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Characterizing the Signless Laplacian H-Spectral Radius for Uniform Directed Hypergraphs

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Abstract

The importance of the signless Laplacian H-spectral radius is highlighted by its known links to connectivity and the maximum cut and independence number. In this paper, we establish signless Laplacian H-spectral radius bounds by simultaneously incorporating both pairwise and higher-order outdegree information. In contrast to the important estimates based solely on the maximum outdegree, our approach yields substantially improved bounds by leveraging richer structural parameters. We further introduce outdegree regularity and semiregularity, under which the tightness of the proposed bounds is fully characterized. Based on similarity transformations, we construct improved bounds using average two-outdegree data. These bounds reveal distinct behaviors between the adjacency and signless Laplacian H-spectral radius of directed hypergraphs. Numerical examples are provided to illustrate the effectiveness of these bounds.

Keywords: directed hypergraph; signless Laplacian H-spectral radius; lower and upper bounds

MSC: 15A18; 15A69; 05C65; 05C20; 05C35

1. Introduction

Let $\mathbb{C}$ and $\mathbb{R}$ denote the sets of complex and real numbers, respectively, and let $\mathbb{C}^n$ and $\mathbb{R}^n$ denote the corresponding n-dimensional complex and real vector spaces, respectively. We write $[n] = \{1, 2, \ldots, n\}$. An mth-order n-dimensional tensor $\mathcal{A}$ is an array of n^m real entries $a_{i_1 i_2 \cdots i_m} \in \mathbb{R}$, where $i_j \in [n]$ for $j = 1, \ldots, m$. If $a_{i_1 \cdots i_m} = a_{i_{p(1)} \cdots i_{p(m)}}$ for every permutation p of $[m]$, then $\mathcal{A}$ is called a symmetric tensor. If all entries are nonnegative (positive), then $\mathcal{A}$ is called a nonnegative (positive) tensor.

A tensor is a higher-order extension of a matrix, which plays an important role in multilinear PageRank [1], higher-order Markov chains [2] and medical imaging [3,4]. Just as matrices serve as a fundamental tool for unveiling graph properties [5], tensors offer an indispensable framework for exploring the structural characteristics of hypergraphs [6,7]. To capture the properties of different undirected hypergraphs, signless Laplacian tensors and adjacency tensors have been established [6,8–13].

Unlike undirected hypergraphs, the edges (called arcs) in directed hypergraphs have directionality, capturing unidirectional relationships among nodes. As a generalization of directed graphs, directed hypergraphs find wide application in image processing [14,15] and social network analysis [16–20]. For example, hypergraph-based representations

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have been successfully used for feature extraction and segmentation in image processing. Formally, a k-uniform directed hypergraph $\vec{\mathbb{G}} = (\mathbb{V}, \mathbb{E})$ consists of a vertex set $\mathbb{V} = [n]$ and an arc set $\mathbb{E} = \{e_1, \ldots, e_m\}$. Each arc $e \in \mathbb{E}$ has the form $e = (v_1, v_2, \ldots, v_k)$, where v_1 is the tail (out-vertex) and $v_2, \ldots, v_k$ are the heads (in-vertices). Accordingly, the associated adjacency or signless Laplacian tensor is assumed to be symmetric with respect to the last $k-1$ indices (or fully symmetric) to ensure a well-defined tensorial representation. The outdegree and indegree of a vertex i are defined as follows:

$$d_i^+ = |\mathbb{E}_i^+|, \qquad d_i^- = \frac{1}{k-1}|\mathbb{E}_i^-|,$$

where $\mathbb{E}_i^+ = \{e \in \mathbb{E} : i \text{ is the tail of } e\}$ and $\mathbb{E}_i^- = \{e \in \mathbb{E} : i \text{ is one of the heads of } e\}$.

The pioneering work in [21] showed that k-uniform directed hypergraphs correspond bijectively to their adjacency tensors. Specifically, for a directed hypergraph $\vec{\mathbb{G}} = (\mathbb{V}, \mathbb{E})$, the adjacency tensor $\mathcal{A}(\vec{\mathbb{G}})$ is a kth-order n-dimensional nonnegative tensor, whose $(i_1, \ldots, i_k)$-entry is defined by

$$\mathcal{A}(\vec{\mathbb{G}})_{i_1 \ldots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } (i_1, \ldots, i_k) = e \in \mathbb{E} \text{ and } i_1 \text{ is the tail of } e, \\ 0, & \text{otherwise.} \end{cases}$$

The tensor is understood to be symmetric with respect to the last $k-1$ indices, and thus the ordering of the heads does not affect the representation.

However, the adjacency tensor reflects only the bare incidence pattern of arcs and ignores the outdegree of each vertex. In many contexts, the outdegree distribution is known to influence invariants such as expansion, the isoperimetric number and connectivity [22]. This shortcoming naturally motivates the introduction of the signless Laplacian tensor:

$$\mathcal{Q}(\vec{\mathbb{G}}) = \mathcal{D}^+ + \mathcal{A}(\vec{\mathbb{G}}),$$

where $\mathcal{D}^+$ is the diagonal tensor of vertex outdegrees [23].

There is an extensive research on the H-spectral radius analysis of adjacency tensors for hypergraphs [8,9,23–26]. In contrast to the adjacency tensor, the signless Laplacian H-spectral radius encodes both the underlying connection structure and the weighted contribution of outdegrees [27], thus providing a more informative spectral measure. Its importance is reinforced by well-known relations to several hypergraph properties, including connectivity and the maximum cut and independence number [28,29]. These observations suggest that the signless Laplacian H-spectral radius is not merely of theoretical interest but also serves as an effective tool for investigating network integrity and extremal properties. From a theoretical perspective, the pursuit of tighter bounds is central to the characterization of extremal hypergraphs [20]. From a practical perspective, directed hypergraphs naturally model complex systems such as citation networks, social networks and biological networks, where spectral bounds are instrumental in assessing robustness and understanding propagation dynamics [19].

To rigorously study the H-spectral radius of tensors, we adopt the framework of H-eigenvalues introduced in [7,30]. A real pair (λ, x) is an H-eigenpair of an mth-order n-dimensional tensor $\mathcal{A}$ if

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where $(\mathcal{A}x^{m-1})_i = \sum_{i_2, \ldots, i_m \in [n]} a_{ii_2 \ldots i_m} x_{i_2} \cdots x_{i_m}$ and $x^{[m-1]} = [x_1^{m-1}, \ldots, x_n^{m-1}]^\top$.

The nonnegativity of $\mathcal{Q}(\vec{\mathbb{G}})$ guarantees that its H-spectral radius is the largest H eigenvalue from the Perron–Frobenius theorem [31]. Owing to the inherent asymmetry

of the signless Laplacian tensor, standard optimization methods are inapplicable to the direct computation of its H-spectral radius [32]. Compounding this difficulty, determining the H-spectral radius for such structures is known to be NP-hard [7]. These obstacles have prompted a shift in research focus toward deriving upper and lower bounds on the H-spectral radius [21,23,25]. In this vein, Xie et al. [23] proposed an estimate for the signless Laplacian H-spectral radius associated with the directed hypergraph $\vec{\mathbb{G}}$ as follows:

$$\Delta^+ \leq \rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq 2\Delta^+,$$

where $\rho(\mathcal{Q}(\vec{\mathbb{G}}))$ is the signless Laplacian H-spectral radius and $\Delta^+ = \max_{i\in[n]} d_i^+$ is the maximum outdegree.

Bounding the signless Laplacian H-spectral radius solely by the maximum outdegree often yields crude estimates. To circumvent this limitation, we present a refined strategy that jointly employs multiple outdegree statistics. This strategy captures local structural correlations beyond the mere global maximum degree, thus leading to sharp bounds on the signless Laplacian H-spectral radius. In directed hypergraphs, multiple arcs with the same tail i may share the same head j, which is a distinctive property absent in ordinary graphs. Accordingly, we define $p_i(j)$ as the number of such arcs and utilize this multiplicity to refine the characterization of the signless Laplacian H-spectral radius, leading to improved bounds. Finally, similarity transformation can preserve tensor eigenvalues and enable a more precise characterization of the eigenvalues [31]. Hence, we employ similarity transformations to introduce the average two-outdegree as an extra structural parameter, which yields sharper bounds. Based on these observations, we establish new bounds of the signless Laplacian H-spectral radius with $O(mk+n)$ complexity and improve the existing results, where $m = |\mathbb{E}|$. Our main contributions are summarized as follows:

(1) We establish improved bounds for the signless Laplacian H-spectral radius in terms of the outdegree sums of two or more vertices and yield equality conditions that precisely characterize extremal directed hypergraphs.

(2) By exploiting parameters unique to directed hypergraphs, we construct new bounds that outperform the previous ones by capturing higher-order structural correlations beyond conventional degree-based approaches.

(3) By applying similarity transformations, we introduce the average two-outdegree as an additional parameter, leading to bounds of a different functional form.

(4) Numerical examples confirm the effectiveness and applicability of our theoretical results.

This paper proceeds as follows. In Section 2, we recall the notation and necessary preliminaries. We present the signless Laplacian H-spectral radius bounds based on double and multiple outdegrees in Section 3. Section 4 derives the new signless Laplacian H-spectral radius bounds via similarity transformations and average two-outdegree information. Section 5 concludes the paper.

2. Notation and Preliminaries

In this section, we review the fundamental definitions and properties concerning tensor eigenvalues and directed hypergraphs. Following [14,23], we define a directed hypergraph as one in which every arc has a single tail.

Definition 1. *For a k-uniform directed hypergraph $\vec{\mathbb{G}} = (\mathbb{V}, \mathbb{E})$, we define the following concepts:*

(1) Each arc has cardinality k.

(2) Two distinct vertices v_i and v_j are weakly connected if there exists a sequence of arcs $(e_1, \ldots, e_m)$ such that v_i belongs to e_1, v_j belongs to e_m and $e_r \cap e_{r+1} \neq \varnothing$ for every $r \in [m-1]$.

The directed hypergraph $\vec{\mathbb{G}}$ is called weakly connected if every pair of distinct vertices in $\mathbb{V}$ is weakly connected.

(3) For two distinct vertices v_i and v_j, we write $v_i \to v_j$ if there exists a sequence of arcs $(e_1, \ldots, e_m)$ such that v_i is the tail of e_1, v_j is one of the heads of e_m and for each $r \in [m-1]$, some head of e_r is the tail of e_{r+1}. The directed hypergraph $\vec{\mathbb{G}}$ is called strongly connected if for every pair of distinct vertices $v_i, v_j \in \mathbb{V}$, we have both $v_i \to v_j$ and $v_j \to v_i$.

(4) For two distinct vertices v_i and v_j, we say $i \sim j$ if there exists an arc with a tail v_i and a head v_j. The out-neighborhood of v_i is defined as $N^+(v_i) = \{v_j : i \sim j\}$, i.e., the set of all out-neighbors of v_i.

