

Article

An Explicit Closed Form for a Half-Integer ${}_3F_2(1)$ Family with Negative Integral Parameter Differences, via Odd Harmonic Sums

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Abstract

We study the half-integer family $F(n, p) = {}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right)$. Although this family belongs to the Karlsson–Minton class of ${}_3F_2(1)$ series with negative integral parameter differences, the most directly relevant symmetric formula of Shpot and Srivastava becomes singular at the free numerator parameter $a = 1$ (here a denotes the first numerator parameter of the Shpot–Srivastava reduction formula, the value of which controls the Beta factors in their evaluation). We show that this apparent singularity is removable and identify its finite value with the independently derived closed form. First, Euler’s integral representation and a recurrence for $J_{m,n}(x) = \int_0^1 v^{2m} (1 - xv^2)^{-n} dv$ yield a closed form for ${}_2F_1\left(-\frac{2n+1}{2}, 1; \frac{2n+5}{2}; x\right)$ as a polynomial in $(1 - x)$ plus $\operatorname{arctanh}(\sqrt{x})/\sqrt{x}$. Rainville’s integral formula then reduces $F(n, p)$ to a rational part and a finite rational linear combination of odd harmonic sums $H_r^{\text{odd}} = \sum_{j=1}^r (2j - 1)^{-1}$. For all integers $n \geq 0$ and $p \geq n + 3$, this proves $F(n, p) \in \mathbb{Q}$. The threshold is sharp: at the adjacent boundary $p = n + 2$ we obtain $F(n, n + 2) = R_n + C_n \pi^2$ with $R_n, C_n \in \mathbb{Q}$ and $C_n \neq 0$, so rationality fails. We also prove, by analyticity and the independently derived formula, that the $a \rightarrow 1$ limit of the Shpot–Srivastava representation equals the present closed form. For computation, the exact assembly uses $O(n + p)$ arithmetic operations under the unit-cost model, while a direct fixed-precision evaluation is ill-conditioned for large n . A stable series/closed-form hybrid for the ${}_2F_1$ factor combined with adaptive positive-kernel Gauss–Jacobi quadrature gives relative errors at the machine-precision scale over the tested range up to $n = 64$.

Keywords: integrals; Gauss hypergeometric functions; generalized hypergeometric functions; numerical analysis

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1. Introduction

Generalized hypergeometric functions occupy a central position in mathematics and its applications, unifying a vast landscape of classical special functions—including Legendre polynomials, Bessel functions, and elliptic integrals—within a single coherent framework [1–3]. Among them, the Gauss hypergeometric function ${}_2F_1$ and its one-step extension ${}_3F_2$ are of particular theoretical and computational importance. The former arises naturally in the study of differential equations, conformal mappings, and orthogonal polynomials, while the latter appears in combinatorial identities [4], quantum angular

momentum theory [5,6], random walk densities [7,8], and Feynman diagram evaluations in theoretical physics [9,10].

Despite this broad relevance, explicit closed-form evaluations of ${}_3F_2$ at unit argument remain comparatively rare and technically demanding [11–13]. Classical results due to Watson [14], Whipple [15], and Dixon [16] cover only special configurations of parameters—typically those satisfying strict symmetry or balancing conditions—and extensions to half-integer parameters, which arise naturally in problems with half-odd-integer angular momenta [5,6], half-integer Pochhammer shifts [9,17], and dimensional regularization in quantum field theory [9], have long resisted systematic treatment [13,18]. The specific half-integer configuration studied here is not merely a formal curiosity: parameters of the form $\pm \frac{2k+1}{2}$ are exactly those produced by the two physical mechanisms most closely associated with ${}_3F_2(1)$ evaluations. In the quantum theory of angular momentum, the Wigner $3j$, $6j$, and $9j$ symbols for half-odd-integer spins reduce to terminating ${}_3F_2(1)$ series whose numerator and denominator parameters are shifted half-integers [5,6]; the family $F(n, p)$ has precisely this half-integer signature, with the shift $2n + 2$ playing the role of an angular-momentum difference. In perturbative quantum field theory, dimensional regularization in $d = 4 - 2\varepsilon$ dimensions generates ε -expansions of one-loop and massive tadpole Feynman integrals whose coefficients are ${}_3F_2(1)$ values at half-integer parameters [9,10]; the transcendental π^2 -contributions isolated at the boundary $p = n + 2$ (Proposition 2) are of exactly the type that appears as the $\zeta(2)$ weight-two piece of such expansions. The closed forms below therefore supply ready-made building blocks for both settings. The difficulty is not that the classical Euler and Pfaff transformations fail for half-integer parameters—they remain valid under their usual hypotheses—but that, for the parameter configuration considered here, they do not by themselves produce the desired tractable closed form. The resulting series also involve delicate cancellations between rational and transcendental contributions [13,19,20], while half-integer expansions can require additional reduction machinery [17]. Partial progress has been made through two-term relations [21], bisection series [22], Whipple quadratic transformations [23], and computer-algebra expansions [9,17], but a self-contained, analytically complete treatment for the specific half-integer family considered here was absent from the literature until recently.

Several recent contributions have made partial but significant progress in this direction. On the combinatorial side, Alzer and Richards [4] established a family of identities linking binomial coefficient sums to specific ${}_3F_2$ evaluations at unit argument, illustrating how combinatorial structure can unlock closed forms that resist purely analytic attack. On the side of explicit evaluations combining hypergeometric theory with fractional calculus, Zaidi and Almuthaybiri [24] derived closed-form expressions for several subfamilies of ${}_3F_2(1)$ by combining fractional integral operator identities with Beta function identities.

The present paper extends the closed-form line of [24]. Whereas that work treated specific low-order instances of the ${}_3F_2$ family considered here, the current paper extends the approach systematically to all integers $n \geq 0$ and $p \geq n + 3$ by deriving explicit closed-form evaluations of

$${}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right)$$

expressed in terms of rational functions of p and finite odd harmonic sums. The derivation proceeds in two analytically non-trivial stages. In the first stage, a new integral representation of ${}_2F_1\left(-\frac{2n+1}{2}, 1; \frac{2n+5}{2}; x\right)$ is established via Euler’s integral formula [1], followed by a chain of variable substitutions, a non-trivial recurrence relation for a family of auxiliary integrals $J_{m,n}(x)$, and partial fraction decompositions, ultimately yielding a closed form involving $\operatorname{arctanh}(\sqrt{x})/\sqrt{x}$ and an explicit polynomial of degree $2n + 1$ in $(1 - x)$. In the second stage, this representation is integrated against a Beta-type kernel via Rainville’s integral formula [2], splitting the computation into a rational component evaluated through

Beta function identities and Pochhammer symbols, and a logarithmic component resolved via a summation identity for series of the form $\sum_{k=0}^{\infty} [(2k+1)(2k+2r+1)]^{-1}$.

Before proceeding, we place the present family precisely within the existing literature. The series ${}_3F_2(1)$ studied here has upper parameters $a = -\frac{2n+1}{2}$, $b = 1$, $c = \frac{2p-1}{2}$ and lower parameters $d = \frac{2n+5}{2}$, $e = \frac{2p+1}{2}$. It is neither Saalschützian ($a + b + c + 1 = p - n + 1$ while $d + e = n + p + 3$) nor well-poised, so the classical theorems of Watson [14], Whipple [15], and Dixon [16] do not apply in their standard statements. This absence of classical applicability does not, however, establish novelty: the family in fact belongs to the extensively studied class of ${}_3F_2(1)$ series with negative integral parameter differences (equivalently, Karlsson–Minton-type series [25,26]), in which one or more lower parameters exceed an upper parameter by a positive integer. Indeed, several lower-minus-upper differences are integers for all admissible n and p ; in particular $d - a = 2n + 3$ and $e - c = 1$. Such series have been investigated systematically since the foundational work of Minton [25] and Karlsson [26], and subsequently by Karlsson [27], Miller and Paris [28], Paris [29], and Shpot and Srivastava [30], who established general summation and reduction formulae. Numerous further generalizations have appeared over the past five decades, including the reduction formulae of Gottschalk and Maslen [31] and the extensions of the Karlsson–Minton theorem obtained by Karp and Prilepkina [32].

It is therefore essential to delineate what these general results do and do not yield for the present family, and where our contribution lies. The most directly relevant is the pair of manifestly symmetric summation formulae of Shpot and Srivastava ([30] Equations (31) and (32)) for

$${}_3F_2(a, b, c; b + 1 + m, c + 1 + n; 1), \quad m, n \in \mathbb{N}_0,$$

in which a, b, c denote the three (free) numerator parameters of their general series and a in particular is the numerator parameter whose value governs the Beta factor $B(1 - a, \cdot)$ appearing in their evaluation. This family specializes, and gives explicit forms of, the reductions of Karlsson [27] and Miller–Paris [28]. Because ${}_3F_2$ is symmetric in its numerator parameters, we may relabel the upper parameters in the notation of Shpot and Srivastava. With n_{SS} denoting their second shift index, the present family is the specialization

$$a = 1, \quad b = -\frac{2n+1}{2}, \quad c = \frac{2p-1}{2}, \quad m = 2n + 2, \quad n_{SS} = 0.$$

Thus, the apparent change in the symbol a is only a relabelling permitted by numerator symmetry, not a change of the hypergeometric function. (In the generic labelling of our own series above we had $a = -\frac{2n+1}{2}$; in the Shpot–Srivastava labelling the roles of the numerator parameters are permuted so that their a equals 1. Both describe the same ${}_3F_2$.) Crucially, however, their evaluation is expressed through Beta factors $B(1 - a, \cdot)$, which are *singular* at $a = 1$: their result degenerates to the indeterminate form $0 \cdot \infty$ precisely for the present family. The authors flag this exact configuration in their Remark 3, noting that when $a = 1$ the terminating ${}_3F_2$ factors on the right-hand side collapse to Gauss ${}_2F_1$ sums and a separate finite-limit argument is required—a limit they do not carry out. The analogous $a = 1$ limit was resolved by Milgram, Miller–Paris, and Paris [29], but only for the different one-parameter family ${}_3F_2(1, 1, c; d, n + 2; 1)$, in which a single pair of parameters is shifted; the present two-parameter half-integer family (with the large shift $m = 2n + 2$ acting on a half-integer parameter) is not covered by those evaluations. Table 1 summarises the position of the present results relative to these summation and reduction frameworks.

Table 1. Position of the present results relative to the most relevant summation frameworks.

Reference/Framework	What It Provides	Limitation for the Present Family	Role in This Paper
Karlsson [27]	General reduction formulas for hypergeometric functions with negative integral parameter differences, generally as finite multiple sums.	Does not by itself display the present half-integer family as a compact odd-harmonic expression.	Establishes the broad reduction framework in which the family is summable.
Miller–Paris and Paris [28,29]	Explicit reductions and selected $a = 1$ limiting cases for shifted ${}_3F_2(1)$ families.	The treated one-parameter configurations do not contain the simultaneous half-integer shift $m = 2n + 2$ considered here.	Provides the closest precedent for handling singular unit-parameter limits.
Shpot–Srivastava [30]	Manifestly symmetric formulas for generic negative integral parameter differences.	At $a = 1$, the Beta factors are singular; their Remark 3 states that the singular terms cancel and that the finite limit must be handled separately.	Supplies the general formula whose removable singularity is identified analytically in Proposition 3; the explicit value itself is obtained independently by the Euler–Rainville derivation.
Present work	A direct Euler–Rainville derivation, an explicit odd-harmonic closed form, the sharp boundary $p = n + 2$ with both coefficients of its rational/ π^2 split given in closed form, and exact/stable evaluators.	—	Turns the singular special case into a directly computable formula and identifies the rationality threshold.

The contribution of the present paper is therefore not the observation that this family is summable in principle—which follows from the Karlsson–Minton framework—but the explicit evaluation of the excluded $a = 1$ case through a self-contained derivation based on the Euler and Rainville integral representations: a single closed form for $F(n, p)$ as a rational function of p together with a finite rational linear combination of partial odd harmonic sums $H_r^{\text{odd}} = \sum_{j=1}^r (2j-1)^{-1}$ (Theorem 3), its exact rationality (Corollary 1), a sharp characterization of the adjacent boundary $p = n + 2$ in which the value $F(n, n + 2) = R_n + C_n \pi^2$ is given with *both* coefficients in closed form (Proposition 2), thereby exhibiting an explicit family of π^2 -series evaluations, and a linear-time evaluation scheme (Section 5). The passage from the digamma expressions implicit in the general theory to the explicit odd harmonic sums rests on the elementary identity $\psi\left(r + \frac{1}{2}\right) - \psi\left(\frac{1}{2}\right) = 2H_r^{\text{odd}}$; Proposition 3 then uses analyticity of the specialized ${}_3F_2(1)$ to prove that the apparent singularity in ([30] Equation (31)) is removable and that its $a \rightarrow 1$ value must coincide with the independently derived closed form. This locates the new formula precisely within

the prior general theory without treating the Shpot–Srivastava representation itself as the derivation of Theorem 3.

While the closed-form results of this study are mathematically complete, their direct numerical implementation is far from straightforward. A naive evaluation involves deeply nested summations, redundant recomputation of double-factorial ratios and binomial coefficients, and a potentially expensive initialization of partial odd harmonic sums. The present work addresses these computational challenges by translating the analytical results of Theorems 2 and 3 into four computational procedures: shared precomputation, stable ${}_2F_1$ evaluation, exact-rational ${}_3F_2$ assembly, and an adaptive positive-kernel quadrature for floating-point work. The first procedure, called PRECOMPUTE, isolates all quantities depending solely on the order n — the scalar Λ_n , the coefficient array v_i , and the switch-point array $S_n(i)$ — and computes them in $O(n)$ time using recurrences and a sliding-window strategy that avoids the quadratic cost of a naïve implementation. The second, called EVAL2F1, then evaluates

$${}_2F_1\left(-\frac{2n+1}{2}, 1; \frac{2n+5}{2}; x\right)$$

at any point $x \in (0, 1)$ by a stable term recurrence (with the Theorem 2 closed form used only sufficiently near $x = 1$). The third, called EVAL3F2, evaluates

$${}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right)$$

for any admissible p in $O(n+p)$ arithmetic operations under the unit-cost model, with the $O(p)$ cost arising from the proposed elementary sequential initialization of the smallest required partial odd harmonic sum, and the remaining $O(n)$ work distributed across the Pochhammer, factorial, and binomial recurrences.

Throughout all three algorithms, every quantity admitting an incremental recurrence—double-factorial ratios [33,34], Pochhammer rising products [1], binomial coefficients [35], and sliding partial sums—is computed that way, without resorting to repeated evaluation from scratch. This design principle keeps the overall complexity linear in the problem parameters and makes the algorithms well-suited to high-throughput evaluations over grids of (n, p) values or dense meshes in x , as required, for instance, in the numerical study of random walk densities [7,8] or in the verification of hypergeometric identities [4,22].

The remainder of this paper is organized as follows. Section 2 collects the necessary preliminaries: the generalized binomial coefficient, the double factorial and its closed-form expressions in terms of the ordinary factorial [33,34], the Euler Gamma and Beta functions [3], the Gauss hypergeometric function ${}_2F_1$ together with Euler’s integral representation [1], the convergence theorem for ${}_3F_2$ at unit argument due to Andrews, Askey and Roy [1], and Rainville’s integral formula [2], which reduces the evaluation of ${}_3F_2$ to a weighted integral of ${}_2F_1$. Section 2 also establishes five auxiliary lemmas: Lemma 1 derives a closed binomial expansion for the iterated integral $J_{m,n}(x)$ via a forward difference operator [35]; Lemma 2 unravels the recurrence for $K_n(x) = J_{0,n}(x)$ into an explicit sum over weighted powers of $(1-x)^{-(k-1)}$; Lemma 3 evaluates $K_1(x)$ in closed form as $\operatorname{arctanh}(\sqrt{x})/\sqrt{x}$; Lemma 4 combines the previous results to express $K_j(x)$ explicitly in terms of the double-factorial coefficients $d_{i,j}$, e_j , and v_i ; and Lemma 5 resolves the infinite series $\sum_{i=0}^{\infty} [(2i+1)(2i+2r+1)]^{-1}$ into a finite odd harmonic sum via a telescoping argument. Section 3 is devoted to the first main result, Theorem 2, which establishes the closed-form representation of ${}_2F_1\left(-\frac{2n+1}{2}, 1; \frac{2n+5}{2}; x\right)$ as a ratio involving the degree- $(2n+1)$ polynomial $P_{2n+1}(x)$ in $(1-x)$ and the logarithmic factor $Q_n(x) \operatorname{arctanh}(\sqrt{x})/\sqrt{x}$; the proof proceeds through Euler’s integral representation, two successive changes of

variables, the identification $H_n(x) = (2n + 3)(1 - x)^{2n+2}J_{n+1,2n+3}(x)$, and the application of Proposition 1. Section 4 presents the second main result, Theorem 3, which gives the explicit closed-form evaluation of ${}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right)$ for all integers $n \geq 0$ and $p \geq n + 3$, together with Corollary 1 establishing the exact rationality of $F(n, p)$: substituting the representation of Theorem 2 into Rainville's formula splits the computation into the rational integral $I_1(n, p)$, evaluated via Beta function identities and expressed through the Pochhammer symbol $(p - n - \frac{3}{2})_{2n+4-i}$, and the logarithmic integral $I_2(n, p)$, resolved by expanding $f_{n,p}(t)$ through the binomial theorem and the Taylor series of $\operatorname{arctanh}$, applying term-by-term integration justified by Tonelli's theorem (applied to the nonnegative $\operatorname{arctanh}$ expansion before the finite binomial is expanded), and invoking Lemma 5 to reduce each resulting infinite series to a finite odd harmonic sum. Section 5 translates Theorems 2 and 3 into linear-time algorithms, proves their operation count in Theorem 4, analyzes the conditioning of their floating-point evaluation and supplies stable evaluators, presents a complete worked example for $(n, p) = (1, 4)$ yielding $F(1, 4) = \frac{7}{10}$, provides a conditioning study together with accuracy tables for the stable evaluators and an extended table of $F(n, p)$ values, and a figure illustrating the monotone decrease of $F(n, p)$ in p for $n = 1, \dots, 5$.

