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Strict Stability of Switched Caputo Fractional Differential Equations

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Abstract

This paper studies a nonlinear system of switched fractional differential equations with generalized Caputo fractional derivatives with respect to another function. The main characteristics of these fractional derivatives are twofold: the lower limit of the fractional derivative equals the switching time, and the applied function in the fractional derivative changes in each interval between two consecutive switching times. Several comparison results and some results for the fractional derivative of convex Lyapunov functions are obtained. Also, strict stability of the zero solution is defined and sufficient conditions for strict stability of the system are obtained. The auxiliary results allow us to avoid the application of fractional derivatives of convex Lyapunov functions in our sufficient conditions. The conditions depend significantly on the type of switching rule and the switching points.

Keywords: strict stability; Caputo derivatives with respect to another function; fractional differential equations; switching rule; Lyapunov functions

MSC: 34A08; 34A34; 34D20

1. Introduction

One type of stability is strict stability, which gives information on the rate of decay of the solutions. Strict stability requires solutions to remain bounded within a strict lower and upper non-zero bound over time, whereas stability only requires solutions to stay within an upper bound without dropping below a lower threshold. Strict stability differs from asymptotic stability since one does not need to approach zero as time approaches infinity. Strict stability is studied for various types of differential equations, for example, for ordinary differential equations in [1], for fractional differential equations with CFF in [2], for delay differential equations with impulses [3–5], for dynamic systems on a time scale [6,7] and for Caputo fractional differential equations [8].

In this paper, we study a nonlinear fractional differential equation with two main characteristics: a switching rule is given and this determines the function from an initially given set of functions, and the fractional derivative is a generalized Caputo fractional derivative with respect to another function (GCFF) (here, both the lower limit and the applied function could change on each interval between two consecutive switching times).

Note that the generalized Caputo fractional derivative with respect to another function, defined in [9], is a generalization of the Caputo fractional derivative [9], the Caputo–Hadamard fractional derivative [10,11] and the Caputo–Erdelyi–Kober fractional

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derivative [12]. Also, this type of derivative is applied and studied in various problems; see, for example, [13–16].

We begin by presenting some comparison results for scalar switched fractional differential equations with GCFF. An explicit formula is obtained for a scalar linear switched fractional differential equation with GCFF. Strict stability is defined for the nonlinear system of switched fractional differential equations with GCFF and studied by using appropriate Lyapunov functions. Some results concerning fractional derivatives of convex Lyapunov functions are obtained.

Strict stability of nonlinear fractional differential equations with GCFF without a switching rule is studied in [2] using Lyapunov functions and their GCFF. In this paper, we prove an inequality about convex Lyapunov functions and their GCFF, and this result allows us to avoid the application of GCFF of Lyapunov functions in our sufficient conditions. As a result, our sufficient conditions are easier to check. Additionally, when there is a switching rule, as in this paper, the applied GCFF could have various functions on each interval between two consecutive switching times. This leads to stronger sufficient conditions but it encompasses more general applications. In [17], a nonlinear switched fractional equations with GCFF is considered and stability properties are studied. In this paper, the lower limit of the fractional derivative changes at any switching time.

The main contributions in the paper can be summarized as follows:

- Strict stability for nonlinear switched systems of fractional differential equations with a piecewise constant switching rule is defined and studied.
- Caputo fractional derivatives with respect to another function are applied with the lower limit and applied function changing at any switching time.
- Lyapunov functions are applied and some inequalities for the Caputo fractional derivatives with respect to another function for convex Lyapunov functions are established.
- Some comparison results for scalar switched fractional differential equations are proven.
- Several sufficient conditions for strict stability of the nonlinear switched system are obtained.
- The influence of the switching rule, of the applied function in the fractional derivative on stability and some of the sufficient conditions are illustrated with examples.

2. Preliminary Results for Fractional Differintegrals

Let $0 \leq \alpha < \beta \leq \infty$ be fixed numbers.

Consider the set

$$\mathcal{P}([\alpha,\beta]) = \{\theta : [\alpha,\beta] \to [0,\infty),\ \theta \in C^1([\alpha,\beta],[0,\infty)),\ \theta'(t) > 0 \text{ for } t \in (\alpha,\beta)\}.$$

Note, if $\beta = \infty$, we will consider the half open interval $[\alpha,\beta)$.

We will give some definitions and results for fractional integrals and derivatives.

Definition 1 ([9])**.** *Let $\rho > 0$, $\theta \in \mathcal{P}([a,b])$. The Riemann fractional integral with respect to another function (FIF) of the function $v : [\alpha,\beta] \to \mathbb{R}$ is defined by (where the integral exists)*

$$I_\alpha^{\rho,\theta} v(t) = \frac{1}{\Gamma(\rho)} \int_\alpha^t \theta'(s)(\theta(t) - \theta(s))^{\rho-1} v(s)\, ds, \quad t \in (\alpha,\beta].$$

Definition 2 ([9])**.** *Let $\rho \in (0,1)$ and the function $\theta \in \mathcal{P}([\alpha,\beta])$. The generalized Caputo fractional derivative with respect to another function (GCFF) of the function $v : [\alpha,\beta] \to \mathbb{R}$ is defined by (where the integral exists)*

$$^{C}D_\alpha^{\rho,\theta} v(t) = \frac{1}{\Gamma(1-\rho)} \int_\alpha^t (\theta(t) - \theta(s))^{-\rho} v'(s)\, ds, \quad t \in (\alpha,\beta].$$

In the case of a vector function, the integral FIF and the derivative GCFF are defined component-wise.

We will use the set of functions ($0 \le \alpha < \beta$):

$$\mathcal{C}^{\rho}((\alpha,\beta],\mathbb{R}^n,\theta) = \{v \in C((\alpha,\beta],\mathbb{R}^n) : \ v' \text{ exists almost everywhere in } (\alpha,\beta] \\ \text{and there exists GCFF } {}^{C}D_{\alpha}^{\rho,\theta}v(t) \text{ for } t \in (\alpha,\beta]\}.$$

We now provide some results for GCFF.

Lemma 1 ([2]). *Let $\rho \in (0,1)$, $\theta \in \mathcal{P}[\alpha,\beta]$, $v : [\alpha,\beta] \to \mathbb{R}$ and there exists a point $\varsigma \in (\alpha,\beta]$ such that $v(\varsigma) = 0$, and $v(s) < 0$ (respectively $v(s) > 0$), for $s \in [\alpha,\varsigma)$ and the GCFF ${}^{C}D_{\alpha}^{\rho,\theta}v(t)|_{t=\varsigma}$ exists. Then, if $\lim_{\sigma\to\varsigma-}\left(\frac{v(\sigma)}{(\theta(\varsigma)-\theta(\sigma))^{\rho}}\right) = 0$, we have ${}^{C}D_{\alpha}^{\rho,\theta}v(t)|_{t=\varsigma} \ge 0$ (respectively ≤ 0).*

Remark 1 ([2]). *Note that the condition $\lim_{\sigma\to\varsigma-}\left(\frac{v(\sigma)}{(\theta(\varsigma)-\theta(\sigma))^{\rho}}\right) = 0$ in Lemma 1 is automatically true if $v \in C^1((\alpha,b],\mathbb{R})$ (because of L'Hopital's rule). Also, this condition about the zero limit is also true if $\lim_{\sigma\to\varsigma-} v'(\sigma)$ exists and is a real number.*

Lemma 2 ([13], Lemma 2). *Let $\rho \in (0,1)$ and $\theta \in \mathcal{P}([\alpha,\beta])$. The solution of the initial value problem for the scalar linear fractional differential equation with GCFF,*

$$ {}^{C}D_{\alpha}^{\rho,\theta}v(t) = \mathcal{L}v(t), \text{ for } t \in (\alpha,\beta], \quad \text{and } \ v(\alpha) = v_0,$$

with $v_0 \in \mathbb{R}$, $\mathcal{L} \in \mathbb{R}$, is the function

$$v(t) = v_0 E_{\rho}(\mathcal{L}(\theta(t)-\theta(\alpha))^{\rho}), \ \ t \in [\alpha,\beta],$$

where $E_{\rho}(t)$ is the Mittag–Leffler function of one parameter.

We will use the following special case (fractional order between 0 and 1) of a result proved in [18]:

Lemma 3. *Let $\rho \in (0,1)$, $\theta \in \mathcal{P}([\alpha,\beta])$ and $f \in C([\alpha,\beta]\times\mathbb{R},\mathbb{R})$. A function $v \in C^{\rho}([\alpha,\beta],\mathbb{R},\theta)$ is a solution of the problem*

$$ {}^{C}D_{\alpha}^{\rho,\theta}v(t) = f(t,v(t)), \ t \in (\alpha,\beta], \quad v(\alpha) = v_0, \tag{1}$$

if and only if v satisfies the integral equation

$$v(t) = v_0 + \frac{1}{\Gamma(\rho)}\int_{\alpha}^{t}\theta'(s)(\theta(t)-\theta(s))^{\rho-1}f(s,v(s))ds, \ \ t \in (\alpha,\beta]. \tag{2}$$

Corollary 1. *Let $\rho \in (0,1)$, $\theta \in \mathcal{P}([\alpha,\beta])$ and $Y \in C([\alpha,\beta],(-\infty,0])$. Then, the solution $v \in C^{\rho}([\alpha,\beta],\mathbb{R},\theta)$ of problem (1) with $f(t,v) = Y(t) \le 0$, $t \in (\alpha,\beta]$, satisfies the inequality $v(t) \le v_0$, $t \in (\alpha,\beta]$.*

Lemma 4. *Assume the following hold:*

1. *$\rho \in (0,1)$, $\theta \in ([\alpha,\beta])$, and $\mathbb{H} \in C([\alpha,\beta],\mathbb{R})$.*
2. *The function $u \in C^{\rho}((\alpha,\beta],\mathbb{R},\theta)$ is a solution of the scalar fractional differential inequality with GCFF*

$$\begin{aligned} &{}^{C}D_{\alpha}^{\rho,\theta}u(t) < \mathbb{H}(u(t)), \ \text{ for } t \in (\alpha,\beta]. \\ &u(\alpha) < u_0. \end{aligned} \tag{3}$$

3. *The function $v \in C^{\rho}((\alpha,\beta],\mathbb{R},\theta)$ is a solution of the IVP for the scalar fractional differential equation with GCFF*

$$\begin{aligned} &{}^{C}D_{\alpha}^{\rho,\theta}v(t) = \mathbb{H}(v(t)), \quad t \in (\alpha,\beta] \\ &v(\alpha) = v_0. \end{aligned} \tag{4}$$

Then, the inequality $u_0 < v_0$ implies

$$u(t) < v(t), \ t \in [\alpha,\beta]. \tag{5}$$

Proof. Assume the contrary; i.e., there exists a point $A \in (\alpha,\beta]$ such that $u(t) < v(t), \ t \in [\alpha, A)$ and $u(A) = v(A)$. According to Lemma 1 and Remark 1, for the function $u(t) - v(t)$ in $[\alpha, A]$, we have ${}^{C}D_{\alpha}^{\rho,\theta}(u(t) - v(t))|_{t=A} \geq 0$, i.e.,

$${}^{C}D_{\alpha}^{\rho,\theta}u(t)|_{t=A} \geq {}^{C}D_{\alpha}^{\rho,\theta}v(t)|_{t=A} = \mathbb{H}(v(A)) = \mathbb{H}(u(A)).$$

The last inequality contradicts (3). □

Lemma 5. *Assume the following hold:*

1. $\rho \in (0,1)$, $\theta \in ([\alpha,\beta])$, *and* $\mathbb{H} \in C([\alpha,\beta],\mathbb{R})$.
2. *The function $u \in C^{\rho}((\alpha,\beta],\mathbb{R},\theta)$ is a solution of the scalar fractional differential inequality with GCFF*

$$\begin{aligned} &{}^{C}D_{\alpha}^{\rho,\theta}u(t) > \mathbb{H}(u(t)), \ \text{for } t \in (\alpha,\beta]. \\ &u(\alpha) > u_0. \end{aligned} \tag{6}$$

3. *The function $v \in C^{\rho}((\alpha,\beta],\mathbb{R},\theta)$ is a solution of the IVP for the scalar fractional differential equation with GCFF (4).*

Then, the inequality $u_0 > v_0$ implies

$$u(t) > v(t), \ t \in [\alpha,\beta]. \tag{7}$$

The proof of Lemma 5 is similar to the proof in Lemma 4.