(5) For a subset $S \subseteq \mathbb{V}$, the directed subhypergraph of $\vec{\mathbb{G}}$ induced by S, denoted by $\vec{\mathbb{G}}_S$, is the directed hypergraph with a vertex set S and arc set $\{e \in \mathbb{E} : \text{all vertices of } e \text{ lie in } S\}$.

Note that the outgoing arc set $\mathbb{E}_i^+$ denotes the set of all arcs with tail a v_i (each of which may contain multiple heads). Hence, $\mathbb{E}_i^+$ is a collection of arcs, while $N^+(v_i)$ is a set of vertices. From the above definitions, it follows that every strongly connected directed hypergraph is also weakly connected. The example below shows that a weakly connected directed hypergraph need not be strongly connected.

Let $\vec{\mathbb{G}}$ be a three-uniform directed hypergraph with a vertex set $\mathbb{V} = \{1,2,3,4,5\}$ and an arc set $\mathbb{E} = \{e_1, e_2\}$, where $e_1 = (1,2,3), \quad e_2 = (2,4,5)$.

Then, $\vec{\mathbb{G}}$ is weakly connected. For any pair of distinct vertices, one can find a sequence of arcs with nonempty intersections between consecutive arcs (e.g., e_1 and e_2 share vertex 2). However, $\vec{\mathbb{G}}$ is not strongly connected. To see this, we compute the out-neighborhoods:

$$N^+(1) = \{2,3\}, \quad N^+(2) = \{4,5\}, \quad N^+(3) = N^+(4) = N^+(5) = \emptyset.$$

Indeed, vertex 1 is the tail of e_1 only, and thus its out-neighbors are two and three; vertex 2 is the tail of e_2 only, so its out-neighbors are four and five; vertices $3, 4, 5$ are not tails of any arc; hence, their out-neighborhoods are empty. Because vertices $3, 4, 5$ have no outgoing arcs, they cannot initiate any directed path. In particular, there is no directed path from five (or four or three) to one, and thus the condition $v_i \to v_j$ fails for some ordered pairs. Therefore, $\vec{\mathbb{G}}$ is not strongly connected. Figure 1 provides a visualization.

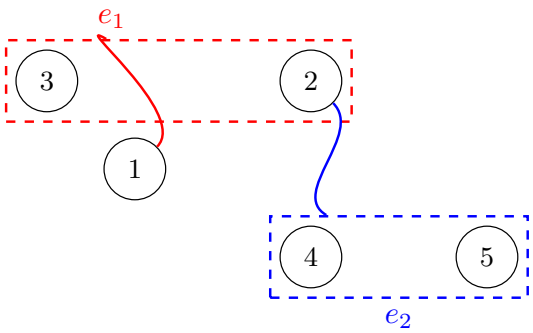


Figure 1. Weakly connected 3-uniform directed hypergraph.

The following defines (generalized) regular directed hypergraphs [23,25].

Definition 2. *For a k-uniform directed hypergraph $\vec{\mathbb{G}} = (\mathbb{V}, \mathbb{E})$, we introduce the following terminology:*

(1) $\vec{\mathbb{G}}$ is called outdegree-regular (in-degree regular) if all d_j^+ (d_j^-) are equal to a common constant w.

(2) $\vec{\mathbb{G}}$ is called outdegree-semiregular if $\mathbb{V} = U \cup W$ with $U \cap W = \varnothing$, the outdegree of every vertex in U is d_1^+, that of every vertex in W is d_2^+ and the subhypergraphs induced by U and W are both empty (i.e., contain no arcs).

Outdegree regularity is clearly a special case of outdegree semiregularity. To facilitate the subsequent analysis, we recall that the signless Laplacian tensor $\mathcal{Q}(\vec{\mathbb{G}})$ is nonnegative and closely related to the hypergraph structure. We therefore summarize below several key properties of nonnegative tensors [10,33] that will be used later. Among these, two facts are particularly relevant. First, $\mathcal{A}(\mathbb{G})$ is weakly irreducible if and only if $\mathbb{G}$ is strongly connected. Second, the weak irreducibility of $\mathcal{A}(\mathbb{G})$ is equivalent to that of $\mathcal{Q}(\mathbb{G})$.

Lemma 1. *Let $\mathcal{A}$ be a nonnegative tensor. Then, $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector. Moreover, if $\mathcal{A}$ is weakly irreducible, then it admits a unique positive eigenvector x associated with $\rho(\mathcal{A})$ up to a positive scalar multiple.*

In the following, we introduce similarity transformations and some important properties of tensors. Let $\mathcal{A}$ be an mth-order n-dimensional tensor and $D = \mathrm{diag}(d_1, \ldots, d_n)$ be a positive diagonal matrix. We define $\mathcal{A}_D$ as the tensor obtained from $\mathcal{A}$ by multiplying each entry by $d_{i_1}^{1-m} d_{i_2} \cdots d_{i_m}$; that is, we have

$$(\mathcal{A}_D)_{i_1 \cdots i_m} = a_{i_1 \cdots i_m} d_{i_1}^{1-m} d_{i_2} \cdots d_{i_m}.$$

Lemma 2 (Theorem 2.8 of [31])**.** *For two mth-order n-dimensional tensors $\mathcal{A}$ and $\mathcal{B}$, if there exists a nonsingular diagonal matrix D such that*

$$\mathcal{B} = \mathcal{A} \cdot D^{1-m} \underbrace{D \cdots D}_{m-1},$$

then $\mathcal{A}$ and $\mathcal{B}$ have identical eigenvalues.

3. Bounds for the Signless Laplacian H-Spectral Radius via Multiple (Two) Outdegrees

This section is devoted to deriving improved bounds for the signless Laplacian H-spectral radius by leveraging two or multiple outdegrees of a directed hypergraph. The proof framework for the subsequent theorems consists of three main steps: (1) selection of eigenvector components and characterization of their properties; (2) derivation of upper and lower bounds based on the chosen components and outdegree data and (3) investigation of the conditions under which equality is attained.

Theorem 1. *Let $\vec{\mathbb{G}}$ be a k-uniform directed hypergraph on n vertices. Then, we have the following:*

$$\text{(i)} \quad \rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \max_{e_i \in \mathbb{E}(\vec{\mathbb{G}})} \max_{j \in e_i} \{d_i^+ + d_j^+\}. \tag{1}$$

If $\vec{\mathbb{G}}$ is strongly connected, then

$$\text{(ii)} \quad \rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq \min_{e_i \in \mathbb{E}(\vec{\mathbb{G}})} \min_{j \in e_i} \{d_i^+ + d_j^+\}. \tag{2}$$

Moreover, if $\vec{\mathbb{G}}$ is strongly connected, then equality holds in both Equations (1) and (2) if and only if $\vec{\mathbb{G}}$ is outdegree-semiregular.

Proof. (i) We establish the upper bound. Under Lemma 1, let $x = (x_1, \ldots, x_n)^\top$ be a nonnegative eigenvector of $\mathcal{Q}(\vec{\mathbb{G}})$ corresponding to $\rho(\mathcal{Q}(\vec{\mathbb{G}}))$. Choose an index t such that $1 = x_t = \max_{i\in[n]} x_i$, and let s be an index satisfying $x_s = \max\{x_j : j \in N^+(t)\}$. From the tth equation of $\mathcal{Q}(\vec{\mathbb{G}})x^{k-1} = \rho(\mathcal{Q}(\vec{\mathbb{G}}))x^{[k-1]}$, we obtain

$$\rho(\mathcal{Q}(\vec{\mathbb{G}}))x_t^{k-1} = d_t^+ x_t^{k-1} + \sum_{(t,i_2,\ldots,i_k)\in e_t} x_{i_2}\ldots x_{i_k}. \tag{3}$$

It follows from $x_t = 1$ and $x_s = \max\{x_j : j \in N^+(t)\}$ that

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_t^+) \le d_t^+ x_s^{k-1}. \tag{4}$$

Similarly, from the sth equation, we have

$$\rho(\mathcal{Q}(\vec{\mathbb{G}}))x_s^{k-1} = d_s^+ x_s^{k-1} + \sum_{(s,i_2,\ldots,i_k)\in e_s} x_{i_2}\ldots x_{i_k} \le d_s^+ x_s^{k-1} + d_s^+,$$

which, after division by $x_s^{k-1} > 0$ (the case $x_s = 0$ is trivial; from the tth equation, we have $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = d_t^+$), yields

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_s^+)x_s^{k-1} \le d_s^+. \tag{5}$$

In Theorem 5.1 of [23], it was shown that

$$\Delta^+ \le \rho(\mathcal{Q}(\vec{\mathbb{G}})) \le 2\Delta^+,$$

which implies $\rho(\mathcal{Q}(\vec{\mathbb{G}})) \ge \max\{d_t^+, d_s^+\}$. Taking the product of Equations (4) and (5) yields

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_t^+)(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_s^+) \le d_t^+ d_s^+.$$

By solving the inequality above, we obtain

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \le d_t^+ + d_s^+,$$

which implies

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \le \max_{e_i\in\mathbb{E}(\vec{\mathbb{G}})} \max_{j\in e_i}\{d_j^+ + d_i^+\}.$$

(ii) We establish the lower bound. According to Lemma 1, strong connectivity of $\vec{\mathbb{G}}$ implies that $\mathcal{Q}(\vec{\mathbb{G}})$ is weakly irreducible, which guarantees a positive eigenvector $x = (x_1, \ldots, x_n)^\top$ associated with the signless Laplacian H-spectral radius $\rho(\mathcal{Q}(\vec{\mathbb{G}}))$. Choose t such that $1 = x_t = \min_{i\in[n]} x_i$, and choose $s \in N^+(t)$ such that $x_s = \min\{x_j : j \in N^+(t)\}$. Applying the same arguments as in the proofs of Equations (4) and (5) yields

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_t^+) \ge d_t^+ x_s^{k-1}, \tag{6}$$

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_s^+)x_s^{k-1} \ge d_s^+. \tag{7}$$

Taking the product of Equations (6) and (7) gives

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_t^+)(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_s^+) \ge d_t^+ d_s^+.$$

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By solving the inequality above, one has

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq d_t^+ + d_s^+.$$

and therefore

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq \min_{e_i \in \mathbb{E}(\vec{\mathbb{G}})} \min_{j \in e_i} \{d_j^+ + d_i^+\}.$$

(iii) It remains to characterize equality. Suppose equality holds in Equation (1). From the derivation of Equation (4), equality forces $x_j = x_s$ for every $j \in N^+(t)$; similarly, equality in Equation (5) forces $x_j = x_t = 1$ for every $j \in N^+(s)$. We distinguish two cases.