2. Preliminaries

To facilitate the understanding of the notation employed above and throughout this manuscript, we introduce in this section several fundamental symbols, definitions, and auxiliary results that will be useful in justifying certain steps in the forthcoming proofs. We summarize the main notations as follows:

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ represent the sets of non-negative integers, integers, rational numbers, real numbers, and complex numbers, respectively;
- $\mathbb{X}^*$ denotes the set $\mathbb{X} \setminus \{0\}$;
- $\mathbb{Z}_0^-$ denotes the set of non-positive integers, i.e., $\mathbb{Z}_0^- = -\mathbb{N} = \{\dots, -n, \dots, -1, 0\}$;
- For all $k \in \mathbb{N}^*$, $\mathbb{H}_k^- = \{z \in \mathbb{C} \mid \Re(z) > -k\}$;
- $D = \{z \in \mathbb{C} \mid |z| < 1\}$ denotes the open unit disk;
- $\overline{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ denotes the closed unit disk;
- $\Re(z)$ denotes the real part of the complex number z .

Definition 1 (See ref. [35] p.168). For all $r \in \mathbb{Q}$, and all $n \in \mathbb{N}$, the generalized binomial coefficient $\binom{r}{n}$ is defined by

$$\binom{r}{n} = \frac{r(r-1) \cdots (r-n+1)}{n!}. \quad (1)$$

Definition 2 (See ref. [33]). The double factorial of a non-negative integer n , denoted $n!!$, is the product of all positive integers up to n sharing the same parity as n :

$$n!! = \begin{cases} n(n-2)(n-4) \cdots (4)(2), & \text{if } n \text{ is even,} \\ n(n-2)(n-4) \cdots (3)(1), & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \in \{0, -1\}. \end{cases} \quad (2)$$

The two cases admit the following closed-form expressions in terms of the ordinary factorial. For all integers $p \geq 0$:

$$(2p)!! = 2^p p!, \quad (3)$$

$$(2p+1)!! = \frac{(2p+1)!}{2^p p!}. \quad (4)$$

These two identities are connected by the relation $(2p+1)! = (2p+1)!! (2p)!!$, which expresses the ordinary factorial as the product of the two double factorials of consecutive integers.

Definition 3 (See ref. [3]). The Euler gamma function, denoted by $\Gamma(z)$, is defined for complex numbers z with positive real part by the integral representation

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \forall z \in \mathbb{C} \mid \Re(z) > 0. \quad (5)$$

An application of integration by parts yields the fundamental recursive property

$$\Gamma(z+1) = z\Gamma(z), \quad \forall z \in \mathbb{C}, \quad \Re(z) > 0. \quad (6)$$

This relation extends the gamma function to the region $\Re(z) \leq 0$ via

$$\Gamma(z) = \frac{\Gamma(z+k)}{(z)_k}, \quad k \in \mathbb{N}^*, \quad z \in \mathbb{H}_k^- \setminus \mathbb{Z}_0^-,$$

where $(z)_k$ denotes the Pochhammer symbol, defined for $z \in \mathbb{C}$ and $k \in \mathbb{N}$ by

$$(z)_0 = 1, \quad (z)_k = z(z+1) \cdots (z+k-1), \quad \forall k \in \mathbb{N}^*. \quad (7)$$

Combining (6) with (7), one obtains

$$\Gamma(k+1) = (1)_k = k!, \quad \forall k \in \mathbb{N}. \quad (8)$$

Definition 4 (See ref. [3]). The Beta function, denoted by $\mathcal{B}(a, b)$, is defined for $a, b > 0$ by the integral representation

$$\mathcal{B}(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt. \quad (9)$$

It admits the explicit expression in terms of the Gamma function:

$$\mathcal{B}(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (10)$$

Definition 5 (See ref. [3]). The Gauss hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; z)$ is defined on the unit disk as the analytic function represented by the following hypergeometric power series:

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{z^k}{k!}, \quad (11)$$

where $\alpha, \beta \in \mathbb{C}$ and $\gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-$. The series converges absolutely for $|z| < 1$ with no further restriction on the parameters. On the boundary $|z| = 1$ (in particular at $z = 1$), it converges absolutely provided $\Re(\gamma - \alpha - \beta) > 0$. Moreover, under the conditions $0 < \Re(\beta) < \Re(\gamma)$ and $|\arg(1-z)| < \pi$, the function ${}_2F_1(\alpha, \beta; \gamma; z)$ admits the following integral representation due to Euler:

$${}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 x^{\beta-1} (1-x)^{\gamma-\beta-1} (1-zx)^{-\alpha} dx. \quad (12)$$

The generalized hypergeometric function ${}_3F_2$ arises as a natural extension of the Gauss hypergeometric function ${}_2F_1$. For the evaluation at $z = 1$, convergence of the generalized hypergeometric series ${}_3F_2$ is governed by the following theorem.

Theorem 1 (Andrews, Askey and Roy [1], p. 69). Let $a, b, c \in \mathbb{C}$ and $d, e \in \mathbb{C} \setminus \mathbb{Z}_0^-$. The generalized hypergeometric series

$${}_3F_2(a, b, c; d, e; 1) = \sum_{k=0}^{+\infty} \frac{(a)_k (b)_k (c)_k}{(d)_k (e)_k k!} \quad (13)$$

converges absolutely if and only if $\Re(d + e - a - b - c) > 0$, unless the series terminates (i.e., one of a, b, c is a non-positive integer), in which case it is a finite sum and converges for all admissible parameters.

For all $p \in \mathbb{N}$, matching the parameters $a = -\frac{2n+1}{2}$, $b = 1$, $c = \frac{2p-1}{2}$, $d = \frac{2n+5}{2}$, $e = \frac{2p+1}{2}$, and $z = 1$ in Rainville's integral representation ([2], Theorem 38), the function

$${}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right)$$

admits the following integral representation:

$${}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right) = \frac{2p-1}{2} \int_0^1 x^{\frac{2p-3}{2}} {}_2F_1\left(-\frac{2n+1}{2}, 1; \frac{2n+5}{2}; x\right) dx, \quad (14)$$

since

$$\Re(d + e - a - b - c) = \frac{2n+5}{2} + \frac{2p+1}{2} + \frac{2n+1}{2} - 1 - \frac{2p-1}{2} = 2n + 3 > 0, \quad \text{for all } n \in \mathbb{N}.$$

Useful Lemmas

The present section presents a collection of intermediate results, stated and proved as lemmas, which will be used as essential building blocks within the proofs of both main results of our study, Theorem 2 and Theorem 3.

Lemma 1. For all $n \in \mathbb{N}^*$ with $n \geq 2$, $0 \leq m \leq n$, let $J_{m,n}(x)$ be defined by the recurrence relation

$$J_{m,n}(x) = \frac{1}{x} \left(J_{m-1,n}(x) - J_{m-1,n-1}(x) \right). \quad (15)$$

Set $K_n(x) = J_{0,n}(x)$ for all $n \in \mathbb{N}$. Then, for all integers $n \geq 2$ and $0 \leq m \leq n$,

$$J_{m,n}(x) = \frac{1}{x^m} \sum_{k=0}^m (-1)^k \binom{m}{k} K_{n-k}(x). \quad (16)$$

Proof of Lemma 1. Let Δ be the forward difference operator acting on the index n , defined by

$$\Delta h_n(x) = h_n(x) - h_{n-1}(x).$$

Then, the recurrence relation for $J_{m,n}(x)$ can be written as

$$J_{m,n}(x) = \frac{1}{x} \Delta J_{m-1,n}(x).$$

By iterating this relation m times, we obtain

$$J_{m,n}(x) = \frac{1}{x^m} \Delta^m J_{0,n}(x).$$

Since $K_n(x) = J_{0,n}(x)$, it follows that

$$J_{m,n}(x) = \frac{1}{x^m} \Delta^m K_n(x).$$

The m -th forward difference admits the binomial expansion

$$\Delta^m K_n(x) = \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} K_{n-m+j}(x).$$

By the change of index $k = m - j$, this expression becomes

$$\Delta^m K_n(x) = \sum_{k=0}^m (-1)^k \binom{m}{k} K_{n-k}(x).$$

Substituting into the previous identity yields

$$J_{m,n}(x) = \frac{1}{x^m} \sum_{k=0}^m (-1)^k \binom{m}{k} K_{n-k}(x).$$

The proof is complete. $\square$

Lemma 2. Let $x \in (0, 1)$, $n \in \mathbb{N}$ with $n \geq 2$, and let $K_n(x)$ be defined by the recurrence relation

$$K_n(x) = \frac{1}{2n-2} \left(\frac{1}{(1-x)^{n-1}} + (2n-3)K_{n-1}(x) \right). \quad (17)$$

Then, for all $n \in \mathbb{N}$

$$K_n(x) = \sum_{k=2}^n \left(\prod_{j=k+1}^n \alpha_j \right) \frac{1}{2(k-1)(1-x)^{k-1}} + \left(\prod_{j=2}^n \alpha_j \right) K_1(x). \quad (18)$$

where

$$\begin{aligned} K_1(x) &= \int_0^1 \frac{dv}{2k-3} \frac{1-xv^2}{xv^2} \\ \alpha_k &= \frac{2k-3}{2(k-1)}. \end{aligned} \quad (19)$$

Proof of Lemma 2. We prove by induction on r that, for $1 \leq r \leq n-1$,

$$K_n(x) = \sum_{k=n-r+1}^n \left(\prod_{j=k+1}^n \alpha_j \right) \frac{1}{2(k-1)(1-x)^{k-1}} + \left(\prod_{j=n-r+1}^n \alpha_j \right) K_{n-r}(x).$$

For $r = 1$, the result follows directly from the recurrence relation.

Assume that the statement holds for some $r \in \{1, \dots, n-2\}$. Using the recurrence

$$K_{n-r}(x) = \frac{1}{2(n-r-1)(1-x)^{n-r-1}} + \alpha_{n-r} K_{n-r-1}(x),$$

and substituting into the induction hypothesis, we obtain

$$\begin{aligned} K_n(x) &= \sum_{k=n-r+1}^n \left(\prod_{j=k+1}^n \alpha_j \right) \frac{1}{2(k-1)(1-x)^{k-1}} \\ &+ \left(\prod_{j=n-r+1}^n \alpha_j \right) \frac{1}{2(n-r-1)(1-x)^{n-r-1}} \\ &+ \left(\prod_{j=n-r}^n \alpha_j \right) K_{n-r-1}(x). \end{aligned}$$

The second term corresponds to the summand with $k = n - r$, so the sum extends from $k = n - r$ to n . Hence, the formula holds for $r + 1$. By induction, the result follows. Taking $r = n - 1$ yields the desired representation. The proof is complete. $\square$

Lemma 3. For all $x \in (0, 1)$, it holds that

$$K_1(x) = \frac{1}{2\sqrt{x}} \ln\left(\frac{1 + \sqrt{x}}{1 - \sqrt{x}}\right) = \frac{\operatorname{arctanh}(\sqrt{x})}{\sqrt{x}}. \quad (20)$$

Proof of Lemma 3. Let $0 < x < 1$. We begin by writing

$$1 - xv^2 = 1 - (\sqrt{x}v)^2,$$

and introduce the change of variables

$$\begin{aligned} w &= \sqrt{x}v \quad ; \quad v = 0 \Leftrightarrow w = 0 \\ dv &= \frac{dw}{\sqrt{x}} \quad ; \quad v = 1 \Leftrightarrow w = \sqrt{x}. \end{aligned}$$

Hence,

$$K_1(x) = \frac{1}{\sqrt{x}} \int_0^{\sqrt{x}} \frac{dw}{1 - w^2}.$$

To evaluate the integral, we use the partial fraction decomposition

$$\frac{1}{1 - w^2} = \frac{1}{(1 - w)(1 + w)} = \frac{1}{2} \left(\frac{1}{1 - w} + \frac{1}{1 + w} \right).$$

Therefore,

$$\int_0^{\sqrt{x}} \frac{dw}{1 - w^2} = \frac{1}{2} \int_0^{\sqrt{x}} \frac{dw}{1 + w} + \frac{1}{2} \int_0^{\sqrt{x}} \frac{dw}{1 - w}.$$

Evaluating each term gives

$$\int_0^{\sqrt{x}} \frac{dw}{1 - w^2} = \frac{1}{2} \ln(1 + \sqrt{x}) - \frac{1}{2} \ln(1 - \sqrt{x}) = \frac{1}{2} \ln\left(\frac{1 + \sqrt{x}}{1 - \sqrt{x}}\right),$$

where the condition $0 < \sqrt{x} < 1$ ensures that the logarithms are well defined.

Substituting back, we obtain

$$K_1(x) = \frac{1}{2\sqrt{x}} \ln\left(\frac{1 + \sqrt{x}}{1 - \sqrt{x}}\right).$$

Using the identity

$$\operatorname{arctanh}(t) = \frac{1}{2} \ln\left(\frac{1 + t}{1 - t}\right)$$

for $|t| < 1$, we conclude that

$$K_1(x) = \frac{\operatorname{arctanh}(\sqrt{x})}{\sqrt{x}}, \quad 0 < x < 1.$$

The proof is complete. $\square$

Lemma 4 (Explicit form of $K_j(x)$). For all $j \geq 2$ and $x \in (0, 1)$,

$$K_j(x) = \sum_{i=2}^j \frac{d_{i,j}}{(1 - x)^{i-1}} + e_j L(x), \quad (21)$$

where

$$d_{i,j} = \frac{(2j-3)!! (2i-2)!!}{(2i-3)!! (2j-2)!! 2(i-1)}, \quad (22)$$

$$e_j = \frac{(2j-3)!!}{(2j-2)!!}. \quad (23)$$

$$L(x) = K_1(x) = \frac{\operatorname{arctanh}(\sqrt{x})}{\sqrt{x}}. \quad (24)$$

Proof of Lemma 4. Throughout this document, we use the double-factorial identities introduced by Meserve [34]:

$$(2m)!! = 2^m m!, \quad (2m-1)!! = \frac{(2m)!}{2^m m!} = \frac{(2m)!}{(2m)!!}, \quad \forall m \in \mathbb{N}^*, \quad (25)$$

with the conventions $(-1)!! = 0!! = 1$.

1. According to relation (18) of Lemma 2, for all $j \geq 2$, and $x \in (0, 1)$, we have:

$$K_j(x) = \sum_{i=2}^j \left(\prod_{\ell=i+1}^j \alpha_\ell \right) \frac{1}{2(i-1)(1-x)^{i-1}} + \left(\prod_{\ell=2}^j \alpha_\ell \right) L(x), \quad \alpha_\ell = \frac{2\ell-3}{2(\ell-1)}. \quad (26)$$

2. Let $A_{i,j} = \prod_{\ell=i+1}^j \alpha_\ell$. Then, by separating the numerator and denominator of α_ℓ , we obtain

$$A_{i,j} = \frac{(2i-1)(2i+1) \cdots (2j-3)}{2^{j-i} i(i+1) \cdots (j-1)} = \frac{(2j-3)!! / (2i-3)!!}{2^{j-i} (j-1)! / (i-1)!}. \quad (27)$$

Applying the identities $(2j-3)!! = \frac{(2j-2)!}{(2j-2)!!}$ and $(2i-3)!! = \frac{(2i-2)!}{(2i-2)!!}$ from (25), as well as $(2j-2)!! = 2^{j-1} (j-1)!$ and $(2i-2)!! = 2^{i-1} (i-1)!$, we obtain

$$A_{i,j} = \frac{(2j-3)!! (2i-2)!!}{(2i-3)!! (2j-2)!!}. \quad (28)$$

3. Let $d_{i,j}$ and e_j be the coefficients defined by

$$d_{i,j} = \frac{A_{i,j}}{2(i-1)} \quad ; \quad e_j = A_{1,j}. \quad (29)$$

Therefore,

$$d_{i,j} = \frac{(2j-3)!! (2i-2)!!}{(2i-3)!! (2j-2)!! 2(i-1)}, \quad (30)$$

and

$$e_j = A_{1,j} = \frac{(2j-3)!! (0)!!}{(-1)!! (2j-2)!!} = \frac{(2j-3)!!}{(2j-2)!!}, \quad (31)$$

4. Substituting $d_{i,j}$ and e_j into (26), we obtain

$$\begin{aligned}
 K_j(x) &= \sum_{i=2}^j \frac{A_{i,j}}{2(i-1)(1-x)^{i-1}} + A_{1,j}L(x), \\
 &= \sum_{i=2}^j \frac{d_{i,j}}{(1-x)^{i-1}} + A_{1,j}L(x), \\
 &= \sum_{i=2}^j \frac{d_{i,j}}{(1-x)^{i-1}} + e_jL(x), \\
 &= e_j \sum_{i=2}^j \frac{v_i}{(1-x)^{i-1}} + e_jL(x),
 \end{aligned} \tag{32}$$

where

$$v_i = \frac{d_{i,j}}{e_j} = \frac{(2i-2)!!}{(2i-3)!! 2(i-1)} = \frac{(2i-4)!!}{(2i-3)!!}. \tag{33}$$

The proof is complete. $\square$

Proposition 1. For all $n \in \mathbb{N}$ and $x \in (0, 1)$,

$$J_{n+1,2n+3}(x) = \frac{1}{x^{n+1}} \left(\sum_{i=2}^{2n+3} \frac{S_n(i) v_i}{(1-x)^{i-1}} + \frac{(-1)^{n+1} (2n+1)!!}{4^{n+1} (2n+2)!!} L(x) \right), \tag{34}$$

where

$$S_n(i) = \begin{cases} \frac{(-1)^{n+1} (2n+1)!!}{4^{n+1} (2n+2)!!}, & 2 \leq i \leq n+2, \\ \sum_{k=0}^{2n+3-i} \binom{n+1}{k} \binom{-\frac{1}{2}}{2n+2-k}, & n+3 \leq i \leq 2n+3. \end{cases} \tag{35}$$

and $\binom{-\frac{1}{2}}{2n+2-k}$ denotes the generalized binomial coefficient given by relation (1).

