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Higher Lie n-Centralizers and Generalized Higher Lie n-Derivations of von Neumann Algebras

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Abstract

Let $\mathcal{U}$ be a von Neumann algebra, $n \geq 2$, $P \in \mathcal{U}$ a projection, and let $p_1(A_1) = A_1$ and $p_n(A_1, \ldots, A_n) = [p_{n-1}(A_1, \ldots, A_{n-1}), A_n]$. Fix a higher Lie n-derivation $\Delta = \{\delta_m\}_{m=0}^{N}$ of $\mathcal{U}$ with $\delta_0 = \mathrm{id}$, of rank $N \in \mathbb{N} \cup \{\infty\}$. We classify the families $\Phi = \{\varphi_m\}_{m=0}^{N}$ of additive maps satisfying $\varphi_m(p_n(A_1, \ldots, A_n)) = \sum_{i_1+\cdots+i_n=m} p_n(\varphi_{i_1}(A_1), \delta_{i_2}(A_2), \ldots, \delta_{i_n}(A_n)) = \sum_{i_1+\cdots+i_n=m} p_n(\delta_{i_1}(A_1), \varphi_{i_2}(A_2), \ldots, \delta_{i_n}(A_n))$ for all $A_1, \ldots, A_n \in \mathcal{U}$ with $A_1A_2 = P$. In terms of the generating series $\Phi(t) = \sum_m \varphi_m t^m$ and $\Delta(t) = \sum_m \delta_m t^m$, these are exactly the families $\Phi(t) = W(t)\Delta(t) + \Xi(t)$ with $W(t)$ a power series over the center of $\mathcal{U}$ and $\Xi(t)$ a series of center-valued maps annihilating the $p_n(A_1, \ldots, A_n)$ with $A_1A_2 = P$. If $\mathcal{U}$ has no central summands of type I_1, then $W(t)$ and $\Xi(t)$ are unique, each ξ_m vanishes on $P\mathcal{U}(I-P)$ and on $(I-P)\mathcal{U}P$, and the solution set splits as the free rank-one module on Δ plus the null families. The underlying reduction of the rank-N problem to the rank-zero problem is valid over any ring, and we apply it also to upper triangular 2×2 matrices over a commutative ring. Moreover, generalized higher Lie n-derivations are characterized as a consequence.

Keywords: higher Lie n-centralizer; generalized higher Lie n-derivation; transfer principle; local action; von Neumann algebra; triangular algebra

MSC: 16W25; 47B47; 46L10

1. Introduction

Let $\mathcal{U}$ be an associative algebra with center $\mathcal{Z}(\mathcal{U})$. A map $\delta: \mathcal{U} \to \mathcal{U}$ is a *centralizer* if $\delta(xy) = \delta(x)y = x\delta(y)$ for all $x, y \in \mathcal{U}$, and a *derivation* if $\delta(xy) = \delta(x)y + x\delta(y)$. For $x_1, \ldots, x_n \in \mathcal{U}$ put

$$p_1(x_1) = x_1, \qquad p_n(x_1, \ldots, x_n) = \big[p_{n-1}(x_1, \ldots, x_{n-1}), x_n\big] \quad (n \geq 2),$$

where $[x, y] = xy - yx$. Replacing the product by these nested commutators yields two much larger classes: δ is a *Lie n-centralizer* if $\delta(p_n(x_1, \ldots, x_n)) = p_n(\delta(x_1), x_2, \ldots, x_n)$, and a *Lie n-derivation* if $\delta(p_n(x_1, \ldots, x_n)) = \sum_{k=1}^{n} p_n(x_1, \ldots, \delta(x_k), \ldots, x_n)$. Every centralizer is a Lie n-centralizer, and the question that organizes the subject is how much larger the second class really is. In every setting in which it has been settled, the answer has the same shape: a Lie n-centralizer differs from a central multiple of the identity by a map into the center that annihilates all $(n-1)$-th commutators. This is known for triangular rings and nest algebras [1], for generalized matrix algebras [2–4], for triangular algebras without

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a unit [5], and, in the local form where the identity is imposed only on arguments tied by a product condition, in [6–9]; the parallel theory of Lie n-derivations is developed in [10–13], and the nonlinear theory on operator algebras in [14,15].

On von Neumann algebras, the local problem was settled only recently. Fadaee and Ghahramani [16,17] treated Lie triple centralizers, and Ashraf, Ansari, Akhter, and Wei [18] obtained the general result: for an arbitrary von Neumann algebra $\mathcal{U}$ carrying a suitable projection P, an additive map φ satisfying

$$\varphi\big(p_n(A_1,\dots,A_n)\big) = p_n\big(\varphi(A_1),A_2,\dots,A_n\big) = p_n\big(A_1,\varphi(A_2),A_3,\dots,A_n\big) \tag{1}$$

for all $A_1,\dots,A_n \in \mathcal{U}$ with $A_1A_2 = P$ is precisely a map $\varphi(A) = WA + \xi(A)$ with $W \in \mathcal{Z}(\mathcal{U})$ and $\xi\colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ additive and vanishing at the relevant p_n's. This is the rank-zero input of Section 4, quoted as Theorem 3.

A second line of work replaces one map by a family. Following Hasse and Schmidt [19], a *higher derivation* of rank $N \in \mathbb{N}\cup\{\infty\}$ is a family $D = \{d_m\}_{m=0}^{N}$ of additive maps with $d_0 = \mathrm{id}$ and $d_m(xy) = \sum_{i+j=m} d_i(x)d_j(y)$; equivalently, $D(t) = \sum_m d_m t^m$ is a unital ring homomorphism of $\mathcal{U}$ into $\mathcal{U}[[t]]$ congruent to the identity modulo t. Generalized versions go back to Nakajima [20]. Higher analogues of Lie-type maps have been studied at length on triangular algebras and their relatives [21–27], and on von Neumann algebras under local hypotheses in [28] and in Kawa, Alsuraiheed, Hasan, Ali, and Wani [29], who prove that a Lie-type higher derivation $\{L_m\}$ of a von Neumann algebra without central summands of type I_1, subject to the higher identity at zero products or at projection products, decomposes as $L_m = \phi_m + \zeta_m$ with $\{\phi_m\}$ an additive higher derivation and ζ_m center-valued and annihilating the relevant p_n's. The centralizer literature has grown alongside it; see for instance Gao and Zhao [30].

Related rigidity phenomena for Lie-type maps have also been investigated beyond the associative setting. Jabeen and Ferreira [31] classify Lie (Jordan) centralizers on alternative algebras, and Andrade, Barreiro, and Ferreira [32] treat $*$-Lie-type maps on alternative $*$-algebras, showing that phenomena of the kind studied here are not confined to associative algebras.

To the best of our knowledge the corresponding relative theory has not been developed, and that is the subject of this paper. A higher Lie n-derivation $\Delta = \{\delta_m\}_{m\in\mathfrak{N}}$, $\delta_0 = \mathrm{id}$, is prescribed once and for all, and we classify the families $\Phi = \{\varphi_m\}_{m\in\mathfrak{N}}$ of additive maps that centralize it:

$$\varphi_m\big(p_n(A_1,\dots,A_n)\big) = \sum_{i_1+\cdots+i_n=m} p_n\big(\varphi_{i_1}(A_1),\delta_{i_2}(A_2),\dots,\delta_{i_n}(A_n)\big), \tag{2}$$

$$\varphi_m\big(p_n(A_1,\dots,A_n)\big) = \sum_{i_1+\cdots+i_n=m} p_n\big(\delta_{i_1}(A_1),\varphi_{i_2}(A_2),\delta_{i_3}(A_3),\dots,\delta_{i_n}(A_n)\big) \tag{3}$$

for every m and all $A_1,\dots,A_n \in \mathcal{U}$ with $A_1A_2 = P$ (Definition 2). The unknown is thus a family measured against a second one, whereas in [29] and in [27] the unknown is a single family subject to a Lie-type identity and the conclusion is a componentwise decomposition of it. Definition 2 is calibrated so that its two lowest levels are the two halves of [18]: at level 0 both displays collapse to (1), so φ_0 is a Lie n-centralizer at P; at level 1, with $\varphi_0 = \mathrm{id}$ and $\delta_1 = L$, the first reads

$$\varphi_1\big(p_n(A_1,\dots,A_n)\big) = p_n\big(\varphi_1(A_1),A_2,\dots,A_n\big) + \sum_{k=2}^{n} p_n\big(A_1,\dots,L(A_k),\dots,A_n\big),$$

which, with its mirror image, is the defining condition of a generalized Lie n-derivation associated with L studied in §3 of [18].

Write $\mathcal{Z}(\mathcal{U})[\![t]\!]_N$ for the ring of truncated power series with coefficients in $\mathcal{Z}(\mathcal{U})$, encode a family by $\Phi(t) = \sum_m \varphi_m t^m$, and call a family $\Xi = \{\xi_m\}$ of additive maps $\mathcal{U} \to \mathcal{Z}(\mathcal{U})$ *null at* P if $\xi_m(p_n(A_1, \dots, A_n)) = 0$ whenever $A_1 A_2 = P$.

Theorem 1. *Let $\mathcal{U}$ be a von Neumann algebra with unit I, let $E_1 + E_2 = I$ be orthogonal central projections with $\mathcal{U}E_1$ of type I_1 and $\mathcal{U}E_2$ without central summands of type I_1, and let $P \in \mathcal{U}E_2$ be a core-free projection with central carrier E_2. Let $n \geq 2$, let $\Delta = \{\delta_m\}_{m=0}^N$ be a higher Lie n-derivation at P, and let $\Phi = \{\varphi_m\}_{m=0}^N$ be a family of additive maps on $\mathcal{U}$. Then Φ is a Δ-higher Lie n-centralizer at P if and only if there are $W(t) \in \mathcal{Z}(\mathcal{U})[\![t]\!]_N$ and a family Ξ null at P with*

$$\Phi(t) = W(t)\,\Delta(t) + \Xi(t), \qquad \textit{that is} \qquad \varphi_m = \sum_{i+j=m} W_i\,\delta_j + \xi_m \quad (0 \leq m \leq N).$$

The answer is a convolution, and the whole tower is governed by a single central power series. What makes this accessible is an identity isolated in Lemma 7. Assume that the conclusion holds at all levels below m and detach from (2) the one term with $i_1 = m$; the remainder is

$$\sum_{a=0}^{m-1} W_a \Big[\delta_{m-a}\big(p_n(A_1, \dots, A_n)\big) - p_n\big(\delta_{m-a}(A_1), A_2, \dots, A_n\big)\Big],$$

in which Φ no longer appears: a central combination of the defects measuring how far the individual δ_j are from being Lie n-centralizers. These cancel against the correction $\sum_{a<m} W_a \delta_{m-a}$, so that $\psi_m = \varphi_m - \sum_{a<m} W_a \delta_{m-a}$ satisfies (1) verbatim and level m is reduced to level 0. Nothing about $\mathcal{U}$, about P, or about the constraint $A_1 A_2 = P$ is used, so we work over an arbitrary ring R and an arbitrary set $\mathcal{S} \subseteq R^n$ of test tuples and obtain the transfer theorem, Theorem 2.

We regard Theorem 2, rather than Theorem 1, as the principal result of the paper. Once property $(\star)$ is known at rank zero, the higher-rank problem is recovered by an induction that depends neither on the specific structure of $\mathcal{U}$ nor on the particular local constraint $A_1 A_2 = P$: the rank-zero classification of Ashraf, Ansari, Akhter, and Wei [18] is the only operator-algebraic input, and Theorem 2 itself is purely algebraic. This is what distinguishes the present approach from the existing higher Lie-type literature [21–27,29], in which the unknown is a single family subject to a Lie-type identity and the conclusion is a componentwise decomposition of that family: here a whole family is classified relative to a second, prescribed one, and the classification is obtained once and for all at rank zero. The two halves of the paper are correspondingly independent: Section 3 uses nothing about operator algebras, whereas Section 4 inherits its rank-zero ingredient from [18]. Two further results show that the principle is reusable. Property $(\star)$ is inherited by every larger set of test tuples, so a single rank-zero input governs a whole range of them; in particular the global counterpart Theorem 6 needs only the one-sided condition (2), since $\mathcal{U}^n$ is closed under interchanging the first two arguments while $\mathcal{S}_P$ is not, and the global half of [18] is recovered from its local half. In Proposition 5, we verify $(\star)$ from first principles for the upper triangular 2×2 matrices over an arbitrary commutative ring, and read off the complete higher classification there, with no operator algebra involved.

Over a von Neumann algebra, the decomposition carries further structure. The data $W(t)$ and Ξ are unique when $\mathcal{U}$ has no central summands of type I_1, and on an arbitrary $\mathcal{U}$ the failure of uniqueness is described exactly: the coefficients may be shifted on the type I_1 summand, and what Φ determines is $W_m E_2$ together with $\xi_m + \sum_{i+j=m} W_i E_1 \delta_j$. The solution set is a $\mathcal{Z}(\mathcal{U})[\![t]\!]_N$-module, and it splits as the direct sum of the free rank-one module generated by Δ and the module of null families. Each ξ_m vanishes on both $P\mathcal{U}(I-P)$ and $(I-P)\mathcal{U}P$. Specializing to the families with $\varphi_0 = \mathrm{id}$ gives the generalized higher Lie

n-derivations, for which Theorem 8 reads $G(t) = W(t)L(t) + X(t)$ with $W(t)$ invertible, whence $L(t) = W(t)^{-1}\big(G(t) - X(t)\big)$: such a family is a central rescaling of its associated higher Lie n-derivation, modulo center-valued maps, and at rank one this returns the theorems of [18]. Neither linearity nor continuity is assumed anywhere, and Proposition 9 shows that whatever regularity φ_m possesses is exactly the regularity of ξ_m.

Section 2 collects the notation and the elementary identities used throughout; although stated in the setting of a von Neumann algebra for convenience, the identities of Section 2.1 are purely algebraic and remain valid over an arbitrary associative ring, which is the generality in which they are used in Section 3. The computational core is Section 3, carried out over an arbitrary ring. Section 4 specializes to von Neumann algebras, and Section 5 contains the applications. We close with some concluding remarks on possible extensions of the method, including to non-associative algebras.

2. Preliminaries

Throughout, $\mathcal{U}$ is a von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$, with unit I and center $\mathcal{Z}(\mathcal{U})$, and $n \geq 2$ is a fixed integer. A projection is a self-adjoint idempotent. The *central carrier* $\overline{T}$ of $T \in \mathcal{U}$ is the smallest central projection Q with $QT = T$; equivalently, $\overline{T}$ is the projection onto the closed linear span of $\{ATh : A \in \mathcal{U},\ h \in \mathcal{H}\}$. For self-adjoint $S \in \mathcal{U}$ the *core* is $\underline{S} = \sup\{W \in \mathcal{Z}(\mathcal{U}) : W = W^*,\ W \leq S\}$, and a projection P is *core-free* when $\underline{P} = 0$; one has $\underline{P} = 0$ if and only if $\overline{I - P} = I$. The algebra $\mathcal{U}$ has no central summands of type I_1 precisely when it contains a projection P with $\underline{P} = 0$ and $\overline{P} = I$. For an arbitrary $\mathcal{U}$ the unit splits as $I = E_1 + E_2$ into orthogonal central projections with $\mathcal{U} = \mathcal{U}E_1 \oplus \mathcal{U}E_2$, where $\mathcal{U}E_1$ is of type I_1 and $\mathcal{U}E_2$ has no central summands of type I_1; consequently, $\mathcal{U}E_2$ contains a core-free projection with central carrier E_2. We use [33,34] for all of this.

