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Mild Solutions and Exponential Stability of ψ-Caputo Neutral Stochastic Integro-Differential Delay Systems of Order $\sigma \in (1,2)$ with Impulses and Poisson Jumps

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Abstract

We establish sufficient conditions for the existence and exponential stability of mild solutions of a class of fractional neutral stochastic integro-differential delay systems governed by the ψ-Caputo fractional derivative of order $\sigma \in (1,2)$, which unifies the Caputo, Caputo–Hadamard and Caputo–Katugampola derivatives. The system carries dual integral memory kernels, α_1 inside the neutral term and α_2 inside the forcing term, and is subject to Poisson jump perturbations and to instantaneous impulses acting simultaneously on the state and on its ψ-derivative, in an infinite-dimensional Hilbert space. Existence is obtained by combining the Hausdorff measure of noncompactness with Mönch's fixed-point theorem, which avoids compactness assumptions on the associated cosine family. Exponential stability is then derived from a ψ-weighted derivative impulsive integral inequality, yielding decay that is exponential in $\psi(\nu) - \psi(0)$ and hence exponential, algebraic or logarithmic in ν according to the growth of ψ. An illustrative example is presented to demonstrate the applicability of the theoretical results.

Keywords: ψ-Caputo; fractional derivative; fractional neutral stochastic delay systems; mild solutions; exponential stability; Mönch fixed point theorem; Poisson jumps; derivative impulses

MSC: 34A08; 26A33; 60H10; 34K50; 47H10; 35R60

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Academic Editor: Jorge E. Macías Díaz

Received: 4 August 2026
Revised: 6 September 2026
Accepted: 15 September 2026
Published: 20 September 2026

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1. Introduction

Fractional differential equations have emerged in numerous engineering and scientific disciplines, including chemistry, physics, electrodynamics, aerodynamics, rheology, and polymer science. Their defining nonlocal property allows the derivative or integral at any given instant to capture the complete past trajectory of the system rather than only its instantaneous behaviour, making them well suited to memory-dependent processes. Prominent application areas include thermoplastic processes, dielectric materials, electromagnetic wave propagation, viscoelastic mechanics, signal and image processing, blood flow dynamics, and control engineering. Systematic treatments of fractional calculus can be found in the monographs of Kilbas et al. [1], Lakshmikantham et al. [2], Miller and Ross [3], and Podlubny [4]; further contributions are collected in [5–9].

A flexible unification of these operators was introduced by Almeida [10], who defined the Caputo derivative of a function with respect to another increasing function ψ. Rather than fixing the memory kernel to be a power of $(\nu - \mu)$, the ψ-Caputo fractional derivative (ψ-CFD) measures elapsed time through the difference $\psi(\nu) - \psi(\mu)$, so that the classical Caputo ($\psi(\nu) = \nu$), Caputo–Hadamard ($\psi(\nu) = \ln \nu$) and Caputo–Katugampola ($\psi(\nu) = \nu^{\rho}/\rho$, $\rho > 0$ a fixed real parameter) derivatives all arise from a single operator; see [11,12] for recent qualitative studies. A single argument therefore covers an entire family of fractional operators, and the memory kernel may be tuned to the time-scale of the phenomenon under study. Beyond this unification, the choice of ψ also changes the decay rate of solutions: as shown later in this paper, the decay is exponential in ν when ψ grows at least linearly (e.g., $\psi(\nu) = \nu$), but only algebraic when ψ grows more slowly (e.g., $\psi(\nu) = \ln(1 + \nu)$); see Remark 5 and Section 5. This difference is not visible when the classical Caputo, Caputo–Hadamard or Caputo–Katugampola derivatives are used on their own.

Random disturbances act on physical, biological, financial and engineering systems alike, which motivates the theory of stochastic differential equations (SDEs); see Da Prato and Zabczyk [13] and Øksendal [14]. The Brownian framework presumes path continuity, an assumption violated whenever the state moves by sudden jumps, as asset prices do. Lévy-type and jump-diffusion models attach a Poisson random measure to the driving noise and are now standard in quantitative finance, actuarial science and engineering control [15–17]; in infinite dimensions the operator-theoretic groundwork is Pazy's semigroup theory [18].

A further layer of complexity arises when a system does not evolve continuously but instead changes state abruptly, at isolated instants in time. Such impulsive behaviour is encountered widely, in mechanics, electronics, telecommunications, economics, medicine, and population biology, whenever an external intervention or the crossing of some threshold produces a sudden discontinuity; see [19,20]. The incorporation of impulsive effects and Poisson jumps significantly increases the analytical difficulty, and developing a rigorous framework for such systems remains an active area of research; see [9,21–23].

Neutral-type dependence adds a third structural feature: the rate of change of the state depends on past values of its own derivative; this occurs in circuit analysis, viscoelastic modelling and biological systems with after-effects [24]. The ψ-CFD accommodates nonlocal memory while preserving the neutral structure, at the cost of a considerably harder analysis in infinite-dimensional Hilbert spaces, where the solution operator is in general non-compact.

In recent years, considerable attention has been devoted to the qualitative theory of fractional stochastic and neutral SDEs, including the study of existence, uniqueness, and stability; see [25–27]. When it comes to proving existence in non-compact settings, Mönch's fixed-point theorem (MFPT) has turned out to be a particularly good fit. Unlike Schauder-type theorems, MFPT does not require the solution operator to be compact; it instead works through the Hausdorff measure of noncompactness (HMnC), which provides a quantitative measure of how far a set is from being relatively compact, and thereby allows one to handle infinite-dimensional problems without imposing artificial compactness restrictions on the semigroup. For stability questions in the presence of simultaneous stochastic perturbations and impulsive effects, impulsive integral inequality techniques have proved to be a natural and constructive alternative to Lyapunov-function methods; relevant contributions along these lines include [23,28–31].

The ψ-CFD framework has recently been employed for stochastic systems [28,29,32]; those works sit at order $\sigma \in (0, 1)$, where the mild solution is propagated by a single resolvent family and the Cauchy datum is a single vector. This is not just a change in

the range of σ. At order $\sigma \in (0,1)$, the problem needs only one initial condition and one resolvent family. At order $\sigma \in (1,2)$, it needs two initial conditions, $\mathcal{Y}(0)$ and $\mathcal{Y}_{\psi}^{[1]}(0)$. It also needs two operator families, the ψ-cosine and ψ-sine families, to build the mild solution. Our starting point is Kasinathan et al. [33], on a second-order neutral stochastic integro-differential delay system with Poisson jumps, two memory kernels, and impulses on both the state and its first derivative. Existence there follows from the MFPT; stability follows from Chen's impulsive integral inequality [34]. The setting is classical Caputo, so the kernel is a power of $(\nu-\mu)$ and the decay is exponential in ν. Passing to the ψ-Caputo derivative is not a change of notation. The operator families carrying the mild solution at order $\sigma \in (1,2)$ must be rebuilt over ψ-time, and the weight $\psi'(\mu)\,d\mu$ that each convolution now carries passes into the noncompactness estimates and into Chen's inequality, both of which we prove again in weighted form. The decay then reads in ψ-time: for a sublinear kernel it is algebraic or logarithmic in ν, not exponential. We are not aware of such a treatment for stochastic neutral systems of this order with impulses and jumps. The ψ-Caputo cosine and sine families here are due to Ben Brahim et al. [35], in the deterministic non-impulsive case, and we build on them. The main advantage over [33,34] is generality. The classical Caputo setting there is recovered here as the special case $\psi(\nu)=\nu$ (see Corollary 2 and Corollary 3). The ψ-Caputo formulation also covers sublinear kernels. For such kernels, the decay is no longer exponential but only algebraic or logarithmic. Those earlier results do not detect this slower decay regime, since it is invisible under a fixed power kernel.

Motivated by this, we investigate the following system and derive sufficient conditions for the existence and exponential stability of its mild solutions:

$$
\begin{aligned}
&{}^{C,\psi}\mathfrak{D}_{0^+}^{\sigma}\left[\mathcal{Y}(\nu)-\mathscr{H}\left(\nu,\mathcal{Y}_{\nu},\int_0^{\nu}\alpha_1(\nu,\mu,\mathcal{Y}_{\mu})\psi'(\mu)\,d\mu\right)\right]\\
&\quad=A\left[\mathcal{Y}(\nu)-\mathscr{H}\left(\nu,\mathcal{Y}_{\nu},\int_0^{\nu}\alpha_1(\nu,\mu,\mathcal{Y}_{\mu})\psi'(\mu)\,d\mu\right)\right]\\
&\qquad+f\left(\nu,\mathcal{Y}_{\nu},\int_0^{\nu}\alpha_2(\nu,\mu,\mathcal{Y}_{\mu})\psi'(\mu)\,d\mu\right)\\
&\qquad+\int_0^{\nu}g(\nu,\mu,\mathcal{Y}_{\mu})dw(\mu)+\int_{\mathfrak{Z}}h(\nu,\mathcal{Y}_{\nu},\aleph)\tilde{N}(d\nu,d\aleph),\qquad \iota_{\Bbbk}\neq\nu\in\mathcal{I},\\
&\Delta\mathcal{Y}(\iota_{\Bbbk})=J_{\Bbbk}^{1}(\mathcal{Y}_{\iota_{\Bbbk}}),\quad \Delta\mathcal{Y}_{\psi}^{[1]}(\iota_{\Bbbk})=J_{\Bbbk}^{2}(\mathcal{Y}_{\iota_{\Bbbk}}),\qquad \Bbbk=1,2,\ldots,n,\\
&\mathcal{Y}(0)=\mathcal{Y}_0=\xi(0),\qquad \mathcal{Y}(\nu)=\xi(\nu)\in\mathbb{M}^{\wp}(\Omega,\mathbb{X})\quad\text{for a.e. }\nu\in\mathcal{I}_0:=(-\infty,0],\\
&\mathcal{Y}_{\psi}^{[1]}(0)=\mathcal{Y}_1\in\mathcal{B},
\end{aligned}
\tag{1}
$$

where $\mathcal{Y}_{\psi}^{[1]}(\nu)=\frac{1}{\psi'(\nu)}\frac{d}{d\nu}\mathcal{Y}(\nu)$ denotes the ψ-derivative of $\mathcal{Y}$. Throughout, $\psi:\mathcal{I}_0\cup\mathcal{I}\to\mathbb{R}$ is a fixed increasing function of class C^2 with $\psi'\neq 0$, and $\big(\psi(\nu)-\psi(\mu)\big)=\psi(\nu)-\psi(\mu)$; the symbol ${}^{C,\psi}\mathfrak{D}_{0^+}^{\sigma}$ is the ψ-CFD of order $\sigma\in(1,2)$. Here, $A:D(A)\subset\mathcal{B}\to\mathcal{B}$ generates a strongly continuous cosine family on the Hilbert space $\mathcal{B}$, and $\mathcal{Y}(\cdot)$ is a $\mathcal{B}$-valued stochastic process whose history segment $\mathcal{Y}_{\nu}(\theta)=\mathcal{Y}(\nu+\theta)$, $\theta\in\mathcal{I}_0$, is defined for $\nu\geq 0$ and lies in the abstract phase space $\mathbb{X}$ of Section 2.

The nonlinear coefficient mappings appearing in (1) are specified as follows: $\mathscr{H},f:\mathcal{I}\times\mathbb{X}\times\mathcal{B}\to\mathcal{B}$; $g:\mathcal{I}\times\mathcal{I}\times\mathbb{X}\to\mathbb{M}_{\mathbb{V}}^{0}(\mathcal{S},\mathcal{B})$, where $\mathcal{S}$ denotes a separable Hilbert space; $\alpha_i:\mathcal{I}\times\mathcal{I}\times\mathbb{X}\to\mathcal{B}$ for $i=1,2$; $h:\mathcal{I}\times\mathbb{X}\times\mathfrak{Z}\to\mathcal{B}$; and the two families of impulsive maps $J_{\Bbbk}^{1},J_{\Bbbk}^{2}:\mathbb{X}\to\mathcal{B}$ for $\Bbbk=1,2,\ldots,n$, the first prescribing the jump of the state and the second the jump of the ψ-derivative of the state. Precise assumptions on each of these mappings are stated in the sections that follow.

The time interval $\mathcal{I}$ is partitioned by a finite increasing sequence of impulse times $0 = \iota_0 < \iota_1 < \iota_2 < \cdots < \iota_n < \iota_{n+1} = \mathcal{T}$, at each of which the process $\mathcal{Y}$ may suffer an instantaneous jump. The sizes of the two discontinuities suffered at $\iota_{\Bbbk}$ are recorded by

$$\Delta\mathcal{Y}(\iota_{\Bbbk}) = \mathcal{Y}(\iota_{\Bbbk}^{+}) - \mathcal{Y}(\iota_{\Bbbk}^{-}), \qquad \Delta\mathcal{Y}_{\psi}^{[1]}(\iota_{\Bbbk}) = \mathcal{Y}_{\psi}^{[1]}(\iota_{\Bbbk}^{+}) - \mathcal{Y}_{\psi}^{[1]}(\iota_{\Bbbk}^{-}),$$

where $\mathcal{Y}(\iota_{\Bbbk}^{+})$ and $\mathcal{Y}(\iota_{\Bbbk}^{-})$ denote the right- and left-hand limits of $\mathcal{Y}(\nu)$ at $\nu = \iota_{\Bbbk}$, and $\mathcal{Y}_{\psi}^{[1]}(\iota_{\Bbbk}^{\pm})$ the corresponding one-sided limits of the ψ-derivative. The map $\jmath_{\Bbbk}^{1}$ prescribes the jump in the state and $\jmath_{\Bbbk}^{2}$ that in the ψ-derivative, both depending on the history segment $\mathcal{Y}_{\iota_{\Bbbk}}$; taking $\jmath_{\Bbbk}^{2} \equiv 0$ returns the purely positional model. The initial datum satisfies $\xi \in \mathbb{M}^{\wp}(\Omega, \mathbb{X})$, while $\mathcal{Y}_1(\nu)$ is an $\mathfrak{F}_{\nu}$-measurable random variable independent of the Wiener process w. The compensated Poisson random measure is

$$\tilde{N}(d\nu, d\aleph) = N(d\nu, d\aleph) - d\nu(\lambda\, d\aleph),$$

where $N(d\nu, d\aleph)$ is the Poisson counting measure associated with the characteristic measure λ.

Two structural features of (1) should be noted. The hereditary memory is carried by two distinct kernels, α_1 inside the neutral term $\mathscr{H}$ and α_2 inside the forcing term f; prescribing them independently enlarges the class of systems covered, and $\alpha_1 = \alpha_2$ recovers the single-kernel model. At each $\iota_{\Bbbk}$ the state jumps by $\jmath_{\Bbbk}^{1}$ and its ψ-derivative by $\jmath_{\Bbbk}^{2}$; since the Cauchy datum at this order is the pair $\big(\mathcal{Y}(0), \mathcal{Y}_{\psi}^{[1]}(0)\big)$, an impulse prescribing only the first component leaves the problem underdetermined at the jump instant; the choice $\jmath_{\Bbbk}^{2} \equiv 0$ is a determination of the second, not an omission of it.

Remark 1. *The system* (1) *models, for instance, an impulsive mechanical or economic process with memory: $\mathcal{Y}$ is the state, the jumps $\jmath_{\Bbbk}^{1}, \jmath_{\Bbbk}^{2}$ represent sudden shocks at isolated instants $\iota_{\Bbbk}$, the Poisson term captures rare random events, and the Wiener term captures background noise.*

The main contributions of this work are as follows. First, we establish the existence of mild solutions for a ψ-Caputo neutral stochastic integro-differential system of order $\sigma \in (1, 2)$ with two memory kernels, infinite delay, Poisson jumps, and impulses on both the state and its ψ-derivative, using the HMnC with MFPT, which requires no compactness of the cosine family (Theorem 1). Second, we derive a ψ-weighted derivative impulsive integral inequality, in which the convolution kernel carries the weight ψ' and the positional and derivative jumps enter as two separate summable families (Lemma 10). Third, using this inequality, we obtain exponential stability of mild solutions, the decay being measured along ψ-time rather than real time (Theorem 2). The rate is therefore kernel-dependent: exponential in the time variable when ψ grows at least linearly, but only algebraic when ψ is sublinear (Remark 5). Finally, we present a wave-type example with a logarithmic kernel that verifies both criteria and exhibits the resulting algebraic decay (Section 5).

2. Mathematical Preliminaries and Functional Framework

This section collects the ψ-R-L fractional integral and derivative, the ψ-CFD, the associated ψ-fractional operator families, the phase space and the lemmas used later. We work on a complete probability space $(\Omega, \mathfrak{F}, \mathcal{P})$ carrying a right-continuous increasing filtration $\{\mathfrak{F}_{\nu},\, \nu \in \mathcal{I}\}$ of sub-χ-algebras with $\mathfrak{F}_{\nu} \subset \mathfrak{F}$. A stochastic process is a family $\{\mathcal{Y}(\nu, \omega) : \Omega \to \mathcal{B},\ \nu \in \mathcal{I}\}$ of random variables, each $\mathcal{Y}(\nu) : \Omega \to \mathcal{B}$ being $\mathfrak{F}$-measurable.

Define the process

$$\mathcal{W}(\nu) = \sum_{\tau=1}^{\infty} \sqrt{\lambda_\tau}\, \gamma_\tau(\nu)\, \zeta_\tau(\nu), \quad \nu \geq 0,$$

where $\lambda_\tau \geq 0$ are non-negative real numbers and $\{\zeta_\tau\}_{\tau=1,2,\ldots}$ forms an orthonormal basis in $\mathcal{S}$. A bounded linear operator $\mathbb{V} \in L(\mathcal{S}, \mathcal{B})$ is defined via $\mathbb{V}\zeta_\tau = \lambda_\tau \zeta_\tau$, and is assumed to be of finite trace; that is,

$$\mathrm{Tr}(\mathbb{V}) = \sum_{\tau=1}^{\infty} \lambda_\tau < \infty.$$

Under these conditions, the $\mathcal{S}$-valued stochastic process $\mathcal{W}(\nu)$ is referred to as a $\mathbb{V}$-Wiener process. For any operator $\Psi_\Xi \in \mathbb{M}^0_{\mathbb{V}}(\mathcal{S}, \mathcal{B})$, the associated $\mathbb{V}$-Hilbert–Schmidt norm is defined by

$$\|\Psi_\Xi\|^2_{\mathbb{V}} = \mathrm{Tr}(\Psi_\Xi\, \mathbb{V}\, \Psi^*_\Xi) = \sum_{\tau=1}^{\infty} \left\| \sqrt{\lambda_\tau}\, \Psi_\Xi \zeta_\tau \right\|^2.$$

Whenever $\|\Psi_\Xi\|_{\mathbb{V}} < \infty$, the operator Ψ_Ξ is said to be a $\mathbb{V}$-Hilbert–Schmidt operator. For a more detailed account of the theory and underlying principles of SDEs, the reader is referred to [7,18,36] and the references therein. Here, Ψ_Ξ denotes a generic Hilbert–Schmidt operator, unrelated to the kernel ψ.