We will prove an inequality for the GCFF of convex fractional derivatives of Lyapunov functions which will be used in the proof of the main results for strict stability. This inequality will allow us to avoid the application of fractional derivatives of convex functions in our sufficient conditions. A similar inequality for convex Lyapunov functions and Caputo fractional derivatives was proved in Theorem 1 [19]. Note that the square Lyapunov function is a convex function and an inequality for a generalized Caputo proportional fractional derivative of a quadratic function is proved in [20].

Lemma 6. *Let $\rho \in (0,1)$, $\theta \in \mathcal{P}([\alpha,\beta])$ and*

1. $x \in C^{\rho}([\alpha,\beta],\mathbb{R}^n,\theta), \ \ x = (x_1, x_2, \ldots, x_n)$.
2. $V \in C^1(\mathbb{R}^n, [0,\infty))$ *is convex.*

Then

$${}^{C}D_{\alpha}^{\rho,\theta}V(x(t)) \leq \sum_{i=1}^{n} \frac{\partial}{\partial x_i}V(x(t))\, {}^{C}D_{\alpha}^{\rho,\theta}x_i(t), \ \ t \in (\alpha,\beta],$$

holds.

Proof. Let $t \in (\alpha, \beta]$ be fixed. Define the function $W(.) : [\alpha, t] \to \mathbb{R}$ by

$$W(s) = V(x(s)) - V(x(t)) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} V(x(t))(x_i(s) - x_i(t)).$$

Since the function $V(.)$ is convex, we have $W(s) \geq 0,\ s \in [\alpha, t]$ and $W(t) = 0$. Note $W'(s) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} V(x(s)) x_i'(s) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} V(x(t)) x_i'(s)$.

Since $W \in C^1([\alpha, t], \mathbb{R})$, then L'Hopital's rule guarantees that

$$\lim_{s \to t-} \left(\frac{W(s)}{(\theta(t) - \theta(s))^{\rho}} \right) = \lim_{s \to t-} \frac{W'(s)}{\rho \theta'(s)} (\theta(t) - \theta(s))^{1-\rho} = 0.$$

Thus,

$$\begin{aligned}
&{}^{C}D_{\alpha}^{\rho,\theta} V(x(t)) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} V(x(t))\, {}^{C}D_{\alpha}^{\rho,\theta} x_i(t) \\
&= \frac{1}{\Gamma(1-\rho)} \int_{\alpha}^{t} (\theta(t) - \theta(s))^{-\rho} \Big[\sum_{i=1}^{n} \frac{\partial}{\partial x_i} V(x(s)) x_i'(s) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} V(x(t)) x_i'(s) \Big] ds \\
&= \frac{1}{\Gamma(1-\rho)} \int_{\alpha}^{t} (\theta(t) - \theta(s))^{-\rho} W'(s) ds \\
&= \frac{1}{\Gamma(1-\rho)} \Big[\frac{W(s)}{(\theta(t) - \theta(s))^{\rho}} \Big|_{s=\alpha}^{s=t} + \frac{\rho}{\Gamma(1-\rho)} \int_{\alpha}^{t} \frac{\theta'(s) W(s)}{(\theta(t) - \theta(s))^{1+\rho}} ds \Big] \\
&= \frac{1}{\Gamma(1-\rho)} \Big[- \frac{W(\alpha)}{(\theta(t) - \theta(\alpha))^{\rho}} + \frac{\rho}{\Gamma(1-\rho)} \int_{\alpha}^{t} \frac{\theta'(s) W(s)}{(\theta(t) - \theta(s))^{1+\rho}} ds \Big] \leq 0.
\end{aligned} \tag{8}$$

Inequality (8) proves the claim. □

3. Some Preliminary Results for Switched Fractional Problems

The switched fractional differential equations depend significantly on the applied fractional derivative. In this paper, we apply the generalized Caputo fractional derivative with respect to another function. This derivative depends significantly on its lower limit. We consider the case when the lower limit changes at each switching time. This approach is used in the study of switched fractional differential equations in [17,21–23].

Suppose m is a given natural number, $\mathcal{M} = \{1, 2, \ldots, m\}$, and let a family $\mathbb{W}$ of functions $\mathbb{F}_j : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $\mathbb{F}_j(t, 0) = 0$, $j \in \mathcal{M}$, be given.

Let the sequence of points $\{\Upsilon_i\}_{i=0}^{\infty} :\ 0 = \Upsilon_0 < \Upsilon_1 < \Upsilon_2 < \cdots < \Upsilon_{k-1} < \Upsilon_k < \ldots$ be given such that $\lim_{k \to \infty} \Upsilon_k = \infty$.

Let the sequence of numbers $\mathbb{K}_k \in \mathcal{M}$, $k = 0, 1, 2, \ldots$, be given such that for any k, we have $\mathbb{K}_k \neq \mathbb{K}_{k+1}$.

Consider the switching rule $\sigma(t) = \mathbb{K}_k \in \mathcal{M}$ for $\Upsilon_k \leq t < \Upsilon_{k+1}$, $k = 0, 1, 2, 3, \ldots$. We will assume that σ is continuous from the right everywhere. Therefore, the switched rule is a piecewise function depending on the initially given points Υ_k, $k = 1, 2, 3, \ldots,$.

Note that the points Υ_k, $k = 1, 2, \ldots$ are points of the switching of the system, i.e., points at which the functions from the set $\mathcal{W}$ are changed and the lower limit and the applied function in the fractional derivative GCFF are changed.

Let $a \in \mathbb{R}$, $\varepsilon > 0$. For any function $\mathbf{x} : [a - \varepsilon, a) \cup (a, a + \varepsilon] \to \mathbb{R}^n$, we define $\mathbf{x}(a-) = \lim_{s \to a, s < a} \mathbf{x}(s)$ and $\mathbf{x}(a+) = \lim_{s \to a, s > a} \mathbf{x}(s)$.

Consider the initial value problem (IVP) for the switched fractional differential equations with GCFF (SFDEGCFF):

$$
\begin{aligned}
&{}^{C}D^{\rho,\theta_k}_{\Upsilon_k}\mathbf{y}(t) = \mathbb{F}_{\sigma(t)}(t,\mathbf{y}(t)), \quad t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0,1,2,\dots,\\
&\mathbf{y}(\Upsilon_k-) = \mathbf{y}(\Upsilon_k+),\ j = 1,2,\dots,\\
&\mathbf{y}(0) = \mathbf{y}_0
\end{aligned}
\tag{9}
$$

where $\mathbf{y}_0 \in \mathbb{R}^n$, $\mathbb{F}_k \in \mathcal{W}$, $k = 0,1,2,\dots$, $\rho \in (0,1)$ and the functions $\theta_k \in \mathcal{P}([\Upsilon_k, \Upsilon_{k+1}])$, $k = 0,1,2,\dots$.

We will say that condition (H) is satisfied if

(H). For any $j \in \mathcal{M}$, the functions $\mathbb{F}_j \in \mathcal{W}$, and for any initial value $\mathbf{x}_0 :\ \mathbf{x}_0 \in \mathbb{R}^n$ and any $k = 0,1,2,3,\dots$, the IVP ${}^{C}D^{\rho,\theta_k}\mathbf{x}(t) = \mathbb{F}_{\mathbb{K}_k}(t,\mathbf{x}(t))$, $t \in (\Upsilon_k, \Upsilon_{k+1}]$ and $\mathbf{x}(\Upsilon_k) = \mathbf{x}_0$ has a unique solution $\mathbf{x}(t)$ defined for $t \in (\Upsilon_k, \Upsilon_{k+1}]$, where $\mathbb{K}_k \in \mathcal{M}$ are values of the switching rule.

Remark 2. *Some sufficient conditions for the existence of Caputo fractional differential equations are given in [24] and for fractional differential equations with GCFF in [18].*

We use the set of functions:

$$
\mathcal{SC}^{\rho}([0,\infty),\mathbb{R}^n) = \{v : [0,\infty) \to \mathbb{R}^n :\ \ v \in \mathcal{C}^{\rho}((\Upsilon_k, \Upsilon_{k+1}], \mathbb{R}^n, \theta)\ \text{ for }\ k = 0,1,2,\dots\}.
$$

Consider the linear scalar case of (9). Let the numbers $\lambda_j \in \mathbb{R}$, $j \in \mathcal{M}$ be given and consider the IVP for a linear scalar SFDEGCFF

$$
\begin{aligned}
&{}^{C}D^{\rho,\theta_k}_{\Upsilon_k} v(t) = \lambda_{\mathbb{K}_k} v,\ \text{for } t \in (\Upsilon_k, \Upsilon_{k+1}],\ \ k = 0,1,2,\dots,\\
&v(\Upsilon_k-) = v(\Upsilon_k+),\ k = 1,2,\dots,\\
&v(0) = v_0
\end{aligned}
\tag{10}
$$

where $v_0 \in \mathbb{R}$.

Lemma 7. *Let $\rho \in (0,1)$, $\theta_k \in \mathcal{P}([\Upsilon_k, \Upsilon_{k+1}])$, $k = 0,1,2,\dots$.*

Then the solution of the IVP for the linear scalar SFDEGCFF (10) is

$$
\begin{aligned}
v(t) = v_0 \left(\prod_{i=0}^{k-1} E_{\rho}(\lambda_{\mathbb{K}_i}(\theta_i(\Upsilon_{i+1}) - \theta_i(\Upsilon_i))^{\rho}) \right) E_{\rho}(\lambda_{\mathbb{K}_k}(\theta_k(t) - \theta_k(\Upsilon_k))^{\rho}),\\
t \in (\Upsilon_k, \Upsilon_{k+1}],\ \ k = 0,1,2,3,\dots,
\end{aligned}
\tag{11}
$$

and

(i) *if $\lambda_j \le 0,\ j \in \mathcal{M}$ then $|v(t)| \le |v_0|$;*

(ii) *if $\lambda_j \ge 0,\ j \in \mathcal{M}$ then $|v(t)| \ge |v_0|$.*

Proof. We use an induction and Lemma 2 with $\mathcal{L} = \lambda_{\mathbb{K}_k}$, $a = \Upsilon_k$, $b = \Upsilon_{k+1}$, $\theta(.) = \theta_k(.)$.

Proof of (i). Let $\lambda_j \le 0,\ j \in \mathcal{M}$. Then $E_{\rho}(\lambda_{\mathbb{K}_i}(\theta_i(\Upsilon_{i+1}) - \theta_i(\Upsilon_i))^{\rho}) \le 1,\ i = 0,1,2,\dots$ and therefore, $|v(t)| \le |v_0|$.

Proof of (ii). Let $\lambda_j \ge 0,\ j \in \mathcal{M}$. Then $E_{\rho}(\lambda_{\mathbb{K}_i}(\theta_i(\Upsilon_{i+1}) - \theta_i(\Upsilon_i))^{\rho}) \ge 1,\ i = 0,1,2,\dots$ and therefore, $|v(t)| \ge |v_0|$. □

Let the sequence of functions $\{\mathbb{G}_i(u)\}_{i=1}^m$, $\mathbb{G}_{\mathbb{K}_k} \in C([\Upsilon_k, \Upsilon_{k+1}], \mathbb{R})$, $k = 0, 1, 2, \ldots$, be given. Consider IVP for the nonlinear scalar SFDEGCFF:

$$\begin{aligned}
&{}^C D_{\Upsilon_k}^{\rho,\theta_k} v(t) = \mathbb{G}_{\sigma(t)}(v(t)), \text{ for } t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0, 1, 2, \ldots, \\
&v(\Upsilon_k -) = v(\Upsilon_k +),\ k = 1, 2, \ldots, \\
&v(0) = v_0
\end{aligned} \tag{12}$$

or its equivalent

$$\begin{aligned}
&{}^C D_{\Upsilon_k}^{\rho,\theta_k} v(t) = \mathbb{G}_{\mathbb{K}_k}(v(t)), \text{ for } t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0, 1, 2, \ldots, \\
&v(\Upsilon_k -) = v(\Upsilon_k +),\ k = 1, 2, \ldots, \\
&v(0) = v_0,
\end{aligned} \tag{13}$$

where $v_0 \in \mathbb{R}$.