Fix a projection $P_1 = P \in \mathcal{U}$ with $\underline{P_1} = 0$ and $\overline{P_1} = I$, put $P_2 = I - P_1$, and set $\mathcal{U}_{ij} = P_i\mathcal{U}P_j$ for $i, j \in \{1, 2\}$, so that

$$\mathcal{U} = \mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}, \qquad A = A_{11} + A_{12} + A_{21} + A_{22}, \qquad A_{ij} = P_iAP_j.$$

We shall use repeatedly the following two facts from Remark 1.1 of [18]:

$$A \in \mathcal{U}, \quad A\mathcal{U}P_1 = \{0\} \implies A = 0, \qquad A \in \mathcal{U}, \quad A\mathcal{U}P_2 = \{0\} \implies A = 0, \tag{4}$$

which hold because $\overline{P_1} = \overline{P_2} = I$, and

$$p_n(A, P_1, P_1, \ldots, P_1) = (-1)^{n-1}P_1AP_2 + P_2AP_1 \qquad \text{for} \quad A \in \mathcal{U}. \tag{5}$$

For completeness we verify Equation (5). For $n = 2$,

$$p_2(A, P_1) = AP_1 - P_1A = (P_1 + P_2)AP_1 - P_1A(P_1 + P_2) = P_2AP_1 - P_1AP_2,$$

which is Equation (5). If Equation (5) holds for some $n \geq 2$, then

$$\begin{aligned} p_{n+1}(A, P_1, \ldots, P_1) &= \big[(-1)^{n-1}P_1AP_2 + P_2AP_1,\ P_1\big] \\ &= (-1)^{n-1}P_1AP_2P_1 + P_2AP_1P_1 - (-1)^{n-1}P_1P_1AP_2 - P_1P_2AP_1 \\ &= P_2AP_1 + (-1)^nP_1AP_2, \end{aligned}$$

which is Equation (5) with $n + 1$ in place of n.

2.1. Some Elementary Identities

Since p_n is a nested commutator it is additive in each argument. The next five lemmas record how central elements interact with it, and how it behaves under interchanging its first two arguments. Although stated here for a von Neumann algebra $\mathcal{U}$, the proofs of Lemmas 1 and 2 use only additivity and the ring axioms, so these two lemmas remain valid, verbatim, for an arbitrary associative ring R in place of $\mathcal{U}$; this is the generality in which they are invoked in Section 3.

Lemma 1. *For every $W \in \mathcal{Z}(\mathcal{U})$, all $A_1, \ldots, A_n \in \mathcal{U}$ and each index $1 \le k \le n$,*

$$p_n(A_1, \ldots, A_{k-1}, WA_k, A_{k+1}, \ldots, A_n) = W\, p_n(A_1, \ldots, A_n).$$

Proof. We use induction on n. For $n = 2$ and $k = 1$, we have

$$p_2(WA_1, A_2) = WA_1A_2 - A_2WA_1 = WA_1A_2 - WA_2A_1 = W\big(A_1A_2 - A_2A_1\big) = W\, p_2(A_1, A_2).$$

Now for $k = 2$, we have

$$p_2(A_1, WA_2) = A_1WA_2 - WA_2A_1 = WA_1A_2 - WA_2A_1 = W\big(A_1A_2 - A_2A_1\big) = W\, p_2(A_1, A_2).$$

Let $n \ge 3$ and assume the assertion for $n-1$. Put $\pi = p_{n-1}(A_1, \ldots, A_{n-1})$ and $\pi_k = p_{n-1}(A_1, \ldots, A_{k-1}, WA_k, A_{k+1}, \ldots, A_{n-1})$ for $1 \le k \le n-1$. If $k \le n-1$, then $\pi_k = W\pi$ by the induction hypothesis, whence

$$\begin{aligned} p_n(A_1, \ldots, WA_k, \ldots, A_n) &= [\pi_k, A_n] = [W\pi, A_n] = W\pi A_n - A_nW\pi \\ &= W\pi A_n - WA_n\pi = W[\pi, A_n] = W\, p_n(A_1, \ldots, A_n). \end{aligned}$$

If $k = n$, then

$$\begin{aligned} p_n(A_1, \ldots, A_{n-1}, WA_n) &= [\pi, WA_n] = \pi WA_n - WA_n\pi \\ &= W\pi A_n - WA_n\pi = W[\pi, A_n] = W\, p_n(A_1, \ldots, A_n). \end{aligned}$$

□

Lemma 2. *Let $C \in \mathcal{Z}(\mathcal{U})$ and $A_1, \ldots, A_n \in \mathcal{U}$ with $n \ge 2$. Then*

$$p_n(C, A_2, A_3, \ldots, A_n) = 0 \qquad \text{and} \qquad p_n(A_1, C, A_3, \ldots, A_n) = 0.$$

Proof. The centrality of C makes the term p_2 vanish in both cases:

$$p_2(C, A_2) = CA_2 - A_2C = 0, \qquad p_2(A_1, C) = A_1C - CA_1 = 0.$$

If $n \ge 3$, then $p_n(x_1, \ldots, x_n) = [p_{n-1}(x_1, \ldots, x_{n-1}), x_n]$, so an induction on n gives $p_{n-1} = 0$ in both cases, hence $p_n = [0, A_n] = 0$. □

Lemma 3. *Suppose $\mathcal{U}$ has no central summands of type I_1, and let $C \in \mathcal{Z}(\mathcal{U})$ be such that $CA \in \mathcal{Z}(\mathcal{U})$ for every $A \in \mathcal{U}$. Then $C = 0$.*

Proof. We first show $\mathcal{Z}(\mathcal{U}) \cap \mathcal{U}_{12} = \{0\}$. Let $T \in \mathcal{Z}(\mathcal{U}) \cap \mathcal{U}_{12}$, so that $T = P_1TP_2$. Then

$$TP_1 = (P_1TP_2)P_1 = P_1T(P_2P_1) = 0, \qquad P_1T = P_1(P_1TP_2) = P_1TP_2 = T,$$

and since T is central, $T = P_1T = TP_1 = 0$.

Now for every $X \in \mathcal{U}$ we have $CP_1XP_2 = P_1(CX)P_2 \in \mathcal{Z}(\mathcal{U}) \cap \mathcal{U}_{12} = \{0\}$, that is,

$$(CP_1)\mathcal{U}P_2 = \{0\},$$

so $CP_1 = 0$ by (4). Interchanging the roles of P_1 and P_2 gives $(CP_2)\mathcal{U}P_1 = \{0\}$ and hence $CP_2 = 0$. Therefore $C = C(P_1 + P_2) = 0$. □

Lemma 4. *Let $\mathcal{U}$ be a von Neumann algebra and let $E \in \mathcal{Z}(\mathcal{U})$ be a central projection such that $\mathcal{U}E$ is abelian. Then*

$$E\,p_n(A_1, A_2, \ldots, A_n) = 0 \qquad \text{for} \quad A_1, \ldots, A_n \in \mathcal{U},\ n \geq 2.$$

Proof. We first show that $\mathcal{U}E \subseteq \mathcal{Z}(\mathcal{U})$. Indeed, for $A, y \in \mathcal{U}$,

$$(EA)y - y(EA) = E^2Ay - yE^2A = (EA)(Ey) - (Ey)(EA) = 0,$$

where the second equality uses $E^2 = E$ and the centrality of E, and the third holds because EA and Ey both lie in the abelian algebra $\mathcal{U}E$. Hence, EA commutes with every element of $\mathcal{U}$.

Consequently, by Lemma 1,

$$E\,p_2(A_1, A_2) = p_2(EA_1, A_2) = (EA_1)A_2 - A_2(EA_1) = 0.$$

For $n \geq 3$, we have

$$E\,p_n(A_1, \ldots, A_n) = \big[E\,p_{n-1}(A_1, \ldots, A_{n-1}), A_n\big],$$

and an induction on n gives $E\,p_{n-1}(A_1, \ldots, A_{n-1}) = 0$, hence $E\,p_n(A_1, \ldots, A_n) = [0, A_n] = 0$. □

Lemma 5. *For all $x_1, \ldots, x_n \in \mathcal{U}$ and all $n \geq 2$,*

$$p_n(x_2, x_1, x_3, \ldots, x_n) = -\,p_n(x_1, x_2, x_3, \ldots, x_n).$$

Proof. For $n = 2$,

$$p_2(x_2, x_1) = x_2x_1 - x_1x_2 = -\big(x_1x_2 - x_2x_1\big) = -\,p_2(x_1, x_2).$$

Let $k \geq 3$ and assume $p_{k-1}(x_2, x_1, x_3, \ldots, x_{k-1}) = -\,p_{k-1}(x_1, x_2, x_3, \ldots, x_{k-1})$. Then

$$\begin{aligned} p_k(x_2, x_1, x_3, \ldots, x_k) &= \big[p_{k-1}(x_2, x_1, x_3, \ldots, x_{k-1}), x_k\big] \\ &= \big[-\,p_{k-1}(x_1, x_2, x_3, \ldots, x_{k-1}), x_k\big] = -\big[p_{k-1}(x_1, x_2, x_3, \ldots, x_{k-1}), x_k\big] \\ &= -\,p_k(x_1, x_2, x_3, \ldots, x_k). \end{aligned}$$

□

2.2. Higher Lie n-Derivations and Their Centralizers

Let $N \in \mathbb{N} \cup \{\infty\}$ and put $\mathfrak{N} = \{0, 1, \ldots, N\}$. A *family of rank N* is a collection $\Phi = \{\varphi_m\}_{m \in \mathfrak{N}}$ of additive maps $\mathcal{U} \to \mathcal{U}$, we encode it by the formal series $\Phi(t) = \sum_{m \in \mathfrak{N}} \varphi_m t^m$. We write $\mathcal{Z}(\mathcal{U})[\![t]\!]_N$ for the commutative ring of formal series $W(t) = \sum_{m \in \mathfrak{N}} W_m t^m$ with $W_m \in \mathcal{Z}(\mathcal{U})$, multiplied by convolution and truncated at t^{N+1} when $N < \infty$. Products such as $W(t)\Phi(t)$ are always understood coefficientwise: $\big(W(t)\Phi(t)\big)_m = \sum_{i+j=m} W_i\varphi_j$.

For a multi-index $\mathbf{i} = (i_1, \dots, i_n) \in \mathfrak{N}^n$, we write $|\mathbf{i}| = i_1 + \cdots + i_n$, and

$$\mathcal{C}(m,n) = \{\mathbf{i} \in \mathbb{N}^n : \ |\mathbf{i}| = m\}$$

for the set of compositions of m into n non-negative parts.

Throughout, $\mathcal{S}$ denotes a nonempty set of *test tuples*, i.e., a subset of $\mathcal{U}^n$. We use two of them: for a fixed projection $P \in \mathcal{U}$,

$$\mathcal{S}_P := \{(A_1, \dots, A_n) \in \mathcal{U}^n : \ A_1 A_2 = P\},$$

and $\mathcal{S} = \mathcal{U}^n$.

Definition 1. *A family $\Delta = \{\delta_m\}_{m \in \mathfrak{N}}$ with $\delta_0 = \mathrm{id}_{\mathcal{U}}$ is a* higher Lie n-derivation *on $\mathcal{S}$ if*

$$\delta_m(p_n(A_1, \dots, A_n)) = \sum_{\mathbf{i} \in \mathcal{C}(m,n)} p_n(\delta_{i_1}(A_1), \delta_{i_2}(A_2), \dots, \delta_{i_n}(A_n)) \tag{6}$$

for every $m \in \mathfrak{N}$ and every $(A_1, \dots, A_n) \in \mathcal{S}$.

Definition 2. *Let Δ be a higher Lie n-derivation on $\mathcal{S}$. A family $\Phi = \{\varphi_m\}_{m \in \mathfrak{N}}$ is a* Δ-higher Lie n-centralizer *on $\mathcal{S}$ if*

$$\varphi_m(p_n(A_1, \dots, A_n)) = \sum_{\mathbf{i} \in \mathcal{C}(m,n)} p_n(\varphi_{i_1}(A_1), \delta_{i_2}(A_2), \dots, \delta_{i_n}(A_n)), \tag{7}$$

$$\varphi_m(p_n(A_1, \dots, A_n)) = \sum_{\mathbf{i} \in \mathcal{C}(m,n)} p_n(\delta_{i_1}(A_1), \varphi_{i_2}(A_2), \delta_{i_3}(A_3), \dots, \delta_{i_n}(A_n)) \tag{8}$$

for every $m \in \mathfrak{N}$ and every $(A_1, \dots, A_n) \in \mathcal{S}$. A family $\Xi = \{\xi_m\}_{m \in \mathfrak{N}}$ of additive maps $\xi_m : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is null on $\mathcal{S}$ *if $\xi_m(p_n(A_1, \dots, A_n)) = 0$ for all m and all $(A_1, \dots, A_n) \in \mathcal{S}$. We write $\mathcal{N}(\mathcal{S})$ for the set of null families and $\mathcal{C}_\Delta(\mathcal{S})$ for the set of Δ-higher Lie n-centralizers on $\mathcal{S}$.*

Remark 1. *Level* 0 *of Equation* (7) *and Equation* (8) *is the single condition*

$$\varphi_0(p_n(A_1, \dots, A_n)) = p_n(\varphi_0(A_1), A_2, \dots, A_n) = p_n(A_1, \varphi_0(A_2), A_3, \dots, A_n),$$

because $\mathcal{C}(0,n) = \{(0, \dots, 0)\}$ and $\delta_0 = \mathrm{id}$; on $\mathcal{S}_P$, this is exactly condition (1). *Level* 1 *of Equation* (7) *reads*

$$\varphi_1(p_n(A_1, \dots, A_n)) = p_n(\varphi_1(A_1), A_2, \dots, A_n) + \sum_{k=2}^{n} p_n(\varphi_0(A_1), A_2, \dots, \delta_1(A_k), \dots, A_n),$$

since $\mathcal{C}(1,n)$ consists of the n multi-indices having a single entry equal to 1.

The two conditions of Definition 2 are independent in general, but they coincide as soon as the test set is closed under interchanging the first two coordinates. This is the case for $\mathcal{S} = \mathcal{U}^n$, and it is not the case for $\mathcal{S}_P$, since $A_1 A_2 = P$ says nothing about $A_2 A_1$.