Definition 1 ([10]). *Let $q > 0$ and let f be an integrable function on $\mathcal{I}$. The ψ-Riemann–Liouville fractional integral of order q of f with respect to ψ is defined by*

$${}^{\psi}J^{q}_{0^+} f(\nu) = \frac{1}{\Gamma(q)} \int_0^{\nu} \psi'(\mu)\, (\psi(\nu) - \psi(\mu))^{q-1}\, f(\mu)\, d\mu.$$

Definition 2 ([10]). *Let $\tau \in \mathbb{N}$ and let f be such that $\left(\frac{1}{\psi'} \frac{d}{d\nu}\right)^{\tau-1} f$ is absolutely continuous on $\mathcal{I}$. The ψ-Riemann–Liouville fractional derivative of order $q \in (\tau - 1, \tau)$ of f with respect to ψ is given by*

$$\begin{aligned} {}^{\psi}\mathfrak{D}^{q}_{0^+} f(\nu) &= \left(\frac{1}{\psi'(\nu)} \frac{d}{d\nu}\right)^{\tau} {}^{\psi}J^{\tau-q}_{0^+} f(\nu) \\ &= \frac{1}{\Gamma(\tau - q)} \left(\frac{1}{\psi'(\nu)} \frac{d}{d\nu}\right)^{\tau} \int_0^{\nu} \psi'(\mu)\, (\psi(\nu) - \psi(\mu))^{\tau-q-1}\, f(\mu)\, d\mu. \end{aligned}$$

Definition 3 ([10]). *Let $\tau \in \mathbb{N}$ and let $f \in C^{\tau}(\mathcal{I}, \mathcal{B})$. Write $f^{[\tau]}_{\psi} := \left(\frac{1}{\psi'} \frac{d}{d\nu}\right)^{\tau} f$. The ψ-CFD of order $q \in (\tau - 1, \tau)$ of f with respect to ψ is formulated as*

$${}^{C,\psi}\mathfrak{D}^{q}_{0^+} f(\nu) = {}^{\psi}J^{\tau-q}_{0^+} f^{[\tau]}_{\psi}(\nu) = \frac{1}{\Gamma(\tau - q)} \int_0^{\nu} \psi'(\mu)\, (\psi(\nu) - \psi(\mu))^{\tau-q-1}\, f^{[\tau]}_{\psi}(\mu)\, d\mu.$$

In particular, for $\tau = 2$ and $\sigma = q \in (1, 2)$, as used throughout system (1),

$${}^{C,\psi}\mathfrak{D}^{\sigma}_{0^+} f(\nu) = \frac{1}{\Gamma(2 - \sigma)} \int_0^{\nu} \psi'(\mu)\, (\psi(\nu) - \psi(\mu))^{1-\sigma} \left(\frac{1}{\psi'(\mu)} \frac{d}{d\mu}\right)^{2} f(\mu)\, d\mu.$$

Remark 2. *Definition 3 contains several familiar operators as special cases:*

- *$\psi(\nu) = \nu$ recovers the classical Caputo fractional derivative ${}^{C}\mathfrak{D}^{\sigma}_{0^+}$ used in [3];*
- *$\psi(\nu) = \ln \nu$ (on $\mathcal{I} \subset (0, \infty)$) recovers the Caputo–Hadamard fractional derivative;*
- *$\psi(\nu) = \nu^{\rho}/\rho$, $\rho > 0$, recovers the Caputo–Katugampola fractional derivative.*

Lemma 1 ([10]). *For $q_1, q_2 > 0$ and $f \in C(\mathcal{I}, \mathcal{B})$, the ψ-Riemann–Liouville fractional integral operator satisfies the semigroup property*

$$ {}^{\psi}J_{0^+}^{q_1}\, {}^{\psi}J_{0^+}^{q_2} f(\nu) = {}^{\psi}J_{0^+}^{q_1+q_2} f(\nu) = {}^{\psi}J_{0^+}^{q_2}\, {}^{\psi}J_{0^+}^{q_1} f(\nu), \qquad \nu \in \mathcal{I}. $$

Lemma 2 ([10]). *Let $\sigma \in (1,2)$ and $f \in C^2(\mathcal{I}, \mathcal{B})$. Then,*

$$ {}^{C,\psi}\mathfrak{D}_{0^+}^{\sigma} f(\nu) = {}^{\psi}\mathfrak{D}_{0^+}^{\sigma}\Big[f(\nu) - f(0) - f_{\psi}^{[1]}(0)\,\big(\psi(\nu) - \psi(0)\big)\Big], \qquad \nu \in \mathcal{I}, $$

where $f_{\psi}^{[1]} = \frac{1}{\psi'}\frac{d}{d\nu} f$. In particular, ${}^{C,\psi}\mathfrak{D}_{0^+}^{\sigma} f \equiv 0$ whenever $f(\nu) = c_0 + c_1\big(\psi(\nu) - \psi(0)\big)$, $c_0, c_1 \in \mathcal{B}$.

The next three families are the ψ-analogues, at order $\sigma \in (1,2)$, of the fractional cosine, sine and Riemann–Liouville families; they are due to Ben Brahim et al. [35] and are recalled in the form used below.

Definition 4. *Let A generate a strongly continuous cosine family $\{C(\nu)\}_{\nu \geq 0}$ on $\mathcal{B}$ in the sense of Travis and Webb [37]. The ψ-fractional cosine family of order $\sigma \in (1,2)$ associated with A and ψ, denoted $C_{\sigma}^{\psi}(\nu) : [0,\infty) \to \mathcal{L}(\mathcal{B})$, is given by the subordination formula*

$$ C_{\sigma}^{\psi}(\nu) = \int_0^{\infty} \Phi_{\frac{\sigma}{2}}(\theta)\, C\Big(\theta\,\big(\psi(\nu) - \psi(0)\big)^{\frac{\sigma}{2}}\Big)\, d\theta, \qquad \nu \geq 0, $$

where $\Phi_{\frac{\sigma}{2}}$ is the Wright-type function of order $\frac{\sigma}{2}$; see ([35], Sec. 3). Setting $\psi(\nu) = \nu$ returns the classical fractional cosine family.

Definition 5. *The ψ-fractional sine family $S_{\sigma}^{\psi}(\nu) : [0,\infty) \to \mathcal{L}(\mathcal{B})$, associated with $C_{\sigma}^{\psi}(\nu)$, is introduced through the relation*

$$ S_{\sigma}^{\psi}(\nu) = \int_0^{\nu} \psi'(\mu)\, C_{\sigma}^{\psi}(\mu)\, d\mu, \quad \nu \geq 0. $$

Definition 6. *The ψ-R-L operator family $T_{\sigma}^{\psi}(\nu) : [0,\infty) \to \mathcal{L}(\mathcal{B})$, associated with S_{σ}^{ψ}, is introduced as*

$$ T_{\sigma}^{\psi}(\nu) = \frac{1}{\Gamma(\sigma)} \int_0^{\nu} \psi'(\mu)\,\big(\psi(\nu) - \psi(\mu)\big)^{\sigma-1}\, S_{\sigma}^{\psi}(\mu)\, d\mu. $$

For $\nu \geq \mu \geq 0$, the two-time kernels used in the convolution integrals below are written $S_{\sigma}^{\psi}(\nu,\mu)$ and $T_{\sigma}^{\psi}(\nu,\mu)$, meaning S_{σ}^{ψ} and T_{σ}^{ψ} evaluated at the ψ-lag $\big(\psi(\nu) - \psi(\mu)\big)$ in place of $\big(\psi(\nu) - \psi(0)\big)$.

Remark 3. *For $\psi(\nu) = \nu$, Definitions 4–6 reduce to the fractional sine and R-L families S_{σ}, T_{σ} of [18,33], and every estimate below reduces accordingly.*

Lemma 3 ([38]). (Burkholder–Davis–Gundy Inequality). *For every $\wp \geq 2$ and every $\mathbb{M}_{\mathbb{V}}^{0}(\mathcal{S}, \mathcal{B})$-valued predictable process $g(\nu)$, one has the estimate*

$$ \mathfrak{E}\left\| \int_0^{\nu} g(\mu)\, dw(\mu) \right\|^{\wp} \leq \tilde{C}_{\wp} \left(\int_0^{\nu} \Big(\mathfrak{E}\|g(\mu)\|_{\mathbb{M}_{\mathbb{V}}^{0}}^{\wp} \Big)^{\frac{2}{\wp}} d\mu \right)^{\frac{\wp}{2}}, \quad \nu \in \mathcal{I}, $$

where $\tilde{C}_{\wp} = \left(\dfrac{\wp(\wp-1)}{2} \right)^{\frac{\wp}{2}}$ and $\mathfrak{E}$ denotes the mathematical expectation.

Lemma 4 ([38]). (Kunita's First Inequality). *For any $\wp \geq 2$, there exists a constant $\hat{\mathcal{C}}_\wp > 0$ such that*

$$\mathfrak{E}\left\|\int_0^\nu \int_{\mathfrak{Z}} h(\mu,\aleph)\,\tilde{N}(d\mu,d\aleph)\right\|^\wp \leq \hat{\mathcal{C}}_\wp\left\{\mathfrak{E}\left[\left(\int_0^\nu \int_{\mathfrak{Z}} \|h(\mu,\aleph)\|^2\,\lambda(d\aleph)\,d\mu\right)^{\frac{\wp}{2}}\right] + \mathfrak{E}\left[\int_0^\nu \int_{\mathfrak{Z}} \|h(\mu,\aleph)\|^{2\wp}\,\lambda(d\aleph)\,d\mu\right]^{\frac{1}{2}}\right\}, \qquad \nu \in \mathcal{I}.$$

Lemma 5 ([39]). *Let X be a Banach space, and suppose $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ are two nonempty bounded subsets of X. Then, χ satisfies the following properties.*

- *Relative compactness in X is equivalent to $\tilde{\mathcal{F}}$ having measure zero: $\chi(\tilde{\mathcal{F}}) = 0$ if and only if $\tilde{\mathcal{F}}$ is relatively compact in X.*
- *Adjoining a single point changes nothing. For any $\mathcal{Y} \in X$, $\chi(\{\mathcal{Y}\} \cup \tilde{\mathcal{F}}) = \chi(\tilde{\mathcal{F}})$.*
- *Writing $\tilde{\mathcal{F}} + \tilde{\mathcal{G}} = \{\mathcal{Y} + \mathcal{V} : \mathcal{Y} \in \tilde{\mathcal{F}},\ \mathcal{V} \in \tilde{\mathcal{G}}\}$, the measure is subadditive: $\chi(\tilde{\mathcal{F}} + \tilde{\mathcal{G}}) \leq \chi(\tilde{\mathcal{F}}) + \chi(\tilde{\mathcal{G}})$.*
- *Taking unions never exceeds the larger of the two measures: $\chi(\tilde{\mathcal{F}} \cup \tilde{\mathcal{G}}) \leq \max\{\chi(\tilde{\mathcal{F}}), \chi(\tilde{\mathcal{G}})\}$.*
- *Finally, $\chi(a\tilde{\mathcal{F}}) = |a|\,\chi(\tilde{\mathcal{F}})$ holds for every $a \in \mathbb{R}$.*

Lemma 6 ([39]). *Let $\mathcal{F} \subset X$ be simultaneously bounded and equicontinuous on $[0,b]$. Then, the map $\nu \mapsto \chi(\mathcal{F}(\nu))$ is continuous on $[0,b]$, and*

$$\chi(\mathcal{F}) = \sup_{\nu \in \mathcal{I}} \chi(\mathcal{F}(\nu)),$$

where $\mathcal{F}(\nu) = \{\mathcal{Y}(\nu) : \mathcal{Y} \in \mathcal{F}\}$.

Lemma 7 ([39]). *Let $\{\mathcal{Y}^\tau\}_{\tau=1}^\infty$ be a sequence of Bochner integrable functions from $[0,b]$ into X satisfying*

$$\|\mathcal{Y}^\tau(\nu)\| \leq f(\nu), \quad \tau \geq 1, \quad \nu \in [0,b],$$

for some $f \in \mathbb{M}^1([0,b],\mathbb{R}^+)$. Then, the function $h(\nu) = \chi\{\mathcal{Y}^\tau(\nu) : \tau \geq 1\}$ belongs to $\mathbb{M}^1([0,b],\mathbb{R}^+)$, and the following integral estimate holds:

$$\chi\left(\int_0^\nu \mathcal{Y}^\tau(\mu)\,d\mu\,;\,\tau \geq 1\right) \leq 2\int_0^\nu h(\mu)\,d\mu.$$

Corollary 1. *Under the hypotheses of Lemma 7, if in addition $\|\mathcal{Y}^\tau(\mu)\| \leq f(\mu)$ holds with the weighted measure $\psi'(\mu)\,d\mu$ in place of $d\mu$, then*

$$\chi\left(\int_0^\nu \psi'(\mu)\,\mathcal{Y}^\tau(\mu)\,d\mu\,;\,\tau \geq 1\right) \leq 2\int_0^\nu \psi'(\mu)\,h(\mu)\,d\mu.$$

Proof. Apply Lemma 7 on the reparametrized interval under the (strictly increasing, C^1) change of variable $\theta = \psi(\mu)$, and transform back using $d\theta = \psi'(\mu)\,d\mu$. □

Lemma 8 (Mönch's fixed-point theorem [40]). *Let $\mathcal{D}$ be a closed convex subset of a Banach space X with $0 \in \mathcal{D}$, and let $\Pi : \mathcal{D} \to X$ be continuous and satisfy the Mönch condition: whenever $m \subset \mathcal{D}$ is countable and $m \subset \overline{\text{conv}}(\{0\} \cup \Pi(m))$, the closure $\overline{m}$ is compact. Then, Π possesses a fixed point in $\mathcal{D}$.*

The phase space $\mathbb{X}$ is constructed in the standard way. Let $q : \mathcal{I}_0 = (-\infty, 0] \to (0, +\infty)$ be a continuous weight function with $l := \int_{-\infty}^{0} q(\nu)\, d\nu < \infty$. For a fixed $\varrho > 0$, the Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is introduced as

$$\mathbb{X} := \Bigg\{ \phi : \mathcal{I}_0 \to \mathcal{B} : (\mathfrak{E}\|\phi(0)\|^{\wp})^{\frac{1}{\wp}} \text{ is a measurable and bounded function on } [-\varrho, 0], \int_{-\infty}^{0} q(\mu) \sup_{\mu \le \theta \le 0} (\mathfrak{E}\|\phi(\theta)\|^{\wp})^{\frac{1}{\wp}} d\mu < \infty \Bigg\},$$

equipped with the norm

$$\|\phi\|_{\mathbb{X}} = \int_{-\infty}^{0} q(\mu) \sup_{\mu \le \theta \le 0} (\mathfrak{E}\|\phi(\theta)\|^{\wp})^{\frac{1}{\wp}} d\mu.$$

We further examine the subspace over $\mathcal{I}_b = (-\infty, b]$, defined by

$$\mathbb{X}_b := \Bigg\{ \mathcal{Y} : \mathcal{I}_b \to \mathcal{B} \mid \mathcal{Y}(\cdot) \text{ is càdlàg with } \mathcal{Y}(0) = \xi(0) = \xi \in \mathbb{M}^{\wp}(\Omega, \mathbb{X}) \text{ on } \mathcal{I}_0, \sup_{0 \le \nu \le \mathcal{T}} \mathfrak{E}(\|\mathcal{Y}(\nu)\|^{\wp}) < \infty \Bigg\}.$$

The seminorm $\|\cdot\|_b$ on $\mathbb{X}_b$ is specified by

$$\|\mathcal{Y}\|_b = \|\xi\|_{\mathbb{X}} + \sup_{\mu \in \mathcal{I}} (\mathfrak{E}\|\mathcal{Y}(\mu)\|^{\wp})^{\frac{1}{\wp}}, \quad \mathcal{Y} \in \mathbb{X}_b.$$

Lemma 9 ([38]). *Let $\mathcal{Y} \in \mathbb{X}_b$. Then, for every $\nu \in \mathcal{I}$, the segment $\mathcal{Y}_\nu$ belongs to $\mathbb{X}$, and the following two-sided estimate holds:*

$$l(\mathfrak{E}\|\mathcal{Y}(\nu)\|^{\wp})^{\frac{1}{\wp}} \le \|\mathcal{Y}_\nu\|_{\mathbb{X}} \le \|\mathcal{Y}_0\|_{\mathbb{X}} + l \sup_{\mu \in \mathcal{I}} (\mathfrak{E}\|\mathcal{Y}(\mu)\|^{\wp})^{\frac{1}{\wp}},$$

where $l = \int_{-\infty}^{0} q(\mu)\, d\mu < \infty$.

Definition 7. *An $\mathfrak{F}_\nu$-adapted stochastic process $\mathcal{Y}(\nu)$, $\nu \in \mathcal{I}_b = (-\infty, \mathcal{T}]$, is called a mild solution of system (1) provided that the following conditions are satisfied:*

1. *$\mathcal{Y}(\nu)$ is $\mathfrak{F}_\nu$-measurable with càdlàg sample paths on $\nu \ge 0$, almost surely.*
2. *For all $\nu \in (-\infty, 0]$, $\mathcal{Y}(\nu) = \xi(\nu) \in \mathcal{B}$.*
3. *For each $\nu \in \mathcal{I}$, $\mathcal{Y}(\nu)$ fulfills the integral equation:*

$$
\mathcal{Y}(\nu)=\begin{cases}
\xi(\nu), \quad \nu\in(-\infty,0],\\
S^{\psi}_{\sigma}(\nu)\,\xi(0)+T^{\psi}_{\sigma}(\nu)[\mathcal{Y}_1-\mathscr{H}(0,\mathcal{Y}(0),0)]+\mathscr{H}\Big(\nu,\mathcal{Y}_{\nu},\int_0^{\nu}\alpha_1(\nu,\varrho,\mathcal{Y}_{\varrho})\psi'(\varrho)\,d\varrho\Big)\\
+\int_0^{\nu}T^{\psi}_{\sigma}(\nu,\mu)\,\psi'(\mu)\Big(f\Big(\mu,\mathcal{Y}_{\mu},\int_0^{\mu}\alpha_2(\mu,\varrho,\mathcal{Y}_{\varrho})\psi'(\varrho)\,d\varrho\Big)\Big)d\mu\\
+\int_0^{\nu}T^{\psi}_{\sigma}(\nu,\mu)\,\psi'(\mu)\Big(\int_0^{\mu}g(\mu,\varrho,\mathcal{Y}_{\varrho})dw(\varrho)\Big)d\mu\\
+\int_0^{\nu}T^{\psi}_{\sigma}(\nu,\mu)\,\psi'(\mu)\int_{\mathfrak{Z}}h(\mu,\mathcal{Y}_{\mu},\aleph)\tilde{N}(d\mu,\,d\aleph), \quad \nu\in[0,\iota_1],\\
\vdots\\
S^{\psi}_{\sigma}(\nu)\,\xi(0)+T^{\psi}_{\sigma}(\nu)[\mathcal{Y}_1-\mathscr{H}(0,\mathcal{Y}(0),0)]+\mathscr{H}\Big(\nu,\mathcal{Y}_{\nu},\int_0^{\nu}\alpha_1(\nu,\varrho,\mathcal{Y}_{\varrho})\psi'(\varrho)\,d\varrho\Big)\\
+\int_{\iota_{\Bbbk}}^{\nu}T^{\psi}_{\sigma}(\nu,\mu)\,\psi'(\mu)\Big(f\Big(\mu,\mathcal{Y}_{\mu},\int_0^{\mu}\alpha_2(\mu,\varrho,\mathcal{Y}_{\varrho})\psi'(\varrho)\,d\varrho\Big)\Big)d\mu\\
+\int_{\iota_{\Bbbk}}^{\nu}T^{\psi}_{\sigma}(\nu,\mu)\,\psi'(\mu)\Big(\int_0^{\mu}g(\mu,\varrho,\mathcal{Y}_{\varrho})dw(\varrho)\Big)d\mu\\
+\int_{\iota_{\Bbbk}}^{\nu}T^{\psi}_{\sigma}(\nu,\mu)\,\psi'(\mu)\int_{\mathfrak{Z}}h(\mu,\mathcal{Y}_{\mu},\aleph)\tilde{N}(d\mu,\,d\aleph)\\
+\sum_{0<\iota_{\Bbbk}<\nu}S^{\psi}_{\sigma}(\nu,\iota_{\Bbbk})J^1_{\Bbbk}(\mathcal{Y}_{\iota_{\Bbbk}})+\sum_{0<\iota_{\Bbbk}<\nu}T^{\psi}_{\sigma}(\nu,\iota_{\Bbbk})J^2_{\Bbbk}(\mathcal{Y}_{\iota_{\Bbbk}}), \quad \forall\,\nu\in(\iota_{\Bbbk},\iota_{\Bbbk+1}],\ \Bbbk=1,2,\ldots,n.
\end{cases}
$$

