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Stability Criteria for Nonlinear Time-Varying Delay Differential Inclusions by Limiting Equations

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Abstract

In this paper, the uniform asymptotic stability (UAS) of nonlinear time-varying delay differential inclusions is investigated. Some new stability criteria are established from the output-to-state perspective for time-varying delay differential inclusions. Furthermore, the limiting systems associated with the time-varying delay differential inclusions are introduced to verify the weakly-zero-state-detectability (WZSD) condition. The proposed criteria can be viewed as extensions of LaSalle's invariance principle to time-varying delay systems with multiple solutions. Finally, a tracking control example is presented to demonstrate the effectiveness of the obtained results.

Keywords: delay differential inclusions; limiting systems; Lyapunov–Krasovskii functional; uniform asymptotic stability

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1. Introduction

The LaSalle invariance principle is one of the most fundamental theoretical tools for analyzing the stability of dynamical systems under weak Lyapunov functions or Lyapunov–Krasovskii functionals [1–3]. However, for time-varying systems, the positive limiting set of a solution is generally no longer an invariant set, and thus the classical LaSalle invariance principle cannot be directly applied to such systems [4]. Motivated by the idea of limiting equations proposed by Artstein [5], this paper aims to provide new characterizations of the uniform asymptotic stability (UAS) of time-varying delay differential inclusions admitting multiple solutions. The limiting systems associated with the time-varying delay differential inclusions are introduced to verify the weakly-zero-state-detectability (WZSD) condition, which has been shown to play an important role in the stability analysis of time-varying ordinary differential (difference) equations [6].

Lyapunov's second theorem has been proved to provide not only sufficient but also necessary conditions for the UAS of ordinary differential equations [2]. For many classes of systems, such as passive systems and adaptive systems, it is often easier to construct a weak Lyapunov function or a Lyapunov–Krasovskii functional [1,2]. The LaSalle invariance principle provides an effective approach for analyzing the UAS of time-invariant ordinary differential equations. Specifically, for time-invariant ordinary differential equations, any bounded solution converges to the largest invariant subset contained in the set where the derivative of the candidate Lyapunov function is zero. If this largest invariant subset reduces to the origin, the UAS of the origin can be concluded. Motivated by this powerful idea, the LaSalle invariance principle has been extended to various classes of dynamical systems, including delay systems [3], discrete-time systems [1], switched systems [7], and hybrid systems [8]. However, for time-varying ordinary differential equations, the LaSalle invariance principle cannot be directly applied because the positive limit set of a solution is generally not an invariant set. To overcome this difficulty, Artstein [5] proposed the concept of limiting equations to extend the LaSalle invariance principle to time-varying systems. It was shown in [5] that the positive limit set of a solution is an invariant set of the corresponding limiting equations. Nevertheless, identifying the invariant sets of time-varying systems remains challenging in stability analysis. To address this issue, the authors in [4,6] introduced detectability conditions to simplify the stability analysis of time-varying ordinary differential equations, where the limiting systems are employed to verify the WZSD condition.

Time delays are inevitable in control engineering and many other practical applications [3]. The existence of a Lyapunov–Krasovskii functional has been shown to be a sufficient and necessary condition for the UAS of time-varying functional differential equations [9]. However, constructing Lyapunov–Krasovskii functionals for general nonlinear delay systems remains a challenging task. Fortunately, for some classes of systems, such as passive systems and adaptive systems, weak Lyapunov–Krasovskii functionals can be constructed [1,3,10,11]. Therefore, the LaSalle invariance principle has been widely employed in the stability analysis of delay systems. Recently, the LaSalle invariance principle has been extended to time-invariant delay differential inclusions by employing Lyapunov–Krasovskii functionals [10]. For time-varying functional differential equations, the idea of limiting equations has also been applied to stability analysis based on Lyapunov–Razumikhin functions [12] and Lyapunov–Krasovskii functionals [13]. However, this approach still requires the identification of invariant sets of the corresponding limiting systems. Motivated by the results in [4,6], this paper aims to extend detectability conditions to time-varying delay differential inclusions admitting multiple solutions. The limiting systems associated with time-varying delay differential inclusions are introduced to verify the WZSD condition. It is worth noting that the results obtained in this paper provide an effective approach for analyzing the UAS of time-varying delay differential inclusions under weak Lyapunov functionals.