Proposition 1. *Let $\mathcal{S} \subseteq \mathcal{U}^n$ be closed under interchanging the first two coordinates, that is,*

$$(A_1, A_2, A_3, \dots, A_n) \in \mathcal{S} \implies (A_2, A_1, A_3, \dots, A_n) \in \mathcal{S};$$

this holds in particular for $\mathcal{S} = \mathcal{U}^n$. Let $\Delta = \{\delta_m\}$ and $\Phi = \{\varphi_m\}$ be arbitrary families of additive maps on $\mathcal{U}$. Then conditions (7) *and* (8) *are equivalent: one holds for every $m \in \mathfrak{N}$ and every tuple in $\mathcal{S}$ if and only if the other does.*

Proof. Assume Equation (7) holds on $\mathcal{S}$, fix m and $(A_1, \dots, A_n) \in \mathcal{S}$, and apply Equation (7) to the tuple $(A_2, A_1, A_3, \dots, A_n)$, which lies in $\mathcal{S}$ by hypothesis:

$$\varphi_m\big(p_n(A_2, A_1, A_3, \dots, A_n)\big) = \sum_{\mathbf{i} \in \mathcal{C}(m,n)} p_n\big(\varphi_{i_1}(A_2), \delta_{i_2}(A_1), \delta_{i_3}(A_3), \dots, \delta_{i_n}(A_n)\big). \quad (9)$$

By Lemma 5 and the additivity of φ_m, the left-hand side of Equation (9) equals

$$\varphi_m\big(-\,p_n(A_1, A_2, A_3, \dots, A_n)\big) = -\,\varphi_m\big(p_n(A_1, A_2, A_3, \dots, A_n)\big). \quad (10)$$

By Lemma 5 again, applied to the tuple $\big(\varphi_{i_1}(A_2), \delta_{i_2}(A_1), \delta_{i_3}(A_3), \dots\big)$, each summand on the right-hand side of Equation (9) equals

$$p_n\big(\varphi_{i_1}(A_2), \delta_{i_2}(A_1), \delta_{i_3}(A_3), \dots, \delta_{i_n}(A_n)\big) = -\,p_n\big(\delta_{i_2}(A_1), \varphi_{i_1}(A_2), \delta_{i_3}(A_3), \dots, \delta_{i_n}(A_n)\big). \quad (11)$$

The transposition $(i_1, i_2, i_3, \dots, i_n) \mapsto (i_2, i_1, i_3, \dots, i_n)$ is a bijection of $\mathcal{C}(m, n)$ onto itself, so summing Equation (11) over $\mathbf{i} \in \mathcal{C}(m, n)$ gives

$$\sum_{\mathbf{i} \in \mathcal{C}(m,n)} p_n\big(\varphi_{i_1}(A_2), \delta_{i_2}(A_1), \delta_{i_3}(A_3), \dots\big) = -\sum_{\mathbf{i} \in \mathcal{C}(m,n)} p_n\big(\delta_{i_1}(A_1), \varphi_{i_2}(A_2), \delta_{i_3}(A_3), \dots, \delta_{i_n}(A_n)\big). \quad (12)$$

Substituting Equation (10) and Equation (12) into Equation (9) and multiplying by -1 yields Equation (8). The converse implication is the same argument with the roles of the first two slots interchanged. □

Higher Lie n-derivations are plentiful, and on a von Neumann algebra they can be produced by conjugation.

Example 1. *Let $T_1, \dots, T_N \in \mathcal{U}$ and set*

$$V(t) = I + T_1 t + T_2 t^2 + \cdots + T_N t^N.$$

Since the constant term is I, the series $V(t)$ is invertible in $\mathcal{U}[\![t]\!]_N$; write $V(t)^{-1} = \sum_j V'_j t^j$, so that $V'_0 = I$ and $V'_j = -\sum_{i=1}^{j} T_i V'_{j-i}$. Define

$$\delta_m(A) = \sum_{a+b=m} V_a\, A\, V'_b, \qquad V_0 = I,\ V_a = T_a\ (a \geq 1).$$

Then $\delta_0 = \mathrm{id}$, each δ_m is additive, and $D(t)(A) := V(t)AV(t)^{-1}$ defines a unital ring homomorphism $\mathcal{U} \to \mathcal{U}[\![t]\!]_N$. Consequently,

$$D(t)\big(p_n(A_1, \dots, A_n)\big) = p_n\big(D(t)A_1, \dots, D(t)A_n\big),$$

and comparing the coefficients of t^m gives Equation (6) for every m, on all of $\mathcal{U}^n$. Explicitly,

$$\delta_1(A) = [T_1, A], \qquad \delta_2(A) = [T_2, A] + AT_1^2 - T_1 A T_1.$$

By the Kadison–Sakai theorem [35,36] every derivation of $\mathcal{U}$ is inner, so δ_1 is necessarily of the above form.

3. Higher Families over an Arbitrary Ring

Let R be a ring with center $\mathcal{Z}(R)$, let $n \geq 2$, and let $\mathcal{S} \subseteq R^n$ be a nonempty set of tuples. We use Lemmas 1 and 2 of Section 2 throughout this section in this generality; as noted there, their proofs use only additivity and the ring axioms and so apply verbatim

with R in place of $\mathcal{U}$. Fix a higher Lie n-derivation Δ on $\mathcal{S}$ and a Δ-higher Lie n-centralizer Φ on $\mathcal{S}$. For $m \in \mathfrak{N}$ and $(A_1, \ldots, A_n) \in \mathcal{S}$ define the two *level-m remainders*

$$R_m(A_1, \ldots, A_n) = \sum_{\substack{\mathbf{i} \in \mathcal{C}(m,n) \\ i_1 \leq m-1}} p_n\big(\varphi_{i_1}(A_1), \delta_{i_2}(A_2), \ldots, \delta_{i_n}(A_n)\big), \tag{13}$$

$$R'_m(A_1, \ldots, A_n) = \sum_{\substack{\mathbf{i} \in \mathcal{C}(m,n) \\ i_2 \leq m-1}} p_n\big(\delta_{i_1}(A_1), \varphi_{i_2}(A_2), \delta_{i_3}(A_3), \ldots, \delta_{i_n}(A_n)\big). \tag{14}$$

Lemma 6. *For every $m \in \mathfrak{N}$ and every $(A_1, \ldots, A_n) \in \mathcal{S}$,*

$$\varphi_m\big(p_n(A_1, \ldots, A_n)\big) = p_n\big(\varphi_m(A_1), A_2, A_3, \ldots, A_n\big) + R_m(A_1, \ldots, A_n), \tag{15}$$

$$\varphi_m\big(p_n(A_1, \ldots, A_n)\big) = p_n\big(A_1, \varphi_m(A_2), A_3, \ldots, A_n\big) + R'_m(A_1, \ldots, A_n). \tag{16}$$

Proof. Let $\mathbf{i} \in \mathcal{C}(m,n)$. Since the entries of $\mathbf{i}$ are non-negative and sum to m, we have $i_1 = m$ if and only if $i_k = 0$ for every $k \geq 2$, and in that case $\delta_0 = \mathrm{id}$ gives

$$p_n\big(\varphi_{i_1}(A_1), \delta_{i_2}(A_2), \ldots, \delta_{i_n}(A_n)\big) = p_n\big(\varphi_m(A_1), \delta_0(A_2), \ldots, \delta_0(A_n)\big) = p_n\big(\varphi_m(A_1), A_2, \ldots, A_n\big).$$

Thus $\mathcal{C}(m,n)$ is the disjoint union of the singleton $\{(m, 0, \ldots, 0)\}$ and $\{\mathbf{i} \in \mathcal{C}(m,n) : i_1 \leq m-1\}$, so the sum in condition (7) breaks into two pieces:

$$\varphi_m\big(p_n(A_1, \ldots, A_n)\big) = p_n\big(\varphi_m(A_1), A_2, \ldots, A_n\big) + \sum_{\substack{\mathbf{i} \in \mathcal{C}(m,n) \\ i_1 \leq m-1}} p_n\big(\varphi_{i_1}(A_1), \delta_{i_2}(A_2), \ldots, \delta_{i_n}(A_n)\big).$$

Likewise $i_2 = m$ if and only if $i_1 = 0$ and $i_k = 0$ for every $k \geq 3$, and in that case

$$\begin{aligned} p_n\big(\delta_{i_1}(A_1), \varphi_{i_2}(A_2), \delta_{i_3}(A_3), \ldots, \delta_{i_n}(A_n)\big) &= p_n\big(\delta_0(A_1), \varphi_m(A_2), \delta_0(A_3), \ldots, \delta_0(A_n)\big) \\ &= p_n\big(A_1, \varphi_m(A_2), A_3, \ldots, A_n\big). \end{aligned}$$

Here $\mathcal{C}(m,n)$ is the disjoint union of the singleton $\{(0, m, 0, \ldots, 0)\}$ and $\{\mathbf{i} \in \mathcal{C}(m,n) : i_2 \leq m-1\}$, so the sum in condition (8) breaks as

$$\varphi_m\big(p_n(A_1, \ldots, A_n)\big) = p_n\big(A_1, \varphi_m(A_2), A_3, \ldots, A_n\big) + \sum_{\substack{\mathbf{i} \in \mathcal{C}(m,n) \\ i_2 \leq m-1}} p_n\big(\delta_{i_1}(A_1), \varphi_{i_2}(A_2), \delta_{i_3}(A_3), \ldots\big).$$

□

The next lemma is the heart of the paper.

Lemma 7. *Let $m \in \mathfrak{N}$ with $m \geq 1$. Suppose there are central elements $W_0, \ldots, W_{m-1}$ in $\mathcal{Z}(R)$ and additive maps $\xi_k \colon R \to \mathcal{Z}(R)$, $0 \leq k \leq m-1$, such that*

$$\varphi_k = \sum_{a+b=k} W_a \delta_b + \xi_k \qquad (0 \leq k \leq m-1). \tag{17}$$

Then, for every $(A_1, \ldots, A_n) \in \mathcal{S}$,

$$R_m(A_1, \ldots, A_n) = \sum_{a=0}^{m-1} W_a \Big[\delta_{m-a}\big(p_n(A_1, \ldots, A_n)\big) - p_n\big(\delta_{m-a}(A_1), A_2, \ldots, A_n\big)\Big], \tag{18}$$

$$R'_m(A_1, \ldots, A_n) = \sum_{a=0}^{m-1} W_a \Big[\delta_{m-a}\big(p_n(A_1, \ldots, A_n)\big) - p_n\big(A_1, \delta_{m-a}(A_2), A_3, \ldots, A_n\big)\Big]. \tag{19}$$

Proof. *Step 1:* Fix $\mathbf{i} \in \mathcal{C}(m,n)$ with $i_1 \leq m-1$. Hypothesis Equation (17) applies at level i_1 and gives

$$\varphi_{i_1}(A_1) = \sum_{a+b=i_1} W_a \delta_b(A_1) + \xi_{i_1}(A_1),$$

so, p_n being additive in its first argument,

$$p_n\big(\varphi_{i_1}(A_1), \delta_{i_2}(A_2), \ldots, \delta_{i_n}(A_n)\big) = \sum_{a+b=i_1} p_n\big(W_a\delta_b(A_1), \delta_{i_2}(A_2), \ldots, \delta_{i_n}(A_n)\big) + p_n\big(\xi_{i_1}(A_1), \delta_{i_2}(A_2), \ldots, \delta_{i_n}(A_n)\big). \quad (20)$$

Now $\xi_{i_1}(A_1) \in \mathcal{Z}(R)$, so Lemma 2 gives

$$p_n\big(\xi_{i_1}(A_1), \delta_{i_2}(A_2), \ldots, \delta_{i_n}(A_n)\big) = 0, \quad (21)$$

while $W_a \in \mathcal{Z}(R)$, so Lemma 1 gives

$$p_n\big(W_a\delta_b(A_1), \delta_{i_2}(A_2), \ldots, \delta_{i_n}(A_n)\big) = W_a\, p_n\big(\delta_b(A_1), \delta_{i_2}(A_2), \ldots, \delta_{i_n}(A_n)\big). \quad (22)$$

With Equations (21) and (22) the right-hand side of Equation (20) becomes

$$p_n\big(\varphi_{i_1}(A_1), \delta_{i_2}(A_2), \ldots, \delta_{i_n}(A_n)\big) = \sum_{a+b=i_1} W_a\, p_n\big(\delta_b(A_1), \delta_{i_2}(A_2), \ldots, \delta_{i_n}(A_n)\big),$$

and summing this over the multi-indices occurring in (13) gives

$$R_m(A_1, \ldots, A_n) = \sum_{\substack{\mathbf{i}\in\mathcal{C}(m,n)\\ i_1\leq m-1}} \sum_{a+b=i_1} W_a\, p_n\big(\delta_b(A_1), \delta_{i_2}(A_2), \ldots, \delta_{i_n}(A_n)\big). \quad (23)$$

Step 2: The double sum in Equation (23) runs over the tuples $(a, b, i_2, \ldots, i_n) \in \mathbb{N}^{n+1}$ subject to

$$a + b + i_2 + \cdots + i_n = m, \qquad a + b \leq m-1.$$

In particular $a \leq m-1$. Collecting the terms according to the value of a, and rewriting the two constraints for fixed a as

$$b + i_2 + \cdots + i_n = m-a, \qquad b \leq m-a-1,$$

we obtain

$$R_m(A_1, \ldots, A_n) = \sum_{a=0}^{m-1} W_a \sum_{\substack{b+i_2+\cdots+i_n=m-a\\ b\leq m-a-1}} p_n\big(\delta_b(A_1), \delta_{i_2}(A_2), \ldots, \delta_{i_n}(A_n)\big). \quad (24)$$

Step 3: Fix a with $0 \leq a \leq m-1$. Among the tuples $(b, i_2, \ldots, i_n)$ with $b + i_2 + \cdots + i_n = m-a$, exactly one has $b = m-a$, namely $(m-a, 0, \ldots, 0)$, and its contribution is

$$p_n\big(\delta_{m-a}(A_1), \delta_0(A_2), \ldots, \delta_0(A_n)\big) = p_n\big(\delta_{m-a}(A_1), A_2, \ldots, A_n\big),$$

since $\delta_0 = \mathrm{id}$. Removing that one tuple therefore leaves

$$\sum_{\substack{b+i_2+\cdots+i_n=m-a\\ b\leq m-a-1}} p_n\big(\delta_b(A_1), \delta_{i_2}(A_2), \ldots, \delta_{i_n}(A_n)\big) = \sum_{\mathbf{j}\in\mathcal{C}(m-a,n)} p_n\big(\delta_{j_1}(A_1), \ldots, \delta_{j_n}(A_n)\big) - p_n\big(\delta_{m-a}(A_1), A_2, \ldots, A_n\big), \quad (25)$$

where $\mathbf{j} = (b, i_2, \dots, i_n)$. Since $0 \le m - a \le N$ and $(A_1, \dots, A_n) \in \mathcal{S}$, the identity (6) at level $m - a$ reads

$$\sum_{\mathbf{j} \in \mathcal{C}(m-a,n)} p_n\big(\delta_{j_1}(A_1), \dots, \delta_{j_n}(A_n)\big) = \delta_{m-a}\big(p_n(A_1, \dots, A_n)\big). \tag{26}$$

Substituting Equation (26) into Equation (25), and the result into (24), we arrive at Equation (18).