Remark 4. *An impulse at $\iota_{\Bbbk}$ restarts the evolution on $(\iota_{\Bbbk},\iota_{\Bbbk+1}]$ from the pair $\big(\mathcal{Y}(\iota_{\Bbbk}^{-})+J^1_{\Bbbk},\,\mathcal{Y}^{[1]}_{\psi}(\iota_{\Bbbk}^{-})+J^2_{\Bbbk}\big)$. Since $\xi(0)$ is carried by S^{ψ}_{σ} and $\mathcal{Y}_1$ by T^{ψ}_{σ}, the positional jump propagates through S^{ψ}_{σ} and the derivative jump through T^{ψ}_{σ}, both evaluated at the ψ-lag $\big(\psi(\nu)-\psi(\iota_{\Bbbk})\big)$. For $J^2_{\Bbbk}\equiv 0$, $\psi(\nu)=\nu$ and $\alpha_1=\alpha_2$, Definition 7 reduces to the single-kernel positional-impulse Caputo representation.*

3. Existence of Mild Solutions

We now introduce the hypotheses needed to establish the existence result:

$(\mathcal{H}_1)$ Let A generate the ψ-fractional sine family $\{S^{\psi}_{\sigma}(\nu)\}_{\nu\geq 0}$ on $\mathcal{B}$, and let $\{T^{\psi}_{\sigma}(\nu)\}_{\nu\geq 0}$ be the associated ψ-R-L family of Definition 6. There is a constant $\mathcal{N}_S>0$ such that, for all $0\leq\mu\leq\nu$ in $\mathcal{I}$,

$$
\big\|S^{\psi}_{\sigma}(\nu,\mu)\big\|\leq\mathcal{N}_S,\qquad \big\|T^{\psi}_{\sigma}(\nu,\mu)\big\|\leq\frac{\mathcal{N}_S\,\big(\psi(\nu)-\psi(\mu)\big)^{\sigma-1}}{\Gamma(\sigma)}. \tag{2}
$$

The second bound follows from the first through Definition 6, since T^{ψ}_{σ} is the ψ-weighted Riemann–Liouville convolution of S^{ψ}_{σ} of order σ. Two consequences of (2) are used repeatedly: for $\nu\in\mathcal{I}$ and $\kappa\geq 1$,

$$
\int_0^{\nu}\big\|T^{\psi}_{\sigma}(\nu,\mu)\big\|\psi'(\mu)\,d\mu\leq\frac{\mathcal{N}_S\,\Psi_{\nu}^{\sigma}}{\Gamma(\sigma+1)},\quad \left(\int_0^{\nu}\big\|T^{\psi}_{\sigma}(\nu,\mu)\big\|^{\kappa}\psi'(\mu)\,d\mu\right)^{\frac{1}{\kappa}}\leq\frac{\mathcal{N}_S}{\Gamma(\sigma)}\left(\frac{\Psi_{\nu}^{\kappa\sigma-\kappa+1}}{\kappa\sigma-\kappa+1}\right)^{\frac{1}{\kappa}}, \tag{3}
$$

where $\Psi_{\nu}:=\big(\psi(\nu)-\psi(0)\big)$; the second requires $\sigma>1-\frac{1}{\kappa}$, which holds for every $\kappa\geq 1$ since $\sigma>1$.

$(\mathcal{H}_2)$ (i) The mapping $\mathscr{H} : \mathcal{I} \times \mathbb{X} \times \mathcal{B} \to \mathcal{B}$ is continuous, and a constant $\mathcal{N}_{\mathscr{H}} > 0$ can be chosen so that, for all $\nu \in \mathcal{I}$, $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathbb{X}$ and $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{B}$,

$$\mathfrak{E}\|\mathscr{H}(\nu, \mathcal{Y}_1, \mathcal{V}_1) - \mathscr{H}(\nu, \mathcal{Y}_2, \mathcal{V}_2)\|^{\wp} \leq \mathcal{N}_{\mathscr{H}}\big(\mathfrak{E}\|\mathcal{Y}_1 - \mathcal{Y}_2\|_{\mathbb{X}}^{\wp} + \mathfrak{E}\|\mathcal{V}_1 - \mathcal{V}_2\|_{\mathcal{B}}^{\wp}\big),$$
$$\mathfrak{E}\|\mathscr{H}(\nu, \mathcal{Y}, \mathcal{V})\|^{\wp} \leq \mathcal{N}_{\mathscr{H}}\big(\mathfrak{E}\|\mathcal{Y}\|_{\mathbb{X}}^{\wp} + \mathfrak{E}\|\mathcal{V}\|_{\mathcal{B}}^{\wp}\big),$$
$$\mathfrak{E}\|\mathscr{H}(\nu, 0, 0)\|^{\wp} = 0.$$

(ii) Assume that, for each pair $(\nu, \mu) \in \mathcal{I} \times \mathcal{I}$, the mappings $\alpha_i : \mathcal{I} \times \mathcal{I} \times \mathbb{X} \to \mathcal{B}$, $i = 1, 2$ are continuous, and that some constant $\mathcal{N}_{\alpha_i} > 0$ satisfies, for all $\nu, \mu \in \mathcal{I}$ and $\mathcal{Y}, \mathcal{V} \in \mathbb{X}$,

$$\mathfrak{E}\left\|\int_0^{\nu} [\alpha_i(\nu, \mu, \mathcal{Y}) - \alpha_i(\nu, \mu, \mathcal{V})]\psi'(\mu)\, d\mu\right\|^{\wp} \leq \mathcal{N}_{\alpha_i}\, \mathfrak{E}\|\mathcal{Y} - \mathcal{V}\|^{\wp}; \qquad \mathfrak{E}\|\alpha_i(\nu, \mu, 0)\|^{\wp} = 0.$$

(iii) Suppose positive functions $\bar{\varkappa}_{\mathscr{H}}, \bar{\varkappa}_{\alpha_1}, \bar{\varkappa}_{\alpha_2} \in \mathbb{M}^1(\mathcal{I} \times \mathcal{I}, \mathbb{R}^+)$ can be found so that, for every bounded set $\mathcal{D}_1 \subset \mathbb{X}$ and every $i = 1, 2$,

$$\chi(\mathscr{H}(\nu, \mathcal{D}_1, \mathcal{D}_1)) \leq \bar{\varkappa}^*_{\mathscr{H}} \sup_{\theta \in \mathcal{I}_0} \chi(\mathcal{D}_1), \qquad \bar{\varkappa}^*_{\mathscr{H}} := \sup_{(\nu,\mu) \in \mathcal{I} \times \mathcal{I}} \bar{\varkappa}_{\mathscr{H}}(\nu, \mu) < \infty,$$
$$\chi(\alpha_i(\nu, \mu, \mathcal{D}_1)) \leq \bar{\varkappa}_{\alpha_i}(\nu, \mu) \sup_{\theta \in \mathcal{I}_0} \chi(\mathcal{D}_1), \qquad \sup_{(\nu,\mu) \in \mathcal{I} \times \mathcal{I}} \bar{\varkappa}_{\alpha_i}(\nu, \mu) = \bar{\varkappa}^*_{\alpha_i} < \infty.$$

$(\mathcal{H}_3)$ The mapping $f : \mathcal{I} \times \mathbb{X} \times \mathcal{B} \to \mathcal{B}$ is continuous, $f(\cdot, \mathcal{Y}, \mathcal{V})$ is measurable for each $(\mathcal{Y}, \mathcal{V}) \in \mathbb{X} \times \mathcal{B}$, and there exist a continuous nondecreasing $\Upsilon_f : \mathbb{R}^+ \to (0, \infty)$ and a continuous $l_f : \mathcal{I} \to \mathbb{R}^+$ with $l_f^* := \sup_{\nu \in \mathcal{I}} l_f(\nu) < \infty$ such that:

(i) $\mathfrak{E}\|f(\nu, \mathcal{Y}, \mathcal{V})\|^{\wp} \leq l_f(\nu)\, \Upsilon_f\big(\mathfrak{E}\|\mathcal{Y}\|_{\mathbb{X}}^{\wp} + \mathfrak{E}\|\mathcal{V}\|_{\mathcal{B}}^{\wp}\big)$, $\mathfrak{E}\|f(\nu, 0, 0)\|^{\wp} = 0$, and $\Upsilon_f^* := \liminf_{r \to \infty} \dfrac{\Upsilon_f(r)}{r} < \infty$.

(ii) A positive function $\bar{\varkappa}_f \in \mathbb{M}^1(\mathcal{I} \times \mathcal{I}, \mathbb{R}^+)$ can be chosen so that, for every bounded set $\mathscr{D}_2 \subset \mathbb{X}$ and all $(\nu, \mu) \in \mathcal{I} \times \mathcal{I}$,

$$\chi(f(\nu, \mathscr{D}_2, \mathscr{D}_2)) \leq \bar{\varkappa}_f(\nu, \mu) \sup_{\theta \in \mathcal{I}_0} \chi(\mathscr{D}_2), \qquad \sup_{(\nu,\mu) \in \mathcal{I} \times \mathcal{I}} \bar{\varkappa}_f(\nu, \mu) = \bar{\varkappa}^*_f < \infty.$$

$(\mathcal{H}_4)$ Suppose the mapping $g : \mathcal{I} \times \mathcal{I} \times \mathbb{X} \to \mathbb{M}^0_{\mathbb{V}}(\mathcal{S}, \mathcal{B})$ has the following properties: for each $\mathcal{Y} \in \mathbb{X}$, the function $g(\cdot, \cdot, \mathcal{Y})$ is measurable on $\mathcal{I} \times \mathcal{I}$, and for every $(\nu, \mu) \in \mathcal{I} \times \mathcal{I}$, the mapping $g(\nu, \mu, \cdot) : \mathbb{X} \to \mathbb{M}^0_{\mathbb{V}}(\mathcal{S}, \mathcal{B})$ is continuous. Assume further that a continuous nondecreasing function $\Upsilon_g : \mathbb{R}^+ \to (0, \infty)$ and a continuous function $l_g : \mathcal{I} \times \mathcal{I} \to \mathbb{R}^+$ with $l_g^* := \sup_{(\nu,\mu)} l_g(\nu, \mu) < \infty$ can be chosen such that:

(i) $\mathfrak{E}\|g(\nu, \mu, \mathcal{Y})\|^{\wp} \leq l_g(\nu, \mu)\, \Upsilon_g\big(\mathfrak{E}\|\mathcal{Y}\|_{\mathbb{X}}^{\wp}\big)$, $\mathfrak{E}\|g(\nu, \mu, 0)\|^{\wp} = 0$, and $\Upsilon_g^* := \liminf_{r \to \infty} \dfrac{\Upsilon_g(r)}{r} < \infty$.

(ii) A positive function $\bar{\varkappa}_g \in \mathbb{M}^1(\mathcal{I} \times \mathcal{I}, \mathbb{R}^+)$ can be found so that, for every bounded set $\mathscr{D}_3 \subset \mathbb{X}$,

$$\chi(g(\nu, \mu, \mathscr{D}_3)) \leq \bar{\varkappa}_g(\nu, \mu) \sup_{\theta \in \mathcal{I}_0} \chi(\mathscr{D}_3), \qquad \text{and} \qquad \sup_{(\nu,\mu) \in \mathcal{I} \times \mathcal{I}} \bar{\varkappa}_g(\nu, \mu) = \bar{\varkappa}^*_g < \infty.$$

$(\mathcal{H}_5)$ Suppose the two families of impulsive mappings $J^1_{\Bbbk}, J^2_{\Bbbk} : \mathbb{X} \to \mathcal{B}$, $\Bbbk = 1, 2, \ldots, n$, satisfy, for all $\mathcal{Y}, \mathcal{V} \in \mathbb{X}$ and each $i = 1, 2$:

(i) $\mathfrak{E}\|j_{\Bbbk}^{i}(\mathcal{Y})-j_{\Bbbk}^{i}(\mathcal{V})\|^{\wp}\le l_{\Bbbk}^{i}\,\mathfrak{E}\|\mathcal{Y}-\mathcal{V}\|^{\wp}$, $\mathfrak{E}\|j_{\Bbbk}^{i}(0)\|^{\wp}=0$, where $L^{i}:=\sum_{\Bbbk=1}^{n}l_{\Bbbk}^{i}<\infty$.

(ii) Constants $\bar{\varkappa}_{\Bbbk}^{1},\bar{\varkappa}_{\Bbbk}^{2}>0$, $\Bbbk=1,2,\dots,n$, can be chosen so that, for every bounded set $\mathscr{D}_4\subset\mathbb{X}$ and each $i=1,2$,

$$\chi\Big(j_{\Bbbk}^{i}(\mathscr{D}_4)\Big)\le\bar{\varkappa}_{\Bbbk}^{i}\sup_{\theta\in\mathcal{I}_0}\chi(\mathscr{D}_4),\qquad\text{and}\qquad\sup_{\Bbbk}\bar{\varkappa}_{\Bbbk}^{i}=\bar{\varkappa}^{*,i}<\infty.$$

Throughout, $i=1$ refers to the impulses acting on the state and $i=2$ to those acting on its ψ-derivative; by Remark 4, their constants enter the criteria through S_{σ}^{ψ} and T_{σ}^{ψ} respectively and are not merged.

$(\mathcal{H}_6)$ The mapping $h:\mathcal{I}\times\mathbb{X}\times\mathfrak{Z}\to\mathcal{B}$ is continuous and admits constants $l_{\bar{h}},l_{\hat{h}}>0$ such that, for all $\nu\in\mathcal{I}$ and $\mathcal{Y}\in\mathbb{X}$,

(i) $\Big(\int_{\mathfrak{Z}}\mathfrak{E}\|h(\nu,\mathcal{Y},\aleph)\|^{2}\lambda(d\aleph)\Big)^{\frac{\wp}{2}}\le l_{\bar{h}}\,\mathfrak{E}\|\mathcal{Y}\|_{\mathbb{X}}^{\wp}$;

(ii) $\Big(\int_{\mathfrak{Z}}\mathfrak{E}\|h(\nu,\mathcal{Y},\aleph)\|^{2\wp}\lambda(d\aleph)\Big)^{\frac{1}{2}}\le l_{\hat{h}}\,\mathfrak{E}\|\mathcal{Y}\|_{\mathbb{X}}^{\wp}$;

(iii) $\mathfrak{E}\|h(\nu,0,\aleph)\|^{\wp}=0$.

In addition, functions $\bar{\varkappa}_{h},\bar{\varkappa}_{\bar{h}}\in\mathbb{M}^{1}(\mathcal{I}\times\mathcal{I},\mathbb{R}^{+})$ can be chosen so that, for every bounded set $\mathscr{D}_5\subset\mathbb{X}$,

$$\Big[\chi\Big(\int_{\mathfrak{Z}}\|h(\nu,\mathscr{D}_5,\aleph)\|^{2}\lambda(d\aleph)\Big)\Big]^{\frac{1}{2}}\le\bar{\varkappa}_{h}(\nu,\mu)\sup_{\theta\in\mathcal{I}_0}\chi(\mathscr{D}_5),\quad\bar{\varkappa}_{h}^{*}:=\sup_{(\nu,\mu)\in\mathcal{I}\times\mathcal{I}}\bar{\varkappa}_{h}(\nu,\mu)<\infty,$$

$$\Big[\chi\Big(\int_{\mathfrak{Z}}\|h(\nu,\mathscr{D}_5,\aleph)\|^{4}\lambda(d\aleph)\Big)\Big]^{\frac{1}{4}}\le\bar{\varkappa}_{\bar{h}}(\nu,\mu)\sup_{\theta\in\mathcal{I}_0}\chi(\mathscr{D}_5),\quad\bar{\varkappa}_{\bar{h}}^{*}:=\sup_{(\nu,\mu)\in\mathcal{I}\times\mathcal{I}}\bar{\varkappa}_{\bar{h}}(\nu,\mu)<\infty.$$

For brevity, the compactness constants generated by $(\mathcal{H}_2)$–$(\mathcal{H}_6)$ are collected once and for all as

$$\begin{aligned}&\varkappa_{\mathscr{H}}^{*}:=\bar{\varkappa}_{\mathscr{H}}^{*}\big(1+2\Psi_{\mathcal{T}}\,\bar{\varkappa}_{\alpha_1}^{*}\big),\qquad\varkappa_{f}^{*}:=\bar{\varkappa}_{f}^{*}\big(1+2\Psi_{\mathcal{T}}\,\bar{\varkappa}_{\alpha_2}^{*}\big),\qquad\varkappa_{g}^{*}:=\bar{\varkappa}_{g}^{*},\\&\varkappa_{\bar{h}}^{*}:=\bar{\varkappa}_{h}^{*},\qquad\varkappa_{\hat{h}}^{*}:=\bar{\varkappa}_{\bar{h}}^{*},\end{aligned}\tag{4}$$

the factor $2\Psi_{\mathcal{T}}$ arising from Corollary 1 applied to the memory convolution $\int_0^{\nu}\alpha_i\,\psi'\,d\varrho$. The two kernels separate already here: $\varkappa_{\mathscr{H}}^{*}$ inherits $\bar{\varkappa}_{\alpha_1}^{*}$ and $\varkappa_{f}^{*}$ inherits $\bar{\varkappa}_{\alpha_2}^{*}$.

Existence requires two independent smallness conditions, one in the Lipschitz and growth constants of $(\mathcal{H}_2)$–$(\mathcal{H}_6)$ and one in their noncompactness constants; neither implies the other.

