i is Archimedean and corresponds to a complex embedding λ_i, then the local degree $d_{\eta_i} = 2$, and we define the *normalized absolute value* as

$$\nu_i(x) = (\eta_i(x))^2, \forall x \in K$$

If η_i is a discrete extension of $|\ |_p$ corresponding to the prime ideal P_i, the local degree is $d_{\eta_i} = e_i f_i$ where $f_i = [R_{P_i}/P_i R_{P_i} : \mathbb{F}_p]$ is the residue class field degree. In this case the *normalized absolute value* is given as

$$\nu_i(x) = (\eta_i(x))^{e_i f_i} = \left(\frac{1}{p^{(t_r - t_s)/e_i}}\right)^{e_i f_i} = \frac{1}{p^{(t_r - t_s) f_i}}$$

where $x = r/s$.

If ν is the normalized absolute value obtained from η, then the ν-topology on K is equal to the η-topology on K since ν and η are equivalent absolute values. In what follows we let $S = \{\nu\}$ denote the set of normalized absolute values on K; K_ν denotes the completion of K with respect to the ν-topology. For ν discrete, we let R_ν denote the ring of integers in K_ν. The absolute value ν extends to an absolute value on K_ν (also denoted by ν.) We consider K_ν to be endowed with the ν-topology.

3. The Adele Ring

Let K be a finite extension of $\mathbb{Q}$ and let $S = \{\nu\}$ denote the set of normalized absolute values on K. For each discrete ν, R_ν is a compact open subset of K_ν. The *adele ring V_K over K* is the topological ring that is the restricted product of the completions K_ν with respect to the collection $\{R_\nu : \ \nu \text{ discrete}\}$, together with the restricted product topology on the completions K_ν with respect to the collection $\{R_\nu : \nu \text{ discrete}\}$. This means that V_K consists of those vectors

$$\{\dots, a_\nu, \dots\} \in \prod_{\nu \in S} K_\nu$$

for which $a_\nu \in R_\nu$ for all but finitely many ν. The ring structure of V_K is given component-wise:

$$\{\dots, a_\nu, \dots\} + \{\dots, b_\nu, \dots\} = \{\dots, a_\nu + b_\nu, \dots\}$$

$$\{\dots, a_\nu, \dots\} \cdot \{\dots, b_\nu, \dots\} = \{\dots, a_\nu b_\nu, \dots\}$$

We write $\{a_\nu\}$ for the adele $\{\dots, a_\nu, \dots\}$. A basis for the topology on V_K consists of open sets of the form

$$\prod_{\nu \in S} U_\nu$$

where U_ν is open in K_ν for all ν and $U_\nu = R_\nu$ for all but finitely many ν.

Let V_K^+ denote the additive group of the adele ring V_K and let K^+ denote the additive group of K.

Proposition 3.1 *Let $b \in K^+$, $b \neq 0$. Then $\prod_{\nu \in S} \nu(b) = 1$.*

Proof. For two proofs, see [1](Chapter II, Theorem, p. 60 and p. 66). ◇

Proposition 3.2 *K^+ embeds into V_K^+ through the map $b \mapsto \{b, b, b, \dots\}$.*

Proof. Let $b \in K^+$ and write $b = a/c$, where $a, c \in R$, $c \neq 0$. Since there are only a finite number of prime divisors of c, $c^{-1} \in R_\nu$ for all but a finite number of ν. Thus $\{b_\nu\}$ with $b_\nu = b$ for all $\nu \in S$ is an adele of K. It is easy to show that the map $b \mapsto \{b, b, b, \dots\}$ is an injection of groups $K^+ \to V_K^+$. $\diamond$

With these preliminaries in mind, we now give two upper bounds for the content bound c_K in the case that K is a real quadratic extension with class number one.