Step 4: Fix $\mathbf{i} \in \mathcal{C}(m, n)$ with $i_2 \le m - 1$. Hypothesis (17) at level i_2, the additivity of p_n in its second argument, and Lemmas 2 and 1 applied in that slot give

$$p_n\big(\delta_{i_1}(A_1), \varphi_{i_2}(A_2), \delta_{i_3}(A_3), \dots, \delta_{i_n}(A_n)\big) = \sum_{a+b=i_2} W_a \, p_n\big(\delta_{i_1}(A_1), \delta_b(A_2), \delta_{i_3}(A_3), \dots, \delta_{i_n}(A_n)\big).$$

Summing over the multi-indices occurring in (14) and collecting the terms according to the value of a, the constraints $i_1 + (a + b) + i_3 + \cdots + i_n = m$ and $a + b \le m - 1$ become, for fixed a,

$$i_1 + b + i_3 + \cdots + i_n = m - a, \qquad b \le m - a - 1,$$

so that

$$R'_m(A_1, \dots, A_n) = \sum_{a=0}^{m-1} W_a \sum_{\substack{i_1 + b + i_3 + \cdots + i_n = m-a \\ b \le m-a-1}} p_n\big(\delta_{i_1}(A_1), \delta_b(A_2), \delta_{i_3}(A_3), \dots, \delta_{i_n}(A_n)\big).$$

For fixed a the unique tuple with $b = m - a$ is the one with $i_1 = 0$ and $i_k = 0$ for every $k \ge 3$, and its contribution is $p_n\big(A_1, \delta_{m-a}(A_2), A_3, \dots, A_n\big)$. Completing the inner sum as in Step 3 and applying (6) at level $m - a$ once more yields (19). □

Proposition 2. *Under the hypotheses of Lemma 7, put*

$$\psi_m = \varphi_m - \sum_{a=0}^{m-1} W_a \, \delta_{m-a}. \tag{27}$$

Then ψ_m is additive and satisfies, for all $(A_1, \dots, A_n) \in \mathcal{S}$,

$$\psi_m\big(p_n(A_1, \dots, A_n)\big) = p_n\big(\psi_m(A_1), A_2, \dots, A_n\big) = p_n\big(A_1, \psi_m(A_2), A_3, \dots, A_n\big). \tag{28}$$

Proof. Additivity is clear, each φ_m and δ_{m-a} being additive and each W_a being a fixed element of $\mathcal{Z}(R)$. Fix $(A_1, \dots, A_n) \in \mathcal{S}$ and write $\pi = p_n(A_1, \dots, A_n)$.

Evaluating Equation (27) at π gives

$$\varphi_m(\pi) = \psi_m(\pi) + \sum_{a=0}^{m-1} W_a \, \delta_{m-a}(\pi). \tag{29}$$

Evaluating Equation (27) at A_1, inserting the result into the first slot of p_n and using Lemma 1 gives

$$p_n\big(\varphi_m(A_1), A_2, \dots, A_n\big) = p_n\big(\psi_m(A_1), A_2, \dots, A_n\big) + \sum_{a=0}^{m-1} W_a \, p_n\big(\delta_{m-a}(A_1), A_2, \dots, A_n\big). \tag{30}$$

Substituting Equations (29), (30), and (18) into Equation (15) we obtain

$$\begin{aligned}\psi_m(\pi)+\sum_{a=0}^{m-1}W_a\delta_{m-a}(\pi) &= p_n\big(\psi_m(A_1),A_2,\dots,A_n\big)+\sum_{a=0}^{m-1}W_a\,p_n\big(\delta_{m-a}(A_1),A_2,\dots,A_n\big)\\ &\quad+\sum_{a=0}^{m-1}W_a\Big[\delta_{m-a}(\pi)-p_n\big(\delta_{m-a}(A_1),A_2,\dots,A_n\big)\Big]\\ &= p_n\big(\psi_m(A_1),A_2,\dots,A_n\big)+\sum_{a=0}^{m-1}W_a\delta_{m-a}(\pi).\end{aligned}$$

Canceling the common sum leaves

$$\psi_m\big(p_n(A_1,\dots,A_n)\big)=p_n\big(\psi_m(A_1),A_2,\dots,A_n\big).$$

The same substitution into Equation (16), using Equation (19) and

$$\begin{aligned}p_n\big(A_1,\varphi_m(A_2),A_3,\dots,A_n\big) &= p_n\big(A_1,\psi_m(A_2),A_3,\dots,A_n\big)\\ &\quad+\sum_{a=0}^{m-1}W_a\,p_n\big(A_1,\delta_{m-a}(A_2),A_3,\dots,A_n\big),\end{aligned}$$

gives $\psi_m(p_n(A_1,\dots,A_n))=p_n(A_1,\psi_m(A_2),A_3,\dots,A_n)$. □

The converse computation is equally explicit and requires no hypothesis beyond those already in force.

Proposition 3. *Let Δ be a higher Lie n-derivation on $\mathcal{S}$, let $W(t)\in\mathcal{Z}(R)[\![t]\!]_N$ and let Ξ be a null family on $\mathcal{S}$. Then $\Phi(t)=W(t)\Delta(t)+\Xi(t)$ is a Δ-higher Lie n-centralizer on $\mathcal{S}$.*

Proof. Fix $m\in\mathfrak{N}$ and $(A_1,\dots,A_n)\in\mathcal{S}$, and write $\pi=p_n(A_1,\dots,A_n)$. By definition $\varphi_k=\sum_{a+b=k}W_a\delta_b+\xi_k$ for every k, so

$$\varphi_m(\pi)=\sum_{a+b=m}W_a\,\delta_b(\pi)+\xi_m(\pi)=\sum_{a+b=m}W_a\,\delta_b(\pi). \tag{31}$$

On the other hand, using Lemmas 1 and 2 exactly as in Step 1 of the proof of Lemma 7,

$$\begin{aligned}\sum_{\mathbf{i}\in\mathcal{C}(m,n)}p_n\big(\varphi_{i_1}(A_1),\delta_{i_2}(A_2),\dots,\delta_{i_n}(A_n)\big) &= \sum_{\mathbf{i}\in\mathcal{C}(m,n)}\ \sum_{a+b=i_1}W_a\,p_n\big(\delta_b(A_1),\delta_{i_2}(A_2),\dots,\delta_{i_n}(A_n)\big)\\ &= \sum_{a=0}^{m}W_a\sum_{\mathbf{j}\in\mathcal{C}(m-a,n)}p_n\big(\delta_{j_1}(A_1),\dots,\delta_{j_n}(A_n)\big)\\ &= \sum_{a=0}^{m}W_a\,\delta_{m-a}(\pi),\end{aligned} \tag{32}$$

where the middle equality is the reindexing $(a,b,i_2,\dots,i_n)\mapsto(a,\mathbf{j})$ with $\mathbf{j}=(b,i_2,\dots,i_n)$. Comparing Equation (31) with Equation (32) gives Equation (7). The computation for Equation (8) is identical, the reindexing being $(a,b,i_1,i_3,\dots,i_n)\mapsto(a,(i_1,b,i_3,\dots,i_n))$. □

We can now state the abstract form of our main result. Its hypothesis is precisely the statement that the rank-zero problem has been solved.

Definition 3. *We say that the pair $(R,\mathcal{S})$ has* property $(\star)$ *if every additive map $\psi\colon R\to R$ satisfying*

$$\psi\big(p_n(A_1,\dots,A_n)\big)=p_n\big(\psi(A_1),A_2,\dots,A_n\big)=p_n\big(A_1,\psi(A_2),A_3,\dots,A_n\big)$$

for all $(A_1, \dots, A_n) \in \mathcal{S}$ *is of the form* $\psi = W\mathrm{id} + \xi$ *with* $W \in \mathcal{Z}(R)$ *and* $\xi\colon R \to \mathcal{Z}(R)$ *additive with* $\xi(p_n(A_1, \dots, A_n)) = 0$ *for all* $(A_1, \dots, A_n) \in \mathcal{S}$.

Shrinking $\mathcal{S}$ makes property $(\star)$ a stronger requirement, not a weaker one: fewer tuples are available to constrain ψ, so more maps have to be shown to have the standard form. It is therefore not evident that $(\star)$ should pass from $\mathcal{S}$ to a larger $\mathcal{S}'$, on which the conclusion also demands more, namely that ξ vanish at every $p_n(\mathcal{S}')$. The next proposition shows that it does, and that the extra vanishing comes for free.

Proposition 4. *Let* $\mathcal{S} \subseteq \mathcal{S}' \subseteq R^n$ *and suppose that* $(R, \mathcal{S})$ *has property* $(\star)$. *Then* $(R, \mathcal{S}')$ *has property* $(\star)$.

Proof. Let $\psi\colon R \to R$ be additive and satisfy

$$\psi\big(p_n(A_1, \dots, A_n)\big) = p_n\big(\psi(A_1), A_2, \dots, A_n\big) = p_n\big(A_1, \psi(A_2), A_3, \dots, A_n\big) \tag{33}$$

for all $(A_1, \dots, A_n) \in \mathcal{S}'$. Since $\mathcal{S} \subseteq \mathcal{S}'$, the same holds for all $(A_1, \dots, A_n) \in \mathcal{S}$, so property $(\star)$ for $(R, \mathcal{S})$ gives $W \in \mathcal{Z}(R)$ and an additive $\xi\colon R \to \mathcal{Z}(R)$ with

$$\psi(A) = WA + \xi(A) \qquad (A \in R). \tag{34}$$

It remains to show that ξ vanishes at p_n on the larger set. Let $(A_1, \dots, A_n) \in \mathcal{S}'$ and put $\pi = p_n(A_1, \dots, A_n)$. Substituting (34) into the first equality of (33) and using the additivity of p_n in its first argument,

$$\psi(\pi) = p_n\big(WA_1 + \xi(A_1), A_2, \dots, A_n\big) = p_n\big(WA_1, A_2, \dots, A_n\big) + p_n\big(\xi(A_1), A_2, \dots, A_n\big) = W\pi,$$

the last equality by Lemma 1 for the first summand and by Lemma 2 for the second, the element $\xi(A_1)$ being central. On the other hand, (34) evaluated at π gives $\psi(\pi) = W\pi + \xi(\pi)$. Comparing the two expressions yields $\xi(\pi) = 0$. □

Theorem 2. *Let* R *be a ring,* $n \geq 2$, $\mathcal{S} \subseteq R^n$, *and suppose* $(R, \mathcal{S})$ *has property* $(\star)$. *Let* $N \in \mathbb{N} \cup \{\infty\}$ *and* Δ *be a higher Lie n-derivation of rank* N *on* $\mathcal{S}$. *Then a family* $\Phi = \{\varphi_m\}_{m \in \mathfrak{N}}$ *of additive maps is a* Δ*-higher Lie n-centralizer on* $\mathcal{S}$ *if and only if there exist* $W(t) \in \mathcal{Z}(R)[\![t]\!]_N$ *and a null family* Ξ *on* $\mathcal{S}$ *with*

$$\Phi(t) = W(t)\Delta(t) + \Xi(t).$$

Proof. Sufficiency is Proposition 3. For necessity, we construct W_m and ξ_m by strong induction on m.

Suppose $m = 0$, by Remark 1 the pair of conditions (7) and (8) at level 0 says that φ_0 is an additive map with

$$\varphi_0\big(p_n(A_1, \dots, A_n)\big) = p_n\big(\varphi_0(A_1), A_2, \dots, A_n\big) = p_n\big(A_1, \varphi_0(A_2), A_3, \dots, A_n\big)$$

for every $(A_1, \dots, A_n) \in \mathcal{S}$. Property $(\star)$ produces $W_0 \in \mathcal{Z}(R)$ and an additive $\xi_0\colon R \to \mathcal{Z}(R)$, vanishing on $p_n(\mathcal{S})$, with $\varphi_0 = W_0\mathrm{id} + \xi_0$. Since $\delta_0 = \mathrm{id}$, this is $\varphi_0 = \sum_{a+b=0} W_a\delta_b + \xi_0$.

Let $1 \leq m \leq N$ and assume that $W_0, \dots, W_{m-1} \in \mathcal{Z}(R)$ and additive maps $\xi_0, \dots, \xi_{m-1}\colon R \to \mathcal{Z}(R)$, each vanishing on $p_n(\mathcal{S})$, have been found with Equation (17). By Proposition 2 the additive map

$$\psi_m = \varphi_m - \sum_{a=0}^{m-1} W_a\delta_{m-a}$$

satisfies Equation (28) on $\mathcal{S}$. Property $(\star)$ therefore yields $W_m \in \mathcal{Z}(R)$ and an additive $\xi_m \colon R \to \mathcal{Z}(R)$ vanishing on $p_n(\mathcal{S})$ with $\psi_m = W_m \mathrm{id} + \xi_m$. Since $\delta_0 = \mathrm{id}$,

$$\varphi_m = \psi_m + \sum_{a=0}^{m-1} W_a \delta_{m-a} = W_m \delta_0 + \sum_{a=0}^{m-1} W_a \delta_{m-a} + \xi_m = \sum_{a+b=m} W_a \delta_b + \xi_m,$$

which is Equation (17) with m in place of $m-1$. The induction produces $W(t) = \sum_m W_m t^m$ and $\Xi = \{\xi_m\}$ with the required properties. □

Combining the two preceding results shows that a single rank-zero input governs an entire range of test sets.

Corollary 1. *Let R be a ring, $n \geq 2$, and let $\mathcal{S} \subseteq R^n$ be such that $(R, \mathcal{S})$ has property $(\star)$. Let $\mathcal{S}'$ satisfy $\mathcal{S} \subseteq \mathcal{S}' \subseteq R^n$ and let Δ be a higher Lie n-derivation of rank N on $\mathcal{S}'$. Then a family Φ of additive maps is a Δ-higher Lie n-centralizer on $\mathcal{S}'$ if and only if $\Phi(t) = W(t)\Delta(t) + \Xi(t)$ for some $W(t) \in \mathcal{Z}(R)[\![t]\!]_N$ and some family Ξ null on $\mathcal{S}'$.*

Proof. By Proposition 4 the pair $(R, \mathcal{S}')$ has property $(\star)$, and Theorem 2 applies to $\mathcal{S}'$. □

An Application Beyond von Neumann Algebras

To show that Theorem 2 is a reusable principle rather than a device adapted to one input, we verify property $(\star)$ from first principles in a setting unrelated to operator algebras, and read off the higher classification there.

Let K be a commutative unital ring and let $T = T_2(K)$ be the ring of upper triangular 2×2 matrices over K, with matrix units e_{11}, e_{12}, e_{22} and unit $I = e_{11} + e_{22}$. Additively $T = Ke_{11} \oplus Ke_{12} \oplus Ke_{22}$, and the brackets we shall use are

$$[e_{11}, e_{12}] = e_{12}, \quad [e_{12}, e_{11}] = -e_{12}, \quad [e_{12}, e_{22}] = e_{12}, \quad [e_{22}, e_{12}] = -e_{12}, \quad [e_{11}, e_{22}] = [e_{12}, e_{12}] = 0. \tag{35}$$

Consequently, $[T, T] \subseteq Ke_{12}$ and $[Ke_{12}, T] \subseteq Ke_{12}$, so that

$$p_n(A_1, \ldots, A_n) \in Ke_{12} \qquad (A_1, \ldots, A_n \in T,\ n \geq 2). \tag{36}$$

Moreover, $\mathcal{Z}(T) = KI$: if $A = \alpha e_{11} + \beta e_{12} + \gamma e_{22}$ is central, then $[e_{11}, A] = \beta e_{12}$ and $[e_{12}, A] = (\gamma - \alpha) e_{12}$ force $\beta = 0$ and $\alpha = \gamma$.