Lemma 8. *Suppose $\rho \in (0,1)$, $\theta_k \in \mathcal{P}([\Upsilon_k, \Upsilon_{k+1}])$, and $\mathbb{G}_{\mathbb{K}_k} \in C([\Upsilon_k, \Upsilon_{k+1}], (-\infty, 0])$, $k = 0, 1, 2, \ldots$. Then the solution $v \in \mathcal{SC}^\rho([0,\infty), \mathbb{R})$ of (12) satisfies the inequality $v(t) \le v_0$, $t \ge 0$.*

Proof. We will use induction.

Consider the interval $[0, \Upsilon_1]$. According to Lemma 3 and the nonpositiveness of the function $\mathbb{G}_{\mathbb{K}_0}(.)$, we have

$$v(t) = v_0 + \frac{1}{\Gamma(\rho)} \int_0^t \theta_0'(s)(\theta_0(t) - \theta_0(s))^{\rho-1} \mathbb{G}_{\mathbb{K}_0}(v(s))ds \le v_0, \quad t \in (0, \Upsilon_1].$$

Consider the interval $(\Upsilon_1, \Upsilon_2]$. According to Lemma 3, we get for $(\Upsilon_1, \Upsilon_2]$ that

$$\begin{aligned}
v(t) &= v(\Upsilon_1+) \\
&+ \frac{1}{\Gamma(\rho)} \int_{\Upsilon_1}^t \theta_1'(s)(\theta_1(t) - \theta_1(s))^{\rho-1} \mathbb{G}_{\mathbb{K}_1}(v(s))ds \le v(\Upsilon_1+) = v(\Upsilon_1-) \le v_0.
\end{aligned}$$

Following this procedure, we obtain the claim. □

Corollary 2. *Suppose $\rho \in (0,1)$, $\theta_k \in \mathcal{P}([\Upsilon_k, \Upsilon_{k+1}])$, and $\Upsilon_{\mathbb{K}_k} \in C([\Upsilon_k, \Upsilon_{k+1}], (-\infty, 0])$, $k = 0, 1, 2, \ldots$. Then, the solution $v \in \mathcal{SC}^\rho([0,\infty), \mathbb{R})$ of problem (12) with $\mathbb{G}_{\mathbb{K}_k}(v(t)) = \Upsilon_{\mathbb{K}_K}(t) \le 0$, $t \in (\Upsilon_k, \Upsilon_{k+1}]$, satisfies the inequality $v(t) \le v_0$, $t \ge 0$.*

The proof of Corollary 2 follows inductively from the application of Corollary 1.

In the case of nonegative functions $\mathbb{G}_k(.)$ in (12), we obtain the following result:

Lemma 9. *Suppose $\rho \in (0,1)$, $\theta_k \in \mathcal{P}([\Upsilon_k, \Upsilon_{k+1}])$, and $\mathbb{G}_{\mathbb{K}_k} \in C([\Upsilon_k, \Upsilon_{k+1}], [0, \infty))$, $k = 0, 1, 2, \ldots$. Then the solution $v \in \mathcal{SC}^\rho([0,\infty), \mathbb{R})$ of (12) satisfies the inequality $v(t) \ge v_0$, $t \ge 0$.*

The proof is similar to the one given in Lemma 8, so we omit it.

Corollary 3. *Suppose $\rho \in (0,1)$, $\theta_k \in \mathcal{P}([\Upsilon_k, \Upsilon_{k+1}])$, and $\Upsilon_{\mathbb{K}_k} \in C([\Upsilon_k, \Upsilon_{k+1}], [0, \infty))$, $k = 0, 1, 2, \ldots$. Then, the solution $v \in \mathcal{SC}^\rho([0,\infty), \mathbb{R})$ of problem (12) with $\mathbb{G}_{\mathbb{K}_k}(v(t)) = \Upsilon_{\mathbb{K}_K}(t) \ge 0$, $t \in (\Upsilon_k, \Upsilon_{k+1}]$, satisfies the inequality $v(t) \ge v_0$, $t \ge 0$.*

Consider the following nonlinear scalar switched fractional differential inequality with GCFF corresponding to (13):

$$\begin{aligned}
&{}^{C}D^{\rho,\theta_k}_{\Upsilon_k}\varsigma(t) < \mathbb{G}_{\mathbb{K}_k}(\varsigma(t)) \text{ for } t\in(\Upsilon_k,\Upsilon_{k+1}],\ k=0,1,2,\dots,\\
&\varsigma(\Upsilon_k+) \le \varsigma(\Upsilon_k-),\ \ k=1,2,\dots,\\
&\varsigma(0) < \varsigma_0,
\end{aligned} \tag{14}$$

where $\varsigma_0 \in \mathbb{R}$.

Lemma 10. *Suppose the following conditions hold:*

1. $\rho\in(0,1)$, $\theta_k\in\mathcal{P}([\Upsilon_k,\Upsilon_{k+1}])$, *and* $\mathbb{G}_{\mathbb{K}_k}\in C([\Upsilon_k,\Upsilon_{k+1}],\mathbb{R})$, $\ k=0,1,2,\dots.$
2. *The function* $\varsigma\in\mathcal{SC}^{\rho}([0,\infty),\mathbb{R})$ *is a solution of inequality* (14).
3. *The function* $v\in\mathcal{SC}^{\rho}([0,\infty),\mathbb{R})$ *is a solution of IVP of the scalar SFDEGCFF* (13).

   Then, $\varsigma_0 < v_0$ *implies* $\varsigma(t) < v(t)$ *for* $t\ge 0$.

Proof. We will use induction.

Consider the interval $[0,\Upsilon_1]$. The function ς satisfies (3) and the function v satisfies (4) with $\mathbb{H}(.) = \mathbb{G}_{\mathbb{K}_0}(.)$, $\alpha=0,\beta=\Upsilon_1$. According to Lemma 4, the inequality $\varsigma(t)<v(t),\ t\in[0,\Upsilon_1]$ holds.

From (13) and (14), we have $\varsigma(\Upsilon_1+)\le\varsigma(\Upsilon_1-)<v(T_1-)=v(T_1+)$.

Now, consider the interval $(\Upsilon_1,\Upsilon_2]$. The function ς satisfies (3) and the function v satisfies (4) with $\mathbb{H}(.) = \mathbb{G}_{\mathbb{K}_1}(.)$, $\alpha=T_1,\ \beta=\Upsilon_2$. According to Lemma 4, the inequality $\varsigma(t)<v(t),\ t\in(\Upsilon_1,T_2]$ holds.

Following this procedure, we prove the claim of Lemma 10 . □

We now consider the case of the opposite inequalities in (14); i.e., consider the following nonlinear scalar switched fractional differential inequality with GCFF:

$$\begin{aligned}
&{}^{C}D^{\rho,\theta_k}_{\Upsilon_k}\varsigma(t) > \mathbb{G}_{\mathbb{K}_k}(\varsigma(t)) \text{ for } t\in(\Upsilon_k,\Upsilon_{k+1}],\ k=0,1,2,\dots,\\
&\varsigma(\Upsilon_k+) \ge \varsigma(\Upsilon_k-),\ \ k=1,2,\dots,\\
&\varsigma(0) > \varsigma_0,
\end{aligned} \tag{15}$$

where $\varsigma_0 \in \mathbb{R}$.

Lemma 11. *Assume that the following conditions hold:*

1. $\rho\in(0,1)$, $\theta_k\in\mathcal{P}([\Upsilon_k,\Upsilon_{k+1}])$, *and* $\mathbb{G}_{\mathbb{K}_k}\in C([\Upsilon_k,\Upsilon_{k+1}],\mathbb{R})$, $\ k=0,1,2,\dots.$
2. *The function* $\varsigma\in\mathcal{SC}^{\rho}([0,\infty),\mathbb{R})$ *is a solution of inequality* (14).
3. *The function* $v\in\mathcal{SC}^{\rho}([0,\infty),\mathbb{R})$ *is a solution of IVP of the scalar SFDEGCFF* (13).

   Then $\varsigma_0 > v_0$ *implies* $\varsigma(t) > v(t)$ *for* $t\ge 0$.

The proof of Lemma 11 is similar to the proof given in Lemma 10, so we omit it.

4. Strict Stability

The main goal of this paper is the study of strict stability of the nonlinear SFDEGCFF (9).

We use the definition of strict stability for ordinary differential equations introduced in [1] to SFDEGCFF (9).

Note that if $x\in\mathbb{R}^n$, we will use the quadratic norm $||x||$ in $\mathbb{R}^n$.

Definition 3. *The zero solution of the nonlinear SFDEGCFF* (9) *is* strictly stable *if for any* $\epsilon_1 > 0$ *there exists* $\delta_1=\delta_1(\epsilon_1)>0$ *and for any* $\delta_2=\delta_2(\epsilon_1)$, $\delta_2\in(0,\delta_1)$ *there exists*

$\epsilon_2 = \epsilon_2(\delta_2) \in (0, \delta_2)$ such that for any initial value $y^{(0)} \in \mathbb{R}^n$, $y^{(0)} = (y_1^{(0)}, y_2^{(0)}, \dots, y_n^{(0)})$, the inequalities $\delta_2 < |y_k^{(0)}| < \delta_1$, $k = 1, 2, \dots, n$, imply $\epsilon_2 < ||y(t)|| < \epsilon_1$ for $t \geq 0$ where $y(.) \in \mathcal{SC}^\rho([0,\infty), \mathbb{R}^n)$ is a solution of the nonlinear SFDEGCFF (9) with initial value $y^{(0)}$.

The applied functions in the fractional derivative and the switching rule have an influence on the strict stability properties of SFDEGCFF.

Example 1. *Suppose $n = 1$, $m = 3$, $\mathcal{M} = \{1,2,3\}$, and the functions $\mathbb{F}_j(t,x) = \lambda_j x$, $t \geq 0$, $x \in \mathbb{R}$, $j = 1,2,3$, where the constants $\lambda_j \in \mathbb{R}$, $j = 1,2,3$, are given.*

Let the points $\Upsilon_i = i$, $i = 0,1,2,\dots$, and $\mathbb{K}_{3k} = 1, \mathbb{K}_{3k+1} = 2, \mathbb{K}_{3k+2} = 3$, $k = 0,1,2,\dots$.

Consider the switching rule $\sigma(t) = \mathbb{K}_k$ for $k \leq t < k+1$, $k = 0,1,2,3,\dots$.

Let $\rho = 0.3$ and $\theta_k \in \mathcal{P}([k,k+1])$, $k = 0,1,2,3,\dots$.

Consider the scalar linear SFDEGCFF

$$\begin{aligned} &{}^C D_k^{0.3,\theta_k} v(t) = \lambda_{\mathbb{K}_k} v, \text{ for } t \in (k,k+1],\ \ k = 0,1,2,\dots, \\ &v(k-) = v(k+),\ k = 1,2,\dots, \\ &v(0) = v_0, \end{aligned} \tag{16}$$

with $v_0 \in \mathbb{R}$ and a solution (see Lemma 7)

$$v(t) = v_0 \left(\prod_{i=1}^{k} E_{0.3}(\lambda_{\mathbb{K}_i} (\theta_i(i) - \theta_i(i-1))^{0.3}) \right) E_{0.3}(\lambda_{\mathbb{K}_k} (\theta_k(t) - \theta_k(k))^{0.3}),\ \ t \in (k,k+1].$$

The functions $\theta_i(.)$ will be defined later.

Case 1. Let the functions $\theta_k(t) = \theta(t)$ with $\theta(t) = \frac{t+1}{t+2}$, $t \geq 0$. We have $\theta_i(i) - \theta_i(i-1) = \frac{i+1}{i+2} - \frac{i}{i+1} \in (0, 0.5)$.

Case 1.1. Let $\lambda_j = 1$, $j = 1,2,3$. Then

$$1 = E_{0.3}(0) \leq E_{0.3}((\theta(i) - \theta(i-1))^{0.3}) < E_{0.3}(0.5^{0.3}) \approx 4.28455,\ \ i = 1,2,\dots.$$

The solution of (30) is

$$v(t) = v_0 \left(\prod_{i=1}^{k-1} E_{0.3}\left(\left(\frac{i+1}{i+2} - \frac{i}{i+1} \right)^{0.3} \right) \right) E_{0.3}\left(\left(\frac{t+1}{t+2} - \frac{k+1}{k+2} \right)^{0.3} \right),\ \ t \in (k,k+1].$$

The solution is unbounded (see Figure 1) and is not strictly stable.