The main contribution of this paper is concluded as the following points. Firstly, this paper establishes that the limiting behavior of solutions to the original delay differential inclusions can be analyzed through the corresponding limiting systems. We prove that the limiting solution of original delay differential inclusions is a solution of limiting systems. An infinite-dimensional extension of the detectability condition is provided which is even new for time-varying delay differential equations. Under an appropriate integral condition, the UAS of the time-varying delay differential inclusions is guaranteed. Secondly, the limiting systems associated with the delay differential inclusions are introduced to verify the detectability conditions. Finally, a tracking control example is presented to demonstrate the effectiveness and applicability of the proposed results.

Throughout this paper, the following notations are adopted. $\mathbb{R}^n$ is the Euclidean space equipped with the 2-norm $|x|$, $x \in \mathbb{R}^n$. $\mathcal{C}_r^n$ denotes the space of continuous functions from $[-r, 0]$ to $\mathbb{R}^n$ equipped with the norm $\|\varphi\|_r = \max_{-r \le s \le 0} |\varphi(s)|$. The space $\mathbb{R} \times \mathcal{C}_r^n$ is equipped with the norm $|(t, \varphi)| = \max\{|t|, \|\varphi\|_r\}$, $(t, \varphi) \in \mathbb{R} \times \mathcal{C}_r^n$. A set-valued functional $G : \mathbb{R} \times \mathcal{C}_r^n \rightrightarrows \mathbb{R}^n$ is upper semicontinuous at the point $y \in \mathbb{R} \times \mathcal{C}_r^n$ if, for any $\varepsilon > 0$, there exists a positive number δ such that $G(z) \subset G(y) + \varepsilon\mathbb{B}$ for all z satisfying $|z - y| < \delta$, and it is further said to be upper semicontinuous if it is upper semicontinuous at each point of $\mathbb{R} \times \mathcal{C}_r^n$. $G : \mathbb{R} \times \mathcal{C}_r^n \rightrightarrows \mathbb{R}^n$ is said to be upper semicontinuous in the second variable, uniformly in the first variable, if for any $\varphi \in \mathcal{C}_r^n$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that $G(t, \phi) \subset G(t, \varphi) + \varepsilon\mathbb{B}$. The uniformly continuous set-valued functional can be given similarly. $\mathbb{B}$ is the unit ball of $\mathbb{R}^n$.

2. Preliminaries

Consider the following time-varying delay differential inclusions:

$$\dot{x} \in \mathcal{F}(t, x_t), \qquad y = h(t, x_t), \tag{1}$$

where D is an open subset of $\mathcal{C}_r^n$, $\mathcal{F} : \mathbb{R}_{\geq 0} \times D \rightrightarrows \mathbb{R}^n$ is a set-valued functional, the continuous functional $h(t, \varphi)$ from $[0, \infty)$ to $\mathbb{R}$ is continuous in φ, uniformly in t, and $h(t, 0) = 0$ for all $t \geq 0$. Moreover, h maps $(0, \infty) \times Y$ into a bounded subset of $\mathbb{R}^n$ for each bounded subset Y of D. Throughout this paper, $\mathcal{F}$ satisfies the following conditions:

(C1) $\mathcal{F}$ is upper semicontinuous and takes nonempty, compact and convex values on its domain;
(C2) For each bounded subset Y of D, $\mathcal{F}$ maps $[0, \infty) \times Y$ into a bounded subset of $\mathbb{R}^n$.

A continuous function $x : [t_0 - r, T) \to \mathbb{R}^n$ for $T > 0$ is said to be a solution of system (1) if it satisfies the following:

(c1) $x(t)$ is locally absolutely continuous on $[t_0, T)$;
(c2) $x_t \in D$ for all $t \in [t_0, T)$;
(c3) $x(t)$ satisfies (1) almost everywhere.

Denote a solution of system (1) with initial time $t_0 \geq 0$ and initial function $\varphi \in D$ by $x(t, t_0, \varphi)$. The existence and continuation of the solutions of system (1) can be found in [14,15].

Proposition 1 ([14]). *Consider the delay differential inclusion (1). Suppose that there exists a locally Lipschitz functional $V(t, x, \varphi) : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathcal{C}_r^n \to \mathbb{R}_{\geq 0}$ that satisfies the following conditions:*

(c1) It possesses basic property;
(c2) The invariant directional derivative exists at each point $(w, \varphi) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}^n) \times \mathcal{C}_r^n$;
(c3) $V'(w, \varphi, z) = V^\circ(w, \varphi, z)$ for each $(w, \varphi) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}^n) \times \mathcal{C}_r^n$ and $z \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$.