4. Two Bounds for c_K

Let d be a square-free positive integer, let $K = \mathbb{Q}(\sqrt{d})$ denote the real quadratic extension with ring of integers R. Let c_K be the content bound for K. We recall some number-theoretic facts about K. If $d \equiv 1 \pmod 4$ then $R = \mathbb{Z}[\frac{1+\sqrt{d}}{2}]$ and if $d \equiv 2, 3 \pmod 4$ then $R = \mathbb{Z}[\sqrt{d}]$. The discriminant $\mathrm{disc}(R) = d$ if $d \equiv 1 \pmod 4$, and $\mathrm{disc}(R) = 4d$ if $d \equiv 2, 3 \pmod 4$. If $d \equiv 1 \pmod 4$, then the only rational primes that ramify are divisors of d. If $d \equiv 2, 3 \pmod 4$, the rational primes that ramify are 2 and the divisors of d.

The set of normalized absolute values on K is computed as follows. The Archimedean absolute value $|\ |_\infty$ on $\mathbb{Q}$ extends to two normalized absolute values, ρ_1, ρ_2, defined as follows. For $a + b\sqrt{d} \in K$,

$$\rho_1\left(a + b\sqrt{d}\right) = \left|a + b\sqrt{d}\right|_\infty$$

and

$$\rho_2\left(a + b\sqrt{d}\right) = \left|\overline{a + b\sqrt{d}}\right|_\infty = \left|a - b\sqrt{d}\right|_\infty$$

The discrete absolute values on $\mathbb{Q}$ extend to K in the following manner. If $p \mid \mathrm{disc}(R)$, then $(p) = P^2$ for some prime ideal P of R. Thus $|\ |_p$ extends to one normalized absolute value ν on K. On the other hand, if $p \nmid \mathrm{disc}(R)$ and $(\frac{d}{p}) = -1$, then $(p) = P$ for P prime, and so, p remains prime in R. In this case, $|\ |_p$ extends to one normalized absolute value ν on K. If $p \nmid \mathrm{disc}(R)$ and $(\frac{d}{p}) = 1$, then $(p) = PQ$ for P, Q prime and so, $|\ |_p$ extends to two normalized absolute values ν, ν'.

Let $S = \{\nu\}$ denote the set of normalized absolute values on K, and let V_K^+ be the additive group of adeles. There is a known bound for c_K due to S. Lang [2].

Proposition 4.1 *(S. Lang) Let d be a positive square-free integer and let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic extension.*

(i) If $d \equiv 1 \pmod 4$, then $c_K \leq (2 + 2\sqrt{d})^2$,
(ii) if $d \equiv 2, 3 \pmod 4$, then $c_K \leq 16d$.

Proof. For a proof see [2](Chapter V, §1, Theorem 0). $\diamond$

To prove our formula for a new bound on c_K, we need some lemmas regarding units in R. The units group of R is

$$\langle -1 \rangle \times \langle h \rangle$$

where h is a fundamental unit in R. Note that $h \in \mathbb{R}$.

Lemma 4.2 *There exists a fundamental unit f in R with $f > 1$.*

Proof. Let h be a fundamental unit. We consider first the case $h > 0$. If $h > 1$, then we set $f = h$ and condition is satisfied. Else, assume that $0 < h < 1$. Then $hh^{-1} = 1$ implies that $h^{-1} > 1$. Of course, h^{-1} is a fundamental unit and so we set $f = h^{-1}$. If $h < 0$, then $-h > 0$ is a fundamental unit and as shown above we may take $f > 1$. $\diamond$

Lemma 4.3 *If h is a fundamental unit of R, then $\overline{h} = \pm h^{-1}$.*

Proof. Let $N_{K/\mathbb{Q}} : K \to \mathbb{Q}$ be the norm map defined as

$$N_{K/\mathbb{Q}}\left(a + b\sqrt{d}\right) = \left(a + b\sqrt{d}\right)\left(\overline{a + b\sqrt{d}}\right) = a^2 - b^2 d$$

for $a, b \in \mathbb{Q}$. The norm map restricts to give a map $N_{K/\mathbb{Q}} : R \to \mathbb{Z}$. Now suppose that h is a fundamental unit with inverse h^{-1}. Then $N_{K/\mathbb{Q}}(h)$ and $N_{K/\mathbb{Q}}(h^{-1})$ are in $\mathbb{Z}$. Moreover, $hh^{-1} = 1$ yields

$$1 = N_{K/\mathbb{Q}}(hh^{-1}) = N_{K/\mathbb{Q}}(h)N_{K/\mathbb{Q}}(h^{-1})$$

Consequently, $N_{K/\mathbb{Q}}(h) = \pm 1$, and thus $h\overline{h} = \pm 1$, or $\overline{h} = \pm h^{-1}$. $\diamond$

We now give the new bound on c_K.