Case 1.2. Let $\lambda_j = (-1)^j$, $j = 1,2,3$. The solution of (30) for $t \in (k,k+1]$, $k = 0,1,2,\dots$ is

$$v(t) = v_0 \left(\prod_{i=1}^{k-1} E_{0.3}\left((-1)^{\mathbb{K}_i} \left(\frac{i+1}{i+2} - \frac{i}{i+1} \right)^{0.3} \right) \right) E_{0.3}\left((-1)^{\mathbb{K}_k} \left(\frac{t+1}{t+2} - \frac{k+1}{k+2} \right)^{0.3} \right).$$

The solution is unbounded on the whole interval (see Figure 2) and is not strictly stable.

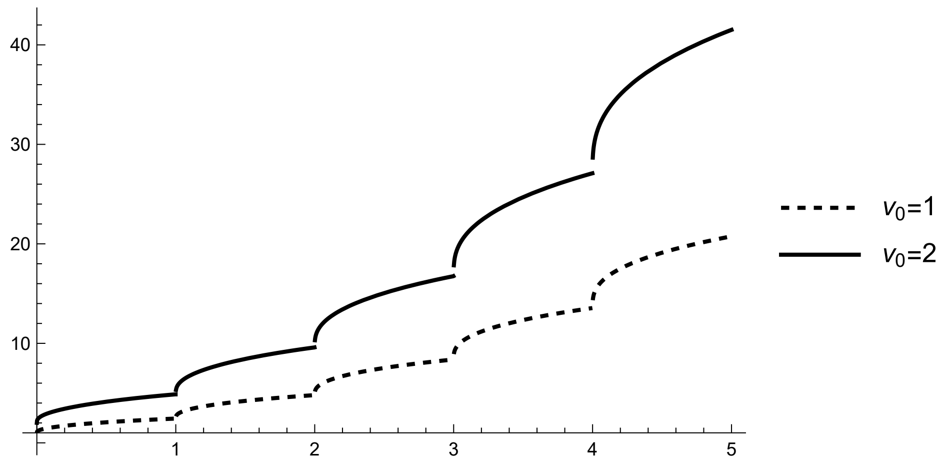

Figure 1. Graphs of sample path solutions of (30) with various initial values, Case 1.1.

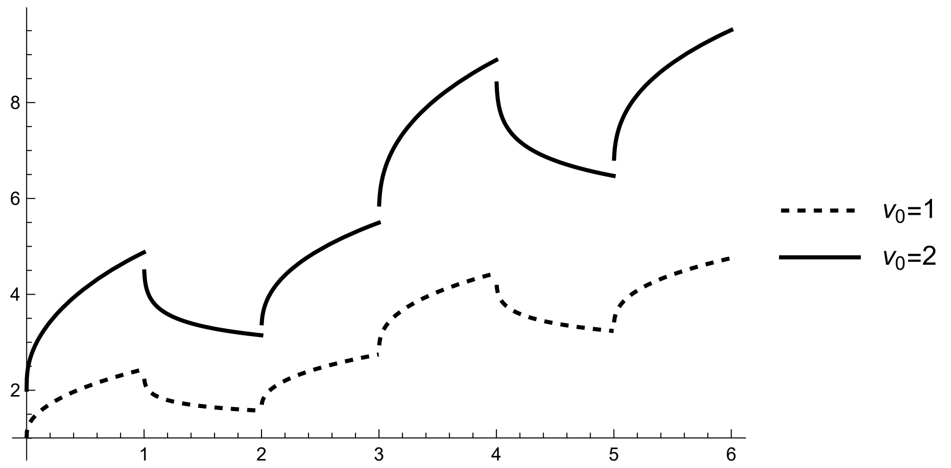

Figure 2. Graphs of sample path solutions of (30) with various initial values, Case 1.2.

Case 1.3. Let $\lambda_j = -1,\ j = 1,2,3$. The solution of (30) is

$$v(t) = v_0\left(\prod_{i=1}^{k-1} E_{0.3}\left(-\left(\frac{i+1}{i+2} - \frac{i}{i+1}\right)^{0.3}\right)\right)E_{0.3}\left(-\left(\frac{t+1}{t+2} - \frac{k+1}{k+2}\right)^{0.3}\right) \leq |v_0|.$$

Then the solution is stable but not strictly stable. The solution approaches 0 as the time goes to infinity (see Figure 3).

Case 1.4. Let $\lambda_j = 1,\ j = 1,2,3$ and the switching rule is not activated for $t > 5$. Then for $t > 5$, the solution is

$$v(t) = v_0\left(\prod_{i=1}^{4} E_{0.3}\left(\left(\frac{i+1}{i+2} - \frac{i}{i+1}\right)^{0.3}\right)\right)E_{0.3}\left(\left(\frac{t+1}{t+2} - \frac{6}{-}\right)^{0.3}\right),\ \ t > 5,$$

and since $\lim_{t\to\infty} E_{0.3}\left(\left(\frac{t+1}{t+2} - \frac{6}{7}\right)^{0.3}\right) = E_{0.3}\left(\left(\frac{1}{7}\right)^{0.3}\right) < \infty$ the solution is strictly stable (see Figure 4).

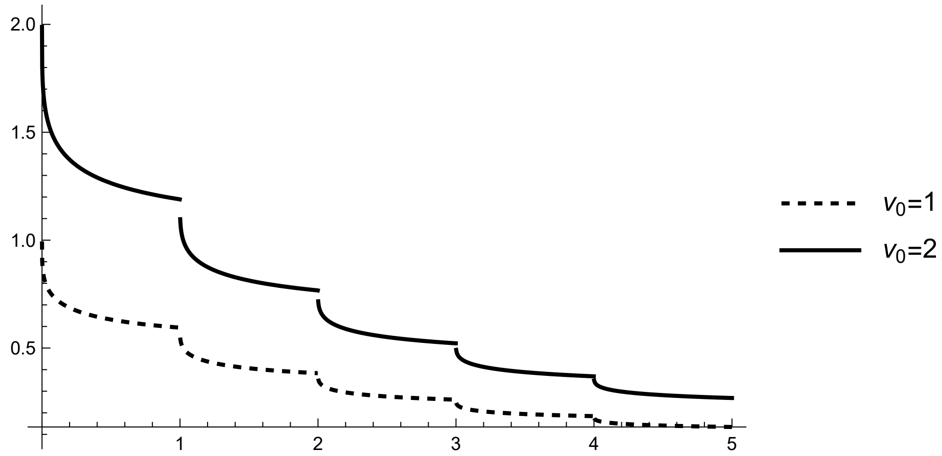

Figure 3. Graphs of sample path solutions of (30) with various initial values, Case 1.3.

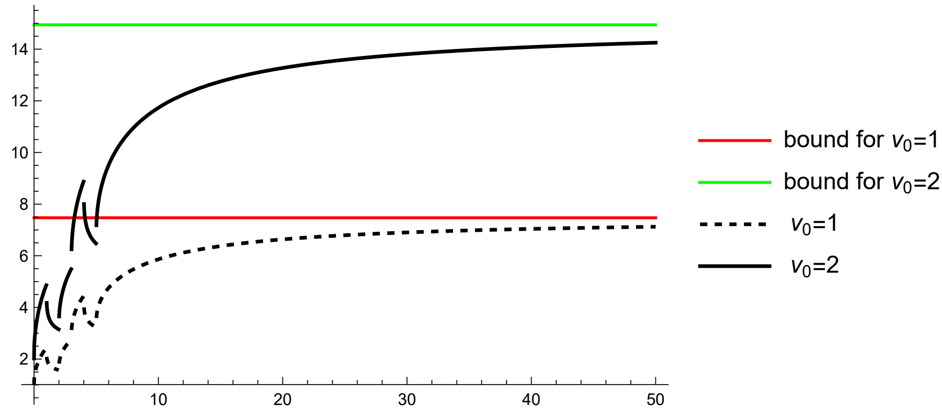

Figure 4. Graphs of sample path solutions of (30) with various initial values, Case 1.4.

Case 2. Let $\theta_k(t) = t$. In this case, GCFF is reduced to a Caputo fractional derivative. Then, $\theta_i(i) - \theta_i(i-1) = 1,\ i = 0,1,2,\ldots$.

Case 2.1. Let $\lambda_j = 1,\ j = 1,2,3$. Then the solution of (30) *is*

$$v(t) = v_0(E_{0.3}(1))^{k-1} E_{0.3}\Big((t-k)^{0.3}\Big),\ \ t \in (k, k+1].$$

The solution is unbounded (see Figure 5). It is not strictly stable.

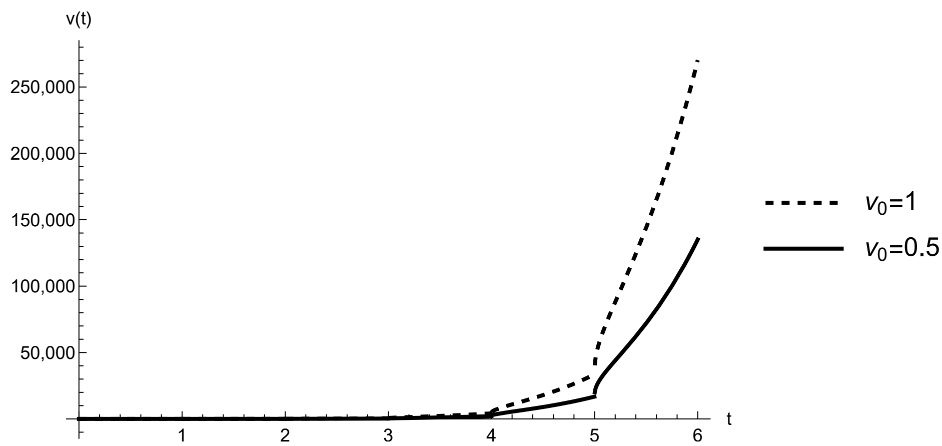

Figure 5. Graphs of sample path solutions of (30) with various initial values, Case 2.1.

Case 2.2. Let $\lambda_j = (-1)^j,\ j = 1,2,3$. The solution of (30) is

$$v(t) = v_0 \left(\prod_{i=1}^{k-1} E_{0.3}\Big((-1)^{\mathbb{K}_i}\Big) \right) E_{0.3}\Big((-1)^{\mathbb{K}_k}(t-k)^{0.3}\Big),\ \ t \in (k, k+1].$$

The solution is unbounded on the whole interval (see Figure 6) and is not strictly stable.

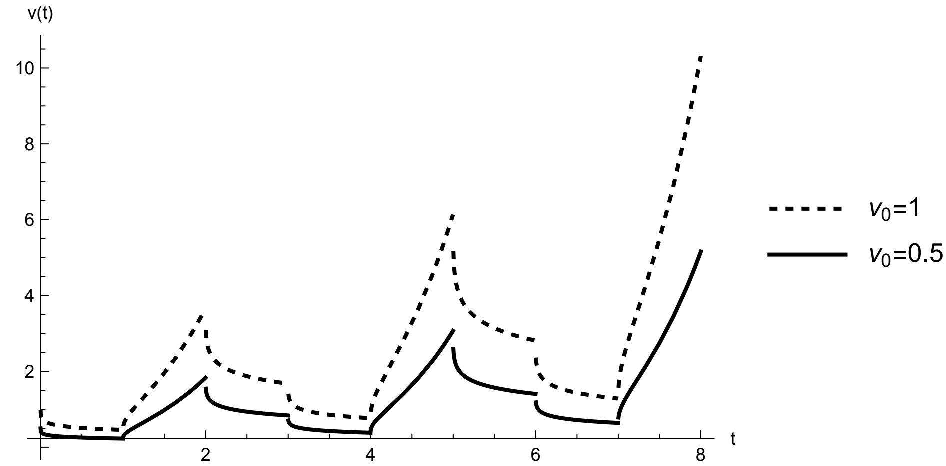

Figure 6. Graphs of sample path solutions of (30) with various initial values, Case 2.2.