Then for any solution $x : [t_0 - r, T)$ of (1), it holds that for almost all $t \in [t_0, T)$,

$$\dot{V}(t, x(t), x_t) \in \tilde{\dot{V}}(t, x(t), x_t)$$

where $\tilde{\dot{V}}(t, x(t), x_t) = \bigcap_{\xi \in \partial V(w, x_t)} \xi \begin{bmatrix} 1 & \mathcal{F}(t, x_t) \end{bmatrix} (t, x(t))$.

3. UAS by Detectability for Time-Varying Delay Differential Inclusions

Let $\hat{h} : [0, \infty) \times D \to \mathbb{R}$ be a continuous functional. Based on the above continuous functional $\hat{h}$, we are in a position to define the output-persistent exciting (out-PE) condition.

Definition 1. *The pair $(\hat{h}, \mathcal{F})$ is said to be output-PE if, for any $\delta > 0$ and any bounded closed subset Y of D, there exist two positive constants $T(\delta, Y)$ and $\rho(\delta, Y)$ such that, for any solution x of system (1) and any $t \geq t_0$, the following holds:*

$$\|x_\tau\|_r \geq \delta,\ \forall t \leq \tau \leq t + T \implies \int_t^{t+T} \left|\hat{h}(\tau, x_\tau)\right|^2 d\tau \geq \rho. \tag{2}$$

Hypothesis 1. *There exists a continuous function $\mu : [0, \infty) \to \mathbb{R}$ satisfying $\mu(0) = 0$ and $\mu(s) > 0$ for all $s > 0$, such that, for any bounded closed subset Y of D and any positive constant*

σ, there exists a positive constant $M(\sigma, Y)$ such that, for every solution $x(t, t_0, \varphi)$ of system (1) satisfying $x_t \in Y$ for all $t \geq t_0$, the following integral inequality holds:

$$\int_{t_0}^{\infty} \big(\mu\big(h(s, x_s)\big) - \sigma\big) ds \leq M(\sigma, Y).$$

Proposition 2. *Consider system (1). Suppose that Hypothesis 1 holds. The origin of system (1) is UAS if the origin is uniformly stable (US) and the pair $(\hat{h}, \mathcal{F})$ is output-PE, where $\hat{h}(t, \varphi) = \sqrt{\mu\big(h(t, \varphi)\big)}$. Furthermore, if $D = C_r^n$ and the origin is uniformly globally stable (UGS), then the origin is uniformly globally asymptotically stable (UGAS).*

Proof of Proposition 2. To prove uniform attractivity, we first establish an intermediate result. By uniform stability (US), there exist two positive constants $\delta > 0$ and $\eta > 0$ such that $B_\eta = \{\varphi \mid \|\varphi\|_r \leq \eta\} \subset D$ and any solution $x(t, t_0, \varphi)$ of system (1) with $\|\varphi\|_r \leq \delta$ satisfies $\|x_t\|_r \leq \eta \; \forall t \geq t_0$. Furthermore, for any $\varepsilon > 0$, another application of US yields a constant $\bar{\delta}$ with $0 \leq \bar{\delta} \leq \delta$ such that any solution with $\|\varphi\|_r \leq \bar{\delta}$ satisfies $\|x_t\|_r \leq \varepsilon$ for all $t \geq t_0$. $x(t, t_0, \varphi)$. Therefore, to establish uniform attractivity, it suffices to show that, for any solution $x(t, t_0, \varphi)$ with $\|\varphi\|_r \leq \delta$, there exists a $\bar{t}$ satisfying $t_0 \leq \bar{t} \leq k_0 T(\bar{\delta}, B_\eta) + t_0$ such that $\|x_{\bar{t}}\|_r \leq \bar{\delta}$, where $k_0 = \left\lceil \frac{2M(\sigma, B_\eta)}{\rho(\bar{\delta}, B_\eta)} \right\rceil + 1$, $M(\sigma, B_\eta)$ comes from Hypothesis 1 with $\sigma = \frac{\rho(\bar{\delta}, B_\eta)}{2T(\bar{\delta}, B_\eta)}$, and $T(\bar{\delta}, B_\eta)$ and $\rho(\bar{\delta}, B_\eta)$ are given by the output-PE property in Definition 1.