Proposition 5. *Let K be a commutative unital ring, $T = T_2(K)$ and $n \geq 2$. Then the pair (T, T^n) has property $(\star)$.*

Proof. Let $X, Y \in T$ and suppose $p_2(X, Y) = \mu e_{12}$ for some $\mu \in K$. Filling the remaining $n-2$ slots with e_{11}, and then with e_{22}, the relations $[\mu e_{12}, e_{11}] = -\mu e_{12}$ and $[\mu e_{12}, e_{22}] = \mu e_{12}$ from (35) give

$$p_n(X, Y, e_{11}, \ldots, e_{11}) = (-1)^n \mu\, e_{12}, \qquad p_n(X, Y, e_{22}, \ldots, e_{22}) = \mu\, e_{12}. \tag{37}$$

Now let $\psi \colon T \to T$ be additive and satisfy

$$\psi\big(p_n(A_1, \ldots, A_n)\big) = p_n\big(\psi(A_1), A_2, \ldots, A_n\big) = p_n\big(A_1, \psi(A_2), A_3, \ldots, A_n\big) \tag{38}$$

for all $A_1, \ldots, A_n \in T$. Additively $T = Ke_{11} \oplus Ke_{12} \oplus Ke_{22}$, so there are additive maps $a, b, c, a', b', c' \colon K \to K$ with

$$\psi(\alpha e_{11}) = a(\alpha)e_{11} + b(\alpha)e_{12} + c(\alpha)e_{22}, \qquad \psi(\gamma e_{22}) = a'(\gamma)e_{11} + b'(\gamma)e_{12} + c'(\gamma)e_{22} \qquad (\alpha, \gamma \in K).$$

Step 1: $b = 0$ and $b' = 0$. Take $A_1 = \alpha e_{11}$ and $A_2 = \cdots = A_n = e_{11}$ in (38). Then $p_2(\alpha e_{11}, e_{11}) = 0$, hence $p_n(A_1, \ldots, A_n) = 0$, while

$$p_2\big(\psi(\alpha e_{11}), e_{11}\big) = b(\alpha)[e_{12}, e_{11}] = -b(\alpha)e_{12},$$

so that (37) turns the first equality of (38) into $0 = (-1)^{n-1} b(\alpha) e_{12}$. Thus $b(\alpha) = 0$. Taking instead $A_1 = \gamma e_{22}$ and $A_2 = \cdots = A_n = e_{22}$, again $p_n(A_1, \ldots, A_n) = 0$, while

$$p_2\big(\psi(\gamma e_{22}), e_{22}\big) = b'(\gamma)[e_{12}, e_{22}] = b'(\gamma)e_{12},$$

so (37) gives $0 = b'(\gamma)e_{12}$ and $b'(\gamma) = 0$.

Step 2: Determining ψ on Ke_{12}. Put $\lambda = a(1) - c(1) \in K$ and take $A_1 = e_{11}$, $A_2 = \beta e_{12}$ and $A_3 = \cdots = A_n = e_{11}$. Here $p_2(e_{11}, \beta e_{12}) = \beta e_{12}$, so $p_n(A_1, \ldots, A_n) = (-1)^n \beta e_{12}$ by (37), while Step 1 gives

$$p_2\big(\psi(e_{11}), \beta e_{12}\big) = \big[a(1)e_{11} + c(1)e_{22},\ \beta e_{12}\big] = \big(a(1) - c(1)\big)\beta e_{12} = \lambda \beta e_{12},$$

whence $p_n(\psi(e_{11}), \beta e_{12}, e_{11}, \ldots, e_{11}) = (-1)^n \lambda \beta e_{12}$, again by (37). The first equality of (38) therefore reads $(-1)^n \psi(\beta e_{12}) = (-1)^n \lambda \beta e_{12}$, that is,

$$\psi(\beta e_{12}) = \lambda \beta e_{12} \qquad (\beta \in K). \tag{39}$$

Step 3: Determining ψ on Ke_{11}. Take $A_1 = \alpha e_{11}$, $A_2 = e_{12}$ and $A_3 = \cdots = A_n = e_{11}$. Here $p_2(\alpha e_{11}, e_{12}) = \alpha e_{12}$, so $p_n(A_1, \ldots, A_n) = (-1)^n \alpha e_{12}$, while Step 1 gives

$$p_2\big(\psi(\alpha e_{11}), e_{12}\big) = \big[a(\alpha)e_{11} + c(\alpha)e_{22},\ e_{12}\big] = \big(a(\alpha) - c(\alpha)\big)e_{12},$$

whence $p_n(\psi(\alpha e_{11}), e_{12}, e_{11}, \ldots, e_{11}) = (-1)^n \big(a(\alpha) - c(\alpha)\big)e_{12}$. The first equality of (38) now gives $\psi(\alpha e_{12}) = \big(a(\alpha) - c(\alpha)\big)e_{12}$, and comparison with (39) yields $a(\alpha) - c(\alpha) = \lambda\alpha$. Substituting $a(\alpha) = \lambda\alpha + c(\alpha)$,

$$\psi(\alpha e_{11}) = a(\alpha)e_{11} + c(\alpha)e_{22} = \lambda\alpha\, e_{11} + c(\alpha)\big(e_{11} + e_{22}\big) = \lambda\,(\alpha e_{11}) + c(\alpha)I. \tag{40}$$

Step 4: Determining ψ on Ke_{22}. Take $A_1 = e_{12}$, $A_2 = \gamma e_{22}$ and $A_3 = \cdots = A_n = e_{22}$. Here $p_2(e_{12}, \gamma e_{22}) = \gamma e_{12}$, so $p_n(A_1, \ldots, A_n) = \gamma e_{12}$ by (37), while Step 1 gives

$$p_2\big(e_{12}, \psi(\gamma e_{22})\big) = \big[e_{12},\ a'(\gamma)e_{11} + c'(\gamma)e_{22}\big] = \big(c'(\gamma) - a'(\gamma)\big)e_{12},$$

whence $p_n(e_{12}, \psi(\gamma e_{22}), e_{22}, \ldots, e_{22}) = \big(c'(\gamma) - a'(\gamma)\big)e_{12}$. The second equality of (38) gives $\psi(\gamma e_{12}) = \big(c'(\gamma) - a'(\gamma)\big)e_{12}$, and comparison with (39) yields $c'(\gamma) - a'(\gamma) = \lambda\gamma$. Substituting $c'(\gamma) = \lambda\gamma + a'(\gamma)$,

$$\psi(\gamma e_{22}) = a'(\gamma)e_{11} + c'(\gamma)e_{22} = \lambda\gamma\, e_{22} + a'(\gamma)\big(e_{11} + e_{22}\big) = \lambda\,(\gamma e_{22}) + a'(\gamma)I. \tag{41}$$

Step 5: Assembling ψ. Put $W = \lambda I$ and, for $A = \alpha e_{11} + \beta e_{12} + \gamma e_{22}$,

$$\xi(A) = \big(c(\alpha) + a'(\gamma)\big)I.$$

Then $W \in \mathcal{Z}(T)$, and ξ is additive, being built from the additive maps c and a' and from the coordinates α and γ, which depend additively on A; its values lie in $KI = \mathcal{Z}(T)$. Adding (40), (39), and (41) and using the additivity of ψ,

$$\psi(A) = \psi(\alpha e_{11}) + \psi(\beta e_{12}) + \psi(\gamma e_{22}) = \lambda\big(\alpha e_{11} + \beta e_{12} + \gamma e_{22}\big) + \big(c(\alpha) + a'(\gamma)\big)I = WA + \xi(A).$$

Finally, let $A_1, \dots, A_n \in T$. By (36) the element $p_n(A_1, \dots, A_n)$ is of the form βe_{12}, and such an element has $\alpha = \gamma = 0$, so $\xi\big(p_n(A_1, \dots, A_n)\big) = 0$. □

Corollary 2. *Let K be a commutative unital ring, $T = T_2(K)$, $n \geq 2$, and $N \in \mathbb{N} \cup \{\infty\}$, and let $\Delta = \{\delta_m\}_{m \in \mathfrak{N}}$ be a higher Lie n-derivation on T^n. A family $\Phi = \{\varphi_m\}_{m \in \mathfrak{N}}$ of additive maps on T is a Δ-higher Lie n-centralizer on T^n if and only if*

$$\varphi_m = \sum_{i+j=m} \lambda_i \, \delta_j + \xi_m \qquad (0 \leq m \leq N),$$

where $\lambda_i \in K$ and $\xi_m \colon T \to KI$ is additive with $\xi_m|_{Ke_{12}} = 0$.

Proof. By Proposition 5 the pair (T, T^n) has property $(\star)$, so Theorem 2 applies with $R = T$ and $\mathcal{S} = T^n$ and gives $\Phi(t) = W(t)\Delta(t) + \Xi(t)$ with $W_i \in \mathcal{Z}(T) = KI$, say $W_i = \lambda_i I$, and Ξ null on T^n. It remains to identify the null families. By (36) every $p_n(A_1, \dots, A_n)$ lies in Ke_{12}, and conversely every element of Ke_{12} arises as such a value, since $p_n(e_{11}, \beta e_{12}, e_{11}, \dots, e_{11}) = (-1)^n \beta e_{12}$ by Step 2 of the proof of Proposition 5. Hence, an additive $\xi_m \colon T \to KI$ is null on T^n precisely when it vanishes on Ke_{12}. The converse implication is precisely Proposition 3. □

4. von Neumann Algebras

Property $(\star)$ has been established for von Neumann algebras by Ashraf, Ansari, Akhter, and Wei. Their result is stated for Lie n-centralizers, which are defined by the single identity

$$\varphi\big(p_n(A_1, \dots, A_n)\big) = p_n\big(\varphi(A_1), A_2, \dots, A_n\big) \qquad (A_1, \dots, A_n \in \mathcal{U}), \tag{42}$$

so we first check that a map satisfying (42) satisfies condition (1) as well. Put $\Delta_0 = \{\mathrm{id}, 0, 0, \dots\}$ and $\Phi = \{\varphi, 0, 0, \dots\}$. If $m \geq 1$ and $\mathbf{i} \in \mathcal{C}(m, n)$, then $i_k \geq 1$ for some k, and the corresponding factor δ_{i_k} is zero; hence Δ_0 is a higher Lie n-derivation on $\mathcal{U}^n$, and Φ satisfies (7) at every level $m \geq 1$, both sides being zero. At level 0 condition (7) is (42). Since $\mathcal{U}^n$ is closed under interchanging the first two coordinates, Proposition 1 gives (8) at every level, and at level 0 this is

$$\varphi\big(p_n(A_1, \dots, A_n)\big) = p_n\big(A_1, \varphi(A_2), A_3, \dots, A_n\big) \qquad (A_1, \dots, A_n \in \mathcal{U}).$$

Every Lie n-centralizer therefore satisfies (1) on $\mathcal{U}^n$, and we may state both parts of the following theorem in terms of (1).

Theorem 3 (Ashraf, Ansari, Akhter, and Wei Theorem 2.1, Corollary 2.2 of [18])**.** *Let $\mathcal{U}$ be a von Neumann algebra with unit I and let $I = E_1 + E_2$, where E_1, E_2 are orthogonal central projections such that $\mathcal{U}E_1$ is of type I_1 and $\mathcal{U}E_2$ has no central summands of type I_1. Let $P \in \mathcal{U}E_2$ be a core-free projection with central carrier E_2, and let $\varphi \colon \mathcal{U} \to \mathcal{U}$ be additive.*

(i) *φ satisfies (1) for all $(A_1, \dots, A_n) \in \mathcal{S}_P$ if and only if $\varphi(A) = WA + \xi(A)$ for all $A \in \mathcal{U}$, where $W \in \mathcal{Z}(\mathcal{U})$ and $\xi \colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is additive with $\xi(p_n(A_1, \dots, A_n)) = 0$ for all $(A_1, \dots, A_n) \in \mathcal{S}_P$.*

(ii) *φ satisfies (1) for all $(A_1, \dots, A_n) \in \mathcal{U}^n$ if and only if $\varphi(A) = WA + \xi(A)$ for all $A \in \mathcal{U}$, where $W \in \mathcal{Z}(\mathcal{U})$ and $\xi \colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is additive with $\xi(p_n(A_1, \dots, A_n)) = 0$ for all $A_1, \dots, A_n \in \mathcal{U}$.*

By Definition 3, part (i) says that $(\mathcal{U}, \mathcal{S}_P)$ has property $(\star)$ and part (ii) says that $(\mathcal{U}, \mathcal{U}^n)$ has property $(\star)$.

Theorem 4 (Theorem 1)**.** *Let $\mathcal{U}$ be a von Neumann algebra with unit I, let $E_1 + E_2 = I$ be orthogonal central projections with $\mathcal{U}E_1$ of type I_1 and $\mathcal{U}E_2$ without central summands of type I_1, and let $P \in \mathcal{U}E_2$ be a core-free projection with central carrier E_2. Let $n \geq 2$, let $\Delta = \{\delta_m\}_{m\in\mathfrak{N}}$ be a higher Lie n-derivation on $\mathcal{S}_P$, and let $\Phi = \{\varphi_m\}_{m\in\mathfrak{N}}$ be a family of additive maps on $\mathcal{U}$. Then Φ is a Δ-higher Lie n-centralizer on $\mathcal{S}_P$ if and only if*

$$\varphi_m = \sum_{i+j=m} W_i\,\delta_j + \xi_m \qquad (0 \leq m \leq N),$$

where $W_m \in \mathcal{Z}(\mathcal{U})$ and each $\xi_m\colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is additive with $\xi_m(p_n(A_1,\dots,A_n)) = 0$ for all $(A_1,\dots,A_n) \in \mathcal{S}_P$; equivalently $\Phi(t) = W(t)\Delta(t) + \Xi(t)$.

Proof. The converse is Proposition 3 with $R = \mathcal{U}$ and $\mathcal{S} = \mathcal{S}_P$.

Assume that Φ is a Δ-higher Lie n-centralizer on $\mathcal{S}_P$. We use induction on m.

Let $m = 0$. Since $\mathcal{C}(0,n) = \{(0,\dots,0)\}$ and $\delta_0 = \mathrm{id}$, conditions (7) and (8) at level 0 become

$$\varphi_0\big(p_n(A_1,\dots,A_n)\big) = p_n\big(\varphi_0(A_1),A_2,\dots,A_n\big) = p_n\big(A_1,\varphi_0(A_2),A_3,\dots,A_n\big)$$

for all $(A_1,\dots,A_n) \in \mathcal{S}_P$. By Theorem 3(i), there exist $W_0 \in \mathcal{Z}(\mathcal{U})$ and an additive map $\xi_0\colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ with $\xi_0(p_n(A_1,\dots,A_n)) = 0$ for all $(A_1,\dots,A_n) \in \mathcal{S}_P$ such that $\varphi_0(A) = W_0A + \xi_0(A)$ for all $A \in \mathcal{U}$. Since $\delta_0 = \mathrm{id}$,

$$\varphi_0 = W_0\,\delta_0 + \xi_0 = \sum_{i+j=0} W_i\,\delta_j + \xi_0.$$

Now let $1 \leq m \leq N$, and suppose that $W_0,\dots,W_{m-1} \in \mathcal{Z}(\mathcal{U})$ and additive maps $\xi_0,\dots,\xi_{m-1}\colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ have been found such that $\xi_k(p_n(A_1,\dots,A_n)) = 0$ for all $(A_1,\dots,A_n) \in \mathcal{S}_P$ and

$$\varphi_k = \sum_{i+j=k} W_i\,\delta_j + \xi_k \qquad (0 \leq k \leq m-1).$$

Put

$$\psi_m = \varphi_m - \sum_{a=0}^{m-1} W_a\,\delta_{m-a}.$$

By Proposition 2,

$$\psi_m\big(p_n(A_1,\dots,A_n)\big) = p_n\big(\psi_m(A_1),A_2,\dots,A_n\big) = p_n\big(A_1,\psi_m(A_2),A_3,\dots,A_n\big)$$

for all $(A_1,\dots,A_n) \in \mathcal{S}_P$. Again, by Theorem 3(i) there exist $W_m \in \mathcal{Z}(\mathcal{U})$ and an additive map $\xi_m\colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ with $\xi_m(p_n(A_1,\dots,A_n)) = 0$ for all $(A_1,\dots,A_n) \in \mathcal{S}_P$ such that

$$\psi_m(A) = W_mA + \xi_m(A) \qquad (A \in \mathcal{U}).$$

Since $\delta_0 = \mathrm{id}$, we get

$$\varphi_m = \psi_m + \sum_{a=0}^{m-1} W_a\,\delta_{m-a} = W_m\,\delta_0 + \sum_{a=0}^{m-1} W_a\,\delta_{m-a} + \xi_m = \sum_{a=0}^{m} W_a\,\delta_{m-a} + \xi_m = \sum_{i+j=m} W_i\,\delta_j + \xi_m,$$

which completes the induction.