Case 2.3. Let $\lambda_j = (-1)^j,\ j = 1,2,3$ and the switching rule is not activated for $t > 3$. The solution of (30) is

$$v(t) = v_0 E_{0.3}\Big((-1)^1\Big) E_{0.3}\Big((-1)^2\Big) E_{0.3}\Big((-1)^3\Big) E_{0.3}\Big((-1)^1(t-3)^{0.3}\Big),\ \ t > 3.$$

Then, since $\lim_{t\to\infty} E_{0.3}\Big(-(t-3)^{0.3}\Big) = 0$, the solution of (30) is approaching 0 as the time $t \to \infty$ (see Figure 7). The solution is not strictly stable.

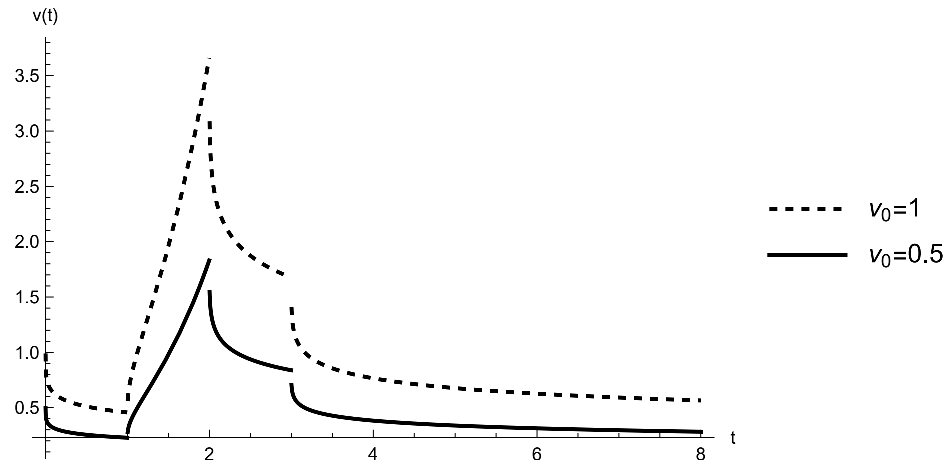

Figure 7. Graphs of sample path solutions of (30) with various initial values, Case 2.3.

Case 3. *We now change the switching rule* $\sigma(t) = k+1$, $t \in [k, k+1)$, $k = 0,1,2,3,\ldots$. *Let* $m = \infty$ *and* $\mathcal{M} = \{0,1,2,3,\ldots\}$, $\theta_k(t) = t$, $\lambda_k = (-0.5)^k$, $k = 0,1,2,\ldots$. *The solution of* (30) *is*

$$v(t) = v_0 \prod_{i=0}^{k-1} E_{0.3}\Big((-0.5)^{i+1}\Big) E_{0.3}\Big((-0.5)^k (t-k)^{0.3}\Big), \; t \in (k, k+1], \; k = 0,1,2,\ldots.$$

Then $\lim_{k\to\infty}(-0.5)^k = 0$, $\lim_{k\to\infty} E_{0.3}((-0.5)^k) = 1$, *and* $E_{0.3}((-0.5)^{k+1}) \geq E_{0.3}(-1)$. *The solution of* (30) *satisfies* $|v_0|E_{0.3}(-1) \leq |v(t)| \leq |v_0|$ *and the zero solution is strictly stable (see Figure 8).*

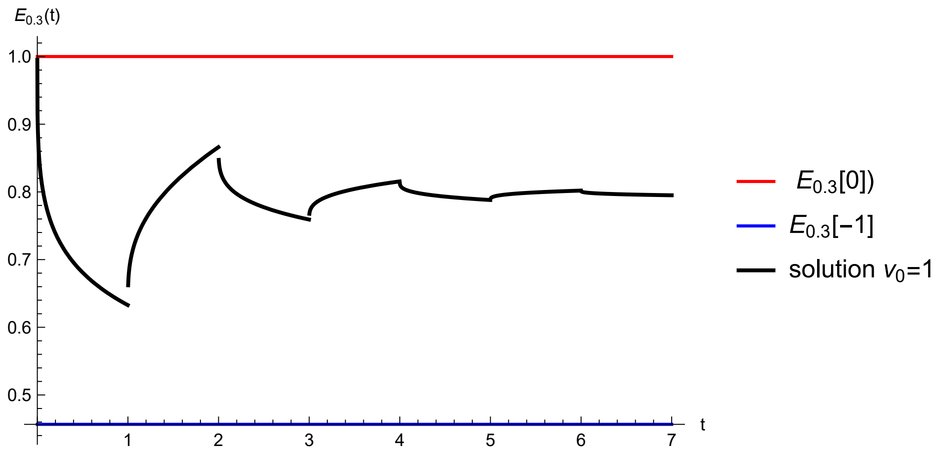

Figure 8. Graphs of sample path solutions of (30) with $v_0 = 1$, Case 3.

5. Main Results

We obtain sufficient conditions for strict stability of SFDEGCFF (9) by Lyapunov functions.

Definition 4. *The function* $\mathcal{V}(.) : \mathbb{R}^n \to [0,\infty)$ *is from the class* $\Lambda(\mathbb{R}^n)$ *if* $\mathcal{V}(.) \in C(\mathbb{R}^n, [0,\infty))$ *is continuously differentiable.*

We will use the followings set:

$$\mathcal{K} = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] : a \text{ is strictly increasing and } a(0) = 0\}.$$

Theorem 1. *Suppose the following conditions are satisfied:*

1. $\rho \in (0,1)$ *and* $\theta_k \in \mathcal{P}([\Upsilon_k, \Upsilon_{k+1}])$, $k = 0,1,2,\ldots$..
2. *Condition (H) holds.*
3. *There exist convex functions* $\mathcal{V}_j^{(1)} \in \Lambda(\mathbb{R}^n)$, $j \in \mathcal{M}$, *such that* $\mathcal{V}_{\mathbb{K}_0}^{(1)}(0) = 0$ *and*
   - (i) *there exist functions* $a_j \in \mathcal{K}$, $j \in \mathcal{M}$ *such that for all* $j \in \mathcal{M}$, *the inequalities* $a_j(||x||) \leq \mathcal{V}_j^{(1)}(x)$, $x \in \mathbb{R}^n$, *hold;*
   - (ii) *there exist functions* $\mu_j \in C([0,\infty),[0,\infty))$, $j \in \mathcal{M}$, *such that for any function* $\mathbf{y}^*(t) \in \mathcal{SC}^\rho([0,\infty), \mathbb{R}^n)$, *the inequalities*

$$\sum_{i=1}^{n} \frac{\partial}{\partial \mathbf{y}_i} \mathcal{V}_{\mathbb{K}_k}^{(1)}(\mathbf{y}^*(t)) \mathbb{F}_{\mathbb{K}_k,i}(t, \mathbf{y}^*(t)) < -\mu_{\mathbb{K}_k}\Big(\mathcal{V}_{\mathbb{K}_k}^{(1)}(\mathbf{y}^*(t))\Big), \; t \in (\Upsilon_k, \Upsilon_{k+1}], \; k = 0,1,2,\ldots,$$

     hold, where $\mathbb{K}_k \in \mathcal{M}$ *are values of the switching rule;*
   - (iii) *there exist constants* $\nu_j \geq 1$, $j \in \mathcal{M}$, *such that* $\mathcal{V}_{\mathbb{K}_k}^{(1)}(x) \leq \nu_{\mathbb{K}_{k+1}} \mathcal{V}_{\mathbb{K}_{k+1}}^{(1)}(x)$ *for any* $x \in \mathbb{R}^n$ *and* $k = 0,1,2,3\ldots$.

4. *There exist functions* $\mathcal{V}_j^{(2)} \in \Lambda(\mathbb{R}^n)$, $j \in \mathcal{M}$, *and for any* $\hat{\delta} > 0$, *there exists* $\hat{\varepsilon} > 0$ *such that for any* $x \in \mathcal{Q}_{\hat{\delta}} = \{x \in \mathbb{R}^n,\ x = (x_1, x_2, \ldots, x_n) :\ |x_i| > \hat{\delta}, i = 1,2,\ldots,n\}$, *the inequality* $V_{\mathbb{K}_0}^{(2)}(x) > \hat{\varepsilon}$ *holds and*
   - (iv) *there exist functions* $b_j \in \mathcal{K}$, $j \in \mathcal{M}$, *such that the inequalities* $\mathcal{V}_j^{(2)}(x) \le b_j(||x||)$ *for* $x \in \mathbb{R}^n$, $j \in \mathcal{M}$, *hold;*
   - (v) *there exist functions* $\eta_j \in C([0,\infty),[0,\infty))$, $j \in \mathcal{M}$, *such that for any solution* $\mathbf{y}^*(t) \in \mathcal{SC}^\rho([0,\infty),\mathbb{R}^n)$ *of* (9) *the inequalities*

$$ {}^C D_{\Upsilon_k}^{\rho,\theta_k} \mathcal{V}_{\mathbb{K}_k}^{(2)}(\mathbf{y}^*(t)) > \eta_{\mathbb{K}_k}\left(\mathcal{V}_{\mathbb{K}_k}^{(2)}(\mathbf{y}^*(t))\right),\ t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0,1,2,\ldots, \tag{17} $$

   hold;
   - (vi) *there exist constants* $\gamma_j \in (0,1]$, $j \in \mathcal{M}$, *such that* $\mathcal{V}_{\mathbb{K}_k}^{(2)}(x) \ge \gamma_{\mathbb{K}_{k+1}} \mathcal{V}_{\mathbb{K}_{k+1}}^{(2)}(x)$ *for any* $x \in \mathbb{R}^n$ *and* $k = 0,1,2,3\ldots.$

Then the zero solution of the SFDEGCFF (9) *is strictly stable.*

Proof. Let $\varepsilon_1 > 0$ be an arbitrary number and $a_{\varepsilon_1} = \min_{k\in\mathcal{M}} a_k(\varepsilon_1)$. Since $\mathcal{V}_{\mathbb{K}_0}^{(1)}(0) = 0$, there exists a $\delta_3 \in (0, a(\varepsilon_1))$ such that $\mathcal{V}_{\mathbb{K}_0}^{(1)}(\mathbf{x}) < 0.5a_{\varepsilon_1}$ for $\mathbf{x} \in \mathbb{R}^n$, $||\mathbf{x}|| < \delta_3$.

Let $\delta_1 = \frac{\delta_3}{\sqrt{n}}$ and $\delta_2 \in (0, \delta_1)$ be an arbitrary number. Choose the initial value $y^{(0)} \in \mathbb{R}^n$, $y^{(0)} = (y_1^{(0)}, y_2^{(0)}, \ldots, y_n^{(0)})$, of SFDEGCFF (9), such that

$$ \delta_2 < |y_j^{(0)}| < \delta_1,\ j = 1,2,\ldots,n, \tag{18} $$

and let $\mathbf{y}^*(t) \in \mathcal{SC}^\rho([0,\infty),\mathbb{R}^n)$ be a solution of (9) for the initial value $y^{(0)}$.

From (18), it follows that $(y_j^{(0)})^2 < \delta_1^2$, $j = 1,2,\ldots,n$, and $||y^{(0)}|| < \sqrt{n}\delta_1 = \delta_3$. Therefore, $\mathcal{V}_{\mathbb{K}_0}^{(1)}(y^{(0)}) < 0.5a_{\varepsilon_1}$.