Case I: $x_s = 1$. Let $\tilde{V} = \{v_k : x_k = 1\}$. If $\tilde{V} \neq \mathbb{V}$, then strong connectivity implies the existence of vertices $r, p \in \tilde{V}$ and $q \notin \tilde{V}$ with $(r \sim p) \in \mathbb{E}$ and $(p \sim q) \in \mathbb{E}$. Therefore, for the tail vertices $r, p \in \tilde{V}$, we obtain

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_r^+) = \sum_{(r,i_2,\ldots,i_k) \in e_r} x_{i_2} \ldots x_{i_k} \leq d_r^+,$$

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_p^+) = \sum_{(p,i_2,\ldots,i_k) \in e_p} x_{i_2} \ldots x_{i_k} < d_p^+.$$

Substituting these relations into the eigenvalue equations gives a strict inequality $\rho(\mathcal{Q}(\vec{\mathbb{G}})) < d_r^+ + d_p^+$, contradicting the equality in Equation (1). Thus, $\tilde{V} = \mathbb{V}$, and so all outdegrees are equal, and $\vec{\mathbb{G}}$ is outdegree-semiregular.

Case II: $x_s < 1$. Then, $x_k = 1$ for every $v_k \in N^+(v_s)$, and $x_j = x_s$ for every $v_j \in N^+(v_t)$. Define

$$U = \{v_k : x_k = 1\} \text{ and } W = \{v_j : x_j = x_s\}.$$

Clearly, $N^+(v_s) \subseteq U$ and $N^+(v_t) \subseteq W$. For any $v_r \in N^+(N^+(v_t))$, choose $v_p \in N^+(v_t)$ with $(t \sim p), (p \sim r) \in \mathbb{E}(\vec{\mathbb{G}})$. Then, using $x_p = x_s$ in the H-eigenvalue equations, we obtain

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_p^+) x_p^{k-1} = \sum_{(p,i_2,\ldots,i_k) \in e_p} x_{i_2} \ldots x_{i_k} \leq d_p^+$$

and

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_t^+) = \sum_{(t,i_2,\ldots,i_k) \in e_t} x_{i_2} \ldots x_{i_k} \leq d_t^+ x_s^{k-1} = d_t^+ x_p^{k-1}.$$

Equivalently, we have

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_p^+) x_p^{k-1} \leq d_p^+ x_t^{k-1} \quad \text{and} \quad (\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_t^+) x_t^{k-1} \leq d_t^+ x_p^{k-1},$$

which imply $\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq d_t^+ + d_p^+$. Suppose that $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = \max_{e_i \in \mathbb{E}(\vec{\mathbb{G}})} \max_{j \in e_i} \{d_i^+ + d_j^+\}$. For any $(t \sim p)$, it holds that

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq d_t^+ + d_p^+.$$

Consequently, $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = d_t^+ + d_p^+$. Therefore, the following equations hold simultaneously:

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_p^+) x_p^{k-1} = \sum_{(p,i_2,\ldots,i_k) \in e_p} x_{i_2} \ldots x_{i_k} = d_p^+ \quad (*)$$

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_t^+) = \sum_{(t,i_2,\ldots,i_k) \in e_t} x_{i_2} \ldots x_{i_k} = d_t^+ x_s^{k-1} = d_t^+ x_p^{k-1}. \quad (**)$$

Since $x_r \in N^+(v_p)$, from $(*)$, we obtain

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_p^+)x_p^{k-1} = \sum_{(p,i_2,r\ldots,i_k)\in e_p} x_{i_2}x_r \ldots x_{i_k} = d_p^+$$

We employ a proof by contradiction to establish that $x_r = 1$. Otherwise, $x_r < 1$, and one has

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_p^+)x_p^{k-1} = \sum_{(p,i_2,r\ldots,i_k)\in e_p} x_{i_2}x_r \ldots x_{i_k} < d_p^+.$$

This, together with $(**)$, yields

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) < d_t^+ + d_p^+,$$

which contradicts $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = d_t^+ + d_p^+$. Thus, $x_r = 1$ and $N^+(N^+(v_t)) \subseteq U$. Arguing analogously for the case $x_r = 1$ yields $N^+(N^+(v_s)) \subseteq W$. By iterating this procedure and using the strong connectivity of $\vec{\mathbb{G}}$, we conclude that $\mathbb{V} = U \cup W$, with the subhypergraphs induced by U and W being arc-free. Moreover, the outdegree is constant on each of the two parts, which implies that $\vec{\mathbb{G}}$ is outdegree-semiregular.

Conversely, under the assumption of strong connectivity, the two-sided inequality in Equations (1) and (2) is satisfied:

$$\min_{e_i\in\mathbb{E}(\vec{\mathbb{G}})} \min_{j\in e_i}\{d_i^+ + d_j^+\} \leq \rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \max_{e_i\in\mathbb{E}(\vec{\mathbb{G}})} \max_{j\in e_i}\{d_i^+ + d_j^+\}.$$

Now, assume further that $\vec{\mathbb{G}}$ is outdegree-semiregular with bipartition $\mathbb{V} = U \cup W$. In this case, every arc must have its tail in one part and all its heads in the other so that $d_i^+ + d_j^+$ is constant over all admissible pairs (i, j) with $j \in e_i$. Consequently, both bounds coincide with $\rho(\mathcal{Q}(\vec{\mathbb{G}}))$, and equality holds. □

Remark 1. *For a k-uniform directed hypergraph, with* $\Delta^+ = \max_{i\in[n]} d_i^+$ *denoting the maximum outdegree, it holds that*

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \max_{e_i\in\mathbb{E}(\vec{\mathbb{G}})} \max_{j\in e_i}\{d_i^+ + d_j^+\} \leq 2\Delta^+.$$

This directly implies that the bounds in Theorem 1 are sharper than those of Theorem 5.1 in [23].

Example 1. *Let* $\vec{\mathbb{G}}_1$ *be a three-uniform directed hypergraph on five vertices with the following outgoing arcs:*

Tail	1	1	2	2	3	4	5
Heads	{4,3}	{5,4}	{3,5}	{4,5}	{1,2}	{1,2}	{1,2}

We verify that $\vec{\mathbb{G}}_1$ is strongly connected and

$$d_1^+ = d_2^+ = 2, d_3^+ = d_4^+ = d_5^+ = 1,$$

which implies $U = \{1,2\}$, $W = \{3,4,5\}$ and $U \cap W = \emptyset$. Furthermore, the subhypergraphs induced by U and W are empty, which shows that $\vec{\mathbb{G}}_1$ is outdegree-semiregular. Since the outdegrees d_i^+ are not all equal to a common constant, $\vec{\mathbb{G}}_1$ is not outdegree-regular. Using Theorem 1, we obtain

$$\rho(\mathcal{Q}(\vec{\mathbb{G}}_1)) = 3.0000.$$

Compared with the precise bounds of Theorem 1, the estimation for $\rho(\mathcal{Q}(\vec{\mathbb{G}}_1))$ provided by Theorem 5.1 of [23] is as follows:

$$2.0000 \leq \rho(\mathcal{Q}(\vec{\mathbb{G}}_1)) \leq 4.0000.$$

This example illustrates the sharpness of our bounds when the structural assumptions (outdegree semiregularity) are met.

Unlike directed graphs, where each pair (i, j) appears in at most one arc, a directed hypergraph can have multiple arcs with the same tail i containing the same head j. To account for this multiplicity, we define $p_i(j)$ as the number of arcs with a tail i that contain j as a head. Selecting the second largest component of the eigenvector is also time-consuming, and thus we choose an arbitrary component x_s of the eigenvector x to estimate the signless Laplacian H-spectral radius. To state the theorem more clearly, we introduce the following definition.

For every arc $e_i \in E(\vec{\mathbb{G}})$ and every vertex $j \in e_i$, define

$$M_{ij} = \max\{d_j^+, 2d_i^+ - p_i(j)\}.$$

Let α_{ij} be the largest real root in $[M_{ij}, \infty)$ of the equation

$$(\lambda - d_j^+)\big(\lambda - (2d_i^+ - p_i(j))\big)^{k-1} = p_i(j)^{k-1} d_j^+.$$

Equivalently, α_{ij} can be written as

$$\alpha_{ij} = \max\Big\{\lambda \in \mathbb{R} : (\lambda - d_j^+)\big(\lambda - (2d_i^+ - p_i(j))\big)^{k-1} \leq p_i(j)^{k-1} d_j^+\Big\}.$$

Theorem 2. *Let $\vec{\mathbb{G}}$ be a k-uniform directed hypergraph on n vertices. Then, we have*

$$\textit{(i)} \quad \rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \max_{e_i \in \mathbb{E}(\vec{\mathbb{G}})} \min_{j \in e_i} \alpha_{ij}. \tag{8}$$

If $\vec{\mathbb{G}}$ is strongly connected, then

$$\textit{(ii)} \quad \rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq \min_{e_i \in \mathbb{E}(\vec{\mathbb{G}})} \max_{j \in e_i} \alpha_{ij}. \tag{9}$$

Moreover, if $\vec{\mathbb{G}}$ is outdegree-regular with a common outdegree w, then $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = 2w$.

Proof. (i) We establish the upper bound. According to Lemma 1, let $x = (x_1, \ldots, x_n)^\top$ be a nonnegative eigenvector of $\mathcal{Q}(\vec{\mathbb{G}})$ corresponding to $\rho(\mathcal{Q}(\vec{\mathbb{G}}))$. Define an index t such that $x_t = \max_{i \in [n]} x_i$ and an arbitrary vertex $s \in N^+(t)$. From the tth equation of $\mathcal{Q}(\vec{\mathbb{G}})x^{k-1} = \rho(\mathcal{Q}(\vec{\mathbb{G}}))x^{[k-1]}$, one has

$$\begin{aligned}
\rho(\mathcal{Q}(\vec{\mathbb{G}}))x_t^{k-1} &= d_t^+ x_t^{k-1} + \sum_{\{t, i_2, \ldots, i_k\} \in e_t} x_{i_2} \cdots x_{i_k} \\
&= d_t^+ x_t^{k-1} + \sum_{\substack{\{t, i_2, \ldots, i_k\} \in e_t \\ s \in \{i_2, \ldots, i_k\}}} x_{i_2} \cdots x_s \cdots x_{i_k} + \sum_{\substack{\{t, i_2, \ldots, i_k\} \in e_t \\ s \notin \{i_2, \ldots, i_k\}}} x_{i_2} \cdots x_{i_k} \\
&\leq d_t^+ x_t^{k-1} + p_t(s) x_s x_t^{k-2} + (d_t^+ - p_t(s)) x_t^{k-1},
\end{aligned}$$

which shows that

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - (2d_t^+ - p_t(s)))x_t \leq p_t(s)x_s. \tag{10}$$

We proceed by considering two cases:

Case I: $x_s = 0$. Then, Equation (10) reduces to

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq 2d_t^+ - p_t(s).$$

By the definition of α_{ts}, we have $\alpha_{ts} \geq M_{ts} \geq 2d_t^+ - p_t(s)$, and hence

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \alpha_{ts}.$$

Case II: $x_s > 0$. From Equation (10), if $\rho(\mathcal{Q}(\vec{\mathbb{G}})) - (2d_t^+ - p_t(s)) \leq 0$, then

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq 2d_t^+ - p_t(s) \leq M_{ts} \leq \alpha_{ts},$$

and the desired inequality for this s follows. Otherwise, the factor is positive, and raising both sides of Equation (10) to the $(k-1)$th power yields