Finally, put $W(t) = \sum_{m\in\mathfrak{N}} W_m t^m$ and $\Xi = \{\xi_m\}$. Then $W(t) \in \mathcal{Z}(\mathcal{U})[\![t]\!]_N$, the family Ξ is null on $\mathcal{S}_P$, and $\Phi(t) = W(t)\Delta(t) + \Xi(t)$. □

Theorem 5. *Let $\mathcal{U}$ be a von Neumann algebra with no central summands of type I_1, let P be a core-free projection with $\overline{P} = I$, let $n \geq 2$, and let $\Delta = \{\delta_m\}_{m\in\mathfrak{N}}$ be a higher Lie n-derivation on $\mathcal{S}_P$. A family $\Phi = \{\varphi_m\}_{m\in\mathfrak{N}}$ of additive maps is a Δ-higher Lie n-centralizer on $\mathcal{S}_P$ if and only if*

$$\varphi_m = \sum_{i+j=m} W_i\,\delta_j + \xi_m \qquad (0 \leq m \leq N),$$

where $W_m \in \mathcal{Z}(\mathcal{U})$ and each $\xi_m \colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is additive with $\xi_m(p_n(A_1,\ldots,A_n)) = 0$ for all $(A_1,\ldots,A_n) \in \mathcal{S}_P$.

Proof. Take $E_1 = 0$ and $E_2 = I$. These are orthogonal central projections with $E_1 + E_2 = I$, and $\mathcal{U}E_1 = \{0\}$, so the requirement that $\mathcal{U}E_1$ be of type I_1 is vacuous, while $\mathcal{U}E_2 = \mathcal{U}$ has no central summands of type I_1 by hypothesis. Moreover, $P \in \mathcal{U} = \mathcal{U}E_2$ is core-free and $\overline{P} = I = E_2$. Thus $\mathcal{U}$, E_1, E_2, and P satisfy the hypotheses of Theorem 4, and Δ is a higher Lie n-derivation on $\mathcal{S}_P$ by assumption.

Theorem 4 now gives: Φ is a Δ-higher Lie n-centralizer on $\mathcal{S}_P$ if and only if there are $W_m \in \mathcal{Z}(\mathcal{U})$ and additive maps $\xi_m \colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ with $\xi_m(p_n(A_1,\ldots,A_n)) = 0$ for all $(A_1,\ldots,A_n) \in \mathcal{S}_P$ such that

$$\varphi_m = \sum_{i+j=m} W_i\,\delta_j + \xi_m \qquad (0 \leq m \leq N).$$

□

Theorem 6. *Let $\mathcal{U}$ be a von Neumann algebra, let $n \geq 2$, and let $\Delta = \{\delta_m\}_{m\in\mathfrak{N}}$ be a higher Lie n-derivation on $\mathcal{U}^n$. A family $\Phi = \{\varphi_m\}_{m\in\mathfrak{N}}$ of additive maps satisfies (7) for all $m \in \mathfrak{N}$ and all $A_1,\ldots,A_n \in \mathcal{U}$ if and only if $\Phi(t) = W(t)\Delta(t) + \Xi(t)$, where $W(t) \in \mathcal{Z}(\mathcal{U})[\![t]\!]_N$ and $\Xi = \{\xi_m\}$ is a family of additive maps $\mathcal{U} \to \mathcal{Z}(\mathcal{U})$ with $\xi_m(p_n(A_1,\ldots,A_n)) = 0$ for all $A_1,\ldots,A_n \in \mathcal{U}$.*

Proof. Suppose first that Φ satisfies (7) for all $m \in \mathfrak{N}$ and all $A_1,\ldots,A_n \in \mathcal{U}$. The set $\mathcal{U}^n$ is closed under interchanging the first two coordinates, so Proposition 1, applied to Δ and Φ with $\mathcal{S} = \mathcal{U}^n$, gives (8) for all $m \in \mathfrak{N}$ and all $A_1,\ldots,A_n \in \mathcal{U}$. Hence, Φ is a Δ-higher Lie n-centralizer on $\mathcal{U}^n$.

By part (ii) of Theorem 3, every additive map $\psi \colon \mathcal{U} \to \mathcal{U}$ satisfying (1) for all $A_1,\ldots,A_n \in \mathcal{U}$ is of the form $\psi = W\mathrm{id} + \xi$ with $W \in \mathcal{Z}(\mathcal{U})$ and $\xi \colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ additive and vanishing on $p_n(\mathcal{U}^n)$; by Definition 3 the pair $(\mathcal{U},\mathcal{U}^n)$ has property $(\star)$. Theorem 2 applies with $R = \mathcal{U}$ and $\mathcal{S} = \mathcal{U}^n$ and gives $W(t) \in \mathcal{Z}(\mathcal{U})[\![t]\!]_N$ and a family Ξ null on $\mathcal{U}^n$ with $\Phi(t) = W(t)\Delta(t) + \Xi(t)$.

Conversely, let $\Phi(t) = W(t)\Delta(t) + \Xi(t)$ with $W(t) \in \mathcal{Z}(\mathcal{U})[\![t]\!]_N$ and Ξ null on $\mathcal{U}^n$. By Proposition 3 with $R = \mathcal{U}$ and $\mathcal{S} = \mathcal{U}^n$, the family Φ is a Δ-higher Lie n-centralizer on $\mathcal{U}^n$; in particular (7) holds for all $m \in \mathfrak{N}$ and all $A_1,\ldots,A_n \in \mathcal{U}$. □

Uniqueness and Further Properties of the Decomposition

Throughout this subsection $\mathcal{U}$ is a von Neumann algebra with no central summands of type I_1, $P = P_1$ is a core-free projection with $\overline{P_1} = I$, $P_2 = I - P_1$, $\mathcal{U}_{ij} = P_i\mathcal{U}P_j$, and $\Delta = \{\delta_m\}_{m\in\mathfrak{N}}$ is a higher Lie n-derivation on $\mathcal{S}_P$. Proposition 7 is the exception and carries its own hypotheses.

Proposition 6. *Let $W_m, W'_m \in \mathcal{Z}(\mathcal{U})$ and let $\xi_m, \xi'_m \colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ be additive maps such that*

$$\sum_{i+j=m} W_i\delta_j + \xi_m = \sum_{i+j=m} W'_i\delta_j + \xi'_m \qquad (0 \leq m \leq N).$$

Then, $W_m = W'_m$ and $\xi_m = \xi'_m$ for every m. In particular, the data $W(t)$ and Ξ of Theorem 5 are determined by Φ.

Proof. We use induction on m. Let $0 \le m \le N$ and suppose $W_k = W'_k$ and $\xi_k = \xi'_k$ for all $k < m$, this being vacuous when $m = 0$. The terms with $i < m$ on the two sides of the level m identity are then equal, and cancelling them leaves

$$W_m\,\delta_0(A) + \xi_m(A) = W'_m\,\delta_0(A) + \xi'_m(A) \qquad (A \in \mathcal{U}).$$

Since $\delta_0 = \mathrm{id}$, this reads

$$(W_m - W'_m)A = \xi'_m(A) - \xi_m(A) \in \mathcal{Z}(\mathcal{U}) \qquad (A \in \mathcal{U}).$$

Put $C = W_m - W'_m \in \mathcal{Z}(\mathcal{U})$. Then $CA \in \mathcal{Z}(\mathcal{U})$ for every $A \in \mathcal{U}$, so $C = 0$ by Lemma 3, and therefore $\xi'_m = \xi_m$. □

The hypothesis on $\mathcal{U}$ cannot be dropped. On an arbitrary von Neumann algebra, the coefficients W_i may be altered on the type I_1 summand without changing Φ, and the next proposition identifies exactly what remains determined.

Lemma 8. *Let $\mathcal{U}$, E_1, E_2, P, and Δ be as in Theorem 4 and put*

$$\delta_j^{(2)}(B) = E_2\,\delta_j(B) \qquad (B \in \mathcal{U}E_2,\ j \in \mathfrak{N}).$$

Then $\mathcal{U}E_2$ is a von Neumann algebra with unit E_2, center $\mathcal{Z}(\mathcal{U}E_2) = \mathcal{Z}(\mathcal{U})E_2$ and no central summands of type I_1; the projection P lies in $\mathcal{U}E_2$, is core-free there and has central carrier E_2; and $\Delta^{(2)} = \{\delta_j^{(2)}\}_{j\in\mathfrak{N}}$ is a higher Lie n-derivation of $\mathcal{U}E_2$ on $\mathcal{S}_P^{(2)} = \{(B_1,\dots,B_n) \in (\mathcal{U}E_2)^n : B_1B_2 = P\}$.

Proof. The assertions about $\mathcal{U}E_2$, its center and P are the defining properties of the decomposition $I = E_1 + E_2$ recalled in Section 2.

Each $\delta_j^{(2)}$ is additive, being δ_j followed by multiplication by E_2, and it maps $\mathcal{U}E_2$ into $\mathcal{U}E_2$, since $E_2\delta_j(B) = E_2^2\delta_j(B) \in \mathcal{U}E_2$. For $B \in \mathcal{U}E_2$, we have $\delta_0^{(2)}(B) = E_2B = B$, so $\delta_0^{(2)} = \mathrm{id}_{\mathcal{U}E_2}$.

Let $(B_1,\dots,B_n) \in \mathcal{S}_P^{(2)}$. Then $B_1,\dots,B_n \in \mathcal{U}E_2$ and $B_1B_2 = P$, so $(B_1,\dots,B_n) \in \mathcal{S}_P$ and (6) holds for Δ at this tuple. Multiplying it by E_2,

$$E_2\,\delta_m\big(p_n(B_1,\dots,B_n)\big) = \sum_{\mathbf{i}\in\mathcal{C}(m,n)} E_2\,p_n\big(\delta_{i_1}(B_1),\dots,\delta_{i_n}(B_n)\big). \tag{43}$$

Consider the left-hand side. Since $\mathcal{U}E_2$ is an ideal of $\mathcal{U}$ and $B_1 \in \mathcal{U}E_2$, the element $p_n(B_1,\dots,B_n)$ lies in $\mathcal{U}E_2$, and therefore

$$E_2\,\delta_m\big(p_n(B_1,\dots,B_n)\big) = \delta_m^{(2)}\big(p_n(B_1,\dots,B_n)\big). \tag{44}$$

Consider a summand on the right-hand side. The element E_2 is central, so Lemma 1 may be applied in each of the n slots in turn, and $E_2^n = E_2$ because E_2 is idempotent:

$$\begin{aligned} p_n\big(\delta_{i_1}^{(2)}(B_1),\dots,\delta_{i_n}^{(2)}(B_n)\big) &= p_n\big(E_2\delta_{i_1}(B_1),\dots,E_2\delta_{i_n}(B_n)\big) \\ &= E_2^n\,p_n\big(\delta_{i_1}(B_1),\dots,\delta_{i_n}(B_n)\big) = E_2\,p_n\big(\delta_{i_1}(B_1),\dots,\delta_{i_n}(B_n)\big). \end{aligned} \tag{45}$$

Substituting (44) and (45) into (43),

$$\delta_m^{(2)}\big(p_n(B_1,\dots,B_n)\big) = \sum_{\mathbf{i}\in\mathcal{C}(m,n)} p_n\big(\delta_{i_1}^{(2)}(B_1),\dots,\delta_{i_n}^{(2)}(B_n)\big),$$

which is (6) for $\Delta^{(2)}$ on $\mathcal{S}_P^{(2)}$. □

Proposition 7. *Let $\mathcal{U}$, E_1, E_2, P, and Δ be as in Theorem 4, let $W_i \in \mathcal{Z}(\mathcal{U})$ and let $\xi_m : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ be additive maps vanishing at every $p_n(A_1,\dots,A_n)$ with $(A_1,\dots,A_n) \in \mathcal{S}_P$. Let $c_i \in \mathcal{Z}(\mathcal{U})E_1$ for $0 \le i \le N$ and put*

$$\eta_m = \sum_{i+j=m} c_i\,\delta_j \qquad (0 \le m \le N).$$

Then, each η_m is additive, takes its values in $\mathcal{Z}(\mathcal{U})$, and vanishes at every $p_n(A_1,\dots,A_n)$ with $(A_1,\dots,A_n) \in \mathcal{S}_P$; moreover

$$\sum_{i+j=m} (W_i + c_i)\,\delta_j + (\xi_m - \eta_m) = \sum_{i+j=m} W_i\,\delta_j + \xi_m \qquad (0 \le m \le N).$$

Two such representations of the same family Φ agree in

$$W_m E_2 \qquad \text{and} \qquad \xi_m + \sum_{i+j=m} W_i E_1\,\delta_j \qquad (0 \le m \le N),$$

and these two quantities are therefore determined by Φ.