Proof of Proposition 1.

1. According to relation (15) of Lemma 1, for all $n \in \mathbb{N}$, and $x \in (0, 1)$, we have:

$$J_{n+1,2n+3}(x) = \frac{1}{x^{n+1}} \left(\underbrace{\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \sum_{i=2}^{2n+3-k} \frac{d_{i,2n+3-k}}{(1-x)^{i-1}}}_{\Omega} + \underbrace{\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} e_{2n+3-k} L(x)}_{\Lambda_n} \right). \tag{36}$$

2. We evaluate Λ_n in three steps.

2.a Applying the identity $(2m-1)!! = \frac{(2m)!}{(2m)!!}$ from (25) with $m = 2n+2-k$, together with $(2m)!! = 2^m m!$ (given in (25)), yields $(4n+4-2k)!! = 2^{2n+2-k} (2n+2-k)!$, and consequently,

$$e_{2n+3-k} = \frac{(4n+3-2k)!!}{(4n+4-2k)!!} = \frac{(4n+4-2k)!}{[(4n+4-2k)!!]^2} = \frac{(4n+4-2k)!}{4^{2n+2-k} [(2n+2-k)!]^2} = \frac{\binom{4n+4-2k}{2n+2-k}}{4^{2n+2-k}}. \tag{37}$$

2.b. The identity

$$\frac{\binom{2m}{m}}{4^m} = (-1)^m \binom{-\frac{1}{2}}{m}, \quad m \in \mathbb{N}, \tag{38}$$

which follows from the definition of the generalized binomial coefficient ([35], p. 186), applied with $m = 2n + 2 - k$, converts each summand:

$$(-1)^k \binom{n+1}{k} \frac{\binom{4n+4-2k}{2n+2-k}}{4^{2n+2-k}} = \binom{n+1}{k} \binom{-\frac{1}{2}}{2n+2-k}. \quad (39)$$

2.c. Since $\binom{n+1}{k} = 0$ for $k > n + 1$, the upper limit of summation may be extended from $n + 1$ to $2n + 2$ without cost. The Vandermonde–Chu identity ([35], p. 169)

$$\sum_{k=0}^r \binom{s}{k} \binom{t}{r-k} = \binom{s+t}{r}, \quad s, t \in \mathbb{C}, \quad r \in \mathbb{N}, \quad (40)$$

applied with $s = n + 1$, $t = -\frac{1}{2}$, $r = 2n + 2$, then gives

$$\Lambda_n = \sum_{k=0}^{n+1} \binom{n+1}{k} \binom{-\frac{1}{2}}{2n+2-k} = \sum_{k=0}^{2n+2} \binom{n+1}{k} \binom{-\frac{1}{2}}{2n+2-k} = \binom{n+\frac{1}{2}}{2n+2}. \quad (41)$$

Finally, expanding $\binom{n+\frac{1}{2}}{2n+2}$ via the definition of the generalized binomial coefficient [1] and applying (25) yields the explicit double-factorial form:

$$\Lambda_n = \binom{n+\frac{1}{2}}{2n+2} = \frac{(-1)^{n+1} (2n+1)!!}{4^{n+1} (2n+2)!!}. \quad (42)$$

3. Setting $j = 2n + 3 - k$ in (22) yields

$$d_{i,2n+3-k} = \frac{(4n+3-2k)!! (2i-2)!!}{(2i-3)!! (4n+4-2k)!! 2(i-1)} = \frac{(4n+3-2k)!! (2i-4)!!}{(2i-3)!! (4n+4-2k)!!}. \quad (43)$$

Substituting (43) into Ω and interchanging the order of summation, with i ranging from 2 to $2n + 3$ as the outer index and k from 0 to $\min(n + 1, 2n + 3 - i)$ as the inner index, gives

$$\Omega = \sum_{i=2}^{2n+3} \frac{(2i-4)!!}{(2i-3)!! (1-x)^{i-1}} \sum_{k=0}^{\min(n+1, 2n+3-i)} (-1)^k \binom{n+1}{k} \frac{(4n+3-2k)!!}{(4n+4-2k)!!}. \quad (44)$$

4. At this stage, it is convenient to name the upper summation limit of the inner sum in (44).

For each outer index $i \in \{2, \dots, 2n + 3\}$, set

$$k_{\max}(i) := \min(n + 1, 2n + 3 - i). \quad (45)$$

We evaluate the inner sum by distinguishing the two cases according to which of the two arguments realizes $k_{\max}(i)$.

4.a If $2 \leq i \leq n + 2$, then $2n + 3 - i \geq n + 1$, so $k_{\max}(i) = n + 1$. The inner sum therefore runs from $k = 0$ to $k = n + 1$. Since $\binom{n+1}{k} = 0$ for every $k > n + 1$, extending the upper limit to $k = 2n + 2$ adds only zero terms, and the three sub-steps of Step 2 apply without change: sub-step (a) converts each double-factorial ratio via (37), sub-step (b) applies the generalized binomial identity (38) with $m = 2n + 2 - k$, and sub-step (c) closes the resulting

full convolution sum by the Vandermonde–Chu identity (40) with $s = n + 1$, $t = -\frac{1}{2}$, $r = 2n + 2$. This yields

$$\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{(4n+3-2k)!!}{(4n+4-2k)!!} = \binom{n+\frac{1}{2}}{2n+2} = \frac{(-1)^{n+1} (2n+1)!!}{4^{n+1} (2n+2)!!}, \quad (46)$$

so that $S_n(i) = \frac{(-1)^{n+1} (2n+1)!!}{4^{n+1} (2n+2)!!}$ for all $i \in \{2, \dots, n+2\}$.

4.b If $n+3 \leq i \leq 2n+3$, then $2n+3-i \leq n$, so $k_{\max}(i) = 2n+3-i < n+1$. The upper limit is now genuinely smaller than $n+1$, so the sum is a *partial* Vandermonde–Chu convolution that does not reduce to a single closed form. Applying sub-steps (a) and (b) of Step 2 only (sub-step (c) is no longer available), the inner sum becomes

$$\sum_{k=0}^{2n+3-i} (-1)^k \binom{n+1}{k} \frac{(4n+3-2k)!!}{(4n+4-2k)!!} = \sum_{k=0}^{2n+3-i} \binom{n+1}{k} \binom{-\frac{1}{2}}{2n+2-k}, \quad (47)$$

which is left in this form.

Combining both cases, the inner sum in (44) equals $S_n(i)$, where

$$\begin{aligned} S_n(i) &= \sum_{k=0}^{k_{\max}(i)} (-1)^k \binom{n+1}{k} \frac{(4n+3-2k)!!}{(4n+4-2k)!!} \\ &= \begin{cases} \frac{(-1)^{n+1} (2n+1)!!}{4^{n+1} (2n+2)!!}, & 2 \leq i \leq n+2, \\ \sum_{k=0}^{2n+3-i} \binom{n+1}{k} \binom{-\frac{1}{2}}{2n+2-k}, & n+3 \leq i \leq 2n+3, \end{cases} \end{aligned} \quad (48)$$

in agreement with (35).

5. Inserting (48) into (44), we obtain

$$\Omega = \sum_{i=2}^{2n+3} \frac{S_n(i) (2i-4)!!}{(2i-3)!! (1-x)^{i-1}}. \quad (49)$$

Combining (49) and (41) with (36) yields (34). The proof is complete. $\square$

Lemma 5. Let $p, m, n \in \mathbb{Z}$ be such that $r = p + m - n - 2 \geq 1$. Then

$$\sum_{i=0}^{+\infty} \frac{1}{(2i+1)(2(p+m+i)-2n-3)} = \frac{1}{2(p+m-n-2)} \sum_{j=1}^{p+m-n-2} \frac{1}{2j-1}. \quad (50)$$

Proof of Lemma 5. Set $r = p + m - n - 2$. Then $r \geq 1$, and $2(p+m+i) - 2n - 3 = 2i + 2r + 1$. Hence,

$$\sum_{i=0}^{+\infty} \frac{1}{(2i+1)(2(p+m+i)-2n-3)} = \sum_{i=0}^{+\infty} \frac{1}{(2i+1)(2i+2r+1)}. \quad (51)$$

Using partial fractions, we have

$$\frac{1}{(2i+1)(2i+2r+1)} = \frac{1}{2r} \left(\frac{1}{2i+1} - \frac{1}{2i+2r+1} \right). \quad (52)$$

Therefore, for $N \geq 0$,

$$\sum_{i=0}^N \frac{1}{(2i+1)(2i+2r+1)} = \frac{1}{2r} \sum_{i=0}^N \left(\frac{1}{2i+1} - \frac{1}{2i+2r+1} \right). \quad (53)$$

The sum telescopes. Indeed,

$$\sum_{i=0}^N \left(\frac{1}{2i+1} - \frac{1}{2i+2r+1} \right) = \sum_{i=0}^{r-1} \frac{1}{2i+1} - \sum_{i=N+1}^{N+r} \frac{1}{2i+1}. \quad (54)$$

Since

$$0 \leq \sum_{i=N+1}^{N+r} \frac{1}{2i+1} \leq \frac{r}{2N+3} \rightarrow 0 \quad \text{as } N \rightarrow +\infty, \quad (55)$$

we obtain

$$\sum_{i=0}^{+\infty} \frac{1}{(2i+1)(2i+2r+1)} = \frac{1}{2r} \sum_{i=0}^{r-1} \frac{1}{2i+1}. \quad (56)$$

Finally, by taking $j = i + 1$, we get

$$\sum_{i=0}^{r-1} \frac{1}{2i+1} = \sum_{j=1}^r \frac{1}{2j-1}. \quad (57)$$

Thus,

$$\sum_{i=0}^{+\infty} \frac{1}{(2i+1)(2(p+m+i)-2n-3)} = \frac{1}{2(p+m-n-2)} \sum_{j=1}^{p+m-n-2} \frac{1}{2j-1}. \quad (58)$$

The proof is complete. $\square$

3. Explicit Form of ${}_2F_1\left(-\frac{2n+1}{2}, 1; \frac{2n+5}{2}; x\right)$

The present section is devoted to the first main result of our study, namely the establishment of a new integral representation of ${}_2F_1\left(-\frac{2n+1}{2}, 1; \frac{2n+5}{2}; x\right)$ for all $x \in (0, 1)$, which constitutes the analytical core of the paper and paves the way for the explicit evaluation of ${}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right)$ carried out in Section 4.

Theorem 2 (First main result). *For all $n \in \mathbb{N}$, $x \in (0, 1)$, the following identity holds:*

$${}_2F_1\left(-\frac{2n+1}{2}, 1; \frac{2n+5}{2}; x\right) = \frac{2n+3}{x^{n+1}} \left(P_{2n+1}(x) + Q_n(x) \frac{\operatorname{arctanh}(\sqrt{x})}{\sqrt{x}} \right) \quad (59)$$

where

$$P_{2n+1}(x) = \sum_{i=2}^{2n+3} \frac{S_n(i) (2i-4)!!}{(2i-3)!!} (1-x)^{2n+3-i}; \quad Q_n(x) = \frac{(-1)^{n+1} (2n+1)!!}{4^{n+1} (2n+2)!!} (1-x)^{2n+2} \quad (60)$$

and

$$S_n(i) = \begin{cases} \frac{(-1)^{n+1} (2n+1)!!}{4^{n+1} (2n+2)!!}, & 2 \leq i \leq n+2, \\ \sum_{k=0}^{2n+3-i} \binom{n+1}{k} \binom{-\frac{1}{2}}{2n+2-k}, & n+3 \leq i \leq 2n+3. \end{cases} \quad (61)$$

Proof of Theorem 2. The proof is structured into a sequence of steps, each denoted by boldface integers.

1. Using Euler's representation (12) with $\alpha = -\frac{2n+1}{2}$, $\beta = 1$, and $\gamma = \frac{2n+5}{2}$, we obtain

$$H_n(x) = {}_2F_1\left(-\frac{2n+1}{2}, 1; \frac{2n+5}{2}; x\right) = C_n \int_0^1 (1-t)^{(2n+1)/2} (1-xt)^{(2n+1)/2} dt, \quad (62)$$

where

$$C_n = \frac{\Gamma(\frac{2n+5}{2})}{\Gamma(1)\Gamma(\frac{2n+3}{2})} = \frac{2n+3}{2}. \quad (63)$$

Thus,

$$H_n(x) = \frac{2n+3}{2} \int_0^1 (1-t)^{(2n+1)/2} (1-xt)^{(2n+1)/2} dt. \quad (64)$$

2. Let u be the variable defined by:

$$u = \sqrt{\frac{1-xt}{1-t}}. \quad (65)$$

Therefore,

$$\begin{aligned} t &= \frac{1-u^2}{x-u^2} ; \quad \lim_{t \rightarrow 0^+} u = 1 \\ \frac{du}{dt} &= \frac{1-x}{2u(1-t)^2} ; \quad \lim_{t \rightarrow 1^-} u = +\infty \end{aligned} \quad (66)$$

Moreover,

$$1-t = \frac{x-1}{x-u^2} \quad \text{and} \quad 1-xt = \frac{u^2(x-1)}{x-u^2}. \quad (67)$$

Thus,

$$\frac{du}{dt} = \frac{(x-u^2)^2}{2u(1-x)}. \quad (68)$$

Applying the transformation in (65), Equation (64) can be rewritten in terms of x and u .

$$\begin{aligned} H_n(x) &= \frac{2n+3}{2} \int_1^{+\infty} \frac{(x-1)^{(2n+1)/2}}{(x-u^2)^{(2n+1)/2}} \frac{u^{2n+1}(x-1)^{(2n+1)/2}}{(x-u^2)^{(2n+1)/2}} \frac{2u(1-x)}{(x-u^2)^2} du \\ &= -(2n+3)(1-x)^{2n+2} \int_1^{+\infty} \frac{u^{2n+2}}{(x-u^2)^{2n+3}} du. \end{aligned} \quad (69)$$

3. Note that the denominator in the above integral can be expressed as:

$$\begin{aligned} (x-u^2)^{2n+3} &= \left[-u^2\left(1-\frac{x}{u^2}\right)\right]^{2n+3} \\ &= -\frac{(1-xv^2)^{2n+3}}{v^{4n+6}}, \quad \text{where } v = \frac{1}{u}. \end{aligned} \quad (70)$$

It follows immediately that,

$$du = -\frac{dv}{v^2} ; \quad \lim_{u \rightarrow 1} v = 1 ; \quad \lim_{u \rightarrow +\infty} v = 0. \quad (71)$$

Consequently, the integral in (69) admits the following representation in terms of v and x :

$$\begin{aligned} H_n(x) &= (2n+3)(1-x)^{2n+2} \int_0^1 \frac{-v^{-2n-2}v^{4n+6}}{(1-xv^2)^{2n+3}} \frac{-dv}{v^2} \\ &= (2n+3)(1-x)^{2n+2} \int_0^1 \frac{v^{2n+2}}{(1-xv^2)^{2n+3}} dv \end{aligned} \quad (72)$$

4. For all $m, n \in \mathbb{N}$, we define the functions $J_{m,n}(x)$ and $K_n(x)$ by:

$$J_{m,n}(x) = \int_0^1 \frac{v^{2m}}{(1-xv^2)^n} dv, \quad (73)$$

and

$$K_n(x) = J_{0,n}(x) = \int_0^1 \frac{dv}{(1-xv^2)^n}. \quad (74)$$

By exploiting the identity

$$v^2 = \frac{1 - (1 - xv^2)}{x}, \quad (75)$$

we derive the recurrence relation:

$$\begin{aligned} J_{m,n}(x) &= \int_0^1 \frac{v^{2(m-1)} v^2}{(1-xv^2)^n} dv \\ &= \frac{1}{x} \int_0^1 \frac{v^{2(m-1)} (1 - (1 - xv^2))}{(1-xv^2)^n} dv \\ &= \frac{1}{x} \left[\int_0^1 \frac{v^{2(m-1)}}{(1-xv^2)^n} dv - \int_0^1 \frac{v^{2(m-1)}}{(1-xv^2)^{n-1}} dv \right] \\ &= \frac{1}{x} (J_{m-1,n}(x) - J_{m-1,n-1}(x)). \end{aligned} \quad (76)$$

5. Given the expression for $H_n(x)$ given by (72), and the definition of $J_{m,n}(x)$ given by (73), we have

$$H_n(x) = (2n+3)(1-x)^{2n+2} J_{n+1,2n+3}(x), \quad \forall n \in \mathbb{N}. \quad (77)$$

6. Applying (34) of Proposition 1 gives an explicit representation of $J_{n+1,2n+3}(x)$:

$$\begin{aligned} H_n(x) &= (2n+3)(1-x)^{2n+2} \frac{1}{x^{n+1}} \left(\sum_{i=2}^{2n+3} \frac{S_n(i) v_i}{(1-x)^{i-1}} + \frac{(-1)^{n+1} (2n+1)!!}{4^{n+1} (2n+2)!!} L(x) \right) \\ &= \frac{2n+3}{x^{n+1}} \left(\sum_{i=2}^{2n+3} S_n(i) v_i (1-x)^{2n+3-i} + \frac{(-1)^{n+1} (2n+1)!!}{(2n+2)!!} (1-x)^{2n+2} L(x) \right) \\ &= \frac{2n+3}{x^{n+1}} \left(P_{2n+1}(x) + Q_n(x) \frac{\operatorname{arctanh}(\sqrt{x})}{\sqrt{x}} \right), \end{aligned} \quad (78)$$

where

$$P_{2n+1}(x) = \sum_{i=2}^{2n+3} S_n(i) v_i (1-x)^{2n+3-i}, \quad (79)$$

and

$$Q_n(x) = \frac{(-1)^{n+1} (2n+1)!!}{4^{n+1} (2n+2)!!} (1-x)^{2n+2}, \quad (80)$$

which is consistent with the result of Theorem 2. This completes the proof. $\square$

4. Explicit Form of ${}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right)$

The present section is devoted to the second main result of our study (Theorem 3), which relies crucially on the integral representation derived in Theorem 2. By integrating the representation of ${}_2F_1\left(-\frac{2n+1}{2}, 1; \frac{2n+5}{2}; x\right)$ against a Beta-type kernel via Rainville's integral Formula [2], we obtain an explicit closed-form evaluation of ${}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right)$ for all integers $p \geq n+3$. The adjacent case $p = n+2$ is also well defined and will be treated separately; it is precisely where the rationality mechanism changes.