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - (2d_t^+ - p_t(s)))^{k-1} x_t^{k-1} \leq p_t(s)^{k-1} x_s^{k-1}. \tag{11}$$

Under the sth equation of $\mathcal{Q}(\vec{\mathbb{G}})x^{[k-1]} = \rho(\mathcal{Q}(\vec{\mathbb{G}}))x^{[k-1]}$, it holds that

$$\rho(\mathcal{Q}(\vec{\mathbb{G}}))x_s^{k-1} = d_s^+ x_s^{k-1} + \sum_{\{s,i_2,\ldots,i_k\} \in e_s} x_{i_2} \cdots x_{i_k} \leq d_s^+ x_t^{k-1},$$

That is, we have

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_s^+) x_s^{k-1} \leq d_s^+ x_t^{k-1}. \tag{12}$$

Multiplying Equations (11) and (12) gives

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_s^+)\big(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - (2d_t^+ - p_t(s))\big)^{k-1} \leq p_t(s)^{k-1} d_s^+. \tag{13}$$

Define

$$f(\lambda) = (\lambda - d_s^+)\big(\lambda - (2d_t^+ - p_t(s))\big)^{k-1}.$$

Then, f is continuous and strictly increasing on $[M_{ts}, \infty)$. If $\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq M_{ts}$, then again, $\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \alpha_{ts}$ because $\alpha_{ts} \geq M_{ts}$. If $\rho(\mathcal{Q}(\vec{\mathbb{G}})) > M_{ts}$. Then, from Equation (13), we have

$$f\big(\rho(\mathcal{Q}(\vec{\mathbb{G}}))\big) \leq p_t(s)^{k-1} d_s^+.$$

Since α_{ts} is the largest real number satisfying

$$f(\lambda) \leq p_t(s)^{k-1} d_s^+,$$

and f is increasing on $[M_{ts}, \infty)$, it follows that

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \alpha_{ts}.$$

Thus, in all cases, we obtain $\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \alpha_{ts}$ for the chosen s. Since s was arbitrary in $N^+(t)$, we get

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \min_{j \in e_t} \alpha_{tj}.$$

Taking the maximum over all arcs $e_i \in \mathbb{E}(\vec{\mathbb{G}})$ yields

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \max_{e_i \in \mathbb{E}(\vec{\mathbb{G}})} \min_{j \in e_i} \alpha_{ij}.$$

(ii) We establish the lower bound. Under Lemma 1, strong connectivity of $\vec{\mathbb{G}}$ implies that $\mathcal{Q}(\vec{\mathbb{G}})$ is weakly irreducible, which guarantees a positive eigenvector $x = (x_1, \ldots, x_n)^\top$ associated with the signless Laplacian H-spectral radius $\rho(\mathcal{Q}(\vec{\mathbb{G}}))$. Define an index t such that $x_t = \min_{i\in[n]} x_i$ and an arbitrary vertex $s \in N^+(t)$. From the same reasoning that led to Equations (10) and (12), we obtain

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_s^+)x_s^{k-1} \geq d_s^+ x_t^{k-1}, \tag{14}$$

and

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - (2d_t^+ - p_t(s)))x_t \geq p_t(s)x_s.$$

Since $x_t > 0$ and $p_t(s)x_s \geq 0$, the factors $\rho(\mathcal{Q}(\vec{\mathbb{G}})) - (2d_t^+ - p_t(s))$ and $\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_s^+$ are nonnegative. Therefore, $\rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq M_{ts}$. Taking the $(k-1)$th power preserves the inequality direction and yields

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - (2d_t^+ - p_t(s)))^{k-1}x_t^{k-1} \geq p_t(s)^{k-1}x_s^{k-1}. \tag{15}$$

The positivity of x_s^{k-1} and x_t^{k-1}, together with $p_t(s) \geq 0$, allows us to multiply Equations (14) and (15), yielding

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_s^+)\big(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - (2d_t^+ - p_t(s))\big)^{k-1} \geq p_t(s)^{k-1}d_s^+. \tag{16}$$

We verify that f is increasing on the interval $[M_{ts}, \infty)$, and Equation (16) is exactly

$$f\big(\rho(\mathcal{Q}(\vec{\mathbb{G}}))\big) \geq p_t(s)^{k-1}d_s^+.$$

By the definition of α_{ts} as the largest real number satisfying

$$f(\lambda) \leq p_t(s)^{k-1}d_s^+,$$

the monotonicity of f on $[M_{ts}, \infty)$ implies

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq \alpha_{ts}.$$

Since s was chosen arbitrarily from $N^+(t)$, the above inequality holds for every $j \in e_t$. Therefore, we have

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq \max_{j\in e_t} \alpha_{tj}.$$

Taking the minimum over all arcs yields

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq \min_{e_i\in\mathbb{E}(\vec{\mathbb{G}})} \max_{j\in e_i} \alpha_{ij}.$$

(iii) It remains to verify that $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = 2w$. If $\vec{\mathbb{G}}$ is outdegree-regular with $d_i^+ = w$, then we confirm that the equation

$$f(\lambda) = (\lambda - w)(\lambda - (2w - p_i(j)))^{k-1} = p_i(j)^{k-1}w$$

has a positive real root, namely $\lambda = 2w$, for any integer $p_i(j) \in [1, w]$. A direct substitution gives

$$f(2w) = w \cdot p_i(j)^{k-1} = p_i(j)^{k-1}w,$$

which shows that $\lambda = 2w$ is indeed a root of the equation. Thus, $\rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq 2w$. Under the assumption that $\vec{\mathbb{G}}$ is outdegree-regular with a common outdegree w, we have $\Delta^+ = w$, and hence $\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq 2w$. Thus, the upper bound is exactly $2w$. □

Remark 2. *(i) In contrast to directed graphs, where $p_i(j) \leq 1$, a directed hypergraph has $p_i(j) \leq d_i^+$, which can be advantageous for bounding the spectral radius.*

(ii) A distinctive feature of our approach is that the proof of Theorem 2 does not require x_s to be the second largest component of the eigenvector; the inequality holds for any chosen component $s \in N^+(t)$. This is achieved by splitting the arcs with tail i into two classes according to whether they contain s or not, which yields the inequality

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - (2d_i^+ - p_i(s)))x_i \leq p_i(s)x_s,$$

This is valid for an arbitrary choice of s. Consequently, no optimality or ordering condition on x_s is needed. This flexibility is a direct mathematical consequence of the classification argument rather than an additional assumption. In contrast to existing approaches that require identifying the second largest eigencomponent (e.g., the proof of Theorem 5.1 in [23]), our method avoids this step entirely, thereby reducing the computational cost while maintaining the validity of the bounds.

(iii) For a k-uniform directed hypergraph, from Equation (8), there exists $i \in [n]$ such that

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_j^+)(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - (2d_i^+ - p_i(j)))^{k-1} \leq p_i(j)^{k-1}d_j^+ \quad \text{for all } j \in e_i.$$

If $p_i(j) = 0$, then the inequality yields either

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) = d_j^+ \leq 2\Delta^+ \quad \text{or} \quad \rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq 2d_i^+ \leq 2\Delta^+.$$

If $d_j^+ = 0$, then we obtain

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq 2d_i^+ - p_i(j) \leq 2\Delta^+.$$

Otherwise, when $p_i(j) > 0$ and $d_j^+ > 0$, the inequality is equivalent to at least one of the following two conditions holding:

$$\frac{\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_j^+}{d_j^+} \leq 1 \quad \text{or} \quad \frac{\rho(\mathcal{Q}(\vec{\mathbb{G}})) - (2d_i^+ - p_i(j))}{p_i(j)} \leq 1.$$

If the first one holds, then $\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq 2d_j^+ \leq 2\Delta^+$. If the second one holds, then $\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq 2d_i^+ \leq 2\Delta^+$. Consequently, the proposed upper bound is no worse than the bound given in Theorem 5.1 of [23]. Moreover, as demonstrated in Examples 2 and 5, it can be strictly tighter in a number of cases.

Example 2. *Let $\vec{\mathbb{G}}_2$ be a three-uniform directed hypergraph on six vertices with the following outgoing arcs:*

Tail	1	1	2	3	3	4	5	6
Heads	{2,3}	{3,4}	{3,5}	{4,6}	{5,6}	{5,6}	{6,1}	{1,2}

It is easy to see that

$$d_1^+ = d_3^+ = 2, d_2^+ = d_4^+ = d_5^+ = d_6^+ = 1.$$

Among the directed arcs with a tail at vertex 1, there is one arc with vertex 2 as a head, two arcs with vertex 3 as a head, and one arc with vertex 4 as a head; all other counts are

zero. These values form the first row of the vector $p_1 = (0,1,2,1,0,0)$. Similarly, we obtain $p_2, \ldots, p_6$. Furthermore, the $p_i(j)$ matrix is given as follows:

$$p_i(j) = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using Theorem 1, we obtain

$$2.0000 \leq \rho(\mathcal{Q}(\vec{\mathbb{G}}_2)) \leq 4.0000.$$

From Theorem 2, we compute

$$2.0000 \leq \rho(\mathcal{Q}(\vec{\mathbb{G}}_2)) \leq \frac{5+\sqrt{5}}{2} \approx 3.6180.$$

The improvement in the upper bound from 4.0000 to 3.6180 demonstrates the advantage of incorporating the multiplicity parameter $p_i(j)$. This is particularly significant in applications where a tighter upper bound is required for spectral analysis.

While the previous theorem relies on the outdegree data of two vertices to bound the spectral radius, the following theorem makes use of the outdegree information of multiple vertices on the same edge, thereby achieving significantly better performance.

For each arc $e = \{j_1, \ldots, j_k\} \in \mathbb{E}(\vec{\mathbb{G}})$, define

$$P_{j_1 \cdots j_k} = \max_{t \in [k]} d^+_{j_t},$$

and let $\beta_{j_1 \cdots j_k}$ be the largest real root in $[P_{j_1 \cdots j_k}, \infty)$ of the equation

$$\prod_{t=1}^{k} (\lambda - d^+_{j_t}) = \prod_{t=1}^{k} d^+_{j_t}.$$

Theorem 3. *Let $\vec{\mathbb{G}}$ be a k-uniform directed hypergraph on n vertices. For each arc $e = \{j_1, \ldots, j_k\} \in \mathbb{E}(\vec{\mathbb{G}})$, the following apply:*

(i) $$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \max_{e \in \mathbb{E}(\vec{\mathbb{G}})} \beta_{j_1 \cdots j_k}, \tag{17}$$

and if $\vec{\mathbb{G}}$ is strongly connected, then

(ii) $$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq \min_{e \in \mathbb{E}(\vec{\mathbb{G}})} \beta_{j_1 \cdots j_k}. \tag{18}$$

Moreover, if $\vec{\mathbb{G}}$ is strongly connected and outdegree-regular, then equality holds in both Equations (17) and (18).