Proposition 4.4 *Let d be a square-free positive integer, let $K = \mathbb{Q}(\sqrt{d})$ and assume that K has class number one. Let $f > 1$ be a fundamental unit in R. Then $c_K \leq f$.*

Proof. We show that if $\{a_\nu\}$ is an adele in V_K^+ with $\mathbf{c}(\{a_\nu\}) \geq f$, then there exists $b \in K^+$, $b \neq 0$, so that $\nu(b) \leq \nu(a_\nu)$ for all $\nu \in S$. For ν discrete, let K_ν denote the completion of K with respect to the ν-topology, and let R_ν denote the ring of integers in K_ν. We have

$$a_\nu = u_\nu \pi_\nu^{m_\nu}$$

where u_ν is a unit in R_ν, $m_\nu \in \mathbb{Z}$, and where π_ν is a uniformizing parameter for R_ν. Since $\{a_\nu\}$ is an adele, $m_\nu \geq 0$ for all but a finite number of ν, and since $\prod_{\nu \in S} \nu(a_\nu) > 0$, $m_\nu = 0$ for all but a finite number of ν. Let $\nu_1, \nu_2, \ldots, \nu_k$ denote the collection of discrete ν for which $m_{\nu_i} \neq 0$, listed so that $\nu_1, \nu_2, \ldots, \nu_l$ are those ν_i with $m_{\nu_i} > 0$ and $\nu_{l+1}, \nu_{l+2}, \ldots, \nu_k$ are the ν_i for which $m_{\nu_i} < 0$.

For $i = 1, 2, \ldots, k$, let P_{ν_i} denote the ideal of R corresponding to the discrete normalized absolute value ν_i. Then the ideal

$$P_{\nu_1}^{m_{\nu_1}} P_{\nu_2}^{m_{\nu_2}} \cdots P_{\nu_l}^{m_{\nu_l}}$$

is principal and generated by an element $\alpha \in R$. Moreover, the ideal

$$P_{\nu_{l+1}}^{-m_{\nu_{l+1}}} P_{\nu_{l+2}}^{-m_{\nu_{l+2}}} \cdots P_{\nu_k}^{-m_{\nu_k}}$$

is principal and generated by an element $\beta \in R$.

Let $\Lambda = \alpha/\beta$. We can assume without loss of generality that $\Lambda > 0$. For all ν discrete, $\nu(a_\nu) = \nu(\Lambda)$. Thus

$$f \leq \prod_{\nu \in S} \nu(a_\nu) \quad = \quad \rho_1(a_{\rho_1})\rho_2(a_{\rho_2}) \prod_{\nu \text{ dis.}} \nu(\Lambda) \tag{1}$$

Since $f > 1$, there exists an integer k for which

$$f^k \leq \frac{\rho_1(a_{\rho_1})}{\Lambda} < f^{k+1} \tag{2}$$

Put $b = f^k\Lambda$. Then $\rho_1(b) \leq \rho_1(a_{\rho_1})$. Now from Inequality (2),

$$\begin{aligned}
\left(|\overline{f}|_\infty\right)^k \frac{\rho_1(a_{\rho_1})}{\Lambda} &< \left(|\overline{f}|_\infty\right)^k f^{k+1} \\
&= (|\pm f^{-1}|_\infty)^k f^k f, \quad \text{by Lemma 4.3} \\
&= (|\pm f^{-1} f|_\infty)^k f \\
&= f
\end{aligned}$$

So,

$$\begin{aligned}
\left(|\overline{f}|_\infty\right)^k \rho_2(\Lambda) &< \frac{f\Lambda}{\rho_1(a_{\rho_1})}\rho_2(\Lambda) \\
&\leq \left(\rho_1(a_{\rho_1})\rho_2(a_{\rho_2}) \prod_{\nu \text{ dis.}} \nu(\Lambda)\right) \frac{\Lambda}{\rho_1(a_{\rho_1})}\rho_2(\Lambda), \quad \text{by (1)} \\
&= \rho_2(a_{\rho_2})\rho_1(\Lambda)\rho_2(\Lambda) \prod_{\nu \text{ dis.}} \nu(\Lambda), \quad \text{since } \rho_1(\Lambda) = \Lambda \\
&= \rho_2(a_{\rho_2}) \prod_{\nu \in S} \nu(\Lambda) \\
&= \rho_2(a_{\rho_2}), \quad \text{by Proposition 3.1.}
\end{aligned}$$