For all integers $n \geq 0$ and $p \geq n+2$, we define the quantity $F(n, p)$ by

$$F(n, p) = {}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right) \quad (81)$$

Substituting the representation of ${}_2F_1\left(-\frac{2n+1}{2}, 1; \frac{2n+5}{2}; x\right)$ obtained in Theorem 2 into the integral formula (14), we obtain

$$\begin{aligned} F(n, p) &= \frac{2p-1}{2} \int_0^1 x^{\frac{2p-3}{2}} \frac{2n+3}{x^{n+1}} \left(P_{2n+1}(x) + Q_n(x) \frac{\operatorname{arctanh}(\sqrt{x})}{\sqrt{x}} \right) dx \\ &= \frac{(2n+3)(2p-1)}{2} \int_0^1 x^{\frac{2p-2n-5}{2}} \left(P_{2n+1}(x) + Q_n(x) \frac{\operatorname{arctanh}(\sqrt{x})}{\sqrt{x}} \right) dx. \end{aligned} \quad (82)$$

where $P_{2n+1}(x)$ and $Q_n(x)$ are defined in (60).

Separating the integral into two components yields the following general representation:

$$F(n, p) = \frac{(2n+3)(2p-1)}{2} (I_1(n, p) + I_2(n, p)), \quad (83)$$

where

$$I_1(n, p) = \int_0^1 x^{p-n-\frac{5}{2}} P_{2n+1}(x) dx, \quad (84)$$

and

$$I_2(n, p) = \int_0^1 x^{p-n-\frac{5}{2}} Q_n(x) \frac{\operatorname{arctanh}(\sqrt{x})}{\sqrt{x}} dx. \quad (85)$$

4.1. Why the Threshold $p \geq n+3$ Appears

The condition $p \geq n+3$ is *not* required for convergence of the original ${}_3F_2(1)$ series: its parametric excess is

$$\frac{2n+5}{2} + \frac{2p+1}{2} - \left(-\frac{2n+1}{2} + 1 + \frac{2p-1}{2} \right) = 2n+3 > 0.$$

Nor is it needed for the rational integral $I_1(n, p)$, which is integrable already for $p \geq n+2$. The stronger threshold enters only in the reduction of $I_2(n, p)$. After the substitution $t = \sqrt{x}$ and the Tonelli interchange used below, the inner series has the form

$$\sum_{j=0}^{\infty} \frac{1}{(2j+1)(2j+2r+1)}, \quad r = p + m - n - 2.$$

Lemma 5 reduces this series to a finite odd harmonic sum when $r \geq 1$. Since m starts at zero, this condition is valid simultaneously for every term exactly when $p \geq n+3$. At the adjacent boundary $p = n+2$, the single $m = 0$ contribution has $r = 0$ and becomes

$$\sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} = \frac{\pi^2}{8}.$$

Consequently, the boundary value contains a nonzero rational multiple of π^2 rather than being rational. Proposition 2 below makes this statement explicit and shows that the rationality threshold in Corollary 1 is sharp.

4.2. Computation of $I_1(n, p)$

We derive an explicit closed-form expression for

$$I_1(n, p) = \int_0^1 x^{p-n-\frac{5}{2}} P_{2n+1}(x) dx, \quad (86)$$

where:

$$P_{2n+1}(x) = \sum_{i=2}^{2n+3} S_n(i) v_i (1-x)^{2n+3-i}, \quad v_i = \frac{(2i-4)!!}{(2i-3)!!} \quad (\text{see (33)}). \quad (87)$$

1. Substituting (87) and exchanging summation with integration (justified by the finiteness of the sum and the integrability of each term for $p \geq n+2$), we obtain

$$I_1(n, p) = \sum_{i=2}^{2n+3} S_n(i) v_i \int_0^1 x^{p-n-\frac{5}{2}} (1-x)^{2n+3-i} dx.$$

Each inner integral is a Beta function:

$$\int_0^1 x^{p-n-\frac{5}{2}} (1-x)^{2n+3-i} dx = \mathcal{B}(p-n-\frac{3}{2}, 2n+4-i), \quad (88)$$

which converges provided

- $p-n-\frac{3}{2} > 0$, i.e., $p \geq n+2$;
- $2n+4-i > 0$, which holds for all $i \leq 2n+3$.

2. Apply $\mathcal{B}(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ with $a = p-n-\frac{3}{2}$ and $b = 2n+4-i$. Introduce the change of index $j = 2n+3-i \in \{0, 1, \dots, 2n+1\}$ (as i decreases from $2n+3$ to 2). Then $b = j+1$ and

$$\Gamma(a+b) = \Gamma(p-n-\frac{3}{2}+j+1) = \Gamma(p-n-\frac{3}{2}) (p-n-\frac{3}{2})_{j+1},$$

where the last step uses $\Gamma(z+k) = \Gamma(z) (z)_k$ with $k = j+1$. Since $\Gamma(b) = \Gamma(j+1) = j!$, we obtain

$$\mathcal{B}(p-n-\frac{3}{2}, j+1) = \frac{j!}{(p-n-\frac{3}{2})_{j+1}} = \frac{(2n+3-i)!}{(p-n-\frac{3}{2})_{2n+4-i}}. \quad (89)$$

3. Recall from (61) that

$$S_n(i) = \begin{cases} \Lambda_n = \frac{(-1)^{n+1}(2n+1)!!}{4^{n+1}(2n+2)!!}, & 2 \leq i \leq n+2, \\ \sum_{k=0}^{2n+3-i} \binom{n+1}{k} \binom{-\frac{1}{2}}{2n+2-k}, & n+3 \leq i \leq 2n+3. \end{cases}$$

4. Combining the steps above, the complete formula in terms of the original index i is:

$$I_1(n, p) = \sum_{i=2}^{2n+3} S_n(i) \frac{(2i-4)!!}{(2i-3)!!} \frac{(2n+3-i)!}{(p-n-\frac{3}{2})_{2n+4-i}}, \quad (90)$$

4.3. Computation of $I_2(n, p)$

For all integers $p \geq n+2$, let $f_{n,p}(t)$ be the function defined by:

$$f_{n,p}(t) = t^{2p-2n-5} (1-t^2)^{2n+2} \operatorname{arctanh}(t), \quad t \in (0, 1). \quad (91)$$

Then, using the variable change $t = \sqrt{x}$, the integral $I_2(n, p)$ can be written as

$$I_2(n, p) = D_n \int_0^1 f_{n,p}(t) dt, \quad \text{where } D_n = 2 \frac{(-1)^{n+1}(2n+1)!!}{4^{n+1}(2n+2)!!}. \quad (92)$$

We first show that $I_2(n, p)$ is convergent already for $p \geq n + 2$. The finite odd-harmonic reduction in Step 3 is then carried out for $p \geq n + 3$; the boundary case $p = n + 2$ is retained separately in Proposition 2.

1. By the binomial theorem and the Taylor series of $\operatorname{arctanh}$:

$$(1 - t^2)^{2n+2} = \sum_{m=0}^{2n+2} (-1)^m \binom{2n+2}{m} t^{2m}, \quad \operatorname{arctanh}(t) = \sum_{j=0}^{+\infty} \frac{t^{2j+1}}{2j+1}, \quad |t| < 1. \quad (93)$$

Multiplying all three factors and collecting powers of t :

$$f_{n,p}(t) = \sum_{m=0}^{2n+2} \sum_{j=0}^{+\infty} \frac{(-1)^m \binom{2n+2}{m}}{2j+1} t^{2(p+m+j)-2n-4} = \sum_{k=0}^{+\infty} c_k t^{2(p+k)-2n-4}, \quad (94)$$

where, after reindexing with $k = m + j$:

$$c_k = \sum_{m=0}^{\min(k, 2n+2)} \frac{(-1)^m \binom{2n+2}{m}}{2(k-m)+1}. \quad (95)$$

The series (94) has radius of convergence $R = 1$ since it is governed by the Taylor series of $\operatorname{arctanh}(t)$.

2. The function $f_{n,p}$ is non-negative on $(0, 1)$. Moreover, for $p \geq n + 2$, near $t = 0$ we have $f_{n,p}(t) \sim t^{2p-2n-4} = O(1)$, and near $t \rightarrow 1^-$, using $(1 - t^2)^{2n+2} = (1 + t)^{2n+2}(1 - t)^{2n+2} \sim 2^{2n+2}(1 - t)^{2n+2}$ and $\operatorname{arctanh}(t) = \frac{1}{2} \ln\left(\frac{1+t}{1-t}\right) \sim -\frac{1}{2} \ln(1 - t)$:

$$f_{n,p}(t) \sim -2^{2n+1}(1 - t)^{2n+2} \ln(1 - t) \rightarrow 0. \quad (96)$$

The function $f_{n,p}$ is therefore integrable on $(0, 1)$, since:

$$\int_0^1 u^{2n+2} |\ln u| du = \frac{1}{(2n+3)^2} < +\infty,$$

by the standard result (see, e.g., ([36], Entry 4.272.1)):

$$\int_0^1 u^\alpha |\ln u| du = \frac{1}{(\alpha+1)^2}, \quad \forall \alpha > -1. \quad (97)$$

The key is to apply Tonelli's theorem *before* expanding the finite binomial factor $(1 - t^2)^{2n+2}$. Writing the $\operatorname{arctanh}$ Taylor series and noting that every term of the integrand is nonnegative on $(0, 1)$, Tonelli's theorem (Theorem 1.38 in [37]) justifies the interchange of sum and integral:

$$J_{n,p} = \int_0^1 t^{2p-2n-5} (1 - t^2)^{2n+2} \operatorname{arctanh}(t) dt = \sum_{j=0}^{+\infty} \frac{1}{2j+1} \int_0^1 t^{2p-2n+2j-4} (1 - t^2)^{2n+2} dt. \quad (98)$$

Only now do we expand the finite binomial factor

$$(1 - t^2)^{2n+2} = \sum_{m=0}^{2n+2} (-1)^m \binom{2n+2}{m} t^{2m}. \quad (99)$$

Because this is a *finite* sum, it may be inserted into (98) and interchanged with the integral and the (already justified) infinite sum without further hypotheses. Integrating $t^{2(p+m+j)-2n-4}$ over $(0, 1)$ gives $\frac{1}{2(p+m+j)-2n-3}$, so

$$J_{n,p} = \sum_{m=0}^{2n+2} (-1)^m \binom{2n+2}{m} \sum_{j=0}^{+\infty} \frac{1}{(2j+1)[2(p+m+j)-2n-3]}. \quad (100)$$

3. Since

$$\sum_{i=0}^{+\infty} \frac{1}{(2i+1)(2(p+m+i)-2n-3)}$$

is a convergent series, Lemma 5 yields

$$\sum_{i=0}^{+\infty} \frac{1}{(2i+1)(2(p+m+i)-2n-3)} = \frac{1}{2(p+m-n-2)} \sum_{j=1}^{p+m-n-2} \frac{1}{2j-1}.$$

Conclusion. Combining all steps and using $I_2(n, p) = D_n \int_0^1 f_{n,p}(t) dt$:

$$I_2(n, p) = \frac{(-1)^{n+1} (2n+1)!!}{4^{n+1} (2n+2)!!} \sum_{m=0}^{2n+2} \frac{(-1)^m \binom{2n+2}{m}}{(p+m-n-2)} \sum_{j=1}^{p+m-n-2} \frac{1}{2j-1} \quad (101)$$

Having established the explicit expressions for $I_1(n, p)$ and $I_2(n, p)$ in (90) and (101), respectively, we are now in a position to state the second main result of this paper.

Theorem 3 (Second main result). *For all integers $n \geq 0$ and $p \geq n+3$, the following closed-form evaluation holds:*

$${}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right) = \frac{(2n+3)(2p-1)}{2} [I_1(n, p) + I_2(n, p)], \quad (102)$$

where the **rational part** $I_1(n, p)$ and the **harmonic part** $I_2(n, p)$ are defined as follows.

Rational part.

$$I_1(n, p) = \sum_{i=2}^{2n+3} S_n(i) \frac{(2i-4)!!}{(2i-3)!!} \frac{(2n+3-i)!}{(p-n-\frac{3}{2})_{2n+4-i}}, \quad (103)$$

where $(a)_k$ is the Pochhammer symbol, $(m)!!$ denotes the double factorial, and $S_n(i)$ is defined by

$$S_n(i) = \begin{cases} \Lambda_n = \frac{(-1)^{n+1} (2n+1)!!}{4^{n+1} (2n+2)!!}, & 2 \leq i \leq n+2, \\ \sum_{k=0}^{2n+3-i} \binom{n+1}{k} \binom{-\frac{1}{2}}{2n+2-k}, & n+3 \leq i \leq 2n+3. \end{cases} \quad (104)$$

Harmonic part.

$$I_2(n, p) = \frac{(-1)^{n+1} (2n+1)!!}{4^{n+1} (2n+2)!!} \sum_{m=0}^{2n+2} (-1)^m \binom{2n+2}{m} \frac{1}{(p+m-n-2)} \sum_{j=1}^{p+m-n-2} \frac{1}{2j-1}, \quad (105)$$

where the inner sum $H_r^{\text{odd}} = \sum_{j=1}^r \frac{1}{2j-1}$ is the r -th partial odd harmonic sum.

Remark 1 (Equivalent digamma form). Using

$$H_r^{\text{odd}} = \frac{1}{2} \left[\psi \left(r + \frac{1}{2} \right) - \psi \left(\frac{1}{2} \right) \right],$$

the harmonic contribution in Theorem 3 may equivalently be written as

$$I_2(n, p) = \frac{\Lambda_n}{2} \sum_{m=0}^{2n+2} (-1)^m \binom{2n+2}{m} \frac{\psi(p+m-n-\frac{3}{2}) - \psi(\frac{1}{2})}{p+m-n-2}. \quad (106)$$

This version makes the connection with general ${}_3F_2(1)$ summation formulae transparent, whereas the odd-harmonic form (105) is better suited to exact rational arithmetic.

Corollary 1 (Rationality of $F(n, p)$). For all integers $n \geq 0$ and $p \geq n + 3$, the value $F(n, p) = {}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right)$ is a rational number.