Suppose, to the contrary, that this assertion does not hold. Then, there exists a solution $x(t, t_0, \varphi)$ of system (1) with $\|\varphi\|_r \leq \delta$ such that $\|x_t\|_r \geq \bar{\delta}$ for all $t_0 \leq t \leq t_0 + k_0 T(\bar{\delta}, B_\eta)$. Combining output-PE and Hypothesis 1 yields

$$\rho \leq \min_{j=0,1,\cdots,k_0-1} \int_{t_0+jT}^{t_0+(j+1)T} \mu\big(h(s, x_s)\big) ds \leq \frac{M}{k_0} + T\sigma < \rho,$$

which is a contradiction. The proof is completed. □

Remark 1. *To verify the UAS of time-varying delay systems, it is sufficient, with the help of Proposition 2, to check the output-PE condition of these systems.*

Definition 2. *The system (1) is said to be WZSD if, for any bounded closed subset Y of D, any unbounded time sequence $\{t_n\}_{n=1}^{\infty}$, and any sequence of solutions $\{x^n(t, t_n, \varphi_n)\}_{n=1}^{\infty}$ satisfying $x^n_{t+t_n} \in Y$ for all $n \in \mathbb{N}$ and all $t \geq 0$, the following condition holds:*

$$\lim_{n \to \infty} h\big(t + t_n, x^n_{t+t_n}\big) = 0, \text{ a.e.}, \tag{3}$$

there exists a subsequence $\{n_m\}_{m=1}^{\infty}$ of $\{n\}_{n=1}^{\infty}$ and a time sequence $\{s_m\}_{m=1}^{\infty}$ with $0 \leq s_m \leq n_m$ such that

$$\lim_{m \to \infty} x^{n_m}_{t_{n_m}+s_m} = 0, \tag{4}$$

Then, the system (1) is locally WZSD if it is WZSD in a neighborhood of the origin.

Remark 2. *Output-persistent excitation means the output function can be used to determine whether the state is near the origin. If the state stays far away the origin, the integral of the output function has a positive lower bound. WZSD means when the output function appears zero, then zero is a limiting point of the solution.*

Proposition 3. *Suppose Hypothesis 1 holds. If the system (1) is WZSD and the origin of system (1) is US, the origin is UAS. Moreover, if $D = C_r^n$ and the origin is UGS, the origin is UGAS.*

Proof of Proposition 3. It is sufficient to assert the pair $\left(\sqrt{\mu(h)}, \mathcal{F}\right)$ is output-PE by Proposition 2. Suppose it is not; then there exist a $\delta > 0$ and a closed bounded subset Y of D such that for any $n > 0$, there exist a $t_n \geq 0$ and a solution $x^n(t, t_n, \varphi_n)$ of system (1) with $x_t^n \in Y$, for all $t \in [t_n, t_n + 2n]$, such that

$$\|x_\tau\|_r \geq \delta,\ \forall\, t_n \leq \tau \leq t_n + 2n \implies \int_{t_n}^{t_n+2n} \mu\big(h(s, x_s)\big) ds \leq \frac{1}{n}. \tag{5}$$

It follows from (5) that

$$\int_0^n \mu\big(h(t + t_n + n, x_{t+t_n+n})\big) dt \leq \frac{1}{n}. \tag{6}$$

Let $\bar{t}_n = t_n + n$. Taking a limit on both sides of (6) derives

$$\lim_{n\to\infty} \int_0^\infty \mu\big(h(t + \bar{t}_n, x_{t+\bar{t}_n})\big) dt = 0.$$

Hence, by Lemma A1 of [16], there exists a subsequence $\{\bar{t}_{n_m}\}_{m=1}^\infty$ of $\{\bar{t}_n\}_{n=1}^\infty$ such that

$$\lim_{m\to\infty} \mu\big(h(t + \bar{t}_{n_m}, x_{t+\bar{t}_{n_m}})\big) = 0, \text{ a.e.}$$

Since μ is a continuous function satisfying $\mu(0) = 0$ and $\mu(s) > 0$ for all $s > 0$, and h is bounded on $[0, \infty) \times Y$, the above relation implies that $\lim_{m\to\infty} h\big(t + \bar{t}_{n_m}, x_{t+\bar{t}_{n_m}}\big) = 0$, a.e. As the system (1) is WZSD, there exist a subsequence $\{\bar{t}_{n_{m_k}}\}_{k=1}^\infty$ of $\{\bar{t}_{n_m}\}_{m=1}^\infty$ and a sequence $\{s_k\}_{k=1}^\infty$ with $s_k \leq m_k \leq n_{m_k}$ that satisfy