Theorem 1. *Assume* $(\mathcal{H}_1)$–$(\mathcal{H}_6)$, *and write* $\Psi_{\mathcal{T}}=\big(\psi(\mathcal{T})-\psi(0)\big)$ *and* $\Lambda_{\sigma}:=\dfrac{\mathcal{N}_S\,\Psi_{\mathcal{T}}^{\sigma}}{\Gamma(\sigma+1)}$. *If the growth condition*

$$\begin{aligned}&28^{\wp-1}l^{\wp}\Big\{\mathcal{N}_{\mathscr{H}}(1+\mathcal{N}_{\alpha_1})+\Lambda_{\sigma}^{\wp}\Big[l_{f}^{*}\Upsilon_{f}^{*}(1+\mathcal{N}_{\alpha_2})+\tilde{C}_{\wp}\,\Psi_{\mathcal{T}}^{\frac{\wp}{2}}l_{g}^{*}\Upsilon_{g}^{*}+\hat{C}_{\wp}\big(\Psi_{\mathcal{T}}^{\frac{\wp}{2}}l_{\bar{h}}+\Psi_{\mathcal{T}}\,l_{\hat{h}}\big)\Big]\\&+\mathcal{N}_{S}^{\wp}\Big[\Big(L^{1}\Big)^{\frac{\wp}{q}+1}+\Big(L^{2}\Big)^{\frac{\wp}{q}+1}\Big]\Big\}<1,\end{aligned}\tag{5}$$

and the noncompactness condition

$$\varkappa^*_{\mathscr{H}}+\frac{2\mathcal{N}_S\,\Psi^{\sigma}_{\mathcal{T}}}{\Gamma(\sigma+1)}\varkappa^*_f+\frac{2\mathcal{N}_S\,\Psi^{\sigma}_{\mathcal{T}}}{\Gamma(\sigma+1)}\sqrt{\Psi_{\mathcal{T}}}\,\varkappa^*_g$$
$$+\frac{2\mathcal{N}_S}{\Gamma(\sigma)}\sqrt{\hat{\mathcal{C}}_2}\left\{\left(\frac{\Psi^{2\sigma-1}_{\mathcal{T}}}{2\sigma-1}\right)^{\frac{1}{2}}\varkappa^*_{\hbar}+\left(\frac{\Psi^{4\sigma-3}_{\mathcal{T}}}{4\sigma-3}\right)^{\frac{1}{4}}\varkappa^*_{\hat{h}}\right\}+\mathcal{N}_S\sum_{\Bbbk=1}^{n}\left(\bar{\varkappa}^1_{\Bbbk}+\bar{\varkappa}^2_{\Bbbk}\right)<1 \quad (6)$$

both hold, then (1) *possesses at least one mild solution on* $\mathcal{I}_b$.

Proof. We now introduce the operator $\varpi:\mathbb{X}_b\to\mathbb{X}_b$ given by

$$(\varpi\mathcal{Y})(\nu)=\begin{cases}\xi(\nu), \quad \nu\in(-\infty,0],\\ S^{\psi}_{\sigma}(\nu)\xi(0)+T^{\psi}_{\sigma}(\nu)\left[\mathcal{Y}_1-\mathscr{H}(0,\mathcal{Y}(0),0)\right]+\mathscr{H}\left(\nu,\mathcal{Y}_\nu,\int_0^\nu\alpha_1(\nu,\varrho,\mathcal{Y}_\varrho)\,\psi'(\varrho)\,d\varrho\right)\\ \quad+\int_0^\nu T^{\psi}_{\sigma}(\nu,\mu)\,\psi'(\mu)\,f\left(\mu,\mathcal{Y}_\mu,\int_0^\mu\alpha_2(\mu,\varrho,\mathcal{Y}_\varrho)\,\psi'(\varrho)\,d\varrho\right)d\mu\\ \quad+\int_0^\nu T^{\psi}_{\sigma}(\nu,\mu)\,\psi'(\mu)\left(\int_0^\mu g(\mu,\varrho,\mathcal{Y}_\varrho)\,dw(\varrho)\right)d\mu\\ \quad+\int_0^\nu T^{\psi}_{\sigma}(\nu,\mu)\,\psi'(\mu)\int_{\mathfrak{Z}}h(\mu,\mathcal{Y}_\mu,\aleph)\,\widetilde{N}(d\mu,d\aleph), \quad \nu\in[0,\iota_1],\\ \vdots\\ S^{\psi}_{\sigma}(\nu)\xi(0)+T^{\psi}_{\sigma}(\nu)\left[\mathcal{Y}_1-\mathscr{H}(0,\mathcal{Y}(0),0)\right]+\mathscr{H}\left(\nu,\mathcal{Y}_\nu,\int_0^\nu\alpha_1(\nu,\varrho,\mathcal{Y}_\varrho)\,\psi'(\varrho)\,d\varrho\right)\\ \quad+\int_{\iota_{\Bbbk}}^\nu T^{\psi}_{\sigma}(\nu,\mu)\,\psi'(\mu)\,f\left(\mu,\mathcal{Y}_\mu,\int_0^\mu\alpha_2(\mu,\varrho,\mathcal{Y}_\varrho)\,\psi'(\varrho)\,d\varrho\right)d\mu\\ \quad+\int_{\iota_{\Bbbk}}^\nu T^{\psi}_{\sigma}(\nu,\mu)\,\psi'(\mu)\left(\int_0^\mu g(\mu,\varrho,\mathcal{Y}_\varrho)\,dw(\varrho)\right)d\mu\\ \quad+\int_{\iota_{\Bbbk}}^\nu T^{\psi}_{\sigma}(\nu,\mu)\,\psi'(\mu)\int_{\mathfrak{Z}}h(\mu,\mathcal{Y}_\mu,\aleph)\,\widetilde{N}(d\mu,d\aleph)\\ \quad+\sum_{0<\iota_{\Bbbk}<\nu}S^{\psi}_{\sigma}(\nu,\iota_{\Bbbk})\,j^1_{\Bbbk}(\mathcal{Y}_{\iota_{\Bbbk}})+\sum_{0<\iota_{\Bbbk}<\nu}T^{\psi}_{\sigma}(\nu,\iota_{\Bbbk})\,j^2_{\Bbbk}(\mathcal{Y}_{\iota_{\Bbbk}}),\\ \qquad\qquad\forall\,\nu\in(\iota_{\Bbbk},\iota_{\Bbbk+1}],\ \Bbbk=1,2,\ldots,n.\end{cases}$$

The existence of a fixed point for ϖ is sufficient to guarantee a mild solution to (1). To this end, we introduce $z\in\mathbb{X}_b$ defined as

$$z_\nu=\begin{cases}\xi(\nu) & \text{if } \nu\in\mathcal{I}_0\\ S^{\psi}_{\sigma}(\nu)\xi(0) & \text{if } \nu\in\mathcal{I}.\end{cases}$$

If $\mathcal{Y}(\cdot)$ satisfies (1), the decomposition $\mathcal{Y}(\nu)=\mathscr{V}(\nu)+z(\nu)$ gives $\mathcal{Y}_\nu=\mathscr{V}_\nu+z_\nu$ on $\mathcal{I}$ if and only if $\mathscr{V}_0=0$, $\mathcal{Y}^{[1]}_{\psi}(0)=\mathcal{Y}_1$ and $\mathscr{V}^{[1]}_{\psi}(0)=\mathscr{V}_1$. Since z is smooth across each $\iota_{\Bbbk}$, both jump relations are inherited unchanged: $\Delta\mathscr{V}(\iota_{\Bbbk})=j^1_{\Bbbk}$ and $\Delta\mathscr{V}^{[1]}_{\psi}(\iota_{\Bbbk})=j^2_{\Bbbk}$.

Let $\mathbb{X}^0_b=\{\mathscr{V}:\mathscr{V}\in\mathbb{X}_b,\mathscr{V}_0=0\}$, equipped with the norm $\|\mathscr{V}\|_b=\|\mathscr{V}_0\|_{\mathbb{X}}+\sup_{\mu\in\mathcal{I}}(\mathfrak{E}\|\mathscr{V}(\mu)\|^{\wp})^{1/\wp}$; $(\mathbb{X}^0_b,\|\cdot\|_b)$ is a Banach space. For $r>0$, set $\mathbb{X}_r=\left\{\mathscr{V}\in\mathbb{X}^0_b:\|\mathscr{V}\|_b\le r\right\}$, a bounded, closed, convex subset of $\mathbb{X}^0_b$. For $\mathscr{V}\in\mathbb{X}_r$,

$$\|\mathscr{V}_\nu+z_\nu\|^{\wp}_{\mathbb{X}}\le 2^{\wp-1}\|\mathscr{V}_\nu\|^{\wp}_{\mathbb{X}}+2^{\wp-1}\|z_\nu\|^{\wp}_{\mathbb{X}}\le 4^{\wp-1}l^{\wp}(r+\mathcal{N}_S\,\mathfrak{E}\|\xi(0)\|^{\wp})+4^{\wp-1}\|z_\nu\|^{\wp}_{\mathbb{X}}=:r^*,$$

so that $\lim_{r\to\infty}r^*/r=4^{\wp-1}l^{\wp}$. The operator ϖ restricts to $\varpi:\mathbb{X}^0_b\to\mathbb{X}^0_b$ by

$$
(\varpi\mathscr{V})(\nu)=\begin{cases}
0, \quad \nu\in\mathcal{I}_0,\\
T_\sigma^{\psi}(\nu)[\mathscr{V}_1-\mathscr{H}(0,\mathscr{V}(0),0)]+\mathscr{H}\Big(\nu,\mathscr{V}_\nu+z_\nu,\int_0^\nu \alpha_1(\nu,\varrho,\mathscr{V}_\varrho+z_\varrho)\psi'(\varrho)\,d\varrho\Big)\\
+\int_0^\nu T_\sigma^{\psi}(\nu,\mu)\,\psi'(\mu)\Big(f\Big(\mu,\mathscr{V}_\mu+z_\mu,\int_0^\mu \alpha_2(\mu,\varrho,\mathscr{V}_\varrho+z_\varrho)\psi'(\varrho)\,d\varrho\Big)\Big)d\mu\\
+\int_0^\nu T_\sigma^{\psi}(\nu,\mu)\,\psi'(\mu)\Big(\int_0^\mu g(\mu,\varrho,\mathscr{V}_\varrho+z_\varrho)dw(\varrho)\Big)d\mu\\
+\int_0^\nu T_\sigma^{\psi}(\nu,\mu)\,\psi'(\mu)\int_{\mathfrak{Z}} h(\mu,\mathscr{V}_\mu+z_\mu,\aleph)\tilde{N}(d\mu,d\aleph), \quad \nu\in[0,\iota_1]\\
\vdots\\
T_\sigma^{\psi}(\nu)[\mathscr{V}_1-\mathscr{H}(0,\mathscr{V}(0),0)]+\mathscr{H}\Big(\nu,\mathscr{V}_\nu+z_\nu,\int_0^\nu \alpha_1(\nu,\varrho,\mathscr{V}_\varrho+z_\varrho)\psi'(\varrho)\,d\varrho\Big)\\
+\int_{\iota_{\Bbbk}}^\nu T_\sigma^{\psi}(\nu,\mu)\,\psi'(\mu)\Big(f\Big(\mu,\mathscr{V}_\mu+z_\mu,\int_0^\mu \alpha_2(\mu,\varrho,\mathscr{V}_\varrho+z_\varrho)\psi'(\varrho)\,d\varrho\Big)\Big)d\mu\\
+\int_{\iota_{\Bbbk}}^\nu T_\sigma^{\psi}(\nu,\mu)\,\psi'(\mu)\Big(\int_0^\mu g(\mu,\varrho,\mathscr{V}_\varrho+z_\varrho)dw(\varrho)\Big)d\mu\\
+\int_{\iota_{\Bbbk}}^\nu T_\sigma^{\psi}(\nu,\mu)\,\psi'(\mu)\int_{\mathfrak{Z}} h(\mu,\mathscr{V}_\mu+z_\mu,\aleph)\tilde{N}(d\mu,d\aleph)\\
+\sum_{0<\iota_{\Bbbk}<\nu} S_\sigma^{\psi}(\nu,\iota_{\Bbbk})\jmath_{\Bbbk}^1(\mathscr{V}_{\iota_{\Bbbk}}+z_{\iota_{\Bbbk}})+\sum_{0<\iota_{\Bbbk}<\nu} T_\sigma^{\psi}(\nu,\iota_{\Bbbk})\jmath_{\Bbbk}^2(\mathscr{V}_{\iota_{\Bbbk}}+z_{\iota_{\Bbbk}}), \quad \forall\,\nu\in(\iota_{\Bbbk},\iota_{\Bbbk+1}],\ \Bbbk=1,2,\ldots,n.
\end{cases}
$$

We proceed to demonstrate the existence of a fixed point for ϖ. The proof is structured into four steps.

Step 1: There exists a positive constant r such that $\varpi(\mathbb{X}_r)\subset\mathbb{X}_r$.

Suppose not. Then, for every $r>0$ there are $\mathscr{V}\in\mathbb{X}_r$ and $\nu\in\mathcal{I}$ with $\mathfrak{E}\|(\varpi\mathscr{V})(\nu)\|^{\wp}>r$. Case (i). For $\nu\in[0,\iota_1]$, we have

$$
\begin{aligned}
r\le&\mathfrak{E}\|(\varpi\mathscr{V})(\nu)\|^{\wp}\\
\le&5^{\wp-1}\Big\{\mathfrak{E}\Big\|T_\sigma^{\psi}(\nu)[\mathscr{V}_1-\mathscr{H}(0,\mathscr{V}(0),0)]\Big\|^{\wp}\\
&+\mathfrak{E}\Big\|\mathscr{H}\Big(\nu,\mathscr{V}_\nu+z_\nu,\int_0^\nu \alpha_1(\nu,\varrho,\mathscr{V}_\varrho+z_\varrho)\psi'(\varrho)\,d\varrho\Big)\Big\|^{\wp}\\
&+\mathfrak{E}\Big\|\int_0^\nu T_\sigma^{\psi}(\nu,\mu)\,\psi'(\mu)\Big(f\Big(\mu,\mathscr{V}_\mu+z_\mu,\int_0^\mu \alpha_2(\mu,\varrho,\mathscr{V}_\varrho+z_\varrho)\psi'(\varrho)\,d\varrho\Big)\Big)d\mu\Big\|^{\wp}\\
&+\mathfrak{E}\Big\|\int_0^\nu T_\sigma^{\psi}(\nu,\mu)\,\psi'(\mu)\int_0^\mu g(\mu,\varrho,\mathscr{V}_\varrho+z_\varrho)dw(\varrho)d\mu\Big\|^{\wp}\\
&+\mathfrak{E}\Big\|\int_0^\nu T_\sigma^{\psi}(\nu,\mu)\,\psi'(\mu)\int_{\mathfrak{Z}} h(\mu,\mathscr{V}_\mu+z_\mu,u)\tilde{N}(d\mu,du)\Big\|^{\wp}\Big\}=:5^{\wp-1}\sum_{i=1}^{5}\jmath_i.
\end{aligned}
$$

Since $\mathscr{V}_0=0$ gives $\mathscr{H}(0,\mathscr{V}(0),0)=0$ by $(\mathcal{H}_2)$(i), hypothesis $(\mathcal{H}_1)$ yields

$$
\jmath_1=\mathfrak{E}\Big\|T_\sigma^{\psi}(\nu)\mathscr{V}_1\Big\|^{\wp}\le\left(\frac{\mathcal{N}_S\,\Psi_\nu^{\sigma-1}}{\Gamma(\sigma)}\right)^{\wp}\mathfrak{E}\|\mathscr{V}_1\|_{\mathcal{B}}^{\wp}=:\mathcal{K}_1,
$$

a constant independent of r. By employing hypotheses $(\mathcal{H}_1)$–$(\mathcal{H}_2)$(i),

$$
\jmath_2=\mathfrak{E}\Big\|\mathscr{H}\Big(\nu,\mathscr{V}_\nu+z_\nu,\int_0^\nu \alpha_1(\nu,\varrho,\mathscr{V}_\varrho+z_\varrho)\psi'(\varrho)\,d\varrho\Big)\Big\|^{\wp}\le\mathcal{N}_{\mathscr{H}}(1+\mathcal{N}_{\alpha_1})r^*.
$$

Upon invoking conditions $(\mathcal{H}_2)$(ii) and $(\mathcal{H}_3)$(i),

$$\begin{aligned} \jmath_3 &= \mathfrak{E}\left\| \int_0^{\nu} T_\sigma^{\psi}(\nu,\mu)\,\psi'(\mu)\left(f\left(\mu, \mathscr{V}_\mu + z_\mu, \int_0^{\mu} \alpha_2(\mu,\varrho,\mathscr{V}_\varrho + z_\varrho)\psi'(\varrho)\,d\varrho\right)\right) d\mu \right\|^{\wp} \\ &\leq \left(\int_0^{\nu} \left\| T_\sigma^{\psi}(\nu,\mu)\right\| \psi'(\mu)\,d\mu \right)^{\wp} \sup_{\mu\in\mathcal{I}} \mathfrak{E}\|f(\mu,\cdot,\cdot)\|^{\wp} \leq \Lambda_\sigma^{\wp}\, l_f^*\, \mathrm{Y}_f((1+\mathcal{N}_{\alpha_2})r^*), \end{aligned}$$

where Hölder's inequality in the measure $\|T_\sigma^{\psi}(\nu,\mu)\|\psi'(\mu)\,d\mu$ and the first estimate of (3) were used. By virtue of assumption $(\mathcal{H}_4)$ (i)–(ii),

$$\begin{aligned} \jmath_4 &\leq \mathfrak{E}\left\| \int_0^{\nu} T_\sigma^{\psi}(\nu,\mu)\,\psi'(\mu) \int_0^{\mu} g(\mu,\varrho,\mathscr{V}_\varrho + z_\varrho)dw(\varrho)d\mu \right\|^{\wp} \\ &\leq \Lambda_\sigma^{\wp}\,\tilde{C}_\wp\,\Psi_\nu^{\frac{\wp}{2}}\, l_g^*\, \mathrm{Y}_g(r^*). \end{aligned}$$

Using $(\mathcal{H}_6)$,

$$\begin{aligned} \jmath_5 &\leq \mathfrak{E}\left\| \int_0^{\nu} T_\sigma^{\psi}(\nu,\mu)\,\psi'(\mu) \int_{\mathfrak{Z}} h(\mu, \mathscr{V}_\mu + z_\mu, \aleph)\tilde{N}(d\mu, d\aleph) \right\|^{\wp} \\ &\leq \Lambda_\sigma^{\wp}\,\hat{\mathcal{C}}_\wp \left\{ \Psi_\nu^{\frac{\wp}{2}} l_{\bar{h}} + \Psi_\nu\, l_{\hat{h}} \right\} r^*. \end{aligned}$$

Combining $\jmath_1$–$\jmath_5$, dividing by r and letting $r \to \infty$ gives

$$1 \leq 20^{\wp-1} l^{\wp} \left\{ \mathcal{N}_{\mathscr{H}}(1+\mathcal{N}_{\alpha_1}) + \Lambda_\sigma^{\wp}\left[l_f^*\, \mathrm{Y}_f^*(1+\mathcal{N}_{\alpha_2}) + \tilde{C}_\wp\, \Psi_{\mathcal{T}}^{\frac{\wp}{2}} l_g^*\, \mathrm{Y}_g^* + \hat{\mathcal{C}}_\wp \left\{ \Psi_{\mathcal{T}}^{\frac{\wp}{2}} l_{\bar{h}} + \Psi_{\mathcal{T}}\, l_{\hat{h}} \right\} \right] \right\},$$

where $\jmath_1 = \mathcal{K}_1$ contributes nothing in the limit, and the definitions of $\mathrm{Y}_f^*, \mathrm{Y}_g^*$ in $(\mathcal{H}_3)$–$(\mathcal{H}_4)$ together with $\lim_{r\to\infty} r^*/r = 4^{\wp-1}l^{\wp}$ were used. This contradicts (5), which is stronger since $28^{\wp-1} > 20^{\wp-1}$. Hence $\varpi(\mathbb{X}_r) \subset \mathbb{X}_r$ on $[0, \iota_1]$ for some $r > 0$.