Observe that

$$\begin{aligned}
\left(|\overline{f}|_\infty\right)^k \rho_2(\Lambda) &= (\rho_2(f))^k \rho_2(\Lambda) \\
&= \rho_2(f^k)\rho_2(\Lambda) \\
&= \rho_2(f^k\Lambda) \\
&= \rho_2(b)
\end{aligned}$$

thus $\rho_2(b) < \rho_2(a_{\rho_2})$. For ν discrete, f^k is a unit in R_ν. Thus $\nu(b) = \nu(\Lambda) = \nu(a_\nu)$ for all ν discrete. Thus b is as required. ◇

5. A Comparison of Bounds

Let d be a square-free positive integer, let $K = \mathbb{Q}(\sqrt{d})$ and let c_K be the content bound for K. In Table 1 below we compare Lang's bound on c_K (Proposition 4.1) with the new bound obtained in Proposition 4.4 for all $d < 50$ for which K has class number one. The values of d were obtained from [4](A003172). The values of the fundamental unit f were computed using an algorithm based on [5](Theorem 11.5) as implemented in [6]. We conclude that our new bound is better than Lang's bound in 17 out of 22 of the cases.

Of course, the fundamental unit f has been proven to be a bound for c_K only in the case that K has class number one. It would be of interest to extend Proposition 4.4 to the case where K has class number greater than one.

Table 1. Lang's bound compared with the fundamental unit.

d	*Lang's bound* (L)	*new bound* (f)	L *vs.* f
2	32	$1+\sqrt{2}$	$f \approx 2.41421$
3	48	$2+\sqrt{3}$	$f \approx 3.73205$
5	$(2+2\sqrt{5})^2 \approx 41.8885$	$\frac{1+\sqrt{5}}{2}$	$f \approx 1.61803$
6	96	$5+2\sqrt{6}$	$f \approx 9.89898$
7	112	$8+3\sqrt{7}$	$f \approx 15.9373$
11	176	$10+3\sqrt{11}$	$f \approx 19.9499$
13	$(2+2\sqrt{13})^2 \approx 84.8444$	$\frac{3+\sqrt{13}}{2}$	$f \approx 3.30278$
14	224	$15+4\sqrt{14}$	$f \approx 29.9666$
17	$(2+2\sqrt{17})^2 \approx 104.985$	$4+\sqrt{17}$	$f \approx 8.12311$
19	304	$170+39\sqrt{19} \approx 339.997$	$L = 304$
21	$(2+2\sqrt{21})^2 \approx 124.661$	$\frac{5+\sqrt{21}}{2}$	$f \approx 4.79129$
22	352	$197+42\sqrt{22} \approx 393.997$	$L = 352$
23	368	$24+5\sqrt{23}$	$f \approx 47.9792$
29	$(2+2\sqrt{29})^2 \approx 163.081$	$\frac{5+\sqrt{29}}{2}$	$f \approx 5.19258$
31	496	$1520+273\sqrt{31} \approx 3040$	$L = 496$
33	$(2+2\sqrt{33})^2 \approx 181.957$	$23+4\sqrt{33}$	$f \approx 45.9783$
37	$(2+2\sqrt{37})^2 \approx 200.662$	$6+\sqrt{37}$	$f \approx 12.0828$
38	608	$37+6\sqrt{38}$	$f \approx 73.9865$
41	$(2+2\sqrt{41})^2 \approx 219.225$	$32+5\sqrt{41}$	$f \approx 64.0156$
43	688	$3482+531\sqrt{43} \approx 6964$	$L = 688$
46	732	$24335+3588\sqrt{46} \approx 48670$	$L = 732$
47	752	$48+7\sqrt{47}$	$f \approx 95.9896$

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