Proof. (i) We establish the upper bound. Under Lemma 1, let $x = (x_1, \ldots, x_n)^\top$ be a nonnegative eigenvector of $\mathcal{Q}(\vec{\mathbb{G}})$ corresponding to $\rho(\mathcal{Q}(\vec{\mathbb{G}}))$. From the i_1th equation of $\mathcal{Q}(\vec{\mathbb{G}})x^{k-1} = \rho(\mathcal{Q}(\vec{\mathbb{G}}))x^{[k-1]}$, we obtain

$$\rho(\mathcal{Q}(\vec{\mathbb{G}}))x_{i_1}^{k-1} = d^+_{i_1} x_{i_1}^{k-1} + \sum_{\{i_1, i_2, \ldots, i_k\} \in e_{i_1}} x_{i_2} \cdots x_{i_k} \tag{19}$$

and

$$\rho(\mathcal{Q}(\vec{\mathbb{G}}))x_{i_1}^k = d_{i_1}^+ x_{i_1}^k + \sum_{\{i_1,i_2,\ldots,i_k\}\in e_{i_1}} x_{i_1}x_{i_2}\cdots x_{i_k}.$$

Define x_β as the maximum of the products $x_{i_1}x_{i_2}\cdots x_{i_k}$ over all arcs. Then, for every $i_1 \in [n]$, the following inequality holds:

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{i_1}^+)x_{i_1}^k = \sum_{\{i_1,i_2,\ldots,i_k\}\in e_{i_1}} x_{i_1}x_{i_2}\cdots x_{i_k} \le \sum_{\{i_1,i_2,\ldots,i_k\}\in e_{i_1}} x_\beta = d_{i_1}^+ x_\beta. \tag{20}$$

We now distinguish two cases.

Case I: $x_\beta = 0$. Since x is nonzero, there exists $p \in [n]$ such that $x_p > 0$. Applying Equation (20) gives

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_p^+)x_p^k \le \sum_{\{p,i_2,\ldots,i_k\}\in e_p} x_\beta = 0,$$

which shows that $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = d_p^+$. Hence, the desired bound holds trivially.

Case II: $x_\beta > 0$. Suppose that $x_\beta = x_{j_1}x_{j_2}\cdots x_{j_k}$ for some edge $\{j_1,\ldots,j_k\} \in \mathbb{E}(\vec{\mathbb{G}})$.

If $d_t^+ = 0$ for some $t \in \{j_1,\ldots,j_k\}$, then Equation (20) yields

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_t^+)x_t^k = 0.$$

Since $x_\beta > 0$, we have $x_t > 0$, and consequently $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = d_t^+ = 0$, confirming the validity of the result.

Now, assume that $d_t^+ > 0$ for all vertices in the chosen edge. By definition, there exists an edge $e = \{j_1,\ldots,j_k\}$ such that $x_{j_1}\cdots x_{j_k} = x_\beta$. For each vertex j_t in this edge, we obtain

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{j_t}^+)x_{j_t}^k \le d_{j_t}^+ x_\beta, t = 1,\ldots,k. \tag{21}$$

Multiplying the k inequalities (Equation (21)) together yields

$$\prod_{t=1}^k(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{j_t}^+)\cdot\prod_{t=1}^k x_{j_t}^k \le \prod_{t=1}^k d_{j_t}^+ \cdot x_\beta^k.$$

Since $\prod_{t=1}^k x_{j_t}^k = x_\beta^k$, we obtain

$$\prod_{t=1}^k(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{j_t}^+) \le \prod_{t=1}^k d_{j_t}^+.$$

Define

$$g(\lambda) = \prod_{t=1}^k(\lambda - d_{j_t}^+).$$

Then, g is continuous and strictly increasing on $[P_{j_1\ldots j_k},\infty)$. Since $\beta_{j_1\ldots j_k}$ is the largest real number satisfying

$$g(\lambda) = \prod_{t=1}^k d_{j_t}^+,$$

and g is increasing on $[P_{j_1\ldots j_k},\infty)$, it follows that

$$g(\rho(\mathcal{Q}(\vec{\mathbb{G}}))) \le \prod_{t=1}^k d_{j_t}^+.$$

This implies that $\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \beta_{j_1\ldots j_k}$. Taking the maximum over all arcs $e \in \mathbb{E}(\vec{\mathbb{G}})$ yields

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \max_{e\in\mathbb{E}(\vec{\mathbb{G}})} \beta_{j_1\ldots j_k}.$$

(ii) We establish the lower bound. From Lemma 1, strong connectivity of $\vec{\mathbb{G}}$ implies that $\mathcal{Q}(\vec{\mathbb{G}})$ is weakly irreducible, which guarantees a positive eigenvector $x = (x_1, \ldots, x_n)^\top$ associated with the signless Laplacian H-spectral radius $\rho(\mathcal{Q}(\vec{\mathbb{G}}))$. Define x_γ as the minimum of the products $x_{i_1}x_{i_2}\cdots x_{i_k}$ over all arcs $\{i_1, i_2, \ldots, i_k\} \in \mathbb{E}(\vec{\mathbb{G}})$. Then, for every $i_1 \in [n]$, the following inequality holds:

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{i_1}^+)x_{i_1}^k = \sum_{\{i_1,i_2,\ldots,i_k\}\in e_{i_1}} x_{i_1}x_{i_2}\cdots x_{i_k} \geq \sum_{\{i_1,i_2,\ldots,i_k\}\in e_{i_1}} x_\gamma = d_{i_1}^+ x_\gamma. \tag{22}$$

Set $x_\gamma = x_{q_1}x_{q_2}\ldots x_{q_k} = \min\{x_{i_1}x_{i_2}\cdots x_{i_k} : \{i_1, i_2, \ldots, i_k\} \in \mathbb{E}(\vec{\mathbb{G}})\}$. Then, we have

$$\begin{aligned}(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{q_1}^+)x_{q_1}^k &\geq d_{q_1}^+ x_\gamma,\\ &\vdots\\ (\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{q_k}^+)x_{q_k}^k &\geq d_{q_k}^+ x_\gamma.\end{aligned}$$

When multiplying the multiple inequalities above, we have

$$\prod_{l=1}^k x_{q_l}^k \cdot \prod_{l=1}^k (\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{q_l}^+) = x_\gamma^k \prod_{l=1}^k (\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{q_l}^+) \geq x_\gamma^k \prod_{l=1}^k d_{q_l}^+.$$

Therefore, we have

$$\prod_{l=1}^k (\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{q_l}^+) \geq \prod_{l=1}^k d_{q_l}^+.$$

Define

$$g(\lambda) = \prod_{l=1}^k (\lambda - d_{q_l}^+).$$

Then, g is continuous and strictly increasing on $[P_{q_1\ldots q_l}, \infty)$. Since $\beta_{q_1\ldots q_l}$ is the largest real number satisfying

$$g(\lambda) = \prod_{l=1}^k d_{q_l}^+,$$

and g is increasing on $[P_{q_1\ldots q_l}, \infty)$, it follows that

$$g(\rho(\mathcal{Q}(\vec{\mathbb{G}}))) \geq \prod_{l=1}^k d_{q_l}^+.$$

This implies that $\rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq \beta_{q_1\ldots q_l}$. Taking the minimum over all arcs $e \in \mathbb{E}(\vec{\mathbb{G}})$ yields

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq \min_{e\in\mathbb{E}(\vec{\mathbb{G}})} \beta_{j_1\ldots j_k}.$$

(iii) If $\vec{\mathbb{G}}$ is outdegree-regular with a common outdegree w, then for every arc $e = \{j_1, \ldots, j_k\}$, we have $d_{j_t}^+ = w$ for all t. Thus, $\beta_{j_1\ldots j_k}$ is the largest root of $(\lambda - w)^k = w^k$, i.e., $\beta_{j_1\ldots j_k} = 2w$. Hence, both $\max_e \beta_{j_1\ldots j_k}$ and $\min_e \beta_{j_1\ldots j_k}$ equal $2w$. Since the all-ones vector gives $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = 2w$, equality holds in both bounds. □

Example 3. *Let $\vec{\mathbb{G}}_3$ be a four-uniform strongly connected directed hypergraph on six vertices with the following outgoing arcs:*

Tail	1	1	2	2	3	4	5	6
Heads	$\{3,4,5\}$	$\{3,4,6\}$	$\{3,5,6\}$	$\{4,5,6\}$	$\{1,5,6\}$	$\{2,3,5\}$	$\{1,4,6\}$	$\{2,3,4\}$

We compute

$$d_1^+ = d_2^+ = 2, d_3^+ = d_4^+ = d_5^+ = d_6^+ = 1.$$

We can verify that $\vec{\mathbb{G}}_3$ is neither outdegree-semiregular nor outdegree-regular. The estimation for $\rho(\mathcal{Q}(\vec{\mathbb{G}}_3))$ provided by Theorem 5.1 of [23] is as follows:

$$2.0000 \leq \rho(\mathcal{Q}(\vec{\mathbb{G}}_3)) \leq 4.0000.$$

For every arc e_i $(i = 1, \ldots, 8)$, Theorem 3 gives the same inequality:

$$2.0000 \leq (\rho(\mathcal{Q}(\vec{\mathbb{G}}_3)) - 2)(\rho(\mathcal{Q}(\vec{\mathbb{G}}_3)) - 1)^3 \leq 2.0000.$$

Solving equality for $(\rho(\mathcal{Q}(\vec{\mathbb{G}}_3)) - 2)(\rho(\mathcal{Q}(\vec{\mathbb{G}}_3)) - 1)^3 = 2$ numerically (using Newton's method) yields the unique root

$$\rho(\mathcal{Q}(\vec{\mathbb{G}}_3)) \approx 2.5437.$$

As stated above, Theorem 3 provides a sufficient condition for equality to hold in both bounds.

4. Bounds for the Signless Laplacian *H*-Spectral Radius via Similarity Transformation

We establish bounds for $\rho(\mathcal{Q}(\vec{\mathbb{G}}))$ by means of similarity transformations. By averaging the outdegree information of adjacent vertices, the similarity transformation allows us to derive more refined bounds without losing spectral information. This method may provide tighter results than the degree-based approach.

Let $\vec{\mathbb{G}}$ be a k-uniform directed hypergraph, and let $d_i^+ \geq 1$ for $i \in [n]$. Moreover, for each vertex i, define

$$m_i^+ = \sum_{\{i,i_2,\ldots,i_k\} \in e_i} \frac{d_{i_2}^+ \cdots d_{i_k}^+}{d_i^{+(k-1)}},$$

which we refer to as the average two-outdegree of i.

Definition 3. *Let $\vec{\mathbb{G}} = (\mathbb{V}, \mathbb{E})$ be a k-uniform directed hypergraph, and let m_j^+ and m_j^- denote the average two-outdegree and average two-in-degree of vertex $j \in [n]$, respectively.*

(i) For some constant w, $\vec{\mathbb{G}}$ is said to be average two-outdegree-regular if $m_j^+ = w$ for all $j \in [n]$. Analogously, it is average two-in-degree-regular if $m_j^- = w$.

(ii) $\vec{\mathbb{G}}$ is called average two-outdegree-semiregular if $\mathbb{V} = U \cup W$ with $U \cap W = \emptyset$, the average two-outdegree of every vertex in U is m_1^+, the average two-outdegree of every vertex in W is m_2^+, and the subhypergraphs induced by U and W are both empty (i.e., contain no arcs).