Case (ii). For $\nu \in (\iota_{\Bbbk}, \iota_{\Bbbk+1}]$ the two impulsive sums are estimated separately. By $(\mathcal{H}_1)$, $(\mathcal{H}_5)$(i) with $i = 1$ and Hölder's inequality in the exponent pair $(\wp, q)$,

$$\jmath_6 = \mathfrak{E}\left\| \sum_{0<\iota_{\Bbbk}<\nu} S_\sigma^{\psi}(\nu,\iota_{\Bbbk})\jmath_{\Bbbk}^1(\mathscr{V}_{\iota_{\Bbbk}} + z_{\iota_{\Bbbk}}) \right\|^{\wp} \leq \mathcal{N}_S^{\wp}\left(L^1\right)^{\frac{\wp}{q}} \sum_{0<\iota_{\Bbbk}<\nu} l_{\Bbbk}^1 r^* \leq \mathcal{N}_S^{\wp}\left(L^1\right)^{\frac{\wp}{q}+1} r^*,$$

and, likewise with $i = 2$, using $\|T_\sigma^{\psi}(\nu,\iota_{\Bbbk})\| \leq \mathcal{N}_S \Psi_{\mathcal{T}}^{\sigma-1}/\Gamma(\sigma) \leq \mathcal{N}_S$ (after absorbing the constant $\Psi_{\mathcal{T}}^{\sigma-1}/\Gamma(\sigma)$ into $\mathcal{N}_S$),

$$\jmath_7 = \mathfrak{E}\left\| \sum_{0<\iota_{\Bbbk}<\nu} T_\sigma^{\psi}(\nu,\iota_{\Bbbk})\jmath_{\Bbbk}^2(\mathscr{V}_{\iota_{\Bbbk}} + z_{\iota_{\Bbbk}}) \right\|^{\wp} \leq \mathcal{N}_S^{\wp}\left(L^2\right)^{\frac{\wp}{q}+1} r^*,$$

The two bounds have the same shape, so the derivative impulses raise the number of additive terms from six to seven and the prefactor to $28^{\wp-1} = (7\cdot 4)^{\wp-1}$, the factor $4^{\wp-1}$ coming from r^*. Combining $\jmath_1$–$\jmath_7$ and letting $r \to \infty$ yields

$$\begin{aligned} 28^{\wp-1} l^{\wp} \Bigg\{ &\mathcal{N}_{\mathscr{H}}(1+\mathcal{N}_{\alpha_1}) + \Lambda_\sigma^{\wp}\left[l_f^*\, \mathrm{Y}_f^*(1+\mathcal{N}_{\alpha_2}) + \tilde{C}_\wp\, \Psi_{\mathcal{T}}^{\frac{\wp}{2}} l_g^*\, \mathrm{Y}_g^* + \hat{\mathcal{C}}_\wp \left\{ \Psi_{\mathcal{T}}^{\frac{\wp}{2}} l_{\bar{h}} + \Psi_{\mathcal{T}}\, l_{\hat{h}} \right\} \right] \\ &+ \mathcal{N}_S^{\wp}\left[\left(L^1\right)^{\frac{\wp}{q}+1} + \left(L^2\right)^{\frac{\wp}{q}+1} \right] \Bigg\} \geq 1, \end{aligned}$$

which is exactly the negation of (5). Taking the larger of the two radii obtained in Cases (i) and (ii), there exists $r > 0$ with $\varpi(\mathbb{X}_r) \subset \mathbb{X}_r$.

Step 2: Continuity of ϖ on $\mathbb{X}_b$.

Let $\{\mathscr{V}^\tau\}_{\tau=1}^{\infty} \subseteq \mathbb{X}_b^0$ converge to $\mathscr{V}$ in $\mathbb{X}_b^0$, with a common bound $r > 0$. By $(\mathcal{H}_2)$–$(\mathcal{H}_6)$, each difference term converges to 0 in $\mathfrak{E}\|\cdot\|^{\wp}$: the neutral term (through α_1), the drift term (through α_2), the diffusion term against dw and the jump term against $\tilde{N}$, all integrated with the weight $\psi'(\mu)$ through $T_\sigma^\psi(\nu,\mu)$, and the impulsive terms $j_{\mathbb{k}}^i\Big(\mathscr{V}_{\iota_{\mathbb{k}}}^\tau + z_{\iota_{\mathbb{k}}}\Big) - j_{\mathbb{k}}^i(\mathscr{V}_{\iota_{\mathbb{k}}} + z_{\iota_{\mathbb{k}}})$, $i = 1, 2$, weighted by $S_\sigma^\psi(\nu,\iota_{\mathbb{k}})$ and $T_\sigma^\psi(\nu,\iota_{\mathbb{k}})$. By dominated convergence,

$$\mathfrak{E}\|(\varpi\mathscr{V}^\tau)(\nu) - (\varpi\mathscr{V})(\nu)\|_{\mathbb{X}_b}^{\wp} \to 0 \quad \text{as} \quad \tau \to \infty,$$

which establishes that the operator ϖ is continuous on $\mathbb{X}_b$.

Step 3: $\varpi(\mathbb{X}_b)$ is equicontinuous on $\mathcal{I}$.

For $0 < \varsigma_1 < \varsigma_2 \leq \mathcal{T}$ and $\mathscr{V} \in \mathbb{X}_b$, writing ς_1, ς_2 for two generic times so as not to clash with the impulse instants,

$$\begin{aligned}
&\mathfrak{E}\|(\varpi\mathscr{V})(\varsigma_2) - (\varpi\mathscr{V})(\varsigma_1)\|_{\mathbb{X}_b}^{\wp} \\
&\leq 8^{\wp-1}\Big\{\mathfrak{E}\Big\|\Big(S_\sigma^\psi(\varsigma_2) - S_\sigma^\psi(\varsigma_1)\Big)\xi(0)\Big\|^{\wp} + \mathfrak{E}\Big\|\Big(T_\sigma^\psi(\varsigma_2) - T_\sigma^\psi(\varsigma_1)\Big)[\mathscr{V}_1 - \mathscr{H}(0,\xi(0),0)]\Big\|^{\wp}\Big\} \\
&\quad + 8^{\wp-1}\Bigg\{\mathfrak{E}\Bigg\|\int_0^{\varsigma_1}\Big(T_\sigma^\psi(\varsigma_2,\mu) - T_\sigma^\psi(\varsigma_1,\mu)\Big)\psi'(\mu)\, f\Big(\mu, \mathscr{V}_\mu + z_\mu, \int_0^\mu \alpha_2(\mu,\varrho,\mathscr{V}_\varrho + z_\varrho)\psi'(\varrho)\, d\varrho\Big) d\mu\Bigg\|^{\wp} \\
&\quad + \mathfrak{E}\Bigg\|\int_{\varsigma_1}^{\varsigma_2} T_\sigma^\psi(\varsigma_2,\mu)\,\psi'(\mu)\, f\Big(\mu, \mathscr{V}_\mu + z_\mu, \int_0^\mu \alpha_2(\mu,\varrho,\mathscr{V}_\varrho + z_\varrho)\psi'(\varrho)\, d\varrho\Big) d\mu\Bigg\|^{\wp} \\
&\quad + \mathfrak{E}\Bigg\|\int_0^{\varsigma_1}\Big(T_\sigma^\psi(\varsigma_2,\mu) - T_\sigma^\psi(\varsigma_1,\mu)\Big)\psi'(\mu)\Big(\int_0^\mu g(\mu,\varrho,\mathscr{V}_\varrho + z_\varrho) dw(\varrho)\Big) d\mu\Bigg\|^{\wp} \\
&\quad + \mathfrak{E}\Bigg\|\int_{\varsigma_1}^{\varsigma_2} T_\sigma^\psi(\varsigma_2,\mu)\,\psi'(\mu)\Big(\int_0^\mu g(\mu,\varrho,\mathscr{V}_\varrho + z_\varrho) dw(\varrho)\Big) d\mu\Bigg\|^{\wp} \\
&\quad + \mathfrak{E}\Bigg\|\int_0^{\varsigma_1}\Big(T_\sigma^\psi(\varsigma_2,\mu) - T_\sigma^\psi(\varsigma_1,\mu)\Big)\psi'(\mu)\int_{\mathfrak{Z}} h(\mu,\mathscr{V}_\mu + z_\mu,\aleph)\tilde{N}(d\mu,d\aleph)\Bigg\|^{\wp} \\
&\quad + \mathfrak{E}\Bigg\|\int_{\varsigma_1}^{\varsigma_2} T_\sigma^\psi(\varsigma_2,\mu)\,\psi'(\mu)\int_{\mathfrak{Z}} h(\mu,\mathscr{V}_\mu + z_\mu,\aleph)\tilde{N}(d\mu,d\aleph)\Bigg\|^{\wp}\Bigg\} \\
&\quad + 8^{\wp-1}\mathfrak{E}\Bigg\|\sum_{0<\iota_{\mathbb{k}}<\varsigma_2}\Big[S_\sigma^\psi(\varsigma_2,\iota_{\mathbb{k}}) - S_\sigma^\psi(\varsigma_1,\iota_{\mathbb{k}})\Big] j_{\mathbb{k}}^1(\mathscr{V}_{\iota_{\mathbb{k}}} + z_{\iota_{\mathbb{k}}})\Bigg\|^{\wp} \\
&\quad + 8^{\wp-1}\mathfrak{E}\Bigg\|\sum_{0<\iota_{\mathbb{k}}<\varsigma_2}\Big[T_\sigma^\psi(\varsigma_2,\iota_{\mathbb{k}}) - T_\sigma^\psi(\varsigma_1,\iota_{\mathbb{k}})\Big] j_{\mathbb{k}}^2(\mathscr{V}_{\iota_{\mathbb{k}}} + z_{\iota_{\mathbb{k}}})\Bigg\|^{\wp}.
\end{aligned}$$

As $\varsigma_2 \to \varsigma_1$, continuity of ψ gives $\big(\psi(\varsigma_2) - \psi(\cdot)\big) \to \big(\psi(\varsigma_1) - \psi(\cdot)\big)$, so every difference of strongly continuous families above converges to 0. The same applies to the last term: by Definition 6, T_σ^ψ is a ψ-weighted convolution of S_σ^ψ against the integrable kernel $\big(\psi(\nu) - \psi(\mu)\big)^{\sigma-1}/\Gamma(\sigma)$ and is therefore strongly continuous in ν. The resulting bound is independent of $\mathscr{V}$, so $\{\varpi\mathscr{V};\ \mathscr{V} \in \mathbb{X}_b\}$ is equicontinuous on $\mathcal{I}$.

Step 4: Mönch conditions are satisfied.

Let $\mathcal{E} \subset \mathbb{X}_b$ be countable and equicontinuous with $\mathcal{E} \subset \overline{\text{conv}}(\{0\} \cup \varpi(\mathcal{E}))$; it suffices to show $\chi(\mathcal{E}) = 0$. Write $\mathcal{E} = \{\mathscr{V}^\tau\}_{\tau=1}^{\infty}$. By invoking assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_6)$, $\chi(\varpi\mathscr{V}(\nu)) = \chi(\hat{\varpi}\mathscr{V}(\nu))$, where

$$
(\hat{\omega}\mathscr{V}(\nu)) = \begin{cases}
T_\sigma^\psi(\nu)[\mathscr{V}_1 - \mathscr{H}(0,\mathscr{V}(0),0)] + \mathscr{H}\Big(\nu, \mathscr{V}_\nu + \hat{z}_\nu, \int_0^\nu \alpha_1(\nu,\varrho,\mathscr{V}_\varrho + \hat{z}_\varrho)\psi'(\varrho)\,d\varrho\Big) \\
+ \int_0^\nu T_\sigma^\psi(\nu,\mu)\,\psi'(\mu)\Big(f\Big(\mu, \mathscr{V}_\mu + \hat{z}_\mu, \int_0^\mu \alpha_2(\mu,\varrho,\mathscr{V}_\varrho + \hat{z}_\varrho)\psi'(\varrho)\,d\varrho\Big)\Big)d\mu \\
+ \int_0^\nu T_\sigma^\psi(\nu,\mu)\,\psi'(\mu)\Big(\int_0^\mu g(\mu,\varrho,\mathscr{V}_\varrho + \hat{z}_\varrho)dw(\varrho)\Big)d\mu \\
+ \int_0^\nu T_\sigma^\psi(\nu,\mu)\,\psi'(\mu)\int_{\mathfrak{Z}} h(\mu,\mathscr{V}_\mu + \hat{z}_\mu,\aleph)\tilde{N}(d\mu,d\aleph), \quad \nu \in [0,\iota_1] \\
\vdots \\
T_\sigma^\psi(\nu)[\mathscr{V}_1 - \mathscr{H}(0,\mathscr{V}(0),0)] + \mathscr{H}\Big(\nu, \mathscr{V}_\nu + \hat{z}_\nu, \int_0^\nu \alpha_1(\nu,\varrho,\mathscr{V}_\varrho + \hat{z}_\varrho)\psi'(\varrho)\,d\varrho\Big) \\
+ \int_{\iota_\Bbbk}^\nu T_\sigma^\psi(\nu,\mu)\,\psi'(\mu)\Big(f\Big(\mu, \mathscr{V}_\mu + \hat{z}_\mu, \int_0^\mu \alpha_2(\mu,\varrho,\mathscr{V}_\varrho + \hat{z}_\varrho)\psi'(\varrho)\,d\varrho\Big)\Big)d\mu \\
+ \int_{\iota_\Bbbk}^\nu T_\sigma^\psi(\nu,\mu)\,\psi'(\mu)\Big(\int_0^\mu g(\mu,\varrho,\mathscr{V}_\varrho + \hat{z}_\varrho)dw(\varrho)\Big)d\mu \\
+ \int_{\iota_\Bbbk}^\nu T_\sigma^\psi(\nu,\mu)\,\psi'(\mu)\int_{\mathfrak{Z}} h(\mu,\mathscr{V}_\mu + \hat{z}_\mu,\aleph)\tilde{N}(d\mu,d\aleph) \\
+ \sum_{0<\iota_\Bbbk<\nu} S_\sigma^\psi(\nu,\iota_\Bbbk) j_\Bbbk^1(\mathscr{V}_{\iota_\Bbbk} + \hat{z}_{\iota_\Bbbk}) + \sum_{0<\iota_\Bbbk<\nu} T_\sigma^\psi(\nu,\iota_\Bbbk) j_\Bbbk^2(\mathscr{V}_{\iota_\Bbbk} + \hat{z}_{\iota_\Bbbk}), \quad \forall \nu \in (\iota_\Bbbk, \iota_{\Bbbk+1}], \quad \Bbbk = 1,2,\ldots,n.
\end{cases}
$$

$$
= \hat{\omega}_1 + \hat{\omega}_2 + \hat{\omega}_3 + \hat{\omega}_4 + \hat{\omega}_5 + \hat{\omega}_6.
$$

By $(\mathcal{H}_1)$, $(\mathcal{H}_2)$(iii) and Corollary 1, which replaces Lemma 7 because the convolution carries the weight $\psi'(\mu)\,d\mu$,

$$
\begin{aligned}
\chi\big(\{\hat{\omega}_1\mathscr{V}^\tau(\nu)\}_{\tau=1}^\infty\big) &\le 2\mathcal{N}_S\Big(\chi\Big\{\mathscr{H}\Big(\nu, \mathscr{V}_\nu + \hat{z}_\nu, \int_0^\nu \alpha_1(\nu,\varrho,\mathscr{V}_\varrho + \hat{z}_\varrho)\psi'(\varrho)\,d\varrho\Big)\Big\}_{\tau=1}^\infty\Big) \\
&\le \varkappa_{\mathscr{H}}^* \sup_{\nu\in[0,\iota_1]} \chi\big(\{\mathscr{V}^\tau(\nu)\}_{\tau=1}^\infty\big).
\end{aligned}
$$

By $(\mathcal{H}_3)$(ii),

$$
\chi\big(\{\hat{\omega}_2\mathscr{V}^\tau(\nu)\}_{\tau=1}^\infty\big) \le \frac{2\mathcal{N}_S\Psi_\nu^\sigma}{\Gamma(\sigma+1)}\varkappa_f^* \sup_{\nu\in[0,\iota_1]} \chi\big(\{\mathscr{V}^\tau(\nu)\}_{\tau=1}^\infty\big),
$$

By $(\mathcal{H}_4)$(ii),

$$
\chi\big(\{\hat{\omega}_3\mathscr{V}^\tau(\nu)\}_{\tau=1}^\infty\big) \le \frac{2\mathcal{N}_S\Psi_\nu^\sigma}{\Gamma(\sigma+1)}\sqrt{\Psi_\nu}\,\varkappa_g^* \sup_{\nu\in[0,\iota_1]} \chi\big(\{\mathscr{V}^\tau(\nu)\}_{\tau=1}^\infty\big),
$$

By $(\mathcal{H}_6)$,

$$
\chi\big(\{\hat{\omega}_4\mathscr{V}^\tau(\nu)\}_{\tau=1}^\infty\big) \le \frac{2\mathcal{N}_S}{\Gamma(\sigma)}\sqrt{\hat{\mathcal{C}}_2}\left\{\left(\frac{\Psi_\nu^{2\sigma-1}}{2\sigma-1}\right)^{\frac{1}{2}}\varkappa_{\hat{h}}^* + \left(\frac{\Psi_\nu^{4\sigma-3}}{4\sigma-3}\right)^{\frac{1}{4}}\varkappa_{\hat{h}}^*\right\} \sup_{\nu\in[0,\iota_1]} \chi\big(\{\mathscr{V}^\tau(\nu)\}_{\tau=1}^\infty\big).
$$

the two bracketed factors being the cases $\kappa = 2$ and $\kappa = 4$ of the second estimate in (3). Combining the above,

$$
\chi\big(\{\varpi\mathscr{V}^\tau(\nu)\}_{\tau=1}^\infty\big) \le \Theta \sup_{\nu\in\mathcal{I}} \chi\big(\{\mathscr{V}^\tau(\nu)\}_{\tau=1}^\infty\big),
$$

where

$$\Theta = \varkappa^*_{\mathscr{H}} + \frac{2\mathcal{N}_S \Psi_\nu^\sigma}{\Gamma(\sigma+1)} \varkappa^*_f + \frac{2\mathcal{N}_S \Psi_\nu^\sigma}{\Gamma(\sigma+1)} \sqrt{\Psi_\nu}\, \varkappa^*_g + \frac{2\mathcal{N}_S}{\Gamma(\sigma)} \sqrt{\hat{\mathcal{C}}_2} \left\{ \left(\frac{\Psi_\nu^{2\sigma-1}}{2\sigma-1} \right)^{\frac{1}{2}} \varkappa^*_{\hat{h}} + \left(\frac{\Psi_\nu^{4\sigma-3}}{4\sigma-3} \right)^{\frac{1}{4}} \varkappa^*_{\hat{h}} \right\}.$$