The conditions of Lemma 6 are satisfied on the interval $[\Upsilon_k, \Upsilon_{k+1}]$, $k = 0,1,2,\ldots,$ because $\mathbf{y}^*(t) \in C^\rho([\Upsilon_k, \Upsilon_{k+1}], \mathbb{R}^n, \theta_k)$, $\mathcal{V}_{\mathbb{K}_k}^{(1)} \in C^1(\mathbb{R}^n, [0,\infty))$ is convex, and therefore, according to Lemma 6, with $a = \Upsilon_k$, $\theta = \theta_k$, $x(.) = \mathbf{y}^*(.)$, $V(x) = \mathcal{V}_{\mathbb{K}_k}^{(1)}(x)$, the inequality ${}^C D_{\Upsilon_k}^{\rho,\theta_k} \mathcal{V}_{\mathbb{K}_k}^{(1)}(\mathbf{y}^*(t)) \le \sum_{i=1}^n \frac{\partial}{\partial x_i} \mathcal{V}_{\mathbb{K}_k}^{(1)}(\mathbf{y}^*(t))\, {}^C D_{\Upsilon_k}^{\rho,\theta_k} \mathbf{y}_i^*(t)$ holds on $(\Upsilon_k, \Upsilon_{k+1}]$. Then apply condition 3(ii); note that the function $\mathbf{y}^*(t)$ is a solution of (9), i.e., ${}^C D_{\Upsilon_k}^{\rho,\theta_k}$ $\mathbf{y}_i^*(t) = \mathbb{F}_{\mathbb{K}_k,i}(t, \mathbf{y}^*(t))$, and so we have

$$ \begin{aligned} {}^C D_{\Upsilon_k}^{\rho,\theta_k} \mathcal{V}_{\mathbb{K}_k}^{(1)}(\mathbf{y}^*(t)) &\le \sum_{i=1}^n \frac{\partial}{\partial \mathbf{y}_i} \mathcal{V}_{\mathbb{K}_k}^{(1)}(\mathbf{y}^*(t)) \mathbb{F}_{\mathbb{K}_k,i}(t, \mathbf{y}^*(t)) \\ &< -\mu_{\mathbb{K}_k}\left(\mathcal{V}_{\mathbb{K}_k}^{(1)}(\mathbf{y}^*(t))\right),\ t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0,1,2,3,\ldots. \end{aligned} $$

Consider the scalar SFDEGCFF (13) with $\mathbb{G}_k(.) = -\mu_k(.) \le 0$ and $v_0 = \mathcal{V}_{\mathbb{K}_0}^{(1)}$ $(y^{(0)}) + 0.5a_{\varepsilon_1} > 0$. Denote its solution by $v(t)$. According to Lemma 8, the inequality

$$ v(t) \le v_0 = V_{\mathbb{K}_0}^{(1)}(y^{(0)}) + 0.5a_{\varepsilon_1} < a_{\varepsilon_1},\ \ t \ge 0 \tag{19} $$

holds.

Define the function $\varsigma_1(t) = \mathcal{V}_{\mathbb{K}_k}^{(1)}(\mathbf{y}^*(t))$, $t \in (\Upsilon_k, \Upsilon_{k+1}]$, $k = 0,1,2,\ldots.$ From inequalities (19), condition 3(iii), $\frac{1}{\nu_k} \le 1$, and $\varsigma_1(\Upsilon_k+) = \mathcal{V}_{\mathbb{K}_k}^{(1)}(\mathbf{y}^*(\Upsilon_k+)) \le \frac{1}{\nu_k}\mathcal{V}_{\mathbb{K}_{k-1}}^{(1)}$ $(\mathbf{y}^*(\Upsilon_k+)) = \frac{1}{\nu_k}\mathcal{V}_{\mathbb{K}_{k-1}}^{(1)}(\mathbf{y}^*(\Upsilon_k-)) \le \mathcal{V}_{\mathbb{K}_{k-1}}^{(1)}(\mathbf{y}^*(\Upsilon_k-)) = \varsigma_1(\Upsilon_k-)$, it follows that the function $\varsigma_1(.) \in \mathcal{SC}^\rho([0,\infty),\mathbb{R})$ satisfies (14) with $\mathbb{G}_k(.) = -\mu_k(.) \le 0$ and $\varsigma_1(0) = \mathcal{V}_{\mathbb{K}_0}^{(1)}$

$(y^{(0)}) < \mathcal{V}^{(1)}_{\mathbb{K}_0}(y^{(0)}) + 0.5a_{\varepsilon_1} = v_0$. According to Lemma 10, the inequality $\varsigma_1(t) < v(t)$ for $t \geq 0$ holds. Therefore, from condition 3(i) and inequality (19), we obtain for any $k = 0, 1, 2, \ldots$

$$a_{\mathbb{K}_k}(||\mathbf{y}^*(t)||) \leq \mathcal{V}^{(1)}_{\mathbb{K}_k}(\mathbf{y}^*(t)) = \varsigma_1(t) < v(t) < a_{\varepsilon_1} \leq a_{\mathbb{K}_k}(\varepsilon_1), \quad t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0, 1, 2, \ldots,$$

or

$$||\mathbf{y}^*(t)|| < \varepsilon_1, \quad t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0, 1, 2, \ldots. \tag{20}$$

From condition 4, it follows that for $\hat{\delta} = \delta_2 > 0$, there exists $\hat{\varepsilon} > 0$ such that for any $x \in \mathcal{Q}_{\delta_2} = \{x \in \mathbb{R}^n,\ x = (x_1, x_2, \ldots, x_n) : |x_i| > \delta_2, i = 1, 2, \ldots, n\}$, the inequality $V^{(2)}_{\mathbb{K}_0}(x) > \hat{\varepsilon}$ holds. From (18), it follows that $y^{(0)} \in \mathcal{Q}_{\delta_2}$ and $V^{(2)}_{\mathbb{K}_0}(y^{(0)}) > \hat{\varepsilon}$. Let $\varepsilon_2 = \frac{\hat{\varepsilon}}{1.5}$ and $\varepsilon_3 = \min_{k \in \mathcal{M}} b_k^{-1}(\varepsilon_2)$.

Consider the scalar SFDEGCFF (13) with $\mathbb{G}_k(.) = \eta_k(.) \geq 0$ and $v_0 = \mathcal{V}^{(2)}_{\mathbb{K}_0}(y^{(0)}) - 0.5\varepsilon_2 > 1.5\varepsilon_2 - 0.5\varepsilon_2 = \varepsilon_2 > 0$. Denote its solution by $w(t)$. According to Lemma 9, the inequality

$$w(t) \geq v_0 = \mathcal{V}^{(2)}_{\mathbb{K}_0}(y^{(0)}) - 0.5\varepsilon_2 > \varepsilon_2, \ \ t \geq 0 \tag{21}$$

holds.

Define the function $\varsigma_2(t) = \mathcal{V}^{(2)}_{\mathbb{K}_k}(\mathbf{y}^*(t)),\ t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0, 1, 2, \ldots$. From condition 4(vi), $\frac{1}{\gamma_k} \geq 1$, and $\varsigma_2(\Upsilon_k+) = \mathcal{V}^{(2)}_{\mathbb{K}_k}(\mathbf{y}^*(\Upsilon_k+)) \geq \frac{1}{\gamma_k}\mathcal{V}^{(2)}_{\mathbb{K}_{k-1}}(\mathbf{y}^*(\Upsilon_k+)) = \frac{1}{\gamma_k} \mathcal{V}^{(2)}_{\mathbb{K}_{k-1}}(\mathbf{y}^*(\Upsilon_k-)) \geq \mathcal{V}^{(2)}_{\mathbb{K}_{k-1}}(\mathbf{y}^*(\Upsilon_k-)) = \varsigma_2(\Upsilon_k-)$ it follows that the function $\varsigma_2(.) \in \mathcal{SC}^\rho((0,\infty), \mathbb{R})$ satisfies (15) with $\mathbb{G}_k(.) = \eta_k(.) \geq 0$ and $\varsigma_0 = \varsigma_2(0) = \mathcal{V}^{(2)}_{\mathbb{K}_0}(y^{(0)}) > \mathcal{V}^{(2)}_{\mathbb{K}_0}(y^{(0)}) - 0.5\varepsilon_2 = v_0$. According to Lemma 11, the inequality $\varsigma_2(t) > w(t),\ t \geq 0$, holds. From condition 4(iv) and inequality (21), we obtain

$$b_{\mathbb{K}_k}(||\mathbf{y}^*(t)||) \geq \mathcal{V}^{(2)}_{\mathbb{K}_k}(\mathbf{y}^*(t)) = \varsigma_2(t) > w(t) > \varepsilon_2 \geq b_{\mathbb{K}_k}(\varepsilon_3), \quad t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0, 1, 2, \ldots.$$

Therefore,

$$||\mathbf{y}^*(t)|| > \varepsilon_3, \quad t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0, 1, 2, \ldots. \tag{22}$$

Inequalities (18), (20), and (22) prove the claim. □

Remark 3. *In the special case when there exists one Lyapunov function in condition 3 of Theorem 1 (i.e., there exist a convex function $\mathcal{V}^{(1)}(x) \equiv \mathcal{V}^{(1)}_j(x)$, $\mathcal{V}^{(1)}(0) = 0$ such that conditions 3(i) and 3(ii) are satisfied), we do not need condition 3(ii). A similar remark applies to conditions 4 (vi).*

In the general case, we need to compare any two consecutive Lyapunov functions $\mathcal{V}^{(1)}_{\mathbb{K}_{k-1}}(x)$ and $\mathcal{V}^{(1)}_{\mathbb{K}_k}(x)$ because in Theorem 1, at any switching time Υ_k, we consider the Lyapunov function $\mathcal{V}^{(1)}_{\mathbb{K}_{k-1}}(\mathbf{y}^*(t))$ for $t \in (\Upsilon_{k-1}, \Upsilon_k]$ and the Lyapunov function $\mathcal{V}^{(1)}_{\mathbb{K}_k}(\mathbf{y}^*(t))$ for $t \in (\Upsilon_k, \Upsilon_{k+1}]$, so a comparison is needed as in condition 3(iii). A similar remark applies to condition 4(vi).

In the case of linear functions in conditions 3(ii) and 4(v) for the Lyapunov functions, we obtain the following result:

Corollary 4. *Suppose all the conditions of Theorem 1 are satisfied with $\mu_k(v) = \alpha_k v$, and $\eta_k(v) = \beta_k v$ for $v \in \mathbb{R}$, $k \in \mathcal{M}$, in conditions 3(ii) and 4(v), respectively, where $\alpha_k > 0, \beta_k > 0$ are constants.*

Then, the zero solution of the SFDEGCFF (9) is strictly stable.

In the case of zero bounds in condition 3(ii) and 4(v) for the Lyapunov functions, we obtain the following result:

Corollary 5. *Suppose all the conditions of Theorem 1 are satisfied with conditions 3(ii) and 4(v) replaced by*

- 3(ii) *for any function* $\mathbf{y}^*(t) \in \mathcal{SC}^\rho([0,\infty),\mathbb{R}^n)$, *the inequalities*

$$\sum_{i=1}^{n} \frac{\partial}{\partial \mathbf{y}_i} \mathcal{V}^{(1)}_{\mathbb{K}_k}(\mathbf{y}^*(t))\mathbb{F}_{\mathbb{K}_k,i}(t,\mathbf{y}^*(t)) \le 0,\ t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0,1,2,\dots,$$

hold, where $\mathbb{K}_k \in \mathcal{M}$ *are values of the switching rule;*

- 4(v) *for any solution* $\mathbf{y}^*(t) \in \mathcal{SC}^\rho([0,\infty),\mathbb{R}^n)$ *of* (9), *the inequalities*

$${}^C D^{\rho,\theta_k}_{\Upsilon_k} \mathcal{V}^{(2)}_{\mathbb{K}_k}(\mathbf{y}^*(t)) \ge 0,\ t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0,1,2,\dots,$$

hold;

respectively.

Then, the zero solution of the SFDEGCFF (9) *is strictly stable.*

Proof. We will follow the ideas in the proof of Theorem 1.

Let $\varepsilon_1 > 0$ be an arbitrary number and $a_{\varepsilon_1} = \min_{k\in\mathcal{M}} a_k(\varepsilon_1)$. Since $\mathcal{V}^{(1)}_{\mathbb{K}_0}(0) = 0$, there exists a $\delta_3 \in (0, a(\varepsilon_1))$ such that $\mathcal{V}^{(1)}_{\mathbb{K}_0}(\mathbf{x}) < a_{\varepsilon_1}$ for $\mathbf{x} \in \mathbb{R}^n,\ ||\mathbf{x}|| < \delta_3$.