Note that the strong connectivity of the directed hypergraph is preserved under the similarity transformation. Through the similarity transformation, we obtain the average two-outdegree information. The subsequent theorems utilize both outdegree and average two-outdegree data to characterize the signless Laplacian H-spectral radius.

Theorem 4. *Let $\vec{\mathbb{G}}$ be a k-uniform directed hypergraph on n vertices with outdegrees $d_i^+ \geq 1$ for $i \in [n]$. Then, we have*

$$(i) \quad \rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \max_{i\in[n]}\{m_i^+ + d_i^+\}. \tag{23}$$

If $\vec{\mathbb{G}}$ is strongly connected, then

$$(ii) \quad \rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq \min_{i\in[n]}\{m_i^+ + d_i^+\}. \tag{24}$$

In the strongly connected and average two-outdegree-regular case, equality in Equations (23) and (24) is attained if and only if $\vec{\mathbb{G}}$ is outdegree-regular.

Proof. (i) We establish the upper bound. Let $D = \mathrm{diag}(d_1^+, \ldots, d_n^+)$, and define $\mathcal{P}(\vec{\mathbb{G}})$ by

$$\mathcal{P}(\vec{\mathbb{G}}) = \mathcal{Q}(\vec{\mathbb{G}}) \cdot D^{1-k}\underbrace{D\cdots D}_{k-1} = \mathcal{D}^+ + \mathcal{A}(\vec{\mathbb{G}})D^{1-k}\underbrace{D\cdots D}_{k-1}.$$

Lemma 2 yields $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = \rho(\mathcal{P}(\vec{\mathbb{G}}))$, while Lemma 1 guarantees a nonnegative eigenvector $x = (x_1, \ldots, x_n)^\top$ associated with the signless Laplacian H-spectral radius $\rho(\mathcal{P}(\vec{\mathbb{G}}))$. Define $1 = x_t = \max_{i\in[n]} x_i$ such that $0 \leq x_i \leq 1$ for $i \neq t$. From the tth equation of $\mathcal{P}(\vec{\mathbb{G}})x^{k-1} = \rho(\mathcal{P}(\vec{\mathbb{G}}))x^{[k-1]}$ and $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = \rho(\mathcal{P}(\vec{\mathbb{G}}))$, we obtain

$$\rho(\mathcal{Q}(\vec{\mathbb{G}}))x_t^{k-1} = \rho(\mathcal{P}(\vec{\mathbb{G}}))x_t^{k-1} = d_t^+ x_t^{k-1} + \sum_{\{t,i_2,\ldots,i_k\}\in e_t} \frac{d_{i_2}^+ \cdots d_{i_k}^+}{d_t^{+(k-1)}} x_{i_2}\cdots x_{i_k}.$$

Using $x_t = 1$ and $x_i \leq 1$, we obtain

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \sum_{\{t,i_2,\ldots,i_k\}\in e_t} \frac{d_{i_2}^+ \cdots d_{i_k}^+}{d_t^{+(k-1)}} x_t^{k-1} + d_t^+ = m_t^+ + d_t^+, \tag{25}$$

which implies

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \max_{i\in[n]}\{m_i^+ + d_i^+\}.$$

(ii) We establish the lower bound. Let $D = \mathrm{diag}(d_1^+, \ldots, d_n^+)$, and define $\mathcal{P}(\vec{\mathbb{G}})$ by

$$\mathcal{P}(\vec{\mathbb{G}}) = \mathcal{Q}(\vec{\mathbb{G}}) \cdot D^{1-k}\underbrace{D\cdots D}_{k-1} = \mathcal{D}^+ + \mathcal{A}(\vec{\mathbb{G}})D^{1-k}\underbrace{D\cdots D}_{k-1}.$$

Lemma 2 yields $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = \rho(\mathcal{P}(\vec{\mathbb{G}}))$. Strong connectivity of $\vec{\mathbb{G}}$ implies that $\mathcal{Q}(\vec{\mathbb{G}})$ is weakly irreducible, which guarantees a positive eigenvector $x = (x_1, \ldots, x_n)^\top$ associated with the signless Laplacian H-spectral radius $\rho(\mathcal{P}(\vec{\mathbb{G}}))$ from Lemma 1. Define $1 = x_t = \min_{i\in[n]} x_i$ and $x_i \geq 1$ for $i \neq t$. An argument parallel to that for Equation (25) gives

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq m_t^+ + d_t^+.$$

Consequently, we have

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq \min_{i\in[n]}\{m_i^+ + d_i^+\}.$$

(iii) It remains to characterize equality. Assume equality holds in Equation (23). Then, Equation (25) implies $x_k = x_t$ for every $k \in N^+(t)$. Define $\tilde{V} = \{v_k : x_k = x_t\}$. If $\tilde{V} \neq \mathbb{V}$,

since $\vec{\mathbb{G}}$ is strongly connected, then there exists an arc from $\tilde{V}$ to $\mathbb{V} \setminus \tilde{V}$. Choose $v_l \in \tilde{V}$ and $v_q \notin \tilde{V}$ with $(l \sim q) \in \mathbb{E}$, and choose $v_r \in \tilde{V}$ with $(r \sim l) \in \mathbb{E}$. The H-eigenvalue equations together with Lemma 2 yield

$$\rho(\mathcal{Q}(\vec{\mathbb{G}}))x_r^{k-1} = d_r^+ x_r^{k-1} + \sum_{\{r,i_2,\ldots,i_k\}\in e_r} \frac{d_{i_2}^+ \cdots d_{i_k}^+}{d_r^{+(k-1)}} x_{i_2} \cdots x_{i_k} < d_r^+ x_r^{k-1} + m_r^+ x_r^{k-1},$$

and thus $\rho(\mathcal{Q}(\vec{\mathbb{G}})) < d_r^+ + m_r^+$, which contradicts the equality condition in Equation (23). Hence, $\tilde{V} = \mathbb{V}$, and

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) = m_i^+ + d_i^+, \forall i \in [n].$$

Since $\vec{\mathbb{G}}$ is average two-outdegree-regular, there exists a constant c such that $m_i^+ = c$ for all $i \in [n]$. It follows that $d_i^+ = \rho(\mathcal{Q}(\vec{\mathbb{G}})) - c$ is independent of i. Thus, $\vec{\mathbb{G}}$ is outdegree-regular.

Conversely, if $\vec{\mathbb{G}}$ is strongly connected and outdegree-regular, then equality in Equations (23) and (24) is attained. □

Remark 3. *(i) Outdegree regularity implies average two-outdegree regularity but not conversely (see Example 4).*

(ii) For a directed hypergraph, average two-outdegree regularity suffices to precisely determine its H-spectral radius [25], whereas an exact characterization of the signless Laplacian H-spectral radius requires outdegree regularity.

Example 4. *Let $\vec{\mathbb{G}}_4$ be a three-uniform directed hypergraph on 12 vertices with the following outgoing arcs:*

Tail	1	2	3	4	5	5	6	6
Heads	{2,6}	{3,5}	{1,6}	{1,7}	{6,8}	{7,8}	{5,7}	{7,8}

Tail	7	7	8	8	9	9	9	10
Heads	{6,8}	{6,5}	{7,6}	{5,7}	{10,5}	{12,6}	{11,7}	{11,5}

Tail	10	10	11	11	11	12	12	12
Heads	{12,6}	{9,8}	{5,12}	{6,10}	{7,9}	{5,9}	{6,11}	{7,10}

It is easy to see that

$$d_1^+ = \cdots = d_4^+ = 1, d_5^+ = \cdots = d_8^+ = 2, d_9^+ = \cdots = d_{12}^+ = 3.$$

Furthermore, we compute

$$m_1^+ = \cdots = m_{12}^+ = 2,$$

which implies that $\vec{\mathbb{G}}_4$ is average two-outdegree-regular. From Theorem 4, we obtain

$$3.0000 \leq \rho(\mathcal{Q}(\vec{\mathbb{G}}_4)) \leq 5.0000.$$

Compared with the bounds of Theorem 1, the estimation for $\rho(\mathcal{Q}(\vec{\mathbb{G}}_4))$ provided by Theorem 5.1 of [23] is as follows:

$$3.0000 \leq \rho(\mathcal{Q}(\vec{\mathbb{G}}_4)) \leq 6.0000.$$

As stated above, similarity transformations can sometimes balance the discrepancies among outdegrees, thereby yielding better estimates for the signless Laplacian H-spectral radius.

Theorem 4 demonstrates that bounds based on a single outdegree and a single average two-outdegree are already effective in certain settings. A natural extension is to consider two outdegrees and two average two-outdegrees, with the expectation of obtaining tighter bounds for the signless Laplacian spectral radius.

Theorem 5. *Let $\vec{\mathbb{G}}$ be a k-uniform directed hypergraph on n vertices with outdegrees $d_i^+ \geq 1$ for $i \in [n]$. Then, we have*

$$(i) \quad \rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \max_{e_i \in \mathbb{E}(\vec{\mathbb{G}})} \max_{j \in e_i} \left\{ \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4m_i^+ m_j^+}}{2} \right\}. \tag{26}$$

If $\vec{\mathbb{G}}$ is strongly connected, then

$$(ii) \quad \rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq \min_{e_i \in \mathbb{E}(\vec{\mathbb{G}})} \min_{j \in e_i} \left\{ \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4m_i^+ m_j^+}}{2} \right\}. \tag{27}$$

Moreover, if $\vec{\mathbb{G}}$ is strongly connected and outdegree-semiregular, then equality holds in both Equations (26) and (27).