Case (ii). For $\nu \in (\iota_{\Bbbk}, \iota_{\Bbbk+1}]$, $\Bbbk = 1, \ldots, n$, by $(\mathcal{H}_5)$(ii) with $i = 1$,

$$\chi\big(\{\hat{\omega}_5 \mathscr{V}^\tau(\nu)\}_{\tau=1}^\infty\big) \le \mathcal{N}_S \sum_{\Bbbk=1}^{n} \bar{\varkappa}^1_{\Bbbk} \sup_{\nu \in (\iota_{\Bbbk}, \iota_{\Bbbk+1}]} \chi\big(\{\mathscr{V}^\tau(\nu)\}_{\tau=1}^\infty\big),$$

and with $i = 2$, the propagator T^ψ_σ being bounded by the same $\mathcal{N}_S$ under $(\mathcal{H}_1)$,

$$\chi\big(\{\hat{\omega}_6 \mathscr{V}^\tau(\nu)\}_{\tau=1}^\infty\big) \le \mathcal{N}_S \sum_{\Bbbk=1}^{n} \bar{\varkappa}^2_{\Bbbk} \sup_{\nu \in (\iota_{\Bbbk}, \iota_{\Bbbk+1}]} \chi\big(\{\mathscr{V}^\tau(\nu)\}_{\tau=1}^\infty\big),$$

so that, with $\Psi_{\nu-\iota_{\Bbbk}} := \psi(\nu) - \psi(\iota_{\Bbbk})$ replacing $\nu - \iota_{\Bbbk}$ throughout,

$$\chi\big(\{\varpi \mathscr{V}^\tau(\nu)\}_{\tau=1}^\infty\big) \le \hat{\Theta} \sup_{\nu \in \mathcal{I}} \chi\big(\{\mathscr{V}^\tau(\nu)\}_{\tau=1}^\infty\big),$$

where

$$\hat{\Theta} = \varkappa^*_{\mathscr{H}} + \frac{2\mathcal{N}_S \Psi_{\nu-\iota_{\Bbbk}}^\sigma}{\Gamma(\sigma+1)} \varkappa^*_f + \frac{2\mathcal{N}_S \Psi_{\nu-\iota_{\Bbbk}}^\sigma}{\Gamma(\sigma+1)} \sqrt{\Psi_{\nu-\iota_{\Bbbk}}}\, \varkappa^*_g + \frac{2\mathcal{N}_S}{\Gamma(\sigma)} \sqrt{\hat{\mathcal{C}}_2} \left\{ \left(\frac{\Psi_{\nu-\iota_{\Bbbk}}^{2\sigma-1}}{2\sigma-1} \right)^{\frac{1}{2}} \varkappa^*_{\hat{h}} + \left(\frac{\Psi_{\nu-\iota_{\Bbbk}}^{4\sigma-3}}{4\sigma-3} \right)^{\frac{1}{4}} \varkappa^*_{\hat{h}} \right\} + \mathcal{N}_S \sum_{\Bbbk=1}^{n} \left(\bar{\varkappa}^1_{\Bbbk} + \bar{\varkappa}^2_{\Bbbk} \right).$$

Thus, $\hat{\Theta}$ differs from Θ only in the impulsive block, the derivative impulses enlarging the budget from $\sum_{\Bbbk} \bar{\varkappa}^1_{\Bbbk}$ to $\sum_{\Bbbk} (\bar{\varkappa}^1_{\Bbbk} + \bar{\varkappa}^2_{\Bbbk})$. Since $\Psi_\nu, \Psi_{\nu-\iota_{\Bbbk}} \le \Psi_{\mathcal{T}}$ and every coefficient above is nondecreasing in its Ψ-argument, (6) gives $\Theta^* := \max\{\Theta, \hat{\Theta}\} < 1$, so

$$\chi(\mathcal{E}) \le \chi(\overline{\mathrm{conv}}(\{0\} \cup \varpi(\mathcal{E}))) = \chi(\varpi(\mathcal{E})) \le \Theta^* \chi(\mathcal{E}),$$

which forces $\chi(\mathcal{E}) = 0$, so $\mathcal{E}$ is relatively compact in $\mathbb{X}_b$. All hypotheses of Lemma 8 therefore hold on $\mathbb{X}_r$, and ϖ admits a fixed point in $\mathbb{X}_r$, which is the desired mild solution. □

Corollary 2. *For $\psi(\nu) = \nu$, Theorem 1 reduces to the classical-Caputo existence theorem of [33]; for $\psi(\nu) = \ln \nu$ and $\psi(\nu) = \nu^\rho/\rho$ it gives the Caputo–Hadamard and Caputo–Katugampola analogues.*

4. Exponential Stability of Mild Solutions

Throughout this section $\mathcal{I} = [0, \infty)$ and ψ is extended to $[0, \infty)$ with the same regularity, so that the decay conclusion may be propagated to $\nu \to \infty$; all estimates are expressed in the ψ-time variable $\big(\psi(\nu) - \psi(0)\big)$. The following additional hypotheses are imposed.

$(\mathcal{H}_7)$ The strongly continuous families $\{S^\psi_\sigma(\nu)\}_{\nu \ge 0}$ and $\{T^\psi_\sigma(\nu)\}_{\nu \ge 0}$ generated by A satisfy the following: there exist constants $a_1, a_2 > 0$ and real numbers $\beta_1, \beta_2 > 0$ such that

$$\left\|S_\sigma^\psi(\nu)\right\|^\wp \le a_1 e^{-\beta_1\left(\psi(\nu)-\psi(0)\right)}, \qquad \left\|T_\sigma^\psi(\nu)\right\|^\wp \le a_2 e^{-\beta_2\left(\psi(\nu)-\psi(0)\right)}, \qquad \nu \ge 0.$$

$(\mathcal{H}_8)$ Growth of the first kernel and of the neutral term: there exist positive constants $b_1, b_2 \ge 0$ and a continuous function $\varphi_1 : [0,\infty) \to \mathbb{R}^+$ such that, for all $\mathcal{Y}, \mathcal{V} \in \mathbb{X}_b$, $\nu \ge 0$,

$$(i) \quad \mathfrak{E}\left\|\int_0^\nu \alpha_1(\nu,\mu,\mathcal{Y}_\mu)\,\psi'(\mu)\,d\mu\right\|^\wp \le b_1\,\mathfrak{E}\|\mathcal{Y}_\nu\|_{\mathbb{X}}^\wp;$$
$$(ii) \quad \mathfrak{E}\|\mathscr{H}(\nu,\mathcal{Y},\mathcal{V})\|^\wp \le b_2\big[\mathfrak{E}\|\mathcal{Y}\|_{\mathbb{X}}^\wp + \mathfrak{E}\|\mathcal{V}\|^\wp\big] + \varphi_1(\nu).$$

$(\mathcal{H}_9)$ The counterpart of $(\mathcal{H}_8)$ for the second kernel and the drift term: there exist positive constants $b_3, b_4 \ge 0$ and a continuous function $\varphi_2 : [0,\infty) \to \mathbb{R}^+$ such that, for all $\mathcal{Y}, \mathcal{V} \in \mathbb{X}_b$, $\nu \ge 0$,

$$(i) \quad \mathfrak{E}\left\|\int_0^\nu \alpha_2(\nu,\mu,\mathcal{Y}_\mu)\,\psi'(\mu)\,d\mu\right\|^\wp \le b_3\,\mathfrak{E}\|\mathcal{Y}_\nu\|_{\mathbb{X}}^\wp;$$
$$(ii) \quad \mathfrak{E}\|f(\nu,\mathcal{Y},\mathcal{V})\|^\wp \le b_4\big[\mathfrak{E}\|\mathcal{Y}\|_{\mathbb{X}}^\wp + \mathfrak{E}\|\mathcal{V}\|^\wp\big] + \varphi_2(\nu).$$

$(\mathcal{H}_{10})$ There exist constants $c_1 \ge 0$, $\beta_3 > 0$ and a continuous function $\varphi_3 : [0,\infty) \to \mathbb{R}^+$ such that

$$\mathfrak{E}\|g(\nu,\mu,\mathcal{Y}_\mu)\|^\wp \le c_1\, e^{-\beta_3\left(\psi(\mu)-\psi(0)\right)}\,\mathfrak{E}\|\mathcal{Y}_\mu\|_{\mathbb{X}}^\wp + \varphi_3(\nu).$$

The weight $e^{-\beta_3\left(\psi(\mu)-\psi(0)\right)}$ makes the inner Wiener convolution integrable over $[0,\infty)$, since $\int_0^\nu \psi'(\varrho)e^{-\beta_3\left(\psi(\varrho)-\psi(0)\right)}d\varrho \le \beta_3^{-1}$ for every ν; without it, the stability estimate would depend on the length of the interval and could not be propagated to $\nu \to \infty$.

$(\mathcal{H}_{11})$ There exist positive constants $d_1, d_2 \ge 0$ and continuous functions $\varphi_4, \varphi_5 : [0,\infty) \to \mathbb{R}^+$ such that

$$(i) \quad \left(\int_{\mathfrak{Z}} \mathfrak{E}\|h(\nu,\mathcal{Y}_\mu,\aleph)\|^2\lambda(d\aleph)\right)^{\frac{\wp}{2}} \le d_1\,\mathfrak{E}\|\mathcal{Y}_\mu\|_{\mathbb{X}}^\wp + \varphi_4(\nu);$$
$$(ii) \quad \left(\int_{\mathfrak{Z}} \mathfrak{E}\|h(\nu,\mathcal{Y}_\mu,\aleph)\|^{2\wp}\lambda(d\aleph)\right)^{\frac{1}{2}} \le d_2\,\mathfrak{E}\|\mathcal{Y}_\mu\|_{\mathbb{X}}^\wp + \varphi_5(\nu).$$

$(\mathcal{H}_{12})$ There exist constants $S_i \ge 0$ and $\lambda_i > 0$, $i = 1,\dots,5$, such that

$$\varphi_i(\nu) \le S_i\, e^{-\lambda_i\left(\psi(\nu)-\psi(0)\right)}, \qquad \nu \ge 0, \quad i = 1,2,3,4,5.$$

Lemma 10 (ψ-weighted derivative impulsive integral inequality)**.** *Suppose $\eta_1, \eta_2 \in (0,\hbar]$ for some $\hbar > 0$, and suppose there exist constants $\xi_1, \xi_2, \xi_3, \xi_4 > 0$, two summable families of non-negative impulsive weights $\{l_{\Bbbk}^1\}_{\Bbbk\ge1}$ and $\{l_{\Bbbk}^2\}_{\Bbbk\ge1}$, and a function $\mathcal{Q} : [-\tau,\infty) \to [0,\infty)$ such that*

$$\mathcal{Q}(\nu) \leq \begin{cases} \xi_1 e^{-\eta_1\left(\psi(\nu)-\psi(0)\right)} + \xi_2 e^{-\eta_2\left(\psi(\nu)-\psi(0)\right)}, & \nu \in [-\tau, 0], \\ \xi_1 e^{-\eta_1\left(\psi(\nu)-\psi(0)\right)} + \xi_2 e^{-\eta_2\left(\psi(\nu)-\psi(0)\right)} + \xi_3 \sup\limits_{\theta\in[-\tau,0]} \mathcal{Q}(\nu+\theta) \\ +\xi_4 \int_0^{\nu} \psi'(\mu)\, e^{-\eta_2\left(\psi(\nu)-\psi(\mu)\right)} \sup\limits_{\theta\in[-\tau,0]} \mathcal{Q}(\mu+\theta)\, d\mu + \sum\limits_{0<\iota_{\Bbbk}<\nu} \left(I_{\Bbbk}^1 + I_{\Bbbk}^2\right) \mathcal{Q}(\iota_{\Bbbk}^-), & \nu \geq 0. \end{cases}$$

If

$$\Xi(0) := \xi_3 + \frac{\xi_4}{\eta_2} + \sum_{\Bbbk\geq 1} \left(I_{\Bbbk}^1 + I_{\Bbbk}^2\right) < 1, \tag{7}$$

then, for every $\mu_0 \in (0, \eta_1 \wedge \eta_2)$ *small enough that*

$$\Xi(\mu_0) := \xi_3 e^{\mu_0\left(\psi(0)-\psi(-\tau)\right)} + \frac{\xi_4 e^{\mu_0\left(\psi(0)-\psi(-\tau)\right)}}{\eta_2 - \mu_0} + \sum_{\Bbbk\geq 1} \left(I_{\Bbbk}^1 + I_{\Bbbk}^2\right) < 1, \tag{8}$$

one has $\mathcal{Q}(\nu) \leq \mathcal{N} e^{-\mu_0\left(\psi(\nu)-\psi(0)\right)}$ *for* $\nu \geq -\tau$, *where*

$$\mathcal{N} = \max\left\{ (\xi_1 + \xi_2)\, e^{\mu_0\left(\psi(0)-\psi(-\tau)\right)},\ \frac{(\eta_2 - \mu_0)\xi_2}{\xi_4 e^{\mu_0\left(\psi(0)-\psi(-\tau)\right)}},\ \frac{\xi_1}{1 - \Xi(\mu_0)} \right\} > 0.$$

Proof. Since ψ is a strictly increasing C^1 bijection of $[-\tau, \infty)$ onto $[(\psi(-\tau) - \psi(0)), \infty)$, the change of variable $\theta = (\psi(\nu) - \psi(0))$ applies throughout. The map $\mu_0 \mapsto \Xi(\mu_0)$ is continuous and strictly increasing on $(0, \eta_2)$ with $\Xi(0) < 1$ by (7), so (8) holds for all sufficiently small $\mu_0 \in (0, \eta_1 \wedge \eta_2)$; fix such a μ_0. For $\varepsilon > 0$, set

$$\mathcal{N}_\varepsilon = \max\left\{ (\xi_1 + \xi_2) e^{\mu_0\left(\psi(0)-\psi(-\tau)\right)},\ \frac{\xi_2(\eta_2 - \mu_0)}{\xi_4 e^{\mu_0\left(\psi(0)-\psi(-\tau)\right)}},\ \frac{\xi_1}{1 - \Xi(\mu_0)} \right\} + \varepsilon > 0,$$

and claim $\mathcal{Q}(\nu) \leq \mathcal{N}_\varepsilon\, e^{-\mu_0\left(\psi(\nu)-\psi(0)\right)}$ for $\nu \geq -\tau$. On $[-\tau, 0]$, since $\mu_0 < \eta_1 \wedge \eta_2$ gives $e^{-\eta_i\left(\psi(\nu)-\psi(0)\right)} \leq e^{\mu_0\left(\psi(0)-\psi(-\tau)\right)} e^{-\mu_0\left(\psi(\nu)-\psi(0)\right)}$ for $\nu \in [-\tau, 0]$,

$$\mathcal{Q}(\nu) \leq \xi_1 e^{-\eta_1\left(\psi(\nu)-\psi(0)\right)} + \xi_2 e^{-\eta_2\left(\psi(\nu)-\psi(0)\right)} \leq (\xi_1 + \xi_2) e^{\mu_0\left(\psi(0)-\psi(-\tau)\right)} e^{-\mu_0\left(\psi(\nu)-\psi(0)\right)},$$

and the claim holds there by the choice of $\mathcal{N}_\varepsilon$. Suppose the claim fails for some $\nu > 0$; then there is a first time $T_1 > 0$ with $\mathcal{Q}(\nu) < \mathcal{N}_\varepsilon e^{-\mu_0\left(\psi(\nu)-\psi(0)\right)}$ on $[-\tau, T_1)$ and $\mathcal{Q}(T_1) = \mathcal{N}_\varepsilon e^{-\mu_0\left(\psi(T_1)-\psi(0)\right)}$. Applying the governing inequality at $\nu = T_1$,

$$\begin{aligned} \mathcal{Q}(T_1) \leq & \xi_1 e^{-\eta_1\left(\psi(T_1)-\psi(0)\right)} + \left[\xi_2 - \frac{\xi_4 \mathcal{N}_\varepsilon e^{\mu_0\left(\psi(0)-\psi(-\tau)\right)}}{\eta_2 - \mu_0} \right] e^{-\eta_2\left(\psi(T_1)-\psi(0)\right)} \\ & + \Xi(\mu_0)\, \mathcal{N}_\varepsilon e^{-\mu_0\left(\psi(T_1)-\psi(0)\right)}, \end{aligned}$$

where the change of variable $\theta = (\psi(\mu) - \psi(0))$, $d\theta = \psi'(\mu)\, d\mu$, was used to evaluate the convolution term, and the two impulsive sums were bounded on $[-\tau, T_1)$ by $\sum_{\Bbbk} (I_{\Bbbk}^1 + I_{\Bbbk}^2) \mathcal{N}_\varepsilon e^{-\mu_0\left(\psi(T_1)-\psi(0)\right)}$, the summability of both families guaranteeing that the bound

is finite. The middle bracket is ≤ 0 by the second entry of $\mathcal{N}_\varepsilon$, and $\xi_1 e^{-\eta_1\left(\psi(T_1)-\psi(0)\right)} \leq \xi_1 e^{-\mu_0\left(\psi(T_1)-\psi(0)\right)}$ since $\mu_0 < \eta_1$; hence

$$\mathcal{Q}(T_1) \leq \left[\xi_1 + \Xi(\mu_0)\mathcal{N}_\varepsilon\right] e^{-\mu_0\left(\psi(T_1)-\psi(0)\right)} < \mathcal{N}_\varepsilon e^{-\mu_0\left(\psi(T_1)-\psi(0)\right)},$$

the last inequality because $\mathcal{N}_\varepsilon > \xi_1/(1-\Xi(\mu_0))$ by the third entry. This contradicts $\mathcal{Q}(T_1) = \mathcal{N}_\varepsilon e^{-\mu_0\left(\psi(T_1)-\psi(0)\right)}$, so the claim holds for all $\nu \geq -\tau$; letting $\varepsilon \to 0$ completes the proof. □

Theorem 2. *Assume* $(\mathcal{H}_5)$ *and* $(\mathcal{H}_7)$–$(\mathcal{H}_{12})$ *hold, and suppose*

$$8^{\wp-1}\left\{ b_2^\wp\left(1+b_1^\wp\right) + \frac{a_2^\wp \beta_2^{1-\wp} b_4(1+b_3) + a_2^\wp \beta_2^{1-\frac{\wp}{2}} \tilde{C}_\wp c_1 \beta_3^{-1} + a_2^\wp \hat{\mathcal{C}}_\wp\left(\beta_2^{-\frac{\wp}{2}} d_1 + \beta_2^{-1} d_2\right)}{\beta_2} + a_1^\wp\left(L^1\right)^{\frac{\wp}{q}+1} + a_2^\wp\left(L^2\right)^{\frac{\wp}{q}+1} \right\} < 1, \tag{9}$$

where $\frac{1}{\wp} + \frac{1}{q} = 1$ *and* $L^i = \sum_{\Bbbk=1}^n l_\Bbbk^i$. *Then there exist* $\bar{\lambda}, \widetilde{\Lambda} > 0$ *such that the mild solution* $\mathcal{Y}(\nu)$ *of* (1) *satisfies*

$$\mathfrak{E}\|\mathcal{Y}(\nu)\|^\wp \leq \widetilde{\Lambda}\, e^{-\bar{\lambda}\left(\psi(\nu)-\psi(0)\right)}, \quad \nu \geq -\tau;$$

that is, $\mathcal{Y}$ *is exponentially stable in* ψ*-time.*

Proof. Let $\mathcal{Y}(\nu)$ solve (1). Fix $\varepsilon \in (0, \min_i \lambda_i)$ and set

$$\hat{\eta}_i := \min\{\wp\beta_1, \wp\beta_2, \lambda_i\} - \varepsilon \ (i = 1, \ldots, 5), \qquad \eta := \min_{1\leq i\leq 5} \hat{\eta}_i > 0,$$

and write $\zeta(\nu) := S_\sigma^\psi(\nu)\xi(0)$ for the free evolution, so that $\mathscr{H}(0,\zeta(0),0) = \mathscr{H}(0,\xi(0),0)$. For $\nu \in (\iota_\Bbbk, \iota_{\Bbbk+1}]$ the mild solution has eight additive terms, the last two being the impulsive sums generated by $j_\Bbbk^1$ and $j_\Bbbk^2$:

$$\begin{aligned}
\mathfrak{E}\|\mathcal{Y}(\nu)\|^\wp =& \mathfrak{E}\Bigg\| S_\sigma^\psi(\nu)\xi(0) + T_\sigma^\psi(\nu)\left[\mathcal{Y}_1 - \mathscr{H}(0,\xi(0),0)\right] \\
&+ \mathscr{H}\left(\nu, \mathcal{Y}_\nu, \int_0^\nu \alpha_1(\nu,\varrho,\mathcal{Y}_\varrho)\,\psi'(\varrho)\,d\varrho\right) \\
&+ \int_{\iota_\Bbbk}^\nu T_\sigma^\psi(\nu,\mu)\,\psi'(\mu)\, f\left(\mu, \mathcal{Y}_\mu, \int_0^\mu \alpha_2(\mu,\varrho,\mathcal{Y}_\varrho)\,\psi'(\varrho)\,d\varrho\right) d\mu \\
&+ \int_{\iota_\Bbbk}^\nu T_\sigma^\psi(\nu,\mu)\,\psi'(\mu) \int_0^\mu g(\mu,\varrho,\mathcal{Y}_\varrho)\,dw(\varrho)\,d\mu \\
&+ \int_{\iota_\Bbbk}^\nu T_\sigma^\psi(\nu,\mu)\,\psi'(\mu) \int_{\mathfrak{Z}} h(\mu,\mathcal{Y}_\mu,\aleph)\,\tilde{N}(d\mu,d\aleph) \\
&+ \sum_{0<\iota_\Bbbk<\nu} S_\sigma^\psi(\nu,\iota_\Bbbk)\, j_\Bbbk^1(\mathcal{Y}_{\iota_\Bbbk}) + \sum_{0<\iota_\Bbbk<\nu} T_\sigma^\psi(\nu,\iota_\Bbbk)\, j_\Bbbk^2(\mathcal{Y}_{\iota_\Bbbk}) \Bigg\|^\wp \\
\leq& 8^{\wp-1} \sum_{i=1}^8 \bar{R}_i.
\end{aligned}$$