Proof. Since $\mathcal{U}E_1$ is of type I_1, it is abelian, and $\mathcal{U}E_1 \subseteq \mathcal{Z}(\mathcal{U})$ by the first paragraph of the proof of Lemma 4. Each η_m is additive, being a finite sum of central multiples of the additive maps δ_j, and since $c_i = c_i E_1$,

$$\eta_m(A) = \sum_{i+j=m} c_i E_1\,\delta_j(A) \in \mathcal{U}E_1 \subseteq \mathcal{Z}(\mathcal{U}) \qquad (A \in \mathcal{U}).$$

Let $(A_1,\dots,A_n) \in \mathcal{S}_P$ and put $\pi = p_n(A_1,\dots,A_n)$. By Definition 1 at level j,

$$E_1\,\delta_j(\pi) = \sum_{\mathbf{i}\in\mathcal{C}(j,n)} E_1\,p_n\big(\delta_{i_1}(A_1),\dots,\delta_{i_n}(A_n)\big) = 0$$

by Lemma 4, and therefore

$$\eta_m(\pi) = \sum_{i+j=m} c_i E_1\,\delta_j(\pi) = 0.$$

The displayed invariance follows at once:

$$\sum_{i+j=m} (W_i + c_i)\,\delta_j + (\xi_m - \eta_m) = \sum_{i+j=m} W_i\,\delta_j + \eta_m + \xi_m - \eta_m = \sum_{i+j=m} W_i\,\delta_j + \xi_m.$$

Now let

$$\sum_{i+j=m} W_i\,\delta_j + \xi_m = \sum_{i+j=m} W_i'\,\delta_j + \xi_m' \qquad (0 \le m \le N)$$

be two representations of the same family. Multiply this identity by E_2 and restrict it to $\mathcal{U}E_2$. With the notation of Lemma 8, and since E_2 is central,

$$E_2 \sum_{i+j=m} W_i\,\delta_j(B) + E_2\xi_m(B) = \sum_{i+j=m} (W_i E_2)\,\delta_j^{(2)}(B) + E_2\xi_m(B) \qquad (B \in \mathcal{U}E_2),$$

and likewise for the primed data. Here, $W_iE_2 \in \mathcal{Z}(\mathcal{U})E_2 = \mathcal{Z}(\mathcal{U}E_2)$, and $B \mapsto E_2\xi_m(B)$ is an additive map of $\mathcal{U}E_2$ into $\mathcal{Z}(\mathcal{U}E_2)$. The identity therefore becomes an equality of two representations of the same family of maps on $\mathcal{U}E_2$, relative to the higher Lie n-derivation $\Delta^{(2)}$ of Lemma 8. Since $\mathcal{U}E_2$ has no central summands of type I_1 and P is a core-free projection of $\mathcal{U}E_2$ with central carrier E_2, Proposition 6 applies inside $\mathcal{U}E_2$ and gives $W_mE_2 = W'_mE_2$ for every m. Put $c_i = (W'_i - W_i)E_1$. Then $W'_i = W_i + c_i$ for every i, so the invariance just proved gives $\xi'_m = \xi_m - \eta_m$ with $\eta_m = \sum_{i+j=m} c_i\delta_j$, and hence

$$\xi'_m + \sum_{i+j=m} W'_iE_1\,\delta_j = \xi_m - \sum_{i+j=m} c_i\,\delta_j + \sum_{i+j=m} (W_i + c_i)E_1\,\delta_j = \xi_m + \sum_{i+j=m} W_iE_1\,\delta_j,$$

where the last equality uses $c_i = c_iE_1$. □

Theorem 7. *The set $\mathcal{C}_\Delta(\mathcal{S}_P)$ is a $\mathcal{Z}(\mathcal{U})[\![t]\!]_N$-module under $V(t)\cdot\Phi := V(t)\Phi(t)$, and*

$$\mathcal{C}_\Delta(\mathcal{S}_P) = \mathcal{Z}(\mathcal{U})[\![t]\!]_N\cdot\Delta \,\oplus\, \mathcal{N}(\mathcal{S}_P),$$

the first summand being free of rank one on Δ.

Proof. Let $\Phi = W(t)\Delta + \Xi \in \mathcal{C}_\Delta(\mathcal{S}_P)$ and let $V(t) \in \mathcal{Z}(\mathcal{U})[\![t]\!]_N$. Then

$$V(t)\Phi(t) = \big(V(t)W(t)\big)\Delta(t) + V(t)\Xi(t),$$

and $V(t)\Xi(t)$ is null on $\mathcal{S}_P$, since its coefficient $\sum_{i+j=m} V_i\xi_j$ is additive, takes values in $\mathcal{Z}(\mathcal{U})$, and vanishes at $p_n(A_1,\dots,A_n)$ for every $(A_1,\dots,A_n) \in \mathcal{S}_P$. Hence, $V(t)\cdot\Phi \in \mathcal{C}_\Delta(\mathcal{S}_P)$ by Proposition 3, and $\mathcal{C}_\Delta(\mathcal{S}_P)$ is a $\mathcal{Z}(\mathcal{U})[\![t]\!]_N$-module.

Consider the map

$$\mathcal{Z}(\mathcal{U})[\![t]\!]_N \times \mathcal{N}(\mathcal{S}_P) \longrightarrow \mathcal{C}_\Delta(\mathcal{S}_P), \qquad (W,\Xi) \longmapsto W(t)\Delta(t) + \Xi(t).$$

It is $\mathcal{Z}(\mathcal{U})[\![t]\!]_N$-linear, it is onto by Theorem 5, and it is one-to-one by Proposition 6. Taking $\Xi = \Xi' = 0$ and $W' = 0$ in Proposition 6 gives $W(t)\Delta(t) = 0 \Rightarrow W(t) = 0$, so the first summand is free of rank one on Δ. □

Lemma 9. *For all $A_{12} \in \mathcal{U}_{12}$ and $A_{21} \in \mathcal{U}_{21}$,*

$$(P_1 + A_{12})P_1 = P_1, \qquad p_n(P_1 + A_{12}, P_1, P_1, \dots, P_1) = (-1)^{n-1}A_{12}, \tag{46}$$

$$P_1(P_1 + A_{21}) = P_1, \qquad p_n(P_1, P_1 + A_{21}, P_1, \dots, P_1) = -A_{21}. \tag{47}$$

Proof. Write $A_{12} = P_1AP_2$. Then $A_{12}P_1 = P_1A(P_2P_1) = 0$, so $(P_1 + A_{12})P_1 = P_1$. Since p_n is additive in its first argument and $p_n(P_1, P_1, \dots, P_1) = 0$,

$$p_n(P_1 + A_{12}, P_1, \dots, P_1) = p_n(A_{12}, P_1, \dots, P_1) = (-1)^{n-1}P_1A_{12}P_2 + P_2A_{12}P_1 = (-1)^{n-1}A_{12},$$

by (5), together with $P_1A_{12}P_2 = A_{12}$ and $P_2A_{12}P_1 = 0$. This proves (46).

Write $A_{21} = P_2AP_1$. Then $P_1A_{21} = (P_1P_2)AP_1 = 0$, so $P_1(P_1 + A_{21}) = P_1$, and by additivity again

$$p_n(P_1, P_1 + A_{21}, P_1, \dots, P_1) = p_n(P_1, A_{21}, P_1, \dots, P_1).$$

Now $p_2(P_1, A_{21}) = P_1A_{21} - A_{21}P_1 = 0 - A_{21} = -A_{21}$, and if $p_k(P_1, A_{21}, P_1, \dots, P_1) = -A_{21}$ for some $k \geq 2$, then

$$p_{k+1}(P_1, A_{21}, P_1, \dots, P_1) = [-A_{21}, P_1] = -A_{21}P_1 + P_1A_{21} = -A_{21} + 0 = -A_{21},$$

so $p_n(P_1, A_{21}, P_1, \dots, P_1) = -A_{21}$ for every $n \geq 2$, which proves (47). □

Proposition 8. *Let $\Phi = \{\varphi_m\}$ be a Δ-higher Lie n-centralizer on $\mathcal{S}_P$, written as $\varphi_m = \sum_{i+j=m} W_i \delta_j + \xi_m$. Then*

$$\xi_m\big|_{\mathcal{U}_{12}} = 0, \qquad \xi_m\big|_{\mathcal{U}_{21}} = 0 \qquad (0 \leq m \leq N),$$

and consequently

$$\varphi_m(A) = \sum_{i+j=m} W_i\, \delta_j(A) \qquad (A \in \mathcal{U}_{12} \cup \mathcal{U}_{21}).$$

Proof. Let $A_{12} \in \mathcal{U}_{12}$. By (46) the tuple $(P_1 + A_{12}, P_1, \dots, P_1)$ lies in $\mathcal{S}_P$, so

$$0 = \xi_m\big(p_n(P_1 + A_{12}, P_1, \dots, P_1)\big) = \xi_m\big((-1)^{n-1} A_{12}\big) = (-1)^{n-1} \xi_m(A_{12}),$$

since an additive map commutes with multiplication by ± 1; hence $\xi_m(A_{12}) = 0$. Let $A_{21} \in \mathcal{U}_{21}$. By (47) the tuple $(P_1, P_1 + A_{21}, P_1, \dots, P_1)$ lies in $\mathcal{S}_P$, so

$$0 = \xi_m\big(p_n(P_1, P_1 + A_{21}, P_1, \dots, P_1)\big) = \xi_m(-A_{21}) = -\xi_m(A_{21}),$$

and $\xi_m(A_{21}) = 0$. The last equation now follows from $\varphi_m = \sum_{i+j=m} W_i \delta_j + \xi_m$. □

5. Applications

Throughout this section $\mathcal{U}$, E_1, E_2 and P are as in Theorem 4, and $\mathcal{S}_P$ is the set of tuples $(A_1, \dots, A_n) \in \mathcal{U}^n$ with $A_1 A_2 = P$.

We begin with the class of families whose level-zero component is the identity.

Definition 4. *Let $L = \{L_m\}_{m \in \mathfrak{N}}$ be a higher Lie n-derivation on $\mathcal{S}$. A family $G = \{G_m\}_{m \in \mathfrak{N}}$ of additive maps with $G_0 = \mathrm{id}$ is a* generalized higher Lie n-derivation associated with L on $\mathcal{S}$ *if G is an L-higher Lie n-centralizer on $\mathcal{S}$, that is, if (7) and (8) hold with $\varphi_m = G_m$ and $\delta_m = L_m$.*

For $N = 1$, so that $L = \{\mathrm{id}, L_1\}$ and $G = \{\mathrm{id}, G_1\}$, level 1 of (7) reads

$$G_1\big(p_n(A_1, \dots, A_n)\big) = p_n\big(G_1(A_1), A_2, \dots, A_n\big) + \sum_{k=2}^{n} p_n\big(A_1, \dots, L_1(A_k), \dots, A_n\big)$$

for $(A_1, \dots, A_n) \in \mathcal{S}_P$, and (8) reads

$$G_1\big(p_n(A_1, \dots, A_n)\big) = p_n\big(L_1(A_1), A_2, \dots, A_n\big) + p_n\big(A_1, G_1(A_2), A_3, \dots, A_n\big) + \sum_{k=3}^{n} p_n\big(A_1, \dots, L_1(A_k), \dots, A_n\big).$$

These are the two conditions studied in § 3 of [18]; without the restriction to $\mathcal{S}_P$ they define a generalized Lie n-derivation in the sense of [10,11].

Theorem 8. *Let $L = \{L_m\}_{m \in \mathfrak{N}}$ be a higher Lie n-derivation on $\mathcal{S}_P$ and let $G = \{G_m\}_{m \in \mathfrak{N}}$ be a generalized higher Lie n-derivation associated with L on $\mathcal{S}_P$. Then there exist $W_m \in \mathcal{Z}(\mathcal{U})$ for $1 \leq m \leq N$ and additive maps $\chi_m \colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ vanishing at every $p_n(A_1, \dots, A_n)$ with $(A_1, \dots, A_n) \in \mathcal{S}_P$, with $\chi_0 = 0$, such that*

$$G_m = L_m + \sum_{a=1}^{m} W_a\, L_{m-a} + \chi_m \qquad (0 \leq m \leq N).$$

Equivalently, $G(t) = W(t)L(t) + X(t)$*, where* $W(t) = I + \sum_{a\geq 1} W_a t^a$ *is invertible in* $\mathcal{Z}(\mathcal{U})[\![t]\!]_N$ *and* $X = \{\chi_m\}$ *is null on* $\mathcal{S}_P$*; consequently*

$$L(t) = W(t)^{-1}\big(G(t) - X(t)\big).$$

Proof. The pair L, G satisfies the hypotheses of Theorem 4 with $\Delta = L$ and $\Phi = G$, so there are $W_m \in \mathcal{Z}(\mathcal{U})$ and additive maps $\chi_m \colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$, vanishing at every $p_n(A_1, \ldots, A_n)$ with $(A_1, \ldots, A_n) \in \mathcal{S}_P$, with

$$G_m = \sum_{i+j=m} W_i\, L_j + \chi_m \qquad (0 \leq m \leq N). \tag{48}$$

At level 0 the only pair with $i + j = 0$ is $(0, 0)$, and $G_0 = \mathrm{id}$, $L_0 = \mathrm{id}$, so (48) becomes

$$A = G_0(A) = W_0\, L_0(A) + \chi_0(A) = W_0 A + \chi_0(A) \qquad (A \in \mathcal{U}).$$

Put $C = I - W_0 \in \mathcal{Z}(\mathcal{U})$. The last display reads

$$CA = \chi_0(A) \in \mathcal{Z}(\mathcal{U}) \qquad (A \in \mathcal{U}). \tag{49}$$

Let $B \in \mathcal{U}E_2$. By (49), $(CE_2)B = CB \in \mathcal{Z}(\mathcal{U}) \cap \mathcal{U}E_2 = \mathcal{Z}(\mathcal{U}E_2)$, and $\mathcal{U}E_2$ has no central summands of type I_1, so Lemma 3, applied inside $\mathcal{U}E_2$, yields

$$CE_2 = 0, \qquad \text{so that} \qquad W_0E_2 = E_2 \quad \text{and} \quad C = CE_1. \tag{50}$$

Define

$$W_0^* = I, \qquad W_a^* = W_a \ \ (1 \leq a \leq N), \qquad \chi_m^* = \chi_m - C\, L_m \ \ (0 \leq m \leq N).$$

Since $W_0^* - W_0 = C$ and $W_a^* = W_a$ for $a \geq 1$, we have $\sum_{i+j=m} W_i^* L_j = \sum_{i+j=m} W_i L_j + C\, L_m$, and therefore

$$\sum_{i+j=m} W_i^* L_j + \chi_m^* = \Big(\sum_{i+j=m} W_i L_j + C\, L_m\Big) + \big(\chi_m - C\, L_m\big) = \sum_{i+j=m} W_i L_j + \chi_m = G_m$$

for $0 \leq m \leq N$. At $m = 0$ this gives

$$\chi_0^*(A) = \chi_0(A) - C\, L_0(A) = CA - CA = 0 \qquad (A \in \mathcal{U}),$$

so $\chi_0^* = 0$.

Each χ_m^* takes its values in $\mathcal{Z}(\mathcal{U})$. Indeed $\mathcal{U}E_1$ is of type I_1, hence abelian, so $\mathcal{U}E_1 \subseteq \mathcal{Z}(\mathcal{U})$ by the first paragraph of the proof of Lemma 4; combining this with $C = CE_1$ from (50),

$$C\, L_m(A) = CE_1\, L_m(A) \in \mathcal{U}E_1 \subseteq \mathcal{Z}(\mathcal{U}) \qquad (A \in \mathcal{U}),$$

while χ_m is center-valued by construction.

Each χ_m^* vanishes at every $p_n(A_1, \ldots, A_n)$ with $(A_1, \ldots, A_n) \in \mathcal{S}_P$. Writing $\pi = p_n(A_1, \ldots, A_n)$ for such a tuple, (6) at level m gives

$$E_1 L_m(\pi) = \sum_{\mathbf{i} \in \mathcal{C}(m,n)} E_1\, p_n\big(L_{i_1}(A_1), \ldots, L_{i_n}(A_n)\big) = 0$$

by Lemma 4, and therefore

$$\chi_m^*(\pi) = \chi_m(\pi) - C\, L_m(\pi) = 0 - CE_1 L_m(\pi) = 0.$$

Replacing W_i by W_i^* and χ_m by χ_m^* in (48), and using $W_0^* = I$ and $L_0 = \mathrm{id}$,

$$G_m = \sum_{i+j=m} W_i^* L_j + \chi_m^* = L_m + \sum_{a=1}^{m} W_a\, L_{m-a} + \chi_m^* \qquad (0 \le m \le N),$$

which is the asserted formula.