Proof. (i) We establish the upper bound. Let $D = \mathrm{diag}(d_1^+, \ldots, d_n^+)$, and define $\mathcal{P}(\vec{\mathbb{G}})$ by

$$\mathcal{P}(\vec{\mathbb{G}}) = \mathcal{Q}(\vec{\mathbb{G}}) \cdot D^{1-k} \underbrace{D \cdots D}_{k-1} = \mathcal{D}^+ + \mathcal{A}(\vec{\mathbb{G}}) D^{1-k} \underbrace{D \cdots D}_{k-1}.$$

Lemma 2 yields $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = \rho(\mathcal{P}(\vec{\mathbb{G}}))$, while Lemma 1 guarantees a nonnegative eigenvector $x = (x_1, \ldots, x_n)^\top$ associated with the signless Laplacian H-spectral radius $\rho(\mathcal{P}(\vec{\mathbb{G}}))$. Choose an index t such that $1 = x_t = \max_{i \in [n]} x_i$, and let s be an index satisfying $x_s = \max\{x_j : j \in N^+(t)\}$. From the tth equation of $\mathcal{P}(\vec{\mathbb{G}})x^{k-1} = \rho(\mathcal{P}(\vec{\mathbb{G}}))x^{[k-1]}$ and $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = \rho(\mathcal{P}(\vec{\mathbb{G}}))$, we obtain

$$\rho(\mathcal{Q}(\vec{\mathbb{G}}))x_t^{k-1} = \rho(\mathcal{P}(\vec{\mathbb{G}}))x_t^{k-1} = d_t^+ x_t^{k-1} + \sum_{\{t, i_2, \ldots, i_k\} \in e_t} \frac{d_{i_2}^+ \cdots d_{i_k}^+}{{d_t^+}^{(k-1)}} x_{i_2} \cdots x_{i_k} \tag{28}$$

It follows from $x_t = 1$ and $x_s = \max\{x_j : j \in N^+(t)\}$ that

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_t^+) \leq m_t^+ x_s^{k-1}. \tag{29}$$

Similarly, from the sth equation, we find that

$$\rho(\mathcal{Q}(\vec{\mathbb{G}}))x_s^{k-1} = \rho(\mathcal{P}(\vec{\mathbb{G}}))x_s^{k-1} = d_s^+ x_s^{k-1} + \sum_{\{s, i_2, \ldots, i_k\} \in e_s} \frac{d_{i_2}^+ \cdots d_{i_k}^+}{{d_s^+}^{(k-1)}} x_{i_2} \cdots x_{i_k}, \tag{30}$$

which, after division by $x_s^{k-1} > 0$ (the case $x_s = 0$ is trivial; from the tth equation, we have $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = d_t^+$), yields

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_s^+)x_s^{k-1} \leq m_s^+ x_t^{k-1} = m_s^+. \tag{31}$$

Taking the product of Equations (29) and (31) yields

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_t^+)(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_s^+) \leq m_t^+ m_s^+.$$

By solving the inequality above, one has

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \frac{d_t^+ + d_s^+ + \sqrt{(d_t^+ - d_s^+)^2 + 4m_t^+ m_s^+}}{2}, \tag{32}$$

which shows that

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \max_{e_i \in \mathbb{E}(\vec{\mathbb{G}})} \max_{j \in e_i} \left\{ \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4m_i^+ m_j^+}}{2} \right\}.$$

(ii) We establish the lower bound. Let $D = \mathrm{diag}(d_1^+, \ldots, d_n^+)$, and define $\mathcal{P}(\vec{\mathbb{G}})$ by

$$\mathcal{P}(\vec{\mathbb{G}}) = \mathcal{Q}(\vec{\mathbb{G}}) \cdot D^{1-k} \underbrace{D \cdots D}_{k-1} = \mathcal{D}^+ + \mathcal{A}(\vec{\mathbb{G}}) D^{1-k} \underbrace{D \cdots D}_{k-1}.$$

Lemma 2 yields $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = \rho(\mathcal{P}(\vec{\mathbb{G}}))$. Strong connectivity of $\vec{\mathbb{G}}$ implies that $\mathcal{Q}(\vec{\mathbb{G}})$ is weakly irreducible, which guarantees a positive eigenvector $x = (x_1, \ldots, x_n)^\top$ associated with the signless Laplacian H-spectral radius $\rho(\mathcal{P}(\vec{\mathbb{G}}))$ from Lemma 1. Choose t such that $1 = x_t = \min_{i \in [n]} x_i$, and choose $s \in N^+(t)$ such that $x_s = \min\{x_j : j \in N^+(t)\}$. Arguing analogously to Equations (29) and (31), we have

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_t^+) \geq m_t^+ x_s^{k-1}, \tag{33}$$

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_s^+) x_s^{k-1} \geq m_s^+. \tag{34}$$

Multiplying Equations (33) and (34) yields

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq \frac{d_t^+ + d_s^+ + \sqrt{(d_t^+ - d_s^+)^2 + 4m_t^+ m_s^+}}{2}$$

and

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq \min_{e_i \in \mathbb{E}(\vec{\mathbb{G}})} \min_{j \in e_i} \left\{ \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4m_i^+ m_j^+}}{2} \right\}.$$

(iii) It remains to characterize equality. Under the assumption of strong connectivity, the two-sided inequality in Equations (26) and (27) is satisfied:

$$\min_{e_i \in \mathbb{E}(\vec{\mathbb{G}})} \min_{j \in e_i} \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4m_i^+ m_j^+}}{2} \leq$$
$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \max_{e_i \in \mathbb{E}(\vec{\mathbb{G}})} \max_{j \in e_i} \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4m_i^+ m_j^+}}{2}. \tag{35}$$

Now, assume further that $\vec{\mathbb{G}}$ is outdegree-semiregular with bipartition $\mathbb{V} = U \cup W$. In this case, every arc must have its tail in one part and all its heads in the other, which implies that $(d_i^+ + d_j^+)$, $(d_i^+ - d_j^+)$ and $m_i^+ m_j^+$ are constant over all admissible pairs (i, j) with $j \in e_i$. Consequently, both bounds coincide with $\rho(\mathcal{Q}(\vec{\mathbb{G}}))$, and equality holds. □

Remark 4. *(i) Outdegree semiregularity implies average two-outdegree semiregularity, but the converse need not hold. Example 4 serves as a counterexample to the reverse implication.*

(ii) For an adjacency hypergraph, average two-outdegree semiregularity suffices to precisely determine its H-spectral radius [25], whereas an exact characterization of the signless Laplacian H-spectral radius requires outdegree semiregularity.

Example 5. *Let $\vec{\mathbb{G}}_5$ be a three-uniform directed hypergraph with vertex set $\mathbb{V} = U \cup W$, where $U = \{1,2\}$ and $W = \{3,4,5\}$, with the arc set*

Tail	1	2	3	4	5
Heads	$\{3,4\},\{3,5\}$	$\{4,5\},\{3,5\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$

We verify that

$$d_1^+ = 2, \quad d_2^+ = 2, \qquad d_3^+ = d_4^+ = d_5^+ = 1.$$

Furthermore, $d_U^+ = 2$ and $d_W^+ = 1$, and the induced subhypergraphs on U and W are empty. Thus, $\vec{\mathbb{G}}_5$ is outdegree-semiregular but not outdegree-regular. Moreover, we have

$$m_1^+ = m_2^+ = \frac{1}{2}, m_3^+ = m_4^+ = m_5^+ = 4.$$

Therefore, $m_U^+ = \frac{1}{2}$ and $m_W^+ = 4$, and the induced subhypergraphs on U and W are empty. Hence, $\vec{\mathbb{G}}_5$ is average two-outdegree-semiregular but not average two-outdegree-regular.

According to Table 1, Theorems 1 and 5 give the exact spectral radius $\rho(\mathcal{Q}(\vec{\mathbb{G}}_5))$. This is because $\vec{\mathbb{G}}_5$ is strongly connected and outdegree-semiregular, which are precisely the equality conditions stated in these theorems. Theorem 2 provides the tighter estimate, with a width of 0.6191, while Theorem 5.1 of [23] and Theorem 4 give the wider intervals. This demonstrates that the multiplicity parameter $p_i(j)$ can significantly improve the crude bounds.

Table 1. Comparison of signless Laplacian H-spectral radius bounds for $\vec{\mathbb{G}}_5$.

Theorem	Lower Bound	Upper Bound	Interval Width
Theorem 5.1 of [23]	2.0000	4.0000	2.0000
Theorem 1	3.0000	3.0000	0.0000
Theorem 2	2.6951	3.3142	0.6191
Theorem 3	2.3724	4.0000	1.6276
Theorem 4	2.5000	5.0000	2.5000
Theorem 5	3.0000	3.0000	0.0000

True value: $\rho(\mathcal{Q}(\vec{\mathbb{G}}_5)) = 3$.

We now proceed to establish refined estimates for the signless Laplacian H-spectral radius by incorporating the average two-outdegree information associated with each arc.

For each arc $e = \{j_1, \ldots, j_k\} \in \mathbb{E}(\vec{\mathbb{G}})$, define

$$U_{j_1 \ldots j_k} = \max_{t \in [k]} m_{j_t}^+,$$

and let $\tau_{j_1 \ldots j_k}$ be the largest real root in $[U_{j_1 \ldots j_k}, \infty)$ of the equation

$$\prod_{t=1}^{k} (\lambda - d_{j_t}^+) = \prod_{t=1}^{k} m_{j_t}^+.$$

Theorem 6. *Let $\vec{\mathbb{G}}$ be a k-uniform directed hypergraph on n vertices. For each arc $e = \{j_1, \ldots, j_k\} \in E(\vec{\mathbb{G}})$, we have*

$$(i) \quad \rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \max_{e \in \mathbb{E}(\vec{\mathbb{G}})} \tau_{j_1 \ldots j_k}, \tag{36}$$

and if $\vec{\mathbb{G}}$ is strongly connected, then

$$(ii) \quad \rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq \min_{e \in \mathbb{E}(\vec{\mathbb{G}})} \tau_{j_1 \ldots j_k}. \tag{37}$$

Moreover, if $\vec{\mathbb{G}}$ is strongly connected and outdegree-regular, then equality holds in both Equations (36) and (37).

Proof. (i) We establish the upper bound. Let $D = \mathrm{diag}(d_1^+, \ldots, d_n^+)$, and define $\mathcal{P}(\vec{\mathbb{G}})$ by

$$\mathcal{P}(\vec{\mathbb{G}}) = \mathcal{Q}(\vec{\mathbb{G}}) \cdot D^{1-k} \underbrace{D \cdots D}_{k-1} = \mathcal{D}^+ + \mathcal{A}(\vec{\mathbb{G}}) D^{1-k} \underbrace{D \cdots D}_{k-1}.$$

Lemma 2 yields $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = \rho(\mathcal{P}(\vec{\mathbb{G}}))$, while Lemma 1 guarantees a nonnegative eigenvector $x = (x_1, \ldots, x_n)^\top$ associated with the signless Laplacian H-spectral radius $\rho(\mathcal{P}(\vec{\mathbb{G}}))$. Define x_β as the maximum of the products $x_{i_1} x_{i_2} \cdots x_{i_k}$ over all arcs $\{i_1, i_2, \ldots, i_k\} \in \mathbb{E}(\vec{\mathbb{G}})$. For every $i_1 \in [n]$, using the i_1th equation of $\mathcal{P}(\vec{\mathbb{G}})x^{k-1} = \rho(\mathcal{P}(\vec{\mathbb{G}}))x^{[k-1]}$ and $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = \rho(\mathcal{P}(\vec{\mathbb{G}}))$, one has

$$\begin{aligned}(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{i_1}^+)x_{i_1}^k =& (\rho(\mathcal{P}(\vec{\mathbb{G}})) - d_{i_1}^+)x_{i_1}^k = \sum_{\{i_1, i_2, \ldots, i_k\} \in e_{i_1}} \frac{d_{i_2}^+ \cdots d_{i_k}^+}{d_{i_1}^{+(k-1)}} x_{i_1} x_{i_2} \cdots x_{i_k} \\ &\leq \sum_{\{i_1, i_2, \ldots, i_k\} \in e_{i_1}} \frac{d_{i_2}^+ \cdots d_{i_k}^+}{d_{i_1}^{+(k-1)}} x_\beta = m_{i_1}^+ x_\beta. \end{aligned} \tag{38}$$

We now distinguish two cases.