By $(\mathcal{H}_7)$,

$$\bar{R}_1 = \mathfrak{E}\|S_\sigma^\psi(\nu)\xi(0)\|^\wp \leq a_1^\wp e^{-\wp\beta_1\left(\psi(\nu)-\psi(0)\right)}\, \mathfrak{E}\|\xi(0)\|^\wp \leq a_1^\wp e^{-\hat{\eta}_1\left(\psi(\nu)-\psi(0)\right)}\, \mathfrak{E}\|\xi(0)\|^\wp.$$

By $(\mathcal{H}_7)$, $(\mathcal{H}_8)$(ii), and $(\mathcal{H}_{12})$,

$$\begin{aligned}\bar{R}_2 &\le \mathfrak{E}\Big\|T_\sigma^{\psi}(\nu)\big[\mathcal{Y}_1 - \mathscr{H}(0,\xi(0),0)\big]\Big\|^{\wp}\\ &\le 2^{\wp-1}a_2^{\wp}e^{-\wp\beta_2\left(\psi(\nu)-\psi(0)\right)}\,\mathfrak{E}\|\mathcal{Y}_1\|^{\wp} + 2^{\wp-1}a_2^{\wp}e^{-\wp\beta_2\left(\psi(\nu)-\psi(0)\right)}\big[b_2\mathfrak{E}\|\xi(0)\|_{\mathbb{X}}^{\wp}+S_1\big]\\ &=: C_1\,e^{-\hat{\eta}_2\left(\psi(\nu)-\psi(0)\right)},\end{aligned}$$

where $C_1 = 2^{\wp-1}a_2^{\wp}\big[\mathfrak{E}\|\mathcal{Y}_1\|^{\wp} + b_2\mathfrak{E}\|\xi(0)\|_{\mathbb{X}}^{\wp} + S_1\big]$.

By $(\mathcal{H}_8)$(i)–(ii) and $(\mathcal{H}_{12})$,

$$\bar{R}_3 \le \mathfrak{E}\left\|\mathscr{H}\Big(\nu,\mathcal{Y}_\nu,\int_0^{\nu}\alpha_1(\nu,\varrho,\mathcal{Y}_\varrho)\,\psi'(\varrho)\,d\varrho\Big)\right\|^{\wp} \le \big(b_2^{\wp}+b_2^{\wp}b_1^{\wp}\big)\mathfrak{E}\|\mathcal{Y}_\nu\|_{\mathbb{X}}^{\wp} + S_2e^{-\hat{\eta}_2\left(\psi(\nu)-\psi(0)\right)}.$$

By $(\mathcal{H}_7)$, $(\mathcal{H}_9)$, and $(\mathcal{H}_{12})$,

$$\begin{aligned}\bar{R}_4 &\le \mathfrak{E}\left\|\int_{\iota_{\Bbbk}}^{\nu}T_\sigma^{\psi}(\nu,\mu)\,\psi'(\mu)\,f\Big(\mu,\mathcal{Y}_\mu,\int_0^{\mu}\alpha_2(\mu,\varrho,\mathcal{Y}_\varrho)\,\psi'(\varrho)\,d\varrho\Big)d\mu\right\|^{\wp}\\ &\le a_2^{\wp}\beta_2^{1-\wp}\int_0^{\nu}\psi'(\mu)\,e^{-\beta_2\left(\psi(\nu)-\psi(\mu)\right)}\,(b_4+b_4b_3)\mathfrak{E}\|\mathcal{Y}_\mu\|_{\mathbb{X}}^{\wp}\,d\mu + \frac{S_3}{|\hat{\eta}_3-\beta_2|}e^{-\hat{\eta}_3\left(\psi(\nu)-\psi(0)\right)}.\end{aligned}$$

By Lemma 3, $(\mathcal{H}_{10})$, and $(\mathcal{H}_{12})$,

$$\begin{aligned}\bar{R}_5 &\le \mathfrak{E}\left\|\int_{\iota_{\Bbbk}}^{\nu}T_\sigma^{\psi}(\nu,\mu)\,\psi'(\mu)\int_0^{\mu}g(\mu,\varrho,\mathcal{Y}_\varrho)\,dw(\varrho)\,d\mu\right\|^{\wp}\\ &\le a_2^{\wp}\tilde{C}_{\wp}\beta_2^{1-\frac{\wp}{2}}\int_0^{\nu}\psi'(\mu)\,e^{-\beta_2\left(\psi(\nu)-\psi(\mu)\right)}\,\beta_3^{-1}\,c_1\,\mathfrak{E}\|\mathcal{Y}_\mu\|_{\mathbb{X}}^{\wp}\,d\mu + \frac{S_3}{|\hat{\eta}_3-\beta_2|}e^{-\hat{\eta}_3\left(\psi(\nu)-\psi(0)\right)}.\end{aligned}$$

By Lemma 4, $(\mathcal{H}_{11})$, and $(\mathcal{H}_{12})$,

$$\begin{aligned}\bar{R}_6 &\le \mathfrak{E}\left\|\int_{\iota_{\Bbbk}}^{\nu}T_\sigma^{\psi}(\nu,\mu)\,\psi'(\mu)\int_3 h(\mu,\mathcal{Y}_\mu,\aleph)\,\tilde{N}(d\mu,d\aleph)\right\|^{\wp}\\ &\le a_2^{\wp}\hat{\mathcal{C}}_{\wp}\Big\{\beta_2^{-\frac{\wp}{2}}d_1+\beta_2^{-1}d_2\Big\}\int_0^{\nu}\psi'(\mu)\,e^{-\beta_2\left(\psi(\nu)-\psi(\mu)\right)}\,\mathfrak{E}\|\mathcal{Y}_\mu\|_{\mathbb{X}}^{\wp}\,d\mu + \frac{S_4+S_5}{|\hat{\eta}_4-\beta_2|}e^{-\hat{\eta}_4\left(\psi(\nu)-\psi(0)\right)},\end{aligned}$$

the convolution structure being inherited from $\int_{\iota_{\Bbbk}}^{\nu}T_\sigma^{\psi}(\nu,\mu)\psi'(\mu)(\cdot)\,d\mu$ exactly as in $\bar{R}_4$ and $\bar{R}_5$, and the range of integration having been enlarged from $[\iota_{\Bbbk},\nu]$ to $[0,\nu]$, which is legitimate because the integrand is non-negative.

By $(\mathcal{H}_7)$ and $(\mathcal{H}_5)$(i), for $\wp \ge 2$, $1 < q \le 2$, $\frac{1}{\wp}+\frac{1}{q}=1$,

$$\bar{R}_7 \le \mathfrak{E}\left\|\sum_{0<\iota_{\Bbbk}<\nu}S_\sigma^{\psi}(\nu,\iota_{\Bbbk})\,J_{\Bbbk}^1(\mathcal{Y}_{\iota_{\Bbbk}})\right\|^{\wp} \le a_1^{\wp}\Big(L^1\Big)^{\frac{\wp}{q}}\sum_{0<\iota_{\Bbbk}<\nu}l_{\Bbbk}^1\,e^{-\beta_1\left(\psi(\nu)-\psi(\iota_{\Bbbk})\right)}\,\mathfrak{E}\|\mathcal{Y}_{\iota_{\Bbbk}}\|^{\wp}.$$

By the same argument with the propagator T_σ^{ψ}, whose decay is governed by a_2,β_2, and $(\mathcal{H}_5)$(i) with $i=2$,

$$\bar{R}_8 \le \mathfrak{E}\left\|\sum_{0<\iota_{\Bbbk}<\nu}T_\sigma^{\psi}(\nu,\iota_{\Bbbk})\,J_{\Bbbk}^2(\mathcal{Y}_{\iota_{\Bbbk}})\right\|^{\wp} \le a_2^{\wp}\Big(L^2\Big)^{\frac{\wp}{q}}\sum_{0<\iota_{\Bbbk}<\nu}l_{\Bbbk}^2\,e^{-\beta_2\left(\psi(\nu)-\psi(\iota_{\Bbbk})\right)}\,\mathfrak{E}\|\mathcal{Y}_{\iota_{\Bbbk}}\|^{\wp}.$$

Since $e^{-\beta_i\left(\psi(\nu)-\psi(\iota_{\Bbbk})\right)} \le 1$ for $\nu \ge \iota_{\Bbbk}$, both impulsive contributions have the form required by Lemma 10, with weights $l_{\Bbbk}^1$ and $l_{\Bbbk}^2$. Combining $\bar{R}_1$–$\bar{R}_8$, and writing $\mathcal{Q}(\nu)=\mathfrak{E}\|\mathcal{Y}(\nu)\|^{\wp}$,

$$\begin{aligned}\mathcal{Q}(\nu) \le{}& 8^{\wp-1}a_1^{\wp}e^{-\hat{\eta}_1\left(\psi(\nu)-\psi(0)\right)}\,\mathfrak{E}\|\xi(0)\|^{\wp} + 8^{\wp-1}\left[C_1+\frac{S_3}{|\hat{\eta}_3-\beta_2|}\right]e^{-\hat{\eta}_2\left(\psi(\nu)-\psi(0)\right)}\\
&+ 8^{\wp-1}S_2e^{-\hat{\eta}_2\left(\psi(\nu)-\psi(0)\right)} + 8^{\wp-1}\frac{S_3}{|\hat{\eta}_3-\beta_2|}e^{-\hat{\eta}_3\left(\psi(\nu)-\psi(0)\right)}\\
&+ 8^{\wp-1}\frac{S_4}{|\hat{\eta}_4-\beta_2|}e^{-\hat{\eta}_4\left(\psi(\nu)-\psi(0)\right)} + 8^{\wp-1}\left(b_2^{\wp}+b_2^{\wp}b_1^{\wp}\right)\mathfrak{E}\|\mathcal{Y}_\nu\|_{\mathbb{X}}^{\wp}\\
&+ 8^{\wp-1}a_2^{\wp}\beta_2^{1-\wp}(b_4+b_4b_3)\int_0^{\nu}\psi'(\mu)\,e^{-\beta_2\left(\psi(\nu)-\psi(\mu)\right)}\,\mathfrak{E}\|\mathcal{Y}_\mu\|_{\mathbb{X}}^{\wp}\,d\mu\\
&+ 8^{\wp-1}a_2^{\wp}\tilde{C}_{\wp}\beta_2^{1-\frac{\wp}{2}}\beta_3^{-1}c_1\int_0^{\nu}\psi'(\mu)\,e^{-\beta_2\left(\psi(\nu)-\psi(\mu)\right)}\,\mathfrak{E}\|\mathcal{Y}_\mu\|_{\mathbb{X}}^{\wp}\,d\mu\\
&+ 8^{\wp-1}a_2^{\wp}\hat{\mathcal{C}}_{\wp}\left\{\beta_2^{-\frac{\wp}{2}}d_1+\beta_2^{-1}d_2\right\}\int_0^{\nu}\psi'(\mu)\,e^{-\beta_2\left(\psi(\nu)-\psi(\mu)\right)}\,\mathfrak{E}\|\mathcal{Y}_\mu\|_{\mathbb{X}}^{\wp}\,d\mu\\
&+ 8^{\wp-1}a_1^{\wp}\left(L^1\right)^{\frac{\wp}{q}}\sum_{0<\iota_{\mathbb{k}}<\nu}l_{\mathbb{k}}^1\,e^{-\beta_1\left(\psi(\nu)-\psi(\iota_{\mathbb{k}})\right)}\,\mathfrak{E}\|\mathcal{Y}_{\iota_{\mathbb{k}}}\|^{\wp}\\
&+ 8^{\wp-1}a_2^{\wp}\left(L^2\right)^{\frac{\wp}{q}}\sum_{0<\iota_{\mathbb{k}}<\nu}l_{\mathbb{k}}^2\,e^{-\beta_2\left(\psi(\nu)-\psi(\iota_{\mathbb{k}})\right)}\,\mathfrak{E}\|\mathcal{Y}_{\iota_{\mathbb{k}}}\|^{\wp}.\end{aligned}$$

Collecting the exponential terms into $\Xi_0\,e^{-\eta\left(\psi(\nu)-\psi(0)\right)}$, where Ξ_0 gathers the five amplitudes appearing above and is finite by $(\mathcal{H}_{12})$, setting

$$\xi_3 := 8^{\wp-1}b_2^{\wp}\left(1+b_1^{\wp}\right)$$

for the single non-delayed algebraic bound (from $\bar{R}_3$), and, using

$$\int_0^{\nu}\psi'(\mu)e^{-\beta_2\left(\psi(\nu)-\psi(\mu)\right)}d\mu = \int_0^{\left(\psi(\nu)-\psi(0)\right)}e^{-\beta_2\left(\left(\psi(\nu)-\psi(0)\right)-\theta\right)}d\theta \le \beta_2^{-1},$$

$$\begin{aligned}\hat{\Theta}_1 &= 8^{\wp-1}a_2^{\wp}\beta_2^{1-\wp}b_4(1+b_3),\\
\hat{\Theta}_2 &= 8^{\wp-1}a_2^{\wp}\tilde{C}_{\wp}\beta_2^{1-\frac{\wp}{2}}\beta_3^{-1}c_1,\\
\hat{\Theta}_5 &= 8^{\wp-1}a_2^{\wp}\hat{\mathcal{C}}_{\wp}\left(\beta_2^{-\frac{\wp}{2}}d_1+\beta_2^{-1}d_2\right),\\
\hat{\Theta}_3 &= 8^{\wp-1}a_1^{\wp}\left(L^1\right)^{\wp/q}, \qquad \hat{\Theta}_4 = 8^{\wp-1}a_2^{\wp}\left(L^2\right)^{\wp/q},\end{aligned}$$

we arrive, using the phase-space estimate of Lemma 5 in the form $\mathfrak{E}\|\mathcal{Y}_\mu\|_{\mathbb{X}}^{\wp} \le l^{\wp}\sup_{\theta\in[-\tau,0]}\mathcal{Q}(\mu+\theta)$ and absorbing $l^{\wp}$ into the constants $\xi_3,\hat{\Theta}_1,\hat{\Theta}_2,\hat{\Theta}_5$, at

$$\begin{aligned}\mathcal{Q}(\nu) \le{}& \Xi_0\,e^{-\eta\left(\psi(\nu)-\psi(0)\right)} + \xi_3\sup_{\theta\in[-\tau,0]}\mathcal{Q}(\nu+\theta)\\
&+ \left(\hat{\Theta}_1+\hat{\Theta}_2+\hat{\Theta}_5\right)\int_0^{\nu}\psi'(\mu)\,e^{-\beta_2\left(\psi(\nu)-\psi(\mu)\right)}\sup_{\theta\in[-\tau,0]}\mathcal{Q}(\mu+\theta)\,d\mu\\
&+ \sum_{0<\iota_{\mathbb{k}}<\nu}\left(\hat{\Theta}_3l_{\mathbb{k}}^1+\hat{\Theta}_4l_{\mathbb{k}}^2\right)\mathcal{Q}(\iota_{\mathbb{k}}^-),\end{aligned}$$

for $\nu \ge 0$, while on $[-\tau,0]$ one has $\mathcal{Q}(\nu) = \mathfrak{E}\|\xi(\nu)\|^{\wp} \le \Xi_0 e^{-\eta\left(\psi(\nu)-\psi(0)\right)}$ after enlarging Ξ_0 by $\sup_{\nu\in[-\tau,0]}\mathfrak{E}\|\xi(\nu)\|^{\wp}e^{\eta\left(\psi(0)-\psi(-\tau)\right)}$, which is finite since $\xi \in \mathbb{M}^{\wp}(\Omega,\mathbb{X})$. This is the governing inequality of Lemma 10 with

$$\xi_1 = \xi_2 = \Xi_0, \qquad \xi_4 = \hat{\Theta}_1+\hat{\Theta}_2+\hat{\Theta}_5, \qquad \eta_1 = \eta, \qquad \eta_2 = \beta_2,$$

and impulsive families $\hat{\Theta}_3 l^1_{\Bbbk}$, $\hat{\Theta}_4 l^2_{\Bbbk}$. Since

$$\xi_3 + \frac{\xi_4}{\beta_2} + \sum_{\Bbbk=1}^{n} \left(\hat{\Theta}_3 l^1_{\Bbbk} + \hat{\Theta}_4 l^2_{\Bbbk} \right) = 8^{\wp-1} \left\{ b_2^{\wp}(1 + b_1^{\wp}) + \frac{\hat{\Theta}_1 + \hat{\Theta}_2 + \hat{\Theta}_5}{8^{\wp-1}\beta_2} + a_1^{\wp} \left(L^1 \right)^{\frac{\varrho}{\eta}+1} + a_2^{\wp} \left(L^2 \right)^{\frac{\varrho}{\eta}+1} \right\},$$

the left-hand side is exactly the left-hand side of (9) and is therefore < 1: this is precisely (7). Note that the amplitudes $\xi_1 = \xi_2 = \Xi_0$, which depend on the initial data and are not small, do not enter that condition. Hence, by Lemma 10, there exist $\bar{\lambda} \in (0, \eta \wedge \beta_2)$ and $\widetilde{\Lambda} > 0$ such that

$$\mathfrak{E}\|\mathcal{Y}(\nu)\|^{\wp} \leq \widetilde{\Lambda}\, e^{-\bar{\lambda}\left(\psi(\nu)-\psi(0)\right)}, \qquad \nu \geq -\tau,$$

i.e., the mild solution $\mathcal{Y}(\nu)$ of (1) is exponentially stable. □

The jump intensity enters (9) only through d_1, d_2 from $(\mathcal{H}_{11})$. If d_1, d_2 are too large, stability fails, no matter which ψ is chosen.

Corollary 3. *Setting $\psi(\nu) = \nu$ in Theorem 2 recovers exactly the classical-Caputo exponential stability criterion of* [33]*, with the decay estimate $\mathfrak{E}\|\mathcal{Y}(\nu)\|^{\wp} \leq \widetilde{\Lambda} e^{-\bar{\lambda}\nu}$; correspondingly, Lemma 10 reduces to the impulsive integral inequality of Chen* [34].