Finally, a series in $\mathcal{Z}(\mathcal{U})[\![t]\!]_N$ with constant term I is invertible: writing $W(t)^{-1} = \sum_j \widetilde{W}_j t^j$, its coefficients are given by

$$\widetilde{W}_0 = I, \qquad \widetilde{W}_j = -\sum_{a=1}^{j} W_a\, \widetilde{W}_{j-a} \qquad (1 \le j \le N).$$

Multiplying $G(t) = W(t)L(t) + X(t)$ by $W(t)^{-1}$ gives $L(t) = W(t)^{-1}\big(G(t) - X(t)\big)$. □

At rank one the conclusion takes two forms, according to whether a summand of type I_1 is present. The first is Theorem 3.3 of [18] and the second is Theorem 3.1 of [18].

Corollary 3. *Let $L, G\colon \mathcal{U} \to \mathcal{U}$ be additive maps such that $\{\mathrm{id}, L\}$ is a higher Lie n-derivation on $\mathcal{S}_P$ and $\{\mathrm{id}, G\}$ is a generalized higher Lie n-derivation associated with it. Then*

$$G(A) = W_1 A + \chi(A) \qquad (A \in \mathcal{U}),$$

where $W_1 \in \mathcal{Z}(\mathcal{U})$ and $\chi\colon \mathcal{U} \to \mathcal{U}$ is an additive map with

$$\chi\big(p_n(A_1, \dots, A_n)\big) = \sum_{k=1}^{n} p_n\big(A_1, \dots, \chi(A_k), \dots, A_n\big) \qquad \big((A_1, \dots, A_n) \in \mathcal{S}_P\big).$$

Proof. At $m = 1$, the formula of Theorem 8 reads

$$G = G_1 = L_1 + W_1 L_0 + \chi_1 = L + W_1 \mathrm{id} + \chi_1,$$

with $W_1 \in \mathcal{Z}(\mathcal{U})$ and $\chi_1\colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ additive and vanishing at every $p_n(A_1, \dots, A_n)$ with $(A_1, \dots, A_n) \in \mathcal{S}_P$. Put $\chi = L + \chi_1$, an additive map on $\mathcal{U}$.

Fix $(A_1, \dots, A_n) \in \mathcal{S}_P$ and $1 \le k \le n$. Since $\chi_1(A_k) \in \mathcal{Z}(\mathcal{U})$, Lemma 2 gives

$$p_n\big(A_1, \dots, \chi_1(A_k), \dots, A_n\big) = 0$$

for $k \in \{1, 2\}$, while for $k \ge 3$,

$$p_k\big(A_1, \dots, A_{k-1}, \chi_1(A_k)\big) = \big[p_{k-1}(A_1, \dots, A_{k-1}), \chi_1(A_k)\big] = 0,$$

again by centrality, so that $p_n(A_1, \dots, \chi_1(A_k), \dots, A_n) = 0$, the remaining brackets being taken with 0. Summing over k,

$$\sum_{k=1}^{n} p_n\big(A_1, \dots, \chi_1(A_k), \dots, A_n\big) = 0.$$

Level 1 of (6) for the family $\{\mathrm{id}, L\}$ gives

$$L\big(p_n(A_1, \dots, A_n)\big) = \sum_{k=1}^{n} p_n\big(A_1, \dots, L(A_k), \dots, A_n\big),$$

and adding the two previous displays, together with $\chi_1(p_n(A_1,\dots,A_n)) = 0$,

$$\chi\big(p_n(A_1,\dots,A_n)\big) = L\big(p_n(A_1,\dots,A_n)\big) + \chi_1\big(p_n(A_1,\dots,A_n)\big) = \sum_{k=1}^{n} p_n\big(A_1,\dots,\chi(A_k),\dots,A_n\big). \quad \square$$

Corollary 4. *Assume in addition that $E_1 = 0$, so that $\mathcal{U}$ has no central summands of type I_1 and $\overline{P} = I$, and let L, G be as in Corollary 3. Then*

$$G(A) = W_1 A + \delta(A) + \chi(A) \qquad (A \in \mathcal{U}),$$

where $W_1 \in \mathcal{Z}(\mathcal{U})$, $\delta\colon \mathcal{U} \to \mathcal{U}$ is an additive derivation, and $\chi\colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is additive with $\chi(p_n(A_1,\dots,A_n)) = 0$ for all $(A_1,\dots,A_n) \in \mathcal{S}_P$.

Proof. At $m = 1$ the formula of Theorem 8 gives $G = L + W_1\mathrm{id} + \chi_1$, with $W_1 \in \mathcal{Z}(\mathcal{U})$ and $\chi_1\colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ additive and vanishing at every $p_n(A_1,\dots,A_n)$ with $(A_1,\dots,A_n) \in \mathcal{S}_P$. Level 1 of (6) says that

$$L\big(p_n(A_1,\dots,A_n)\big) = \sum_{k=1}^{n} p_n\big(A_1,\dots,L(A_k),\dots,A_n\big) \qquad \big((A_1,\dots,A_n) \in \mathcal{S}_P\big).$$

Since $\mathcal{U}$ has no central summands of type I_1, we may write $L = \delta + \gamma$ with $\delta\colon \mathcal{U} \to \mathcal{U}$ an additive derivation and $\gamma\colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ additive and vanishing at every $p_n(A_1,\dots,A_n)$ with $(A_1,\dots,A_n) \in \mathcal{S}_P$ Theorem 3.11 of [37]. Put $\chi = \chi_1 + \gamma$. $\square$

We turn to the two smallest values of n and to two extreme choices of Δ.

Corollary 5. *Let $\Delta = \{\delta_m\}_{m\in\mathfrak{N}}$ be a higher Lie n-derivation on $\mathcal{S}_P$ and let $\Phi = \{\varphi_m\}_{m\in\mathfrak{N}}$ be a family of additive maps on $\mathcal{U}$.*

(i) *Let $n = 2$. Then Φ satisfies*

$$\varphi_m\big([A,B]\big) = \sum_{i+j=m} \big[\varphi_i(A), \delta_j(B)\big] = \sum_{i+j=m} \big[\delta_i(A), \varphi_j(B)\big] \qquad (A,B \in \mathcal{U},\ AB = P)$$

for all $m \in \mathfrak{N}$ if and only if $\varphi_m = \sum_{i+j=m} W_i\delta_j + \xi_m$ with $W_m \in \mathcal{Z}(\mathcal{U})$ and $\xi_m\colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ additive and vanishing at every $[A,B]$ with $AB = P$.

(ii) *Let $n = 3$. Then Φ satisfies*

$$\begin{aligned}\varphi_m\big([[A,B],C]\big) &= \sum_{i+j+k=m} \big[[\varphi_i(A), \delta_j(B)], \delta_k(C)\big] \\ &= \sum_{i+j+k=m} \big[[\delta_i(A), \varphi_j(B)], \delta_k(C)\big]\end{aligned} \qquad (A,B,C \in \mathcal{U},\ AB = P)$$

for all $m \in \mathfrak{N}$ if and only if $\varphi_m = \sum_{i+j=m} W_i\delta_j + \xi_m$ with $W_m \in \mathcal{Z}(\mathcal{U})$ and $\xi_m\colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ additive and vanishing at every $[[A,B],C]$ with $AB = P$.

Proof. Write $p_2(A,B) = [A,B]$ and $p_3(A,B,C) = [[A,B],C]$. For each of $n = 2$ and $n = 3$, the two displayed identities are Equations (7) and (8), and the assertion is the corresponding case of Theorem 4. $\square$

Corollary 6. *Let $\Delta_0 = \{\mathrm{id}, 0, 0, \dots\}$. Then Δ_0 is a higher Lie n-derivation on $\mathcal{U}^n$, and a family $\Phi = \{\varphi_m\}$ of additive maps is a Δ_0-higher Lie n-centralizer on $\mathcal{S}_P$ if and only if*

$$\varphi_m = W_m\mathrm{id} + \xi_m \qquad (0 \le m \le N)$$

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with $W_m \in \mathcal{Z}(\mathcal{U})$ and $\xi_m \colon \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ additive and vanishing at every $p_n(A_1, \ldots, A_n)$ with $(A_1, \ldots, A_n) \in \mathcal{S}_P$.

Proof. Let $m \geq 1$ and $\mathbf{i} \in \mathcal{C}(m, n)$. Then $i_k \geq 1$ for some k, so the k-th argument of the corresponding term of (6) is $\delta_{i_k}(A_k) = 0$, and the term vanishes because p_n is additive in that argument. Both sides of (6) are thus zero for $m \geq 1$, while at $m = 0$ it reads $p_n(A_1, \ldots, A_n) = p_n(A_1, \ldots, A_n)$; hence Δ_0 is a higher Lie n-derivation on $\mathcal{U}^n$.

Write $\Delta_0 = \{\delta_j^{(0)}\}_{j \in \mathfrak{N}}$, so that $\delta_0^{(0)} = \mathrm{id}$ and $\delta_j^{(0)} = 0$ for $j \geq 1$. Taking $\Delta = \Delta_0$ in Theorem 4, the family Φ is a Δ_0-higher Lie n-centralizer on $\mathcal{S}_P$ if and only if $\varphi_m = \sum_{i+j=m} W_i \delta_j^{(0)} + \xi_m$ with W_m and ξ_m as stated, and this sum reduces to its term with $j = 0$, namely $W_m \mathrm{id}$. □

Corollary 7. *Let $\mathcal{U}$ be a factor with $\mathcal{U} \neq \mathbb{C}I$ and let $P \in \mathcal{U}$ be a core-free projection with $\overline{P} = I$. Then every Δ-higher Lie n-centralizer Φ on $\mathcal{S}_P$ has the form*

$$\varphi_m(A) = \sum_{i+j=m} \lambda_i \, \delta_j(A) + \xi_m(A) \qquad (A \in \mathcal{U}),$$

where $\lambda_i \in \mathbb{C}$ and $\xi_m \colon \mathcal{U} \to \mathbb{C}I$ is additive and vanishes at every $p_n(A_1, \ldots, A_n)$ with $(A_1, \ldots, A_n) \in \mathcal{S}_P$. If, moreover, the additive subgroup of $\mathcal{U}$ generated by $\{p_n(A_1, \ldots, A_n) : (A_1, \ldots, A_n) \in \mathcal{S}_P\}$ is all of $\mathcal{U}$, then $\xi_m = 0$ for every m and $\Phi(t) = \left(\sum_i \lambda_i t^i\right)\Delta(t)$.

Proof. A factor has center $\mathcal{Z}(\mathcal{U}) = \mathbb{C}I$ and its only central projections are 0 and I; as $\mathcal{U} \neq \mathbb{C}I$, the algebra $\mathcal{U}$ is not of type I_1, and so it has no central summands of type I_1. Theorem 5 therefore gives $\varphi_m = \sum_{i+j=m} W_i \delta_j + \xi_m$ with $W_i \in \mathcal{Z}(\mathcal{U}) = \mathbb{C}I$, say $W_i = \lambda_i I$ with $\lambda_i \in \mathbb{C}$, and with ξ_m mapping into $\mathcal{Z}(\mathcal{U}) = \mathbb{C}I$.

Assume now that the additive subgroup generated by the elements $p_n(A_1, \ldots, A_n)$, $(A_1, \ldots, A_n) \in \mathcal{S}_P$, is all of $\mathcal{U}$. Being additive and vanishing on those elements, ξ_m vanishes on the subgroup they generate, that is, on $\mathcal{U}$. Hence, $\xi_m = 0$ and $\varphi_m = \sum_{i+j=m} \lambda_i \delta_j$, which is the coefficient of t^m in $\left(\sum_i \lambda_i t^i\right)\Delta(t)$. □

Finally we record when the maps φ_m inherit linearity or continuity.

Proposition 9. *Assume $E_1 = 0$, let $\Phi = \{\varphi_m\}$ be a Δ-higher Lie n-centralizer on $\mathcal{S}_P$, written as $\varphi_m = \sum_{i+j=m} W_i \delta_j + \xi_m$, and let $0 \leq m \leq N$.*

(i) *If $\delta_0, \ldots, \delta_m$ are linear, then φ_m is linear if and only if ξ_m is linear.*
(ii) *If $\delta_0, \ldots, \delta_m$ are continuous for a topology on $\mathcal{U}$ in which multiplication by a fixed element of $\mathcal{Z}(\mathcal{U})$ is continuous, then φ_m is continuous if and only if ξ_m is continuous.*

Proof. Write

$$\Theta_m = \sum_{i+j=m} W_i \, \delta_j, \qquad \text{so that} \qquad \varphi_m = \Theta_m + \xi_m \quad \text{and} \quad \xi_m = \varphi_m - \Theta_m.$$

The sum defining Θ_m has one term for each of the finitely many pairs (i, j) with $i + j = m$, and each term is δ_j followed by multiplication by the fixed central element W_i.

(i) Let $\delta_0, \ldots, \delta_m$ be linear. For $A, B \in \mathcal{U}$ and $\lambda \in \mathbb{C}$,

$$\Theta_m(\lambda A + B) = \sum_{i+j=m} W_i \, \delta_j(\lambda A + B) = \sum_{i+j=m} W_i\big(\lambda \, \delta_j(A) + \delta_j(B)\big) = \lambda \, \Theta_m(A) + \Theta_m(B),$$

so Θ_m is linear. If ξ_m is linear, then $\varphi_m = \Theta_m + \xi_m$ is a sum of two linear maps and is linear; if φ_m is linear, then $\xi_m = \varphi_m - \Theta_m$ is a difference of two linear maps and is linear.

(ii) Let $\delta_0, \dots, \delta_m$ be continuous for a topology on $\mathcal{U}$ in which multiplication by a fixed element of $\mathcal{Z}(\mathcal{U})$ is continuous. Each map $A \mapsto W_i\, \delta_j(A)$ is then a composition of two continuous maps, and Θ_m, a finite sum of such maps, is continuous. If ξ_m is continuous, then so is $\varphi_m = \Theta_m + \xi_m$; if φ_m is continuous, then so is $\xi_m = \varphi_m - \Theta_m$. □

6. Concluding Remarks

The transfer theorem, Theorem 2, is formulated for an arbitrary ring R and an arbitrary set of test tuples $\mathcal{S} \subseteq R^n$, and its proof uses only additivity, the ring axioms, and property $(\star)$ at rank zero; nothing in the argument uses commutativity of scalars or the existence of an involution. It is therefore natural to ask whether an analogue of the transfer theorem holds beyond associative rings, for instance for a unital alternative algebra A with the left-normed commutators $p_1(x_1) = x_1$, $p_n(x_1, \dots, x_n) = [p_{n-1}(x_1, \dots, x_{n-1}), x_n]$: if an appropriate rank-zero version of property $(\star)$ is available on A, does every Δ-higher Lie n-centralizer necessarily decompose as $\Phi(t) = W(t)\Delta(t) + \Xi(t)$, with central coefficients and a center-valued null family, as in Theorem 2? Rigidity phenomena of this kind have already been observed for Lie-type maps on alternative and alternative $*$-algebras [31,32], and alternative division algebras and split octonion algebras would be natural first cases to examine. We leave this direction, and the corresponding non-associative counterpart of Theorem 1, for future work.

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