Case I: $x_\beta = 0$. Since x is nonzero, there exists $p \in [n]$ such that $x_p > 0$. Applying Equation (38) gives

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_p^+)x_p^k \leq \sum_{\{p, i_2, \ldots, i_k\} \in e_p} \frac{d_{i_2}^+ \cdots d_{i_k}^+}{d_p^{+(k-1)}} x_\beta = 0,$$

which shows that $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = d_p^+$. Hence, the desired bound holds trivially.

Case II: $x_\beta > 0$. Suppose that $x_\beta = x_{j_1} x_{j_2} \cdots x_{j_k}$ for some edge $\{j_1, \ldots, j_k\} \in \mathbb{E}(\vec{\mathbb{G}})$.

If $m_t^+ = 0$ for some $t \in \{j_1, \ldots, j_k\}$, then Equation (38) yields

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_t^+)x_t^k = 0.$$

Since $x_\beta > 0$, we have $x_t > 0$, and consequently $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = d_t^+$, confirming the validity of the result.

Now, assume that $m_t^+ > 0$ for all vertices in the chosen edge. By definition, there exists an edge $e = \{j_1, \ldots, j_k\}$ such that $x_{j_1} \cdots x_{j_k} = x_\beta$. For each vertex j_t in this edge, we obtain

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{j_t}^+)x_{j_t}^k \leq m_{j_t}^+ x_\beta, t = 1, \ldots, k. \tag{39}$$

Multiplying the k inequalities (Equation (39)) together yields

$$\prod_{t=1}^{k}(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{j_t}^{+}) \cdot \prod_{t=1}^{k} x_{j_t}^{k} \leq \prod_{t=1}^{k} m_{j_t}^{+} \cdot x_{\beta}^{k}.$$

Since $\prod_{t=1}^{k} x_{j_t}^{k} = x_{\beta}^{k}$, we obtain

$$\prod_{t=1}^{k}(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{j_t}^{+}) \leq \prod_{t=1}^{k} m_{j_t}^{+}.$$

Define

$$g(\lambda) = \prod_{t=1}^{k}(\lambda - d_{j_t}^{+}).$$

Then, g is continuous and strictly increasing on $[U_{j_1 \cdots j_k}, \infty)$. Since $\tau_{j_1 \cdots j_k}$ is the largest real number satisfying

$$g(\lambda) = \prod_{t=1}^{k} m_{j_t}^{+},$$

and g is increasing on $[U_{j_1 \cdots j_t}, \infty)$, it follows that

$$g(\rho(\mathcal{Q}(\vec{\mathbb{G}}))) \leq \prod_{t=1}^{k} m_{j_t}^{+}.$$

This implies that $\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \tau_{j_1 \cdots j_t}$. Taking the maximum over all arcs $e \in \mathbb{E}(\vec{\mathbb{G}})$ yields

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \leq \max_{e \in \mathbb{E}(\vec{\mathbb{G}})} \tau_{j_1 \cdots j_k}.$$

(ii) We establish the lower bound. Let $D = \operatorname{diag}(d_1^{+}, \ldots, d_n^{+})$, and define $\mathcal{P}(\vec{\mathbb{G}})$ by

$$\mathcal{P}(\vec{\mathbb{G}}) = \mathcal{Q}(\vec{\mathbb{G}}) \cdot D^{1-k} \underbrace{D \cdots D}_{k-1} = \mathcal{D}^{+} + \mathcal{A}(\vec{\mathbb{G}}) D^{1-k} \underbrace{D \cdots D}_{k-1}.$$

Lemma 2 yields $\rho(\mathcal{Q}(\vec{\mathbb{G}})) = \rho(\mathcal{P}(\vec{\mathbb{G}}))$. Strong connectivity of $\vec{\mathbb{G}}$ implies that $\mathcal{Q}(\vec{\mathbb{G}})$ is weakly irreducible, which guarantees a positive eigenvector $x = (x_1, \ldots, x_n)^{\top}$ associated with the signless Laplacian H-spectral radius $\rho(\mathcal{P}(\vec{\mathbb{G}}))$ from Lemma 1. Define x_γ as the minimum of the products $x_{i_1} x_{i_2} \cdots x_{i_k}$ over all arcs $\{i_1, i_2, \ldots, i_k\} \in \mathbb{E}(\vec{\mathbb{G}})$. Then, for every $i_1 \in [n]$, the following inequality holds:

$$\begin{aligned}(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{i_1}^{+}) x_{i_1}^{k} &= \sum_{\{i_1, i_2, \ldots, i_k\} \in e_{i_1}} \frac{d_{i_2}^{+} \cdots d_{i_k}^{+}}{(d_{i_1}^{+})^{k-1}} x_{i_1} x_{i_2} \cdots x_{i_k} \\ &\geq \sum_{\{i_1, i_2, \ldots, i_k\} \in e_{i_1}} \frac{d_{i_2}^{+} \cdots d_{i_k}^{+}}{(d_{i_1}^{+})^{k-1}} x_{\gamma} = m_{i_1}^{+} x_{\gamma}. \end{aligned} \tag{40}$$

Let $\{q_1, \ldots, q_k\}$ be an arc such that $x_{q_1} \cdots x_{q_k} = x_\gamma$. Then, from applying Equation (40) to each q_t, we have

$$(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{q_t}^{+}) x_{q_t}^{k} \geq m_{q_t}^{+} x_{\gamma}, \quad t = 1, \ldots, k.$$

By multiplying the multiple inequalities above, we have

$$\prod_{l=1}^{k} x_{q_l}^{k} \cdot \prod_{l=1}^{k}(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{q_l}^{+}) = x_{\gamma}^{k} \prod_{l=1}^{k}(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{q_l}^{+}) \geq x_{\gamma}^{k} \prod_{l=1}^{k} m_{q_l}^{+}.$$

Therefore, we have

$$\prod_{l=1}^{k}(\rho(\mathcal{Q}(\vec{\mathbb{G}})) - d_{q_l}^{+}) \geq \prod_{l=1}^{k} m_{q_l}^{+}.$$

Define

$$g(\lambda) = \prod_{t=1}^{k}(\lambda - d_{q_t}^{+}).$$

Let $U_{q_1\ldots q_k} = \max\{d_{q_l}^{+} : l = 1, \ldots, k\}$. On the interval $[U_{q_1\ldots q_k}, \infty)$, each factor $\lambda - d_{q_l}^{+}$ is nonnegative and strictly increasing, and thus g is continuous and strictly increasing. Let $\tau_{q_1\ldots q_k}$ be the unique real number in $[U_{q_1\ldots q_k}, \infty)$ satisfying

$$g(\tau_{q_1\ldots q_k}) = \prod_{l=1}^{k} m_{q_l}^{+}.$$

Under the monotonicity of g, we have

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq \tau_{q_1\ldots q_k}.$$

Taking the minimum over all arcs $e \in \mathbb{E}(\vec{\mathbb{G}})$ yields

$$\rho(\mathcal{Q}(\vec{\mathbb{G}})) \geq \min_{e\in\mathbb{E}(\vec{\mathbb{G}})} \tau_{j_1\ldots j_k}.$$

(iii) It remains to characterize equality. Under the assumptions of strong connectivity and outdegree regularity, it follows that $d_i^{+} = \mu$ and $m_i^{+} = \nu$ for every $i \in [n]$. Hence, $\tau_{j_1\ldots j_k}$ is constant, which immediately yields the equalities. □

Example 6. *Let $\vec{\mathbb{G}}_6$ be a four-uniform directed hypergraph with a vertex set $\mathbb{V} = U \cup W$, where $U = \{1, 2\}$ and $W = \{3, 4, 5, 6\}$, with the arc set*

Tail	1	1	2	2	3	4	5	6
Heads	$\{2,3,4\}$	$\{2,5,6\}$	$\{1,3,5\}$	$\{1,4,6\}$	$\{1,2,4\}$	$\{1,2,5\}$	$\{1,2,6\}$	$\{1,2,3\}$

We verify that

$$d_1^{+} = d_2^{+} = 2, \qquad d_3^{+} = d_4^{+} = d_5^{+} = d_6^{+} = 1.$$

Thus, $d_U^{+} = 2$ and $d_W^{+} = 1$. Although the vertex set admits a partition $\mathbb{V} = U \cup W$ such that the outdegrees are constant on each part, the induced subhypergraphs on U and W are not empty. Therefore, $\vec{\mathbb{G}}_6$ is not outdegree-semiregular. Moreover, we have

$$m_1^{+} = m_2^{+} = \frac{1}{2}, \qquad m_3^{+} = m_4^{+} = m_5^{+} = m_6^{+} = 4.$$

Thus, $d_U^{+} = 2$ and $d_W^{+} = 1$. Since the induced subhypergraphs on U and W are not empty, $\vec{\mathbb{G}}_6$ is not average two-outdegree-semiregular.

As illustrated by this example, the failure of outdegree semiregularity prevents Theorems 1 and 5 from attaining the exact spectral radius. However, Table 2 reveals that even in the absence of regularity or even semiregularity, the product-type bounds established in Theorems 3 and 6 may still capture the precise value of the spectral radius. This underscores the complementary nature and versatility of the proposed approaches.

Table 2. Comparison of signless Laplacian H-spectral radius bounds for $\vec{\mathbb{G}}_6$.

Theorem	Lower Bound	Upper Bound	Interval Width
Theorem 5.1 of [23]	2.0000	4.0000	2.0000
Theorem 1	2.0000	4.0000	2.0000
Theorem 2	2.3276	4.0000	1.6724
Theorem 3	3.0000	3.0000	0.0000
Theorem 4	2.5000	5.0000	2.5000
Theorem 5	2.5000	4.5000	2.0000
Theorem 6	3.0000	3.0000	0.0000

True value: $\rho(\mathcal{Q}(\vec{\mathbb{G}}_6)) = 3$.

As illustrated by this example, the failure of outdegree semiregularity prevents Theorems 1 and 5 from attaining the exact spectral radius. However, the same example reveals that even in the absence of regularity or even semiregularity, the product-type bounds established in Theorems 3 and 6 may still capture the precise value of the spectral radius. This underscores the complementary nature and versatility of the proposed approaches.

5. Conclusions

We derived sharp bounds for the signless Laplacian H-spectral radius of a directed hypergraph using both pairwise and higher-order outdegree parameters, including the average two-outdegree. Necessary and sufficient conditions for equality were also characterized. In contrast to adjacency hypergraphs, where average two-outdegree regularity (or semiregularity) completely determines the H-spectral radius [25], the analogous situation for signless Laplacian hypergraphs fails, highlighting the additional complexity of their spectral behavior. In light of the unique nonnegative structure of the signless Laplacian tensor, future research should take full advantage of this structure to devise efficient algorithms, such as projection algorithms [34] and conjugate gradient methods [35], for computing the signless Laplacian H-spectral radius. Meanwhile, by leveraging techniques such as arc rewiring or the identification of critical arcs, we intend to achieve targeted control over network robustness or spreading behavior, while preserving the essential structural properties of the underlying network.

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