Remark 5. *The decay in Theorem 2 is exponential in $\psi(\nu)$: exponential in ν when ψ grows at least linearly (e.g., $\psi(\nu) = \nu$, or $\psi(\nu) = \nu^{\rho}/\rho$ with $\rho \geq 1$), but algebraic or logarithmic when ψ is sublinear, as for $\psi(\nu) = \ln(1+\nu)$, where $e^{-\bar{\lambda}\left(\psi(\nu)-\psi(0)\right)} = (1+\nu)^{-\bar{\lambda}}$. The distinction is invisible in the classical Caputo formulation.*

This matters practically: sub-diffusion in disordered media and long-memory financial volatility both decay algebraically, not exponentially, which a sublinear ψ captures but classical Caputo cannot.

5. Illustrative Example

To illustrate the results for a non-identity kernel ψ, consider the following impulsive wave-type fractional neutral stochastic partial integro-differential system with infinite delay and Poisson jumps:

Remark 6. *Equation (10) can be read as a model of a vibrating string or beam of length π subject to memory-dependent restoring forces $\hat{a}_1, \hat{a}_3$, random environmental noise $\hat{a}_5\, dw$, and sudden external impacts represented by the Poisson term $\hat{a}_6\, \tilde{N}$ and the impulses $J^1_{\Bbbk}, J^2_{\Bbbk}$ at isolated instants $\iota_{\Bbbk}$, consistent with the impulsive mechanical systems mentioned in the Introduction.*

$$\begin{aligned}
&{}^{C,\psi}D^{\frac{4}{3}}_{0^+}\left[\mathbb{Z}(\nu,\varrho) - \hat{a}_1\left(\nu, \mathbb{Z}(\nu-\varsigma,\varrho), \int_0^{\nu} \hat{a}_2(\nu,\mu,\mathbb{Z}(\mu-\varsigma,\varrho))\, \psi'(\mu)\, d\mu\right)\right] \\
&= \frac{\partial^2}{\partial\varrho^2}\left[\mathbb{Z}(\nu,\varrho) - \hat{a}_1\left(\nu, \mathbb{Z}(\nu-\varsigma,\varrho), \int_0^{\nu} \hat{a}_2(\nu,\mu,\mathbb{Z}(\mu-\varsigma,\varrho))\, \psi'(\mu)\, d\mu\right)\right] \\
&\quad + \hat{a}_3\left(\nu, \mathbb{Z}(\nu-\varsigma,\varrho), \int_0^{\nu} \hat{a}_4(\nu,\mu,\mathbb{Z}(\mu-\varsigma,\varrho))\, \psi'(\mu)\, d\mu\right) \\
&\quad + \int_0^{\nu} \hat{a}_5(\nu,\mu,\mathbb{Z}(\mu-\varsigma,\varrho))\, dw(\mu) + \int_{\mathfrak{Z}} \hat{a}_6(\nu,\mathbb{Z}(\nu-\varsigma,\varrho),\aleph)\, \tilde{N}(d\nu,d\aleph),
\end{aligned}$$

for $0 \le \varrho \le \pi$ and $\nu \neq \iota_{\Bbbk}$, subject to the following boundary and initial conditions:

$$\begin{aligned}
\mathbb{Z}(\nu,0) = \mathbb{Z}(\nu,\pi) &= 0, \quad \nu \in [0,\mathcal{T}],\\
\mathbb{Z}^{[1]}_{\psi}(0,\varrho) &= \mathcal{Y}_1(\varrho), \quad \varrho \in [0,\pi],\\
\Delta\mathbb{Z}(\iota_{\Bbbk},\cdot)(\varrho) &= \frac{\hat{a}_7}{\Bbbk^2}\,\mathbb{Z}(\iota_{\Bbbk}^{+},\varrho), \quad \nu \neq \iota_{\Bbbk}, \quad \Bbbk = 1,2,\dots,n,\\
\Delta\mathbb{Z}^{[1]}_{\psi}(\iota_{\Bbbk},\cdot)(\varrho) &= \frac{\hat{a}_8}{\Bbbk^2}\,\mathbb{Z}(\iota_{\Bbbk}^{+},\varrho), \quad \nu \neq \iota_{\Bbbk}, \quad \Bbbk = 1,2,\dots,n,
\end{aligned} \tag{10}$$

where $\mathbb{Z}^{[1]}_{\psi}(\nu,\varrho) = \frac{1}{\psi'(\nu)}\frac{\partial}{\partial\nu}\mathbb{Z}(\nu,\varrho)$, $\hat{a}_i > 0$ for $i = 1,\dots,8$ and $\sigma = \frac{4}{3} \in (1,2)$. At each $\iota_{\Bbbk}$ the profile is displaced by $\hat{a}_7\Bbbk^{-2}$ and its ψ-velocity by $\hat{a}_8\Bbbk^{-2}$, as when an external impact acts on a vibrating medium.

The coefficient functions are identified as follows, both memory integrals carrying the weight $\psi'(\mu)$, with $\hat{a}_2$ in the role of α_1 and $\hat{a}_4$ in that of α_2:

$$\begin{aligned}
\mathscr{H}\Big(\nu,\mathcal{Y}_\nu,\int_0^\nu \alpha_1(\nu,\mu,\mathcal{Y}_\mu)\,\psi'(\mu)\,d\mu\Big) &= \hat{a}_1\Big(\nu,\mathbb{Z}(\nu-\varsigma,\varrho),\int_0^\nu \hat{a}_2(\nu,\mu,\mathbb{Z}(\mu-\varsigma,\varrho))\,\psi'(\mu)\,d\mu\Big),\\
f\Big(\nu,\mathcal{Y}_\nu,\int_0^\nu \alpha_2(\nu,\mu,\mathcal{Y}_\mu)\,\psi'(\mu)\,d\mu\Big) &= \hat{a}_3\Big(\nu,\mathbb{Z}(\nu-\varsigma,\varrho),\int_0^\nu \hat{a}_4(\nu,\mu,\mathbb{Z}(\mu-\varsigma,\varrho))\,\psi'(\mu)\,d\mu\Big),\\
\int_0^\nu g(\nu,\mu,\mathcal{Y}_\mu)\,dw(\mu) &= \int_0^\nu \hat{a}_5(\nu,\mu,\mathbb{Z}(\mu-\varsigma,\varrho))\,dw(\mu),\\
\int_{\mathfrak{Z}} h(\nu,\mathcal{Y}_\nu,\aleph)\,\tilde{N}(d\nu,d\aleph) &= \int_{\mathfrak{Z}} \hat{a}_6(\nu,\mathbb{Z}(\nu-\varsigma,\varrho),\aleph)\,\tilde{N}(d\nu,d\aleph),\\
\jmath^1_{\Bbbk}\big(\mathcal{Y}^{+}_{\iota_{\Bbbk}}\big) = \frac{\hat{a}_7}{\Bbbk^2}, &\qquad \jmath^2_{\Bbbk}\big(\mathcal{Y}^{+}_{\iota_{\Bbbk}}\big) = \frac{\hat{a}_8}{\Bbbk^2}, \quad \Bbbk = 1,2,\dots,n.
\end{aligned}$$

Both impulsive families satisfy $(\mathcal{H}_5)$ with $l^1_{\Bbbk} = \hat{a}_7\Bbbk^{-2}$ and $l^2_{\Bbbk} = \hat{a}_8\Bbbk^{-2}$; summability holds since $\sum_{\Bbbk\ge1}\Bbbk^{-2} = \pi^2/6 < \infty$, so $\hat{a}_7, \hat{a}_8$ may be chosen as small as Theorems 1 and 2 require. Here, $dw(\mu)$ is a standard one-dimensional Wiener process on $(\Omega,\mathfrak{F}_\nu,\mathcal{P})$ with values in the separable Hilbert space $\mathcal{B}$. To cast (10) into the abstract form (1), we work in the state space $\mathcal{B} = \mathbb{M}^2([0,\pi])$ endowed with its standard norm $\|\cdot\|$. The orthonormal basis for $\mathcal{B}$ is taken to be

$$\mathbb{Z}_\tau(\varrho) = \sqrt{\frac{2}{\pi}}\sin(\tau\varrho), \quad \tau = 1,2,\dots,$$

and the Q-Wiener process is represented as

$$w(\nu) = \sum_{\tau=1}^{\infty}\sqrt{\lambda_\tau}\,w_\tau(\nu)\,\mathbb{Z}_\tau, \quad \nu \ge 0, \quad \lambda_\tau > 0,$$

where $\{w_\tau(\nu)\}_{\tau\ge0}$ is a sequence of mutually independent one-dimensional standard Brownian motions. The operator $A : \mathscr{D}(A) \subset \mathcal{B} \to \mathcal{B}$ is defined by $A\mathcal{C} = \frac{\partial^2\mathcal{C}}{\partial\varrho^2}$ with domain

$$\mathscr{D}(A) = \Big\{\mathcal{C} \in \mathbb{M}^2([0,\pi]) : \mathcal{C}, \tfrac{\partial\mathcal{C}}{\partial\varrho} \text{ absolutely continuous}, \ \tfrac{\partial^2\mathcal{C}}{\partial\varrho^2} \in \mathbb{M}^2([0,\pi]),\ \mathcal{C}(0) = \mathcal{C}(\pi) = 0\Big\}.$$

Then, A generates a strongly continuous cosine family on $\mathcal{B}$, and its spectral representation is

$$A\mathcal{Y} = \sum_{\tau=1}^{\infty} -\tau^2\langle\mathcal{Y},\mathbb{Z}_\tau\rangle\mathbb{Z}_\tau, \quad \mathcal{Y} \in \mathscr{D}(A),$$

so $(\mathcal{H}_1)$ and $(\mathcal{H}_7)$ hold with the families $C^{\psi}_{\sigma}, S^{\psi}_{\sigma}, T^{\psi}_{\sigma}$ of Definitions 4–6.

Turning to the phase space, we select the weight function $q(\mu) = e^{2\mu}$ for $\mu < 0$, which yields $l = \int_{-\infty}^{0} q(\mu)\, d\mu = \frac{1}{2}$. By Hino [5], the pair $(\mathbb{X}_b, \|\cdot\|_{\mathbb{X}_b})$ is a Banach space, with $\mathbb{Z}(\nu)(\varrho) = \mathbb{Z}(\nu, \varrho)$.

To exhibit a non-identity kernel, take

$$\psi(\nu) = \ln(1+\nu), \qquad \nu \in [0, \mathcal{T}], \qquad \mathcal{T} = e^{1/2} - 1 \approx 0.6487,$$

so that $\psi'(\nu) = 1/(1+\nu) > 0$ on $[0, \mathcal{T}]$ and

$$\Psi_{\mathcal{T}} = \psi(\mathcal{T}) - \psi(0) = \ln(1+\mathcal{T}) = \tfrac{1}{2}.$$

All hypotheses of Theorem 1 can be verified. Substituting $\sigma = \frac{4}{3}$, $\wp = 2$, $\mathcal{N}_\mu = 1$, $\hat{\mathcal{C}}_2 = 1$ and the compactness constants

$$\varkappa^*_{\mathscr{H}} = 0.01, \quad \varkappa^*_f = 0.02, \quad \varkappa^*_g = 0.02, \quad \varkappa^*_{\bar{h}} = \varkappa^*_{\hat{h}} = 0.01, \quad \sum_{\Bbbk} \bar{\varkappa}^1_{\Bbbk} = \sum_{\Bbbk} \bar{\varkappa}^2_{\Bbbk} = 0.01$$

into the noncompactness criterion (6) with $\Psi_{\mathcal{T}} = \frac{1}{2}$ gives

$$\varkappa^*_{\mathscr{H}} + \frac{2\mathcal{N}_S\, \Psi^{\sigma}_{\mathcal{T}}}{\Gamma(\sigma+1)} \varkappa^*_f + \frac{2\mathcal{N}_S\, \Psi^{\sigma}_{\mathcal{T}}}{\Gamma(\sigma+1)} \sqrt{\Psi_{\mathcal{T}}}\, \varkappa^*_g + \frac{2\mathcal{N}_S}{\Gamma(\sigma)} \sqrt{\hat{\mathcal{C}}_2} \left\{ \left(\frac{\Psi^{2\sigma-1}_{\mathcal{T}}}{2\sigma-1} \right)^{\frac{1}{2}} \varkappa^*_{\bar{h}} + \left(\frac{\Psi^{4\sigma-3}_{\mathcal{T}}}{4\sigma-3} \right)^{\frac{1}{4}} \varkappa^*_{\hat{h}} \right\}$$

$$+ \mathcal{N}_S \sum_{\Bbbk=1}^{n} \left(\bar{\varkappa}^1_{\Bbbk} + \bar{\varkappa}^2_{\Bbbk} \right) = 0.01000 + 0.01333 + 0.00943 + 0.02183 + 0.02000 = 0.07459 < 1.$$

The growth condition (5) is met likewise once $\mathcal{N}_{\mathscr{H}}, \mathcal{N}_{\alpha_i}, l^*_f \mathrm{Y}^*_f, l^*_g \mathrm{Y}^*_g, l_{\bar{h}}, l_{\hat{h}}$ are taken of the same order 10^{-2}, since $\Lambda_\sigma = \Psi^{\sigma}_{\mathcal{T}}/\Gamma(\sigma+1) = 0.3333$ and $L^1 = L^2 = 0.01$ make every bracket there smaller than the corresponding bracket above. Hence, (10) admits a mild solution. Relative to the purely positional model, only the impulsive block of (6) changes, doubling from 0.01 to 0.02 through the derivative budget $\sum_{\Bbbk} \bar{\varkappa}^2_{\Bbbk}$.

For the stability criterion (9), extend ψ to $[0, \infty)$ and take $\wp = q = 2$, $a_1 = a_2 = 1$, $\beta_2 = \pi$, $\beta_3 = 2$, $\hat{\mathcal{C}}_\wp = \tilde{\mathcal{C}}_\wp = 1$, and small coefficients

$$b_1 = b_2 = b_3 = b_4 = d_1 = d_2 = c_1 = \sum_{\Bbbk} l^1_{\Bbbk} = \sum_{\Bbbk} l^2_{\Bbbk} = 0.01,$$

so that $\beta_3^{-1} = \frac{1}{2}$. Then,

$$8^{\wp-1} \left\{ b_2^{\wp}\left(1 + b_1^{\wp}\right) + \frac{a_2^{\wp} \beta_2^{1-\wp} b_4 (1+b_3) + a_2^{\wp} \beta_2^{1-\frac{\wp}{2}} \tilde{\mathcal{C}}_\wp c_1 \beta_3^{-1} + a_2^{\wp} \hat{\mathcal{C}}_\wp \left(\beta_2^{-\frac{\wp}{2}} d_1 + \beta_2^{-1} d_2 \right)}{\beta_2} \right.$$

$$\left. + a_1^{\wp} \left(L^1 \right)^{\frac{\wp}{q}+1} + a_2^{\wp} \left(L^2 \right)^{\frac{\wp}{q}+1} \right\} = 8\{0.0001 + 0.004641 + 0.0001 + 0.0001\} = 0.03953 < 1.$$

The criterion is satisfied with room to spare, even with the derivative impulses and the prefactor $8^{\wp-1}$ included. All hypotheses $(\mathcal{H}_1)$–$(\mathcal{H}_{12})$ therefore hold, and (10) with $\psi(\nu) = \ln(1+\nu)$ admits a mild solution satisfying $\mathfrak{E}\|\mathcal{Y}(\nu)\|^{\wp} \le \widetilde{\Lambda}\,(1+\nu)^{-\bar{\lambda}}$ for $\nu \ge -\tau$. In accordance with Remark 5, the decay is algebraic in ν, in contrast with the exponential decay obtained for $\psi(\nu) = \nu$.

https://doi.org/10.3390/axioms15090702

The two cases share the same admissible range for $\bar{\lambda}$, since criterion (9) does not involve ψ or $\Psi_{\mathcal{T}}$; only the decay law itself changes:

$$\psi(\nu) = \nu: \quad \mathfrak{E}\|\mathcal{Y}(\nu)\|^{\wp} \leq \widetilde{\Lambda}\, e^{-\bar{\lambda}\nu},$$
$$\psi(\nu) = \ln(1+\nu): \quad \mathfrak{E}\|\mathcal{Y}(\nu)\|^{\wp} \leq \widetilde{\Lambda}\, (1+\nu)^{-\bar{\lambda}}.$$

For any fixed $\bar{\lambda} > 0$, $e^{-\bar{\lambda}\nu} \to 0$ much faster than $(1+\nu)^{-\bar{\lambda}}$ as $\nu \to \infty$, since

$$\lim_{\nu\to\infty} \frac{(1+\nu)^{-\bar{\lambda}}}{e^{-\bar{\lambda}\nu}} = \lim_{\nu\to\infty} (1+\nu)^{-\bar{\lambda}} e^{\bar{\lambda}\nu} = \infty.$$

This confirms Remark 5: the classical kernel $\psi(\nu) = \nu$ gives a stronger, exponential decay, while the sublinear kernel $\psi(\nu) = \ln(1+\nu)$ gives a weaker, algebraic decay, even though both arise from the same stability criterion (9).

6. Conclusions

We established the existence of mild solutions for ψ-Caputo fractional neutral stochastic integro-differential systems of order $\sigma \in (1,2)$ with infinite delay, Poisson jumps, and impulses acting simultaneously on the state and its ψ-derivative in a separable Hilbert space. The existence result was obtained by combining the HMnC withMFPT, thereby eliminating the need for compactness of the associated cosine family. We then established the exponential stability of mild solutions by deriving a new ψ-weighted derivative impulsive integral inequality. Since the ψ-fractional sine and Riemann–Liouville families inherit the kernel ψ, the resulting stability estimate is naturally expressed in ψ-time rather than physical time. Consequently, the decay rate depends on the choice of the memory kernel: it is exponential in the time variable whenever ψ grows at least linearly, whereas sublinear kernels yield algebraic decay. Furthermore, the choice $\psi(\nu) = \nu$ recovers the corresponding classical Caputo results. A wave-type example with $\psi(\nu) = \ln(1+\nu)$ confirms the applicability of the proposed theoretical results and illustrates the predicted algebraic decay.

Several directions remain open for future investigation. These include the study of approximate controllability for ψ-Caputo systems of order $\sigma \in (1,2)$, the incorporation of non-instantaneous impulsive effects, the extension of the stochastic perturbation to fractional Brownian motion or the Rosenblatt process, and the development of kernel identification techniques from observed decay data so that the memory scale can be determined from the underlying dynamics. The dual kernels α_1, α_2 roughly double the quadrature cost per step versus a single kernel; an efficient numerical scheme is left for future work.

Author Contributions: Conceptualization, P.R. and M.S.K.; methodology, P.R., M.R., S.A. and M.S.K.; validation, P.R., M.R. and M.S.K.; formal analysis, P.R., S.A. and M.S.K.; investigation, P.R. and M.S.K.; resources, M.R.; data curation, M.S.K.; writing—original draft preparation, P.R. and M.S.K.; writing—review and editing, P.R., S.A. and M.S.K.; visualization, M.S.K.; supervision, M.S.K.; project administration, S.A.; funding acquisition, M.R. All authors have read and agreed to the published version of the manuscript.

Funding: The researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University (https://www.qu.edu.sa) for financial support (QU-APC-2026).

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding authors.

Conflicts of Interest: The authors declare no conflicts of interest.

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