Let $\delta_2 \in (0,\delta_1)$ be an arbitrary number and $\delta_1 = \frac{\delta_3}{\sqrt{n}}$. Choose the initial value $y^{(0)} \in \mathbb{R}^n,\ y^{(0)} = (y^{(0)}_1, y^{(0)}_2, \dots, y^{(0)}_n)$, of SFDEGCFF (9), such that

$$\delta_2 < |y^{(0)}_j| < \delta_1,\ j = 1,2,\dots,n, \tag{23}$$

and let $\mathbf{y}^*(t) \in \mathcal{SC}^\rho([0,\infty),\mathbb{R}^n)$ be a solution of (9) for the initial value $y^{(0)}$. From inequalities (23), we have $||y^{(0)}|| < \sqrt{n}\delta_1 = \delta_3$, and therefore, $\mathcal{V}^{(1)}_{\mathbb{K}_0}(y^{(0)}) < a_{\varepsilon_1}$.

The conditions of Lemma 6 are satisfied on the interval $[\Upsilon_k, \Upsilon_{k+1}],\ k = 0,1,2,\dots,$ because $\mathbf{y}^*(t) \in C^\rho([\Upsilon_k,\Upsilon_{k+1}],\mathbb{R}^n,\theta_k)$, $\mathcal{V}^{(1)}_{\mathbb{K}_k} \in C^1(\mathbb{R}^n,[0,\infty))$ is convex and therefore according to Lemma 6 with $a = \Upsilon_k,\ \theta = \theta_k,\ \ x(.) = \mathbf{y}^*(.),\ V(x) = \mathcal{V}^{(1)}_{\mathbb{K}_k}(x)$ the inequality ${}^C D^{\rho,\theta_k}_{\Upsilon_k} \mathcal{V}^{(1)}_{\mathbb{K}_k}(\mathbf{y}^*(t)) \le \sum_{i=1}^n \frac{\partial}{\partial x_i}\mathcal{V}^{(1)}_{\mathbb{K}_k}(\mathbf{y}^*(t))\, {}^C D^{\rho,\theta_k}_{\Upsilon_k} \mathbf{y}^*_i(t)$ holds on $(\Upsilon_k, \Upsilon_{k+1}]$. From condition 3(ii) of Corollary 5, we have

$${}^C D^{\rho,\theta_k}_{\Upsilon_k} \mathcal{V}^{(1)}_{\mathbb{K}_k}(\mathbf{y}^*(t)) \le \sum_{i=1}^{n} \frac{\partial}{\partial \mathbf{y}_i} \mathcal{V}^{(1)}_{\mathbb{K}_k}(\mathbf{y}^*(t))\mathbb{F}_{\mathbb{K}_k,i}(t,\mathbf{y}^*(t)) \le 0,\ \ t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0,1,2,3,\dots.$$

Therefore, for any $k = 0,1,2,\dots,$ there exists a function $Y_k \in C([\Upsilon_k,\Upsilon_{k+1}],[0,\infty))$ such that ${}^C D^{\rho,\theta_k}_{\Upsilon_k} \mathcal{V}^{(1)}_{\mathbb{K}_k}(\mathbf{y}^*(t)) + Y_k(t) = 0,\ \ t \in (\Upsilon_k, \Upsilon_{k+1}]$, or ${}^C D^{\rho,\theta_k}_{\Upsilon_k} \mathcal{V}^{(1)}_{\mathbb{K}_k}(\mathbf{y}^*(t)) = -Y_k(t)$ for $t \in (\Upsilon_k, \Upsilon_{k+1}]$. Define the function $\varsigma_1(t) = \mathcal{V}^{(1)}_{\mathbb{K}_k}(\mathbf{y}^*(t)),\ t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0,1,2,\dots.$ According to Corollary 2, we have $\varsigma_1(t) \le \mathcal{V}^{(1)}_{\mathbb{K}_0}(y^{(0)}) < a_{\varepsilon_1}$.

Therefore, from condition 3(i), we obtain for any $k = 0,1,2,\dots$

$$a_{\mathbb{K}_k}(||\mathbf{y}^*(t)||) \le \mathcal{V}^{(1)}_{\mathbb{K}_k}(\mathbf{y}^*(t)) = \varsigma_1(t) < a_{\varepsilon_1} \le a_{\mathbb{K}_k}(\varepsilon_1),\ \ \ t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0,1,2,\dots,$$

or

$$||\mathbf{y}^*(t)|| < \varepsilon_1,\ \ \ t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0,1,2,\dots. \tag{24}$$

From condition 4 of Theorem 1, it follows that for $\hat{\delta} = \delta_2 > 0$ there exists $\hat{\varepsilon} > 0$ such that for any $x \in \mathcal{Q}_{\delta_2} = \{x \in \mathbb{R}^n,\ x = (x_1, x_2, \dots, x_n) :\ |x_i| > \delta_2, i = 1, 2, \dots, n\}$ the inequality $V^{(2)}_{\mathbb{K}_0}(x) > \hat{\varepsilon}$ holds. From inequalities (23), it follows that $y^{(0)} \in \mathcal{Q}_{\delta_2}$ and $V^{(2)}_{\mathbb{K}_0}(y^{(0)}) > \hat{\varepsilon}$. Let $\varepsilon_2 = \tilde{\varepsilon}$ and $\varepsilon_3 = \min_{k \in \mathcal{M}} b_k^{-1}(\varepsilon_2)$.

Consider the scalar SFDEGCFF (13) with $\mathbb{G}_k(.) = \eta_k(.) \geq 0$ and $v_0 = \mathcal{V}^{(2)}_{\mathbb{K}_0}(y^{(0)}) > 0$. Denote its solution by $w(t)$. According to Lemma 9, the inequality

$$w(t) \geq \mathcal{V}^{(2)}_{\mathbb{K}_0}(y^{(0)}) > \varepsilon_2,\ \ t \geq 0 \tag{25}$$

holds.

Define the function $\varsigma_2(t) = \mathcal{V}^{(2)}_{\mathbb{K}_k}(\mathbf{y}^*(t)),\ t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0, 1, 2, \dots$. From condition 4(v) of Corollary 5, it follows that for any $k = 0, 1, 2, \dots$, there exists a function $Y_k \in C([\Upsilon_k, \Upsilon_{k+1}], [0, \infty))$ such that ${}^{C}D^{\rho,\theta_k}_{\Upsilon_k}\varsigma_2(t) - Y_k(t) = 0,\ \ t \in (\Upsilon_k, \Upsilon_{k+1}]$, or ${}^{C}D^{\rho,\theta_k}_{\Upsilon_k}\varsigma_2(t) = Y_k(t)$, for $t \in (\Upsilon_k, \Upsilon_{k+1}]$. According to Corollary 3, we have $\varsigma_2(t) \geq \mathcal{V}^{(2)}_{\mathbb{K}_0}(y^{(0)}) > \varepsilon_2$, $t \in (\Upsilon_k, \Upsilon_{k+1}]$.

From condition 4(iv) and inequality (25), we obtain

$$b_{\mathbb{K}_k}(||\mathbf{y}^*(t)||) \geq \mathcal{V}^{(2)}_{\mathbb{K}_k}(\mathbf{y}^*(t)) = \varsigma_2(t) \geq \mathcal{V}^{(2)}_{\mathbb{K}_0}(y^{(0)}) > \varepsilon_2 \geq b_{\mathbb{K}_k}(\varepsilon_3),\ \ \ t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0, 1, 2, \dots.$$

Therefore,

$$||\mathbf{y}^*(t)|| > \varepsilon_3,\ \ \ \ t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0, 1, 2, \dots. \tag{26}$$

Inequalities (23), (24), and (26) prove the claim. □

We now combine Corollary 4 and Corollary 5.

Corollary 6. *Suppose all the conditions of Theorem 1 are satisfied with conditions 3(ii) and 4(v) replaced by*

- *3(ii) for any function $\mathbf{y}^*(t) \in \mathcal{SC}^{\rho}([0, \infty), \mathbb{R}^n)$, the inequalities*

$$\sum_{i=1}^{n} \frac{\partial}{\partial \mathbf{y}_i} \mathcal{V}^{(1)}_{\mathbb{K}_k}(\mathbf{y}^*(t)) \mathbb{F}_{\mathbb{K}_k, i}(t, \mathbf{y}^*(t)) < -\alpha_k V^{(1)}_{\mathbb{K}_k}(\mathbf{y}^*(t)),\ t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0, 1, 2, \dots,$$

hold, where $\mathbb{K}_k \in \mathcal{M}$ are values of the switching rule and $\alpha_k > 0,\ k \in \mathcal{M}$, are constants;

- *4(v) for any solution $\mathbf{y}^*(t) \in \mathcal{SC}^{\rho}([0, \infty), \mathbb{R}^n)$ of (9), the inequalities*

$${}^{C}D^{\rho,\theta_k}_{\Upsilon_k} \mathcal{V}^{(2)}_{\mathbb{K}_k}(\mathbf{y}^*(t)) \geq 0,\ t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0, 1, 2, \dots,$$

hold, where $g\beta_k > 0,\ k = 0, 1, 2, \dots$, are constants;

respectively.

Then, the zero solution of the SFDEGCFF (9) is strictly stable.

Corollary 7. *Suppose all the conditions of Theorem 1 are satisfied with conditions 3(ii) and 4(v) replaced by*

- *3(ii) for any function $\mathbf{y}^*(t) \in \mathcal{SC}^{\rho}([0, \infty), \mathbb{R}^n)$, the inequalities*

$$\sum_{i=1}^{n} \frac{\partial}{\partial \mathbf{y}_i} \mathcal{V}^{(1)}_{\mathbb{K}_k}(\mathbf{y}^*(t)) \mathbb{F}_{\mathbb{K}_k, i}(t, \mathbf{y}^*(t)) \leq 0,\ t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0, 1, 2, \dots,$$

hold, where $\mathbb{K}_k \in \mathcal{M}$ are values of the switching rule;

- 4(v) *for any solution* $\mathbf{y}^*(t) \in \mathcal{SC}^{\rho}([0,\infty),\mathbb{R}^n)$ *of* (9), *the inequalities*

$$ {}^C D^{\rho,\theta_k}_{\Upsilon_k} \mathcal{V}^{(2)}_{\mathbb{K}_k}(\mathbf{y}^*(t)) > \beta_k \mathcal{V}^{(2)}_{\mathbb{K}_k}(\mathbf{y}^*(t)),\ t \in (\Upsilon_k, \Upsilon_{k+1}],\ k = 0,1,2,\ldots, $$

hold, where $\beta_k > 0,\ k \in \mathcal{M}$, *are constants,*

respectively.

Then, the zero solution of the SFDEGCFF (9) *is strictly stable.*

We will now illustrate some of the sufficient conditions obtained above.

Example 2. *Let* $n = 2,\ m = 2,\ \mathcal{M} = \{1,2\}$ *and the functions* $\mathbb{F}_k : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbb{F}_k(t,0) = 0,\ k = 1,2,\ \ldots,\ \mathbb{F}_1 = (\mathbb{F}_{1,1}, \mathbb{F}_{1,2}),\quad \mathbb{F}_2 = (\mathbb{F}_{2,1}, \mathbb{F}_{2,2})$, *are defined by*

$$ \mathbb{F}_{1,1}(t,x,y) = x, \quad \mathbb{F}_{1,2}(t,x,y) = -y + 2yx^2 \frac{e^{y^2}}{1 - e^{y^2}}. $$
$$ \mathbb{F}_{2,1}(t,x,y) = 2x, \quad \mathbb{F}_{2,2}(t,x,y) = -y + 1.5y^3x^2 \frac{e^{y^4}}{1 - e^{y^4}}. $$

Let the sequence $\{\mathbb{K}_k\}_{k=0}^{\infty}$ *be defined by*

$$ \mathbb{K}_k = \begin{cases} 1 & \text{if } k \text{ is an even,} \\ 2 & \text{if } k \text{ is an odd.} \end{cases} $$

Let $\Upsilon_k = k,\ k = 0,1,2,3,\ldots$ *and the switching rule* $\sigma(t) = \mathbb{K}_k \in \mathcal{M}$ *for* $k \le t < k+1$, $k = 0,1,2,3,\ldots$.

Let $\theta_k(t) = \theta(t) \in \mathcal{P}([0,\infty)),\ k = 1,2$ *and* $\rho = 0.3$.