$$\lim_{k\to\infty} \left\|x_{\bar{t}_{n_{m_k}}+s_k}\right\|_r = 0,$$

which is a contradiction. The proof is completed. □

Definition 3. *A function $x\colon [-r, \infty) \to \mathbb{R}^n$ is said to be a limit solution of system (1) with respect to an unbounded sequence $\{t_n \mid t_n \geq 0\}_{n=1}^\infty$ if there exists a bounded closed subset Y of D and a solution sequence $\{x^n\colon [t_n - r, \infty) \to \mathbb{R}^n \mid x_t^n \in Y,\ \forall t \in [t_n, \infty)\}_{n=1}^\infty$ of system (1) such that the associated sequence $\{\hat{x}^n\colon t \mapsto x^n(t + t_n)\}_{n=1}^\infty$ converges uniformly to x on each compact subset of $[-r, \infty)$.*

Lemma 1. *For any bounded closed subset Y of D and any solution sequence $\{x^n\colon [t_n - r, \infty) \to \mathbb{R}^n \mid x_t^n \in Y,\ \forall t \in [t_n, \infty)\}_{n=1}^\infty$ with $\lim_{n\to\infty} t_n \to \infty$, there exist a limit solution x of system (1) and a subsequence $\{t_{n_m}\}_{m=1}^\infty$ of $\{t_n\}_{n=1}^\infty$ such that the associated sequence $\{\hat{x}^{n_m}\colon t \mapsto x^{n_m}(t + t_{n_m})\}_{m=1}^\infty$ converges uniformly to x on each compact subset of $(-r, \infty)$.*

Remark 3. *In Lemma 1, x^{n_m} is Lipschitz continuous on every compact subset of $(-r, \infty)$, since its derivative is bounded in accordance of the conditions. The existence of the subsequence $\{x^{n_m}\colon t \mapsto x^{n_m}(t + t_{n_m})\}_{m=1}^\infty$ follows from the Arzela–Ascoli theorem, as the sequence $\{x^n\colon t \mapsto x^n(t + t_n)\}_{n=1}^\infty$ is uniformly bounded and $\mathcal{F}$ maps $\mathbb{R}_{\geq 0} \times Y$ into a bounded subset of $\mathbb{R}^n$.*

For a continuous function $x\colon [-r, \infty) \to \mathbb{R}^n$, $\varphi \in C_r^n$ is said to be a (ω)-limit point of $\{x_t \mid t \in [0, \infty)\}$ if there exists an unbounded time sequence $\{t_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} x_{t_n} = \varphi$.

Assumption 1. *For any limit solution $\bar{x}$ with respect to an unbounded sequence $\{t_n\}_{n=1}^{\infty}$ in $[0,\infty)$, if the equation*

$$\lim_{n\to\infty} h(t+t_n,\bar{x}_t) = 0, \text{ a.e.},$$

holds, then we have either $\bar{x}_{t_0} = 0$ for some $t_0 \geq 0$ or the origin is a ω-limit point of $\{\bar{x}_t \mid t \in [0,\infty)\}$.

Let $G\colon [a,b] \rightrightarrows \mathbb{R}^n$ be an upper semicontinuous set-valued function. As defined in [17], it has

$$\int_a^b G(s)ds = \left\{ \int_a^b f(s)ds \,\middle|\, f(t) \text{ is Lebesgue integrable}, f(t) \in G(t),\ \forall t \in [a,b] \right\}. \tag{7}$$

Proposition 4. *Consider the system (1). Suppose additionally that $\mathcal{F}(t,\varphi)$ is upper semicontinuous in φ uniformly in t. Let $\bar{x}\colon [-r,T) \to \mathbb{R}^n$ be a limit solution of (1) associated with an unbounded time sequence $\{t_n\}_{n=1}^{\infty}$ and $\{x^n\colon t \mapsto x^n(t+t_n)\}$; then we have*

$$\bar{x}(t) \in \bar{x}(0) + \lim_{n\to\infty} \int_0^t \mathcal{F}(s+t_n,\bar{x}_s)ds.$$

Proof of Proposition 4. By the definition of limit solutions, there exists a sequence of solutions $x^n(t,t_n,\varphi_n)$ of system (1) that converges uniformly to $\bar{x}$ on every compact subset of $[0,T)$. Therefore, for any $t \in [0,T)$, it follows that

$$x^n(t,t_n,\varphi_n) \in x^n(t_n,t_n,\varphi_n) + \int_0^t