Proof. This is a structural consequence of Theorem 3. The part $I_1(n, p)$ is a finite sum of rational quantities. The part $I_2(n, p)$ is a finite rational linear combination of partial odd harmonic sums H_r^{odd} , each of which is rational for integer $r \geq 1$. Hence, the prefactor in (102) preserves rationality. $\square$

Proposition 2 (Sharp boundary of rationality). For every integer $n \geq 0$, the adjacent value $p = n + 2$ satisfies

$$F(n, n+2) = R_n + C_n \pi^2, \quad (107)$$

where the transcendental coefficient is

$$C_n = \frac{(2n+3)^2}{8} \Lambda_n = \frac{(-1)^{n+1} (2n+3)^2 (2n+1)!!}{8 \cdot 4^{n+1} (2n+2)!!} \neq 0, \quad (108)$$

and the rational coefficient $R_n \in \mathbb{Q}$ is given in closed form by

$$R_n = \frac{(2n+3)^2}{2} \left[I_1(n, n+2) + \Lambda_n \sum_{m=1}^{2n+2} (-1)^m \binom{2n+2}{m} \frac{H_m^{\text{odd}}}{m} \right], \quad (109)$$

in which $\Lambda_n = \frac{(-1)^{n+1} (2n+1)!!}{4^{n+1} (2n+2)!!}$, $H_m^{\text{odd}} = \sum_{j=1}^m (2j-1)^{-1}$, and the rational part $I_1(n, n+2)$ is the specialization of (103) at $p = n + 2$,

$$I_1(n, n+2) = \sum_{i=2}^{2n+3} S_n(i) \frac{(2i-4)!!}{(2i-3)!!} \frac{(2n+3-i)!}{(\frac{1}{2})_{2n+4-i}}, \quad (110)$$

the Pochhammer symbol being evaluated at $p - n - \frac{3}{2} = \frac{1}{2}$. Since every quantity on the right-hand side of (109)–(110) is rational, $R_n \in \mathbb{Q}$. Consequently, $F(n, n+2)$ is transcendental; in particular, the threshold $p \geq n + 3$ in Corollary 1 is sharp.

Proof. Equation (100) remains valid for $p = n + 2$, because the integrand in (98) is nonnegative and integrable already at that boundary. For $m \geq 1$, Lemma 5 applies with $r = m$ and gives $\frac{1}{2m} H_m^{\text{odd}}$. For $m = 0$, however,

$$\sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} = \frac{\pi^2}{8}.$$

Since the factor relating I_2 to $J_{n,p}$ is $2\Lambda_n$, we obtain

$$I_2(n, n+2) = \Lambda_n \left[\frac{\pi^2}{4} + \sum_{m=1}^{2n+2} (-1)^m \binom{2n+2}{m} \frac{H_m^{\text{odd}}}{m} \right]. \quad (111)$$

The second term in brackets is rational, and $I_1(n, n+2)$ is rational because its Beta/Pochhammer representation is valid at $p - n - \frac{3}{2} = \frac{1}{2}$. Multiplying by the prefactor $\frac{(2n+3)^2}{2}$ in (83) yields (107)–(108). Since C_n is a nonzero rational number and π^2 is transcendental, $R_n + C_n\pi^2$ is transcendental.

Collecting the rational contributions explicitly, the prefactor $\frac{(2n+3)^2}{2}$ multiplies $I_1(n, n+2)$ together with the rational bracket of (111), which gives precisely the closed form (109) for R_n . Thus, R_n is not merely known to be rational: it is available in closed form for every n , on the same footing as C_n . $\square$

Remark 2 (Explicit rational parts). Formula (109) reproduces the rational entries of Table 2. Evaluating it for the first few orders gives

$$R_0 = \frac{9}{4}, \quad R_1 = 0, \quad R_2 = \frac{343}{360}, \quad R_3 = \frac{18}{35},$$

in agreement with the values $F(0, 2) = \frac{9}{4} - \frac{9\pi^2}{64}$, $F(1, 3) = \frac{75\pi^2}{1024}$ (so that $R_1 = 0$), $F(2, 4) = \frac{343}{360} - \frac{245\pi^2}{8192}$, and $F(3, 5) = \frac{18}{35} + \frac{2835\pi^2}{262,144}$. In particular, the case $n = 1$ shows that the rational part may vanish, leaving a purely transcendental value.

Table 2. First boundary values illustrating Proposition 2.

n	$F(n, n+2)$
0	$\frac{9}{4} - \frac{9\pi^2}{64}$
1	$\frac{75\pi^2}{1024}$
2	$\frac{343}{360} - \frac{245\pi^2}{8192}$
3	$\frac{18}{35} + \frac{2835\pi^2}{262,144}$

Proposition 3 (Removal of the Shpot–Srivastava singularity). Let

$$\mathcal{S}_{n,p}(a) = {}_3F_2\left(a, -\frac{2n+1}{2}, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right), \quad p \geq n+3.$$

Thus, $\mathcal{S}_{n,p}(a)$ is the one-parameter family obtained by promoting the fixed numerator entry 1 of $F(n, p)$ to a free variable a ; in particular $\mathcal{S}_{n,p}(1) = F(n, p)$, as we verify in the proof. After the substitutions $b = -\frac{2n+1}{2}$, $c = \frac{2p-1}{2}$, $m = 2n+2$, and $n_{\text{SS}} = 0$, the right-hand side of Shpot and Srivastava ([30], Equation (31)) has a removable singularity at $a = 1$. Its continuous extension at $a = 1$ satisfies

$$\lim_{a \rightarrow 1} \{\text{right-hand side of Shpot and Srivastava}\} = \mathcal{S}_{n,p}(1) = F(n, p), \quad (112)$$

where $F(n, p)$ is the explicit odd-harmonic expression obtained independently in Theorem 3.

Proof. For the specialized parameters, the parametric excess of $\mathcal{S}_{n,p}(a)$ is

$$\frac{2n+5}{2} + \frac{2p+1}{2} - \left(a - \frac{2n+1}{2} + \frac{2p-1}{2}\right) = 2n+4-a.$$

Hence, for a in a sufficiently small complex neighborhood of 1, the parametric excess has positive real part. The defining ${}_3F_2(1)$ series therefore converges locally uniformly there and defines an analytic function of a .

For $a \neq 1$ in a punctured neighborhood of 1, the specialized Shpot–Srivastava right-hand side equals this analytic function. Consequently, any pole appearing in its separate Beta-function factors is removable in the complete expression, and

$$\lim_{a \rightarrow 1} \{\text{right-hand side}\} = \mathcal{S}_{n,p}(1).$$

It remains to identify the value $\mathcal{S}_{n,p}(1)$ with $F(n, p)$, and we do this explicitly rather than by appeal to symmetry alone. Setting $a = 1$ in the definition of $\mathcal{S}_{n,p}(a)$ gives

$$\mathcal{S}_{n,p}(1) = {}_3F_2\left(\underbrace{1, -\frac{2n+1}{2}, \frac{2p-1}{2}}_{\text{numerator parameters}}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right). \quad (113)$$

Recall that for any admissible parameters, the generalized hypergeometric series is defined by

$${}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; 1) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k (\alpha_3)_k}{(\beta_1)_k (\beta_2)_k} \frac{1}{k!},$$

where $(\cdot)_k$ is the Pochhammer symbol. The numerator contribution of the k -th term is the product $(\alpha_1)_k (\alpha_2)_k (\alpha_3)_k$, which is invariant under any permutation of $\alpha_1, \alpha_2, \alpha_3$; the denominator and the factor $1/k!$ do not involve the numerator parameters at all. Hence, ${}_3F_2$ is symmetric in its three numerator parameters, term by term. Comparing (113) with the definition (81) of $F(n, p)$,

$$F(n, p) = {}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right),$$

we see that the two series have the same multiset of numerator parameters $\{1, -\frac{2n+1}{2}, \frac{2p-1}{2}\}$ —the passage from $\mathcal{S}_{n,p}(1)$ to $F(n, p)$ merely moves the entry 1 from the first numerator slot to the second—and the identical pair of denominator parameters $\frac{2n+5}{2}, \frac{2p+1}{2}$. By the term-by-term symmetry just noted, the two series agree summand by summand, so

$$\mathcal{S}_{n,p}(1) = F(n, p) \quad (\text{an exact identity, not merely a limiting coincidence}). \quad (114)$$

This identity is what allows the analytic limit computed above to be read off from the independently derived closed form: the singular Shpot–Srivastava representation and our Euler–Rainville evaluation are two expressions for one and the same number. By numerator symmetry, $\mathcal{S}_{n,p}(1)$ is exactly the function $F(n, p)$ defined in (81); Theorem 3, derived independently from Euler’s and Rainville’s integral representations, gives its explicit finite expression. This proves (112).

The local structure of the removable singularity is also consistent with the odd-harmonic form. Writing $\varepsilon = 1 - a$, the Beta factors satisfy

$$B(\varepsilon, z) = \frac{1}{\varepsilon} + \psi(1) - \psi(z) + O(\varepsilon), \quad \varepsilon \rightarrow 0. \quad (115)$$

Because the complete right-hand side is analytic, the $1/\varepsilon$ terms cancel in the full two-term combination. The finite parts therefore involve digamma differences; at half-integer arguments these reduce through

$$\psi\left(r + \frac{1}{2}\right) - \psi\left(\frac{1}{2}\right) = 2 \sum_{j=1}^r \frac{1}{2j-1} = 2H_r^{\text{odd}}, \quad r \in \mathbb{N}. \quad (116)$$

Thus, the form predicted by the singular general representation is consistent with the finite odd-harmonic expression of Theorem 3. We stress that Proposition 3 identifies the limiting value by analytic continuation; the explicit coefficients in Theorem 3 come from the independent Euler–Rainville derivation of Sections 3 and 4. $\square$

5. Numerical Evaluation of $H_n(x)$ and $F(n, p)$

The closed forms of Theorems 2 and 3 are exact algebraic identities; their value as computational tools depends on *how* they are evaluated. This section makes that dependence precise. We (i) record the linear arithmetic-operation structure of the closed forms, which yields exact values in rational arithmetic; (ii) analyze the conditioning of their floating-point evaluation, showing that a naïve implementation suffers catastrophic cancellation as the order n grows; and (iii) provide stable floating-point evaluators whose accuracy is assessed over an explicit, finite test grid. No parameter-uniform quadrature theorem is asserted. All numerical values reported in this section were generated in exact rational arithmetic or verified against a 50-digit reference, as detailed in Section 5.10.

The computational framework is organized into four procedures. Algorithm 1 (PRECOMPUTE) isolates all quantities that depend solely on the order parameter n : the scalar Λ_n , the coefficient array v_i , the switch-point array $S_n(i)$, and auxiliary binomial tables. These are computed once using $O(n)$ arithmetic operations and shared by the subsequent algorithms. Algorithm 2 (EVAL2F1) evaluates ${}_2F_1$ stably by a term recurrence away from $x = 1$, with the well-conditioned closed form available near $x = 1$. Algorithm 3 (EVAL3F2) assembles the exact closed form in $O(n + p)$ arithmetic operations under the unit-cost model. Algorithm 4 (EVAL3F2STABLE) provides a separate adaptive floating-point route based on the positive Rainville kernel. Throughout, quantities that can be computed by simple recurrences—double-factorial ratios, Pochhammer products, binomial coefficients and sliding partial sums—are handled incrementally. As shown in Section 5.3, the closed form for $H_n(x)$ should not be evaluated naïvely in fixed precision for large n or small x .

5.1. Notation and Key Quantities

Throughout, $n \in \mathbb{N}$, $x \in (0, 1)$, and $p \geq n + 3$ is an integer. The main building blocks are:

- **Double-factorial coefficients.**

$$v_i = \frac{(2i-4)!!}{(2i-3)!!}, \quad i \geq 2, \quad \Lambda_n = \frac{(-1)^{n+1}(2n+1)!!}{4^{n+1}(2n+2)!!}.$$

- **Partial odd harmonic sum.**

$$H_r^{\text{odd}} = \sum_{j=1}^r \frac{1}{2j-1}, \quad r \geq 1.$$

- **Switch-point coefficients $S_n(i)$** (Equations (42), (61) and (104)):

$$S_n(i) = \begin{cases} \Lambda_n, & 2 \leq i \leq n+2, \\ \sum_{k=0}^{2n+3-i} \binom{n+1}{k} \left(-\frac{1}{2}\right)_{2n+2-k}, & n+3 \leq i \leq 2n+3. \end{cases}$$

- **Pochhammer product** $(a)_m = a(a+1) \cdots (a+m-1)$.

Remark 3 (Notational convention for Landau symbols). *Throughout this paper, the computational complexity of the proposed algorithms is expressed using the Landau notation $O(\cdot)$. In particular, $O(n)$ denotes a number of arithmetic operations growing linearly with the parameter n ; doubling n approximately doubles the number of arithmetic operations. This type of complexity is generally*

considered computationally efficient and suitable for large-scale computations. To avoid ambiguity, we reserve the two Landau symbols for two distinct purposes: the upper-bound symbol $O(\cdot)$ is used exclusively for all cost statements, that is, for counts of elementary arithmetic operations, storage, and bit-operations; the exact-order symbol $\Theta(\cdot)$ is used only for the intrinsic magnitude of numerical quantities in the conditioning analysis of Section 5.3 (for example, an intermediate term that is genuinely of order 4^n), where a two-sided estimate is essential to the cancellation argument. No cost in this paper is expressed with $\Theta(\cdot)$.

5.2. Algorithm 1: Precompute Shared Quantities for Fixed n

Many quantities depend only on n and are reused in both ${}_2F_1$ (for every x) and ${}_3F_2$ (for every p). Algorithm 1 computes them once.

Algorithm 1 PRECOMPUTE(n) – shared quantities for a fixed order n

Input: integer $n \geq 0$

Output: arrays $S[2 \dots 2n+3]$, $v[2 \dots 2n+3]$, scalar Λ

Local: scratch arrays $b[0 \dots 2n+2]$ (generalized binomials $\binom{-1/2}{j}$) and $c[0 \dots n+1]$ (binomials $\binom{n+1}{k}$), used only to assemble $S[\cdot]$ and not returned

```

1: num  $\leftarrow 1$ ; den  $\leftarrow 1$ 
2: for  $k = 1$  to  $n+1$  do %  $(2n+1)!! = (1)(3) \cdots (2n+1)$ ,  $(2n+2)!! = (2)(4) \cdots (2n+2)$ 
3:   num  $\leftarrow \text{num} \times (2k-1)$ 
4:   den  $\leftarrow \text{den} \times (2k)$ 
5: end for
6:  $\Lambda \leftarrow (-1)^{n+1} \text{num} / (4^{n+1} \text{den})$ 
7:  $v[2] \leftarrow 1$  %  $v_2 = (2)!! / ((1)!! 2(1)) = 2 / (1 \times 2) = 1$ 
8: for  $i = 3$  to  $2n+3$  do %  $v_i = v_{i-1} \frac{2i-4}{2i-3}$  (recurrence)
9:    $v[i] \leftarrow v[i-1] \times (2i-4) / (2i-3)$ 
10: end for
11: for  $i = 2$  to  $n+2$  do
12:    $S[i] \leftarrow \Lambda$ 
13: end for
14:  $b[0] \leftarrow 1$ 
15: for  $j = 1$  to  $2n+2$  do %  $\binom{-1/2}{j} = \binom{-1/2}{j-1} \frac{-1/2-j+1}{j}$ 
16:    $b[j] \leftarrow b[j-1] \times (-1/2-j+1) / j$ 
17: end for
18:  $c[0] \leftarrow 1$ 
19: for  $k = 1$  to  $n+1$  do %  $\binom{n+1}{k} = \binom{n+1}{k-1} (n+2-k)/k$ 
20:    $c[k] \leftarrow c[k-1] \times (n+2-k) / k$ 
21: end for
22:  $i \leftarrow 2n+3$ ; val  $\leftarrow c[0] \times b[2n+2]$  %  $k=0$  term, limit =  $2n+3-i=0$ 
23:  $S[2n+3] \leftarrow \text{val}$ 
24: for  $i = 2n+2$  downto  $n+3$  do % each step adds one more  $k$ -term
25:    $k \leftarrow 2n+3-i$  % newly added index
26:   val  $\leftarrow \text{val} + c[k] \times b[2n+2-k]$ 
27:    $S[i] \leftarrow \text{val}$ 
28: end for
   return  $S[\cdot]$ ,  $v[\cdot]$ ,  $\Lambda$ 

```

Complexity. $O(n)$ arithmetic operations and $O(n)$ storage.

5.3. Conditioning of the Naïve Closed-Form Evaluation

The closed form of Theorem 2 has the shape

$$H_n(x) = \frac{2n+3}{x^{n+1}} B_n(x), \quad B_n(x) = P_{2n+1}(x) + Q_n(x) \frac{\operatorname{arctanh} \sqrt{x}}{\sqrt{x}}. \quad (117)$$

Since H_n is analytic at the origin with $H_n(0) = 1$, the bracket satisfies $B_n(x) = \Theta(x^{n+1})$ as $x \rightarrow 0^+$: it is a difference of terms of size $\Theta(1)$ that cancel to order x^{n+1} . Forming $B_n(x)$ in finite precision therefore loses relative accuracy through subtractive cancellation, and the subsequent multiplication by $x^{-(n+1)}$ amplifies the resulting absolute error by the same factor. The loss worsens both as $x \rightarrow 0^+$ and as n increases.

Table 3 quantifies this. At the fixed, moderate abscissa $x = 0.15$ —well away from the origin—the double-precision relative error of the naïve evaluation (117) against a 50-digit reference grows by roughly one order of magnitude per unit increase in n , exceeding 1 (total loss of accuracy) near $n \approx 15$ and reaching 10^{16} by $n = 30$. The stable series evaluator introduced below (last column) remains at the scale of machine epsilon $\varepsilon_{\text{mach}} \approx 2.22 \times 10^{-16}$ throughout.

Table 3. Double-precision relative error of the naïve closed-form evaluation (117) of $H_n(x)$ at $x = 0.15$, against a 50-digit reference, compared with the stable series evaluator (118). The naïve closed form loses all significant digits near $n \approx 15$; the stable evaluator does not.

n	Naïve Closed Form (117)	Stable Series (118)
1	1.6×10^{-16}	3.4×10^{-17}
2	6.5×10^{-15}	1.3×10^{-17}
4	2.9×10^{-13}	2.9×10^{-17}
8	4.5×10^{-9}	1.1×10^{-17}
12	3.5×10^{-4}	3.7×10^{-17}
16	2.4×10^1	6.5×10^{-18}
20	6.6×10^4	3.8×10^{-18}
30	9.5×10^{16}	4.2×10^{-18}

The same mechanism affects $F(n, p)$ through Theorem 3: the alternating binomial sum in the harmonic part $I_2(n, p)$ and the Pochhammer ratios in the rational part $I_1(n, p)$ involve intermediate quantities of magnitude $\Theta(4^n)$ whose signed combination is $\Theta(1)$. Consequently, a direct double-precision evaluation of the $F(n, p)$ closed form degrades in the same way for moderate to large n ; already at $p = n + 3$ (where the exact value is $\frac{2n+5}{4n+6}$) such an evaluation departs from the true value in the second significant figure by $n = 50$. We emphasize that this is a property of the *evaluation*, not of the identities: carried out in exact rational arithmetic, or through the stable schemes below, the very same closed forms are correct to full precision (Corollary 1 and Table 5).

5.4. Stable Evaluators

Two observations yield stable evaluators that avoid the cancellation mechanisms identified above.

(a) Stable $H_n(x)$ by its defining series. The Gauss series

$$H_n(x) = {}_2F_1\left(-\frac{2n+1}{2}, 1; \frac{2n+5}{2}; x\right) = \sum_{k \geq 0} t_k, \quad \frac{t_{k+1}}{t_k} = \frac{-\frac{2n+1}{2} + k}{\frac{2n+5}{2} + k} x = \frac{2k - (2n+1)}{2k + (2n+5)} x, \quad (118)$$

where the upper parameter $b = 1$ has cancelled the factorial in the series denominator (the factors $(1+k)$ and $(k+1)$ coincide). The series converges for all $x \in (0, 1)$, and each term follows from the previous by an $O(1)$ update. Because $|x| < 1$ and the parameters keep the partial sums well scaled, none of the cancellation analyzed in Section 5.3 occurs. In our experiments (118) reaches a relative residual below $\varepsilon_{\text{mach}}$ in about 36 terms for $x \leq 0.5$ and about 97 terms for $x \leq 0.8$ (uniformly for all n tested up to 100), the term count being governed by the geometric decay $\sim x^k$ and growing only as $x \rightarrow 1^-$. This yields a stable

evaluator whose term count is essentially independent of n ; near $x = 1$ one may instead use the closed form, which is well conditioned there, or the endpoint value $H_n(1) = \frac{2n+3}{4n+4}$.

(b) Stable $F(n, p)$ by an adaptive positive-kernel quadrature. Rainville's representation (14) reads

$$F(n, p) = \left(p - \frac{1}{2}\right) \int_0^1 x^{p-3/2} H_n(x) dx, \quad (119)$$

whose integrand is smooth and strictly positive on $(0, 1)$. Rather than placing the increasingly concentrated factor $x^{p-3/2}$ inside an unweighted quadrature rule, we absorb it into a Gauss–Jacobi weight. Writing $q = p - \frac{3}{2}$ and $x = (1 + z)/2$ gives

$$F(n, p) = \left(p - \frac{1}{2}\right) 2^{-(q+1)} \int_{-1}^1 (1+z)^q H_n\left(\frac{1+z}{2}\right) dz. \quad (120)$$

The implementation doubles the Jacobi order until two successive estimates satisfy a prescribed relative stopping test. This avoids any unproved claim that one fixed quadrature order works uniformly for all n and p .

Remark 4. For floating-point work, scheme (b) is the recommended evaluator and the exact rational assembly of Theorem 3 (Corollary 1) is recommended whenever an exact fraction is required. The naïve closed-form evaluation is accurate only for small n (say $n \leq 8$; see Table 3) or in exact arithmetic. The operation count for the exact route is given by Theorem 4; the adaptive floating-point cost additionally depends on the final quadrature order.

5.5. Algorithm 2: Evaluate ${}_2F_1\left(-\frac{2n+1}{2}, 1; \frac{2n+5}{2}; x\right)$

Algorithm 2 uses a two-branch strategy. Away from $x = 1$ it sums the defining Gauss series using a term recurrence. Near $x = 1$ it switches to the closed form of Theorem 2, where the cancellation that makes the formula unstable near $x = 0$ is no longer severe. In the numerical experiments we use the explicit switch

$$x_{\text{sw}}(n) = \max\left\{0.90, 1 - \frac{1}{2(n+1)}\right\}. \quad (121)$$

The switch is a numerical implementation choice and is not needed in the analytical results.

For $x < x_{\text{sw}}(n)$, let t_k denote the k th term of the defining series. Since the magnitude of the consecutive-term ratio is bounded by $x < 1$, the remaining tail is bounded conservatively by a geometric majorant. Accordingly, the stopping test uses $|t_k|/(1-x)$ rather than $|t_k|$ alone. For $x \geq x_{\text{sw}}(n)$, the algorithm evaluates

$$H_n(x) = \frac{2n+3}{x^{n+1}} \left(P_{2n+1}(x) + Q_n(x) \frac{\operatorname{arctanh}(\sqrt{x})}{\sqrt{x}} \right),$$

with

$$P_{2n+1}(x) = \sum_{i=2}^{2n+3} S_n(i) v_i (1-x)^{2n+3-i}, \quad Q_n(x) = \Lambda_n (1-x)^{2n+2},$$

using the shared data returned by Algorithm 1.

Algorithm 2 EVAL2F1(n, x) – stable hybrid evaluation of ${}_2F_1\left(-\frac{2n+1}{2}, 1; \frac{2n+5}{2}; x\right)$

Input: integer $n \geq 0$, real $x \in (0, 1)$; tolerance τ (default $\tau = \varepsilon_{\text{mach}}$)

Output: approximation to ${}_2F_1\left(-\frac{2n+1}{2}, 1; \frac{2n+5}{2}; x\right)$

```

1:  $x_{\text{sw}} \leftarrow \max\{0.90, 1 - [2(n+1)]^{-1}\}$ 
2: if  $x \geq x_{\text{sw}}$  then
3:    $\{S[\cdot], \nu[\cdot], \Lambda\} \leftarrow \text{PRECOMPUTE}(n)$ 
4:    $y \leftarrow 1 - x; \quad P \leftarrow 0$ 
5:   for  $i = 2$  to  $2n + 3$  do
6:      $P \leftarrow P + S[i] \nu[i] y^{2n+3-i}$ 
7:   end for
8:    $Q \leftarrow \Lambda y^{2n+2}$ 
9:    $L \leftarrow \text{arctanh}(\sqrt{x}) / \sqrt{x}$ 
10:  return  $\frac{2n+3}{x^{n+1}}(P + QL)$ 
11: else
12:   $t \leftarrow 1; \quad s \leftarrow 1; \quad k \leftarrow 0$ 
13:  while true do
14:     $t \leftarrow t \times \frac{2k - (2n+1)}{2k + (2n+5)} x$ 
15:     $s \leftarrow s + t; \quad k \leftarrow k + 1$ 
16:    if  $\frac{|t|}{1-x} \leq \tau \max(1, |s|)$  and  $k > 1$  then
17:      break
18:    end if
19:  end while
20:  return  $s$ 
21: end if
```

Cost. The series branch uses $O(1)$ arithmetic per term and, on each fixed subinterval $x \leq x_0 < 1$, a bounded number of terms for a prescribed floating-point tolerance. The near-one branch uses $O(n)$ arithmetic operations after the shared precomputation. Thus, the algorithm makes the series/closed-form hybrid described in the abstract and in Algorithm 4 explicit. Figure 1 illustrates the curves of $H_n(x)$ for $n = 1, \dots, 4$ over $(0.15, 0.99)$, computed via Algorithm 2 (EVAL2F1).

Table 4. $F(n, p) = {}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right)$ for $n \in \{1, 2, 3, 4\}$ and $p \in \{7, 8, 9, 10\}$.

$n \setminus p$	$p = 7$	$p = 8$	$p = 9$	$p = 10$
$n = 1$	5408	83,689	623,696	77,355,821
	8085	126,126	945,945	117,927,810
$n = 2$	15,613	44,719	916,147	74,354,087
	24,948	72,072	1,486,485	121,297,176
$n = 3$	299	5651	42,092	18,502,219
	495	9438	70,785	31,286,970
$n = 4$	13	335	5406	2,558,939
	22	572	9295	4,424,420

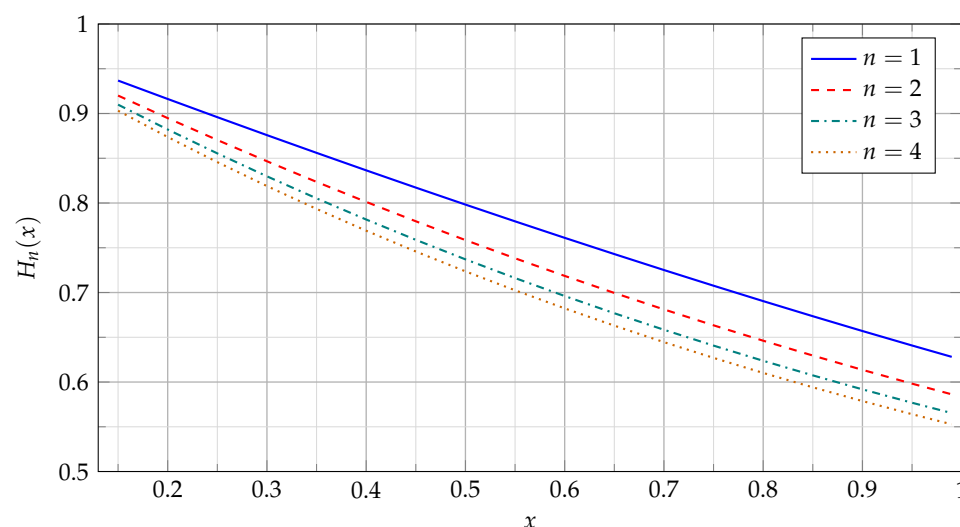


Figure 1. Graphs of $H_n(x) = {}_2F_1\left(-\frac{2n+1}{2}, 1; \frac{2n+5}{2}; x\right)$ for $n = 1, 2, 3, 4$, computed via Algorithm 2 and verified against an independent 50-digit direct summation of the defining Gauss series. Each curve satisfies $H_n(0) = 1$ with initial slope $H'_n(0) = -\frac{2n+1}{2n+5} < 0$, so every curve starts at 1 and decreases monotonically on $(0, 1)$. For the plotted values $n = 1, 2, 3, 4$, the curves appear strictly ordered by n , and as $x \rightarrow 1^-$ they approach the finite endpoints $H_n(1) = \frac{2n+3}{4n+4}$, i.e., $H_1(1) = 0.625$, $H_2(1) = 0.583$, $H_3(1) = 0.5625$, $H_4(1) = 0.55$.

5.6. Algorithm 3: Evaluate ${}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right)$

Building on the closed form established in Theorem 3, this algorithm computes the value of ${}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right)$ for any integers $n \geq 0$ and $p \geq n + 3$. The result splits into two independent contributions: a rational part $I_1(n, p)$, built from Beta function evaluations via Pochhammer symbols and double factorials, and a harmonic part $I_2(n, p)$, which involves a finite alternating binomial sum weighted by partial odd harmonic sums $H_r^{\text{odd}} = \sum_{j=1}^r \frac{1}{2j-1}$. Both parts draw on the arrays $S[\cdot]$, $v[\cdot]$, and Λ produced by PRECOMPUTE, so Algorithm 1 should be called first. The dominant computational cost is $O(p)$ for initializing the smallest required odd harmonic sum, followed by $O(n)$ work for all remaining loops, giving an overall operation count of $O(n + p)$. We emphasize that this $O(p)$ term reflects the proposed elementary sequential initialization of the partial odd harmonic sum; it is not an intrinsic lower bound, since the identity $H_r^{\text{odd}} = H_{2r} - \frac{1}{2}H_r$ permits evaluation through digamma values, which may be preferable for large p .

Algorithm 3 assembles the closed form directly. Executed in exact rational arithmetic, it returns $F(n, p)$ as an irreducible fraction with no rounding error (Corollary 1); this is the mode used to generate Tables 4 and 7. In fixed-precision floating point, however, the alternating binomial sum in I_2 (and the Pochhammer ratios in I_1) exhibit the cancellation analyzed in Section 5.3, so for floating-point evaluation the stable positive-kernel quadrature (119)—with H_n supplied by the stable Algorithm 2—is used instead; its accuracy is documented in Table 5. The exact route has the $O(n + p)$ arithmetic-operation count of Theorem 4, whereas the adaptive floating-point route has a cost proportional to its final quadrature order and to the cost of evaluating H_n at the quadrature nodes. The closed form is (Theorem 3):

$${}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right) = \frac{(2n+3)(2p-1)}{2} (I_1(n, p) + I_2(n, p)),$$

$$I_1(n, p) = \sum_{i=2}^{2n+3} S_n(i) v_i \frac{(2n+3-i)!}{(p-n-3/2)_{2n+4-i}},$$

$$I_2(n, p) = \frac{(-1)^{n+1}(2n+1)!!}{4^{n+1}(2n+2)!!} \sum_{m=0}^{2n+2} (-1)^m \binom{2n+2}{m} \frac{H_{p+m-n-2}^{\text{odd}}}{p+m-n-2}.$$

Algorithm 3 EVAL3F2(n, p) – evaluates ${}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right)$

Input: integers $n \geq 0, p \geq n+3$

Output: value of ${}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right)$

```

1:  $\{S[\cdot], \nu[\cdot], \Lambda\} \leftarrow \text{PRECOMPUTE}(n)$  % Algorithm 1
  – Compute  $I_1(n, p)$  –
2:  $a \leftarrow p - n - 3/2$  % base of Pochhammer, a rising factorial
3: Precompute Pochhammer products  $\Pi[j]$  for  $j = 0, \dots, 2n+2$ :
4:    $\Pi[0] \leftarrow 1$ 
5: for  $j = 1$  to  $2n+2$  do %  $\Pi[j] = (a)_j = \Pi[j-1] \times (a+j-1)$ 
6:    $\Pi[j] \leftarrow \Pi[j-1] \times (a+j-1)$ 
7: end for % Now  $(a)_{2n+4-i} = \Pi[2n+4-i]$  for each  $i$ 
8: Precompute factorials  $f[j]$  for  $j = 0, \dots, 2n+1$ :
9:    $f[0] \leftarrow 1$ 
10: for  $j = 1$  to  $2n+1$  do
11:    $f[j] \leftarrow f[j-1] \times j$ 
12: end for %  $f[2n+3-i] = (2n+3-i)!$ 
13:  $I_1 \leftarrow 0$ 
14: for  $i = 2$  to  $2n+3$  do
15:    $I_1 \leftarrow I_1 + S[i] \times \nu[i] \times f[2n+3-i] / \Pi[2n+4-i]$ 
16: end for
  – Build partial odd harmonic sums  $H_r^{\text{odd}}$  for  $r = p - n - 2, \dots, p + n -$  % We need
     $r = p + m - n - 2$  for  $m = 0, \dots, 2n+2$ , i.e.  $r \in [p - n - 2, p + n]$ 
17:  $r_{\min} \leftarrow p - n - 2$ ;  $r_{\max} \leftarrow p + n$ ;  $H \leftarrow 0$ 
18: for  $j = 1$  to  $2r_{\min} - 1$  step 2 do % fill  $H_{r_{\min}}^{\text{odd}} = \sum_{j=1}^{r_{\min}} 1/(2j-1)$ 
19:    $H \leftarrow H + 1/(j)$  %  $j$  takes odd values  $1, 3, \dots, 2r_{\min} - 1$ 
20: end for
21:  $H^{\text{odd}}[r_{\min}] \leftarrow H$ 
22: for  $r = r_{\min} + 1$  to  $r_{\max}$  do % extend by one term at a time:  $H_r^{\text{odd}} = H_{r-1}^{\text{odd}} + 1/(2r-1)$ 
23:    $H \leftarrow H + 1/(2r-1)$ ;  $H^{\text{odd}}[r] \leftarrow H$ 
24: end for
  – Binomial coefficients  $\binom{2n+2}{m}$  for  $m = 0, \dots, 2n+2 -$ 
25:  $\beta[0] \leftarrow 1$ 
26: for  $m = 1$  to  $2n+2$  do %  $\binom{2n+2}{m} = \binom{2n+2}{m-1} (2n+3-m)/m$ 
27:    $\beta[m] \leftarrow \beta[m-1] \times (2n+3-m) / m$ 
28: end for
  – Sum for  $I_2$  –
29:  $\text{sum} \leftarrow 0$ 
30: for  $m = 0$  to  $2n+2$  do
31:    $r \leftarrow p + m - n - 2$ ;  $\text{sum} \leftarrow \text{sum} + (-1)^m \times \beta[m] \times H^{\text{odd}}[r] / r$ 
32: end for
33:  $I_2 \leftarrow \Lambda \times \text{sum}$  %  $\Lambda = (-1)^{n+1}(2n+1)!! / (4^{n+1}(2n+2)!!)$ , from PRECOMPUTE
  return  $\frac{(2n+3)(2p-1)}{2} (I_1 + I_2)$ 

```

For floating-point evaluation, Algorithm 4 uses the positive Rainville kernel through the weighted representation (120). The factor $x^{p-3/2}$ is incorporated into the Gauss–Jacobi weight rather than sampled as part of the integrand. To avoid asserting a parameter-independent quadrature order, the implementation is adaptive: it starts with M_0 nodes, doubles the order, and stops when successive estimates agree to a prescribed tolerance. The ${}_2F_1$ factor is evaluated by the hybrid Algorithm 2: the stable series is used for $x < x_{\text{sw}}(n)$

and the well-conditioned closed form of Theorem 2 is used for $x \geq x_{\text{sw}}(n)$, with the switch specified in (121). The switching rule is numerical and does not enter any analytical result.

Algorithm 4 EVAL3F2STABLE(n, p) – adaptive stable floating-point evaluation of $F(n, p)$

Input: integers $n \geq 0$, $p \geq n + 3$; tolerance τ ; initial order M_0

Output: stable floating-point approximation to $F(n, p)$

```

1:  $M \leftarrow M_0$ 
2:  $F_{\text{old}} \leftarrow \text{GAUSSJACOBIESTIMATE}(n, p, M)$ 
3: while true do
4:    $M \leftarrow 2M$ 
5:    $F_{\text{new}} \leftarrow \text{GAUSSJACOBIESTIMATE}(n, p, M)$ 
6:   if  $|F_{\text{new}} - F_{\text{old}}| \leq \tau \max(1, |F_{\text{new}}|)$  then
7:     return  $F_{\text{new}}$ 
8:   end if
9:    $F_{\text{old}} \leftarrow F_{\text{new}}$ 
10: end while
```

Here GAUSSJACOBIESTIMATE(n, p, M) obtains the M nodes and weights $\{(z_\ell, w_\ell)\}_{\ell=1}^M$ for the Jacobi weight $(1+z)^q$ on $(-1, 1)$, where $q = p - \frac{3}{2}$, and returns

$$\left(p - \frac{1}{2}\right) 2^{-(q+1)} \sum_{\ell=1}^M w_\ell H_n\left(\frac{1+z_\ell}{2}\right).$$

In the reference implementation used for Table 5, $M_0 = 16$ and the stopping tolerance is 5×10^{-14} . On the reported grid with $n \geq 1$, the adaptive procedure terminates at $M = 32$ except for $n = 1$, where $M = 64$ is used. This is an empirical performance statement for the tested range, not a uniform quadrature theorem.

Cost. At a final adaptively selected order M , the quadrature requires $O(M)$ evaluations of H_n . On any fixed subinterval bounded away from $x = 1$, each series evaluation uses a bounded number of terms; nodes sufficiently near $x = 1$ use the $O(n)$ closed-form branch. No parameter-independent asymptotic bound on the final M is claimed.

5.7. Optimization Notes

This subsection summarizes the main optimization strategies employed in the proposed algorithms, with particular emphasis on reducing redundant computations and achieving linear-time complexity.

1. Shared precomputation. When both ${}_2F_1$ and ${}_3F_2$ are evaluated for the same n , call PRECOMPUTE once and pass its output to both Algorithms 2 and 3.
2. $S_n(i)$ sliding window. The partial sums defining $S_n(i)$ for $n+3 \leq i \leq 2n+3$ are built by starting from the shortest sum ($i = 2n+3$, one term) and adding one term per step as i decreases, avoiding quadratic cost.
3. v_i recurrence. The ratio $v_i = (2i-4)!! / (2i-3)!!$ satisfies $v_i = v_{i-1} (2i-4) / (2i-3)$, computed in $O(1)$ per step.
4. Pochhammer products. $(a)_j$ is computed incrementally: $\Pi[j] = \Pi[j-1] \times (a+j-1)$.
5. Odd harmonic sums. The $2n+3$ required values $H_{p-n-2}^{\text{odd}}, H_{p-n-1}^{\text{odd}}, \dots, H_{p+n}^{\text{odd}}$ form a contiguous range; they are built in one forward pass after initializing the smallest value.
6. Repeated calls in p . If ${}_3F_2$ must be evaluated for several values of p with the same n , the arrays $S[\cdot], v[\cdot], \Lambda, f[\cdot], \beta[\cdot]$ need not be recomputed. Only the Pochhammer array $\Pi[\cdot]$ (which depends on $a = p - n - 3/2$) and the odd-harmonic range change with p .

5.8. Complexity Analysis

Theorem 4 (Complexity of Algorithms 1–3). *Under the unit-cost arithmetic model (each arithmetic operation counted as $O(1)$), the algorithms satisfy the following operation-count and space bounds.*

1. PRECOMPUTE(n) uses $O(n)$ arithmetic operations and $O(n)$ space.
2. EVAL2F1(n, x) (Algorithm 2) uses a bounded number of series terms on any fixed subinterval $x \leq x_0 < 1$ for a prescribed tolerance; its near-one closed-form branch uses $O(n)$ operations after PRECOMPUTE(n).
3. EVAL3F2(n, p) (Algorithm 3) uses $O(n + p)$ arithmetic operations after PRECOMPUTE(n) under the unit-cost model. The stable quadrature variant (Algorithm 4) uses $O(M)$ evaluations of the ${}_2F_1$ factor at its final, adaptively selected quadrature order M ; no parameter-independent bound on M is asserted.

Under exact rational arithmetic the arithmetic-operation counts above are unchanged, but they are not bit-complexity bounds: the numerator and denominator lengths grow with n and p , so the cost of an individual integer operation is not constant. We therefore state the complexity result only in the unit-cost arithmetic model. A separate bit-complexity analysis would require explicit bounds for coefficient and denominator growth. The naïve implementation of $S_n(i)$ for $n + 3 \leq i \leq 2n + 3$ —recomputing each partial sum from scratch—requires $\sum_{i=n+3}^{2n+3} (2n + 4 - i) = \frac{(n+1)(n+2)}{2}$ multiplications, giving $O(n^2)$ total cost. The sliding-window construction in Algorithm 1 reduces this to $O(n)$.

Proof.

Algorithm 1. The loop computing Λ_n (lines 2–5) executes $n + 1$ steps. The loop for v_i (lines 8–10) executes $2n + 1$ steps. The constant-region fill (lines 11–13) executes $n + 1$ steps. Building the auxiliary arrays $b[\cdot]$ (lines 14–16) and $c[\cdot]$ (lines 18–20) each take $O(n)$ steps. The sliding-window loop (lines 22–28) executes $n + 1$ steps, each adding exactly one new term $c[k] \times b[2n + 2 - k]$ to the running sum. The total operation count is therefore $O(n)$; storage is $O(n)$ for the arrays $S[\cdot]$, $v[\cdot]$, $b[\cdot]$, $c[\cdot]$.

Algorithm 2. The series loop performs one $O(1)$ ratio update and one addition per term. The number of terms needed to reach a relative residual below τ is $O(\log \tau / \log x)$, governed by the geometric decay $\sim x^k$ and essentially independent of n ; it is bounded by a constant on any fixed subinterval $x \leq x_0 < 1$. Total: $O(1)$ terms per evaluation on such subintervals. When the well-conditioned closed form of Theorem 2 is used near $x = 1$, building the power array and accumulating P_{2n+1} take $O(n)$ steps, with the single evaluation of $\operatorname{arctanh}(\sqrt{x})/\sqrt{x}$ a fixed $O(1)$ cost.

Algorithm 3. The Pochhammer array $\Pi[\cdot]$ (lines 4–7) takes $2n + 2$ steps; the factorial array (lines 9–12) takes $2n + 1$ steps; the I_1 loop takes $2n + 2$ steps; the binomial array $\beta[\cdot]$ (lines 26–28) takes $2n + 2$ steps; the I_2 loop (lines 30–32) takes $2n + 3$ steps. All these contribute $O(n)$ operations. The dominant cost is the initialization of $H_{r_{\min}}^{\text{odd}}$ (lines 18–21), which sums $r_{\min} = p - n - 2$ odd-reciprocal terms, contributing $O(p)$ operations. The extension loop (lines 22–24) adds $2n + 2$ more terms, which is $O(n)$. The total is therefore $O(n + p)$.

Quadratic cost of naïve $S_n(i)$ construction. A direct implementation would compute $S_n(i) = \sum_{k=0}^{2n+3-i} \binom{n+1}{k} \binom{-1/2}{2n+2-k}$ for each $i \in \{n + 3, \dots, 2n + 3\}$ independently. The sum for index i has $2n + 4 - i$ terms, so the total work is $\sum_{i=n+3}^{2n+3} (2n + 4 - i) = \sum_{j=1}^{n+1} j = \frac{(n+1)(n+2)}{2} = O(n^2)$. The sliding-window strategy increments a running total by one term per step, reducing this to $O(n)$. $\square$

5.9. When Exact Arithmetic Ceases to Be Preferable

The operation counts of Theorem 4 are stated in the unit-cost model. In exact rational arithmetic the same $O(n + p)$ operation count holds, but each operation acts on integers whose bit-length grows with the parameters, so the exact scheme is not uniformly preferable to the adaptive floating-point evaluator (Algorithm 4). We make the crossover quantitative.

The denominators produced by EVAL3F2 are controlled by three factors: the coefficient Λ_n , whose denominator $4^{n+1}(2n+2)!! = 2^{3n+3}(n+1)!$ has bit-length

$$\log_2(4^{n+1}(2n+2)!!) = (3n+3) + \log_2((n+1)!) = O(n \log n)$$

by Stirling's formula; the Pochhammer array $(p - n - \frac{3}{2})_k$, whose entries have bit-length $O(n \log p)$; and the odd-harmonic sums H_r^{odd} for $r \leq p + n$, whose reduced denominators are of bit-length $O(p)$ (their least common multiple is $\text{lcm}(1, 3, \dots, 2r-1) = e^{O(r)}$). The exact value $F(n, p)$ therefore carries integers of bit-length

$$B(n, p) = O(n \log n) + O(p) = O(n \log n + p), \quad (122)$$

and a single big-integer multiply/add on operands of B bits costs $O(B \log B)$ time by schoolbook-to-fast integer arithmetic. The exact assembly thus runs in $O((n + p)B(n, p) \log B(n, p))$ bit-operations, i.e., it is *super-linear* in the parameters, whereas Algorithm 4 runs in $O(M)$ evaluations at fixed IEEE 754 double-precision cost independent of B .

Two practical thresholds follow. First, exact arithmetic remains the method of choice whenever a *provably correct rational value* is required (for identity verification, or as the reference column of Tables 5–7), and its cost is negligible for the moderate ranges $n \lesssim 30$, $p \lesssim 60$, where $B(n, p)$ stays below a few hundred bits—comfortably within one or two machine words per limb and far cheaper than a high-order quadrature. Second, once $B(n, p)$ exceeds the working precision one is willing to carry—in double precision, once $B(n, p) \gtrsim 53$ bits, which by (122) occurs already around $n \approx 16$ or $p - n \approx 40$ —a *fixed-precision* floating-point evaluation of the closed form is no longer safe because of the subtractive cancellation documented in Section 5.3 (Table 3), and the adaptive positive-kernel scheme of Algorithm 4 becomes preferable: it delivers near machine-epsilon accuracy at $O(M)$ cost with $M \leq 64$ across the entire grid of Table 5, without any growth in operand length. The regime $n \gtrsim 30$ combined with large p is precisely where the exact denominators' bit-length explosion makes rational arithmetic slower than the stable quadrature while the naïve fixed-precision closed form has already lost all significant digits. In short: use exact arithmetic for correctness and for moderate (n, p) ; switch to Algorithm 4 once $B(n, p)$ outgrows the available precision.

5.10. Numerical Experiments

This subsection reports four sets of experiments that validate the exact assembly and assess the floating-point procedures Algorithms 1–4 over the stated finite test grids. All floating-point computations were performed in IEEE 754 double precision (machine epsilon $\epsilon_{\text{mach}} \approx 2.22 \times 10^{-16}$). Exact rational reference values were obtained from EVAL3F2 under exact integer/fraction arithmetic and independently cross-checked by direct high-precision summation of the defining ${}_3F_2$ series using its consecutive-term recurrence. This avoids relying on a general-purpose hypergeometric routine as the universal reference for the largest parameter values. For the stable ${}_3F_2$ calculations we used the adaptive Gauss–Jacobi scheme of Algorithm 4, with $M_0 = 16$ and stopping tolerance 5×10^{-14} . The numerical statements below are empirical statements for the displayed parameter ranges, not parameter-uniform error theorems.

5.10.1. Experiment 1: Worked Example for $n = 1, p = 4$

To illustrate the algorithm concretely, we trace EVAL3F2(1,4) step by step. With $n = 1$, PRECOMPUTE yields

$$\Lambda_1 = \frac{(-1)^2 \cdot 3!!}{4^2 \cdot 4!!} = \frac{3}{16 \cdot 8} = \frac{3}{128},$$

and the coefficient array

$$v_2 = 1, \quad v_3 = \frac{2}{3}, \quad v_4 = \frac{8}{15}, \quad v_5 = \frac{16}{35},$$

(computed via $v_i = v_{i-1} \cdot \frac{2i-4}{2i-3}$), and the switch-point array ($n + 3 = 4, 2n + 3 = 5$):

$$S_1(2) = S_1(3) = \frac{3}{128}, \quad S_1(4) = \sum_{k=0}^1 \binom{2}{k} \binom{-\frac{1}{2}}{4-k} = \frac{-45}{128}, \quad S_1(5) = \sum_{k=0}^0 \binom{2}{k} \binom{-\frac{1}{2}}{4-k} = \frac{35}{128}.$$

With $p = 4$, the base Pochhammer argument is $a = p - n - \frac{3}{2} = \frac{3}{2}$, so $\Pi[0] = 1$, $\Pi[1] = \frac{3}{2}$, $\Pi[2] = \frac{15}{4}$, $\Pi[3] = \frac{105}{8}$, $\Pi[4] = \frac{945}{16}$. Assembling $I_1(1,4)$ from (103):

$$\begin{aligned} I_1(1,4) &= \frac{3/128 \cdot 1 \cdot 6}{945/16} + \frac{3/128 \cdot (2/3) \cdot 2}{105/8} + \frac{(-45/128) \cdot (8/15) \cdot 1}{15/4} + \frac{(35/128) \cdot (16/35) \cdot 1}{3/2} \\ &= \frac{1}{420} + \frac{1}{420} - \frac{1}{20} + \frac{1}{12} = \frac{4}{105}. \end{aligned} \quad (123)$$

For $I_2(1,4)$ from (105), the required partial odd harmonic sums are $H_1^{\text{odd}} = 1$, $H_2^{\text{odd}} = \frac{4}{3}$, $H_3^{\text{odd}} = \frac{23}{15}$, $H_4^{\text{odd}} = \frac{176}{105}$, $H_5^{\text{odd}} = \frac{563}{315}$, giving

$$I_2(1,4) = \frac{3}{128} \left[\frac{H_1^{\text{odd}}}{1} - 4 \cdot \frac{H_2^{\text{odd}}}{2} + 6 \cdot \frac{H_3^{\text{odd}}}{3} - 4 \cdot \frac{H_4^{\text{odd}}}{4} + \frac{H_5^{\text{odd}}}{5} \right] = \frac{1}{525}.$$

Hence, using (102):

$$F(1,4) = {}_3F_2\left(-\frac{3}{2}, 1, \frac{7}{2}; \frac{7}{2}, \frac{9}{2}; 1\right) = \frac{5 \cdot 7}{2} \left[\frac{4}{105} + \frac{1}{525} \right] = \frac{35}{2} \cdot \frac{1}{25} = \frac{7}{10}.$$

Direct series evaluation $\sum_{k=0}^{\infty} \frac{(-3/2)_k (1)_k (7/2)_k}{(7/2)_k (9/2)_k k!}$ truncated at $k = 50$ gives 0.70000..., confirming the result.

5.10.2. Experiment 2: Accuracy of the Stable Evaluator over a Wide Parameter Range

We assess the stable evaluator (Algorithm 4) against the exact rational value

$$F_{\text{exact}}(n, p) = \frac{(2n+3)(2p-1)}{2} [I_1(n, p) + I_2(n, p)]$$

of Theorem 3, computed in exact fraction arithmetic (integer numerator and denominator, no rounding at any step) and independently cross-checked by direct high-precision summation of the defining series using its term recurrence. Table 5 reports the relative error $\delta(n, p) = |F_{\text{stab}}(n, p) - F_{\text{exact}}(n, p)| / |F_{\text{exact}}(n, p)|$ in IEEE 754 double precision over a range chosen to be genuinely demanding: orders up to $n = 64$ and parameter gaps $p - n \in \{3, 5, 8, 12, 20\}$. The errors remain within roughly one order of machine epsilon over this grid. The adaptive rule terminates at $M = 32$ for $n = 2, 4, 8, 16, 32, 64$ and at $M = 64$ for $n = 1$, in sharp contrast to the naïve closed form (Table 3), which has already lost all significant digits by $n = 16$.

Table 5. Relative error $\delta(n, p) = |F_{\text{stab}}(n, p) - F_{\text{exact}}(n, p)| / |F_{\text{exact}}(n, p)|$ of the adaptive stable evaluator (Algorithm 4) in IEEE 754 double precision, against the exact rational value of Theorem 3. The algorithm uses $M_0 = 16$ and stopping tolerance 5×10^{-14} ; the final order is $M = 64$ for $n = 1$ and $M = 32$ for the remaining rows.

$n \setminus (p - n)$	3	5	8	12	20
1	7.0×10^{-16}	2.0×10^{-16}	9.6×10^{-17}	8.4×10^{-16}	2.5×10^{-16}
2	6.2×10^{-16}	2.6×10^{-16}	5.7×10^{-16}	2.9×10^{-16}	9.9×10^{-17}
4	2.4×10^{-16}	4.5×10^{-16}	1.4×10^{-18}	2.7×10^{-16}	5.6×10^{-16}
8	1.1×10^{-16}	3.2×10^{-16}	1.6×10^{-16}	1.2×10^{-15}	1.2×10^{-16}
16	4.4×10^{-16}	9.0×10^{-16}	7.8×10^{-16}	4.8×10^{-16}	3.9×10^{-16}
32	7.8×10^{-16}	8.6×10^{-17}	2.3×10^{-16}	6.7×10^{-16}	6.5×10^{-16}
64	9.0×10^{-16}	1.2×10^{-15}	2.9×10^{-16}	7.8×10^{-16}	8.6×10^{-16}

The reviewer-style stress test at small argument is addressed directly in Table 6, which reports the relative error of the stable ${}_2F_1$ evaluator (Algorithm 2) at abscissae as small as $x = 10^{-8}$ —exactly the regime in which the naïve closed form (117) becomes numerically unusable. The errors remain at the machine-precision scale over the displayed values of x and n .

Table 6. Double-precision relative error of the stable ${}_2F_1$ evaluator (Algorithm 2) at small x , against a 50-digit reference.

$n \setminus x$	10^{-8}	10^{-6}	10^{-4}	10^{-2}
1	5.8×10^{-17}	1.3×10^{-17}	7.7×10^{-17}	2.3×10^{-17}
4	1.2×10^{-18}	2.1×10^{-17}	8.9×10^{-17}	9.1×10^{-18}
8	1.0×10^{-16}	5.7×10^{-17}	1.1×10^{-16}	1.5×10^{-16}
16	2.2×10^{-17}	4.5×10^{-17}	1.1×10^{-16}	1.1×10^{-16}
32	1.0×10^{-18}	2.6×10^{-17}	9.0×10^{-17}	3.1×10^{-17}

5.10.3. Experiment 3: Extended Table of $F(n, p)$ Values

Table 7 presents the decimal values of $F(n, p)$ (to ten significant figures) for $n \in \{0, 1, 2, 3, 4, 5\}$ and $p \in \{n + 3, n + 5, n + 8, n + 12, n + 20\}$, illustrating the strictly monotonic decrease in p for each fixed $n \geq 1$. Each row converges as $p \rightarrow \infty$ to $H_n(1) = \frac{2n+3}{4n+4}$, a limit that depends on n .

The row for $n = 0$ can be independently verified: since the parameter $c = \frac{2p-1}{2}$ equals the lower parameter $d = \frac{2n+5}{2} = \frac{5}{2}$ only when $p = 3$, the general $n = 0$ values do not admit a single elementary closed form, and the entries in the table are the exact rational outputs of Algorithm 3 for each listed p . This provides an independent consistency check of the formulas. All entries in Table 7 for rows $n = 1, \dots, 5$ have been regenerated to ensure consistency with Algorithm 3 and the worked example verification in Experiment 1.

Table 7. Decimal values of $F(n, p)$ to ten significant figures for selected (n, p) . All entries are exactly rational; the decimal expansion is shown for readability.

$n \setminus p$	$n+3$	$n+5$	$n+8$	$n+12$	$n+20$
$n = 0$	0.8333333333	0.8057142857	0.7875814662	0.7763749672	0.7666278917
$n = 1$	0.7000000000	0.6759637188	0.6593364308	0.6488956872	0.6398431270
$n = 2$	0.6428571429	0.6258217092	0.6129910807	0.6044497323	0.5967181729
$n = 3$	0.6111111111	0.5987497351	0.5886992337	0.5816099413	0.5748897941
$n = 4$	0.5909090909	0.5816030124	0.5735382860	0.5675507961	0.5616257921
$n = 5$	0.5769230769	0.5696861309	0.5630748778	0.5579436844	0.5526642908

5.10.4. Experiment 4: Monotonic Decrease in $F(n, p)$ —Graphical Illustration

Figure 2 plots $F(n, p)$ as a function of $p - n$ for $n = 1, 2, 3, 4, 5$ over the range $p - n \in [3, 33]$. The strict decrease in p can be proved rigorously from Rainville's representation $F(n, p) = \left(p - \frac{1}{2}\right) \int_0^1 x^{p-3/2} H_n(x) dx$. We first note that H_n is strictly decreasing on $(0, 1)$: Euler's integral representation, applied with $a = -\frac{2n+1}{2}$, $b = 1$, $c = \frac{2n+5}{2}$ (so that $-a = \frac{2n+1}{2} > 0$ and $c - 1 = \frac{2n+3}{2}$), gives

$$H_n(x) = \frac{2n+3}{2} \int_0^1 (1-t)^{(2n+1)/2} (1-xt)^{(2n+1)/2} dt,$$

and differentiating under the integral sign yields

$$H'_n(x) = -\frac{(2n+3)(2n+1)}{4} \int_0^1 t(1-t)^{(2n+1)/2} (1-xt)^{(2n-1)/2} dt < 0 \quad \text{for } x \in (0, 1),$$

since the integrand is strictly positive on $(0, 1)$. Writing $d\mu_p(x) = \left(p - \frac{1}{2}\right) x^{p-3/2} dx$, each μ_p is a probability measure on $(0, 1)$, and μ_{p+1} first-order stochastically dominates μ_p ; since H_n is strictly decreasing, it follows that $F(n, p+1) < F(n, p)$. The ordering of the curves by n that is visible in the figure is an empirical observation and is not claimed as a theorem here. This complements the tabular evidence of Table 7.

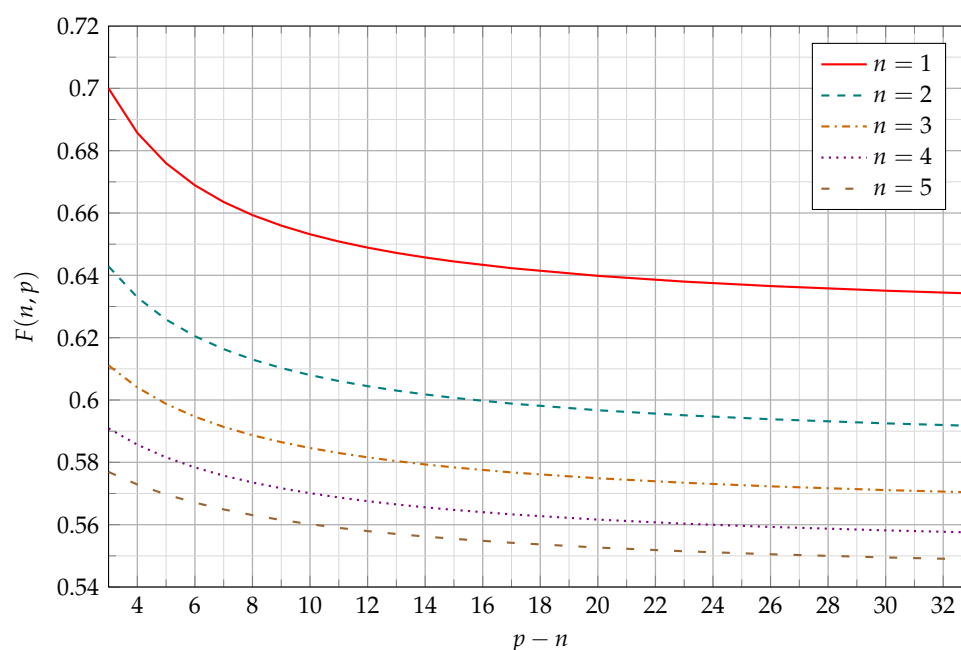


Figure 2. Strict decrease in $F(n, p)$ with respect to p , shown as a function of $p - n$, for $n = 1, 2, 3, 4, 5$, computed in exact rational arithmetic via Theorem 3 (Algorithm 3) and verified against an independent high-precision direct summation of the defining ${}_3F_2$ series. Each curve decreases strictly as p increases and converges to $\lim_{p \rightarrow \infty} F(n, p) = H_n(1) = \frac{2n+3}{4n+4}$.

5.10.5. Summary of the Numerical Behavior

The experiments above, together with the conditioning analysis of Section 5.3, give a complete and consistent picture of the numerical behavior of the closed forms. Three modes of evaluation are available, each appropriate to a different setting.

- **Exact rational arithmetic.** Assembling the closed form of Theorem 3 over $\mathbb{Q}$ returns $F(n, p)$ as an irreducible fraction with no rounding error, at the $O(n + p)$ operation count of Theorem 4 (Tables 4 and 7). This is the recommended mode when an exact value is required.

- **Naïve floating-point closed form.** As shown in Section 5.3, the factor $x^{-(n+1)}$ multiplying the cancelling bracket $B_n(x)$, and the $\Theta(4^n)$ -scale intermediates in I_1, I_2 , make this route reliable only for small n (Table 3); it should not be used in double precision for $n \gtrsim 15$.
- **Stable floating-point evaluators.** The directly summed Gauss series (Algorithm 2) and the adaptive positive-kernel Gauss–Jacobi quadrature (Algorithm 4) attain machine-precision-scale accuracy over the tested range—orders up to $n = 64$, gaps $p - n$ up to 20, and abscissae down to $x = 10^{-8}$ (Tables 5 and 6)—and are the recommended mode for floating-point work.

A secondary, unrelated overflow consideration concerns the intermediate double-factorial products: although $\Lambda_n = (-1)^{n+1}(2n+1)!!/[4^{n+1}(2n+2)!!]$ decays like $\frac{1}{4^{n+1}\sqrt{\pi(n+1)}}$ by Stirling’s approximation, the products $(2n+1)!!$ and $(2n+2)!!$ grow like $(2n/e)^n$ and overflow IEEE 754 double precision near $n = 84$ if formed explicitly. This is avoided entirely by the incremental recurrence $\Lambda_{n+1} = -\Lambda_n \cdot \frac{2n+3}{8n+16}$ from $\Lambda_0 = -\frac{1}{8}$, which keeps every intermediate bounded, and does not arise at all in exact rational arithmetic.

6. Conclusions

This paper has established explicit closed-form evaluations of the generalized hypergeometric function ${}_3F_2\left(-\frac{2n+1}{2}, 1, \frac{2p-1}{2}; \frac{2n+5}{2}, \frac{2p+1}{2}; 1\right)$ for all integers $n \geq 0$ and $p \geq n+3$, expressed in terms of rational functions of p and finite partial odd harmonic sums $H_r^{\text{odd}} = \sum_{j=1}^r (2j-1)^{-1}$. This half-integer family belongs to the class of ${}_3F_2(1)$ series with negative integral parameter differences; in the notation of Shpot and Srivastava [30] it is precisely the case $a = 1$ (with $m = 2n+2$ and their n equal to 0) that their general summation formula excludes, because the governing Beta factors are singular there. The main analytical contribution of the paper is an independent explicit evaluation of this excluded case: a single closed form for $F(n, p)$ in which the digamma structure suggested by the general theory is realized as finite odd harmonic sums, together with its exact rationality (Corollary 1) and a linear arithmetic-operation evaluation scheme. Proposition 3 proves that the apparent $a = 1$ singularity in ([30], Equation (31)) is removable and, by analyticity, identifies its $a \rightarrow 1$ value with the independently derived closed form. The derivation rests on two pillars. The first is Theorem 2, which provides a new closed-form representation of ${}_2F_1\left(-\frac{2n+1}{2}, 1; \frac{2n+5}{2}; x\right)$ as a ratio involving a degree- $(2n+1)$ polynomial $P_{2n+1}(x)$ in $(1-x)$ and a logarithmic factor $Q_n(x) \operatorname{arctanh}(\sqrt{x})/\sqrt{x}$. The proof combines Euler’s integral representation, two successive changes of variables, the recurrence relation for the auxiliary integrals $J_{m,n}(x)$, and the Vandermonde–Chu identity, which closes the full convolution sum arising in the constant branch of the switch-point array $S_n(i)$. The second pillar is Theorem 3, which integrates the representation of Theorem 2 against a Beta-type kernel via Rainville’s formula, splitting the computation into the rational part $I_1(n, p)$, evaluated through Beta function identities and Pochhammer symbols, and the logarithmic part $I_2(n, p)$, resolved via Tonelli’s theorem applied to the nonnegative Taylor expansion of $\operatorname{arctanh}$ and the telescoping identity of Lemma 5. Corollary 1 establishes that $F(n, p)$ is exactly rational for $p \geq n+3$. Proposition 2 shows that this threshold is sharp: at $p = n+2$ one has $F(n, n+2) = R_n + C_n\pi^2$ with $R_n, C_n \in \mathbb{Q}$ and $C_n \neq 0$. On the computational side, Section 5 translated the exact closed form into an algorithm with an $O(n+p)$ arithmetic-operation count under the unit-cost model (Theorem 4), and analyzed the conditioning of its floating-point evaluation. The central algorithmic contribution is the sliding-window construction in PRECOMPUTE, which reduces the otherwise $O(n^2)$ cost of building the switch-point array $S_n(i)$ to linear time, together with the single-pass initialization of the contiguous range of partial odd harmonic sums required by EVAL3F2.

A distinct and equally important point concerns *how* the closed forms are evaluated: in exact rational arithmetic they return $F(n, p)$ with no rounding error, but a naïve fixed-precision evaluation suffers catastrophic cancellation as n grows (the factor $x^{-(n+1)}$ multiplies a bracket that cancels to order x^{n+1} , and the harmonic part involves $\Theta(4^n)$ -scale intermediates). We therefore supply stable evaluators—a directly summed Gauss series for $H_n(x)$ and an adaptive positive-kernel Gauss–Jacobi quadrature for $F(n, p)$ —which attain machine-precision-scale accuracy over the tested range. The numerical experiments of Section 5.10 include a fully traced worked example for $(n, p) = (1, 4)$ yielding $F(1, 4) = \frac{7}{10}$; a conditioning study (Table 3); accuracy tables for the stable evaluators extending to order $n = 64$, parameter gaps $p - n$ up to 20, and abscissae down to $x = 10^{-8}$ (Tables 5 and 6); an extended decimal table of $F(n, p)$ values (Table 7); and a graphical illustration of the strictly monotonic decrease in $F(n, p)$ in p for $n = 1, \dots, 5$ (Figure 2).

Several directions for future research naturally emerge from this work.

1. **Extension to adjacent half-integer families.** The recurrence-based framework developed here—auxiliary integrals $J_{m,n}(x)$, the two-branch switch-point formula for $S_n(i)$, and Rainville’s integral reduction—is not tied to the specific parameter configuration treated in this paper. The same pipeline applies in principle to any ${}_3F_2$ family whose upper parameters include a negative half-integer and whose lower parameters are chosen so that Rainville’s integral formula yields a tractable Beta-type kernel. Identifying the full class of half-integer parameter configurations for which this approach produces closed forms in terms of rational functions and odd harmonic sums, and determining the structural conditions on the parameters that govern the complexity of the resulting switch-point array, constitute natural directions for future investigation.
2. **Higher-order generalization to ${}_4F_3$ and beyond.** Rainville’s integral formula applies iteratively: the representation of ${}_3F_2$ derived here can itself be integrated against a second Beta-type kernel to produce a closed form for a ${}_4F_3$ at unit argument with half-integer parameters. The feasibility of this approach depends on whether the resulting integrals remain expressible in terms of elementary functions and harmonic sums, a question that could be explored using the integral recurrence technique of Section 2.
3. **Polylogarithmic reduction of the harmonic part.** The harmonic part $I_2(n, p)$ involves a finite alternating sum of partial odd harmonic sums $H_{p+m-n-2}^{\text{odd}}$ weighted by binomial coefficients. For specific arithmetic progressions in p —for instance, $p = n + 3 + k$ for fixed small k —it may be possible to express $I_2(n, p)$ in terms of polylogarithm values at rational arguments, Catalan’s constant, or other known constants, reducing the harmonic sum to a single closed-form expression, as was done for the special case $n = 3$ in [24].
4. **Applications to random walk densities and Feynman integrals.** General and related ${}_3F_2$ functions at unit argument occur in the analysis of short uniform random walks in higher dimensions [7,8] and in ε -expansions of Feynman diagrams in dimensional regularization [9,17]. A useful next step is to identify concrete parameter specializations in those applications that reduce *exactly* to the half-integer family treated here. Until such a reduction is established, these subjects should be viewed as motivations for further work rather than direct applications of Theorem 3.

Supplementary Materials: The following supporting information can be downloaded at <https://www.mdpi.com/article/10.3390/axioms15090664/s1>. Supplementary File S1 provides a Python reference implementation of Algorithms 1–4, including exact fraction arithmetic, the boundary decomposition of Proposition 2, and the adaptive Gauss–Jacobi evaluator used for the numerical checks.

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