Consider the following IVP for SFDEGCFF

$$ \begin{aligned} &{}^C D^{0.3,\theta(t)}_k x(t) = \mathbb{F}_{\sigma(t),1}(t,x(t),y(t)), \\ &{}^C D^{0.3,\theta(t)}_k y(t) = \mathbb{F}_{\sigma(t),2}(t,x(t),y(t)), \quad t \in (k,k+1],\ k = 0,1,2,\ldots, \\ &x(k-) = x(k+), \quad y(k-) = y(k+), \quad k = 1,2,\ldots, \\ &x(0) = x_0,\ y(0) = y_0, \end{aligned} \tag{27} $$

or its equivalent

$$ \begin{aligned} &{}^C D^{0.3,\theta(t)}_k x(t) = x(t), \quad t \in (k,k+1],\ k = 0,2,4\ldots, \\ &{}^C D^{0.3,\theta(t)}_k y(t) = -y(t) + 2y(t)x^2(t) \frac{e^{y^2(t)}}{1 - e^{y^2(t)}}, \quad t \in (k,k+1],\ k = 0,2,4\ldots, \\ &{}^C D^{0.3,\theta(t)}_k x(t) = 2x(t), \quad t \in (k,k+1],\ k = 1,3,5\ldots, \\ &{}^C D^{0.3,\theta(t)}_k y(t) = -y(t) + y^3(t)x^2(t) \frac{e^{y^4(t)}}{1 - e^{y^4(t)}}, \quad t \in (k,k+1],\ k = 1,3,5\ldots, \\ &x(k-) = x(k+), \quad y(k-) = y(k+), \quad k = 1,2,\ldots, \\ &x(0) = x_0,\ y(0) = y_0. \end{aligned} \tag{28} $$

Let $(x(t),y(t))$ *be a solution of* (27).

Consider the convex Lyapunov functions $\mathcal{V}^{(1)}_1(x,y) = x^2 + y^2$ *and* $\mathcal{V}^{(1)}_2(x,y) = 0.5x^2 + 2y^2 \ge 0.5(x^2 + y^2)$ *for* $x,y \in \mathbb{R}$. *Then condition 3(i) of Theorem 1 is satisfied with* $a_k(u) = 0.5u^2$.

Condition 3(iii) is satisfied with $\nu_{\mathbb{K}_1} = 1$ *and* $\nu_{\mathbb{K}_2} = 2$.

Note for $k = 0, 2, 4, 6, \ldots$ *applying* $y^2 \frac{e^{y^2}}{1-e^{y^2}} \leq -1$, *we have*

$$\begin{aligned}
&\frac{\partial}{\partial x}\mathcal{V}^{(1)}_{\mathbb{K}_k}(x(t), y(t))\mathbb{F}_{\mathbb{K}_k,1}(t, x(t), y(t)) + \frac{\partial}{\partial y}\mathcal{V}^{(1)}_{\mathbb{K}_k}(x(t), y(t))\mathbb{F}_{\mathbb{K}_k,2}(t, x(t), y(t)) \\
&= \frac{\partial}{\partial x}\mathcal{V}^{(1)}_{1}(x(t), y(t))\mathbb{F}_{1,1}(t, x(t), y(t)) + \frac{\partial}{\partial y}\mathcal{V}^{(1)}_{1}(x(t), y(t))\mathbb{F}_{1,2}(t, x(t), y(t)) \\
&= 2x^2(t) + 2y(t)(-y(t) + 2y(t)x^2(t)\frac{e^{y^2(t)}}{(1 - e^{y^2(t)})}) \\
&= 2x^2(t) - 2y^2(t) + 4y^2(t)x^2(t)\frac{e^{y^2(t)}}{1 - e^{y^2(t)}} \leq 2x^2(t) - 2y^2(t) - 4x^2(t) \leq 0,
\end{aligned} \tag{29}$$

i.e., condition 3(ii) of Corollary 5 is satisfied for the functions $\mathcal{V}^{(1)}_{\mathbb{K}_k}$, $k = 0, 2, 4, \ldots$.

Similarly, for $k = 1, 3, 5, 7, \ldots$, *applying* $y^4 \frac{e^{y^4}}{1-e^{y^4}} \leq -1$, *we have*

$$\begin{aligned}
&\frac{\partial}{\partial x}\mathcal{V}^{(1)}_{\mathbb{K}_k}(x(t), y(t))\mathbb{F}_{\mathbb{K}_k,1}(t, x(t), y(t)) + \frac{\partial}{\partial x}\mathcal{V}^{(1)}_{\mathbb{K}_k}(x(t), y(t))\mathbb{F}_{\mathbb{K}_k,2}(t, x(t), y(t)) \\
&= \frac{\partial}{\partial x}\mathcal{V}^{(1)}_{2}(x(t), y(t))\mathbb{F}_{2,1}(t, x(t), y(t)) + \frac{\partial}{\partial y}\mathcal{V}^{(1)}_{2}(x(t), y(t))\mathbb{F}_{2,2}(t, x(t), y(t)) \\
&= x^2(t) + 4y(t)(-y(t) + y^3(t)x^2(t)\frac{e^{y^4(t)}}{1 - e^{y^4(t)}}) \\
&= x^2(t) - 4y^2(t) + 4y^4(t)x^2(t)\frac{e^{y^4(t)}}{(1 - e^{y^4(t)})}) \\
&\leq x^2(t) - 4y^2(t) - 4x^2(t) \leq 0,
\end{aligned}$$

i.e., condition 3(ii) of Corollary 5 is satisfied for the functions $\mathcal{V}^{(1)}_{\mathbb{K}_k}$, $k = 1, 3, 5, 7, \ldots$.

Consider the Lyapunov functions $\mathcal{V}^{(2)}_1(x, y) = x^2$ *and* $\mathcal{V}^{(2)}_2(x, y) = e^{x^2} - 1$ *for* $x, y \in \mathbb{R}$; *i.e., condition 4(iv) of Theorem 1 is satisfied with* $b(v) = v^2$, $k = 0, 1, 2, \ldots$.

Also, condition 4 of Theorem 1 holds for the function $\mathcal{V}^{(2)}_1(x, y)$. *Indeed, let* $\tilde{\delta} > 0$ *be an arbitrary number,* $\tilde{\varepsilon} = \tilde{\delta}^2$, *and let* $(x, y) \in \mathcal{Q}_{\tilde{\delta}} = \{(x, y) : |x| > \tilde{\delta}, |y| > \tilde{\delta}\}$. *Then* $\mathcal{V}^{(2)}_1(x, y) = x^2 > \tilde{\delta}^2 = \tilde{\varepsilon}$.

Note that the solution of the equation

$$^{C}D^{0.3,t}_{k}x(t) = x(t) \tag{30}$$

is (see Lemma 7 and Figure 9)

$$x(t) = x_0\left(\prod_{i=1}^{k-1} E_{0.3}\Big((\theta(i) - \theta(i-1))^{0.3}\Big)\right)E_{0.3}\Big((\theta(t) - \theta(k))^{0.3}\Big), \quad t \in (k, k+1].$$

Apply $\frac{d}{dt}E_{0.3}\Big((\theta(t) - \theta(k))^{0.3}\Big) = \theta'(t)\frac{E_{0.3,0.3}\big((\theta(t)-\theta(k))^{0.3}\big)}{(\theta(t)-\theta(k))^{0.7}}$ *and get for* $t \in (k, k+1]$

$$\frac{d}{dt}x(t) = x_0\left(\prod_{i=1}^{k-1} E_{0.3}\Big((\theta(i) - \theta(i-1))^{0.3}\Big)\right)\theta'(t)\frac{E_{0.3,0.3}\Big((\theta(t) - \theta(k))^{0.3}\Big)}{(\theta(t) - \theta(k))^{0.7}}.$$

Then for $k = 0, 2, 4, 6, \ldots$ *we have*

$$\frac{d}{dt}\mathcal{V}^{(2)}_{\mathbb{K}_k}(x(t),y(t)) = \frac{d}{dt}\mathcal{V}^{(2)}_1(x(t),y(t)) = 2x(t)\frac{d}{dt}x(t)$$
$$= 2x_0^2\left(\prod_{i=1}^{k-1}E_{0.3}\Big((\theta(i)-\theta(i-1))^{0.3}\Big)\right)E_{0.3}\Big((\theta(t)-\theta(k))^{0.3}\Big)$$
$$\times\left(\prod_{i=1}^{k-1}E_{0.3}\Big((\theta(i)-\theta(i-1))^{0.3}\Big)\right)\theta'(t)\frac{E_{0.3,0.3}\Big((\theta(t)-\theta(k))^{0.3}\Big)}{(\theta(t)-\theta(k))^{0.7}} \geq 0,\ t\in(k,k+1],\ k=0,1,\ldots.$$

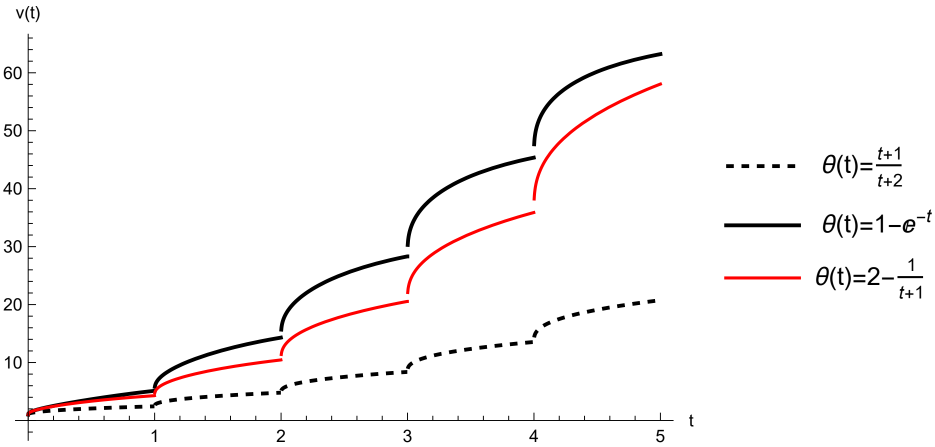

Figure 9. Graphs of solutions of (30) with various weights $\theta(.)$ in GCFF.

Similarly, for $k = 1,3,5,7,\ldots$ we have

$$\frac{d}{dt}\mathcal{V}^{(2)}_{\mathbb{K}_k}(x(t),y(t)) = \frac{d}{dt}\mathcal{V}^{(2)}_2(x(t),y(t)) = 2x^2(t)e^{x^2(t)}\frac{d}{dt}x(t) \geq 0,$$

where $x(t)$ is a solution of ${}^{C}D_k^{0.3,\theta(t)}x(t) = 2x(t) \quad t\in(k,k+1],\ k=1,3,5\ldots$, (see Lemma 7), i.e.,

$$x(t) = x_0\left(\prod_{i=1}^{k-1}E_{0.3}\Big(2(\theta(i)-\theta(i-1))^{0.3}\Big)\right)E_{0.3}\Big(2(\theta(t)-\theta(k))^{0.3}\Big),\quad t\in(k,k+1]$$

and

$$\frac{d}{dt}x(t) = 2x_0\left(\prod_{i=1}^{k-1}E_{0.3}\Big(2(\theta(i)-\theta(i-1))^{0.3}\Big)\right)\theta'(t)\frac{E_{0.3,0.3}\Big(2(\theta(t)-\theta(k))^{0.3}\Big)}{(\theta(t)-\theta(k))^{0.7}}.$$

Therefore, ${}^{C}D_k^{0.3,\theta(t)}\mathcal{V}^{(2)}_2(x(t),y(t)) \geq 0$; i.e., condition 4(v) of Corollary 5 is satisfied. According to Corollary 5, the zero solution of SFDEGCFF (27) is strictly stable.

6. Conclusions

In this paper, a nonlinear system of switched fractional differential equations with GCFF is studied. The switching rule is a piecewise constant function activated at any of its points of discontinuity, called switching times. The applied GCFF changes its lower limits at any switching time. Also, the applied function in GCFF changes at any interval of two consecutive switching times. We define and study strict stability for the given system. We apply Lyapunov functions and initially we prove some inequalities for the GCFF of convex Lyapunov functions. These inequalities allow us to avoid the application of the GCFF of

Lyapunov functions in the sufficient conditions. Some examples are given to illustrate the theoretical results.

It would be interesting to consider and study the case of a fixed lower limit of GCFF at the initial time. This changes not only the statement of the problem but also the study of stability and the Lyapunov functions.

It would also be interesting to extend our results to various switching models of neural networks and some population models. This topic goes beyond the scope of this paper and will be a challenging issue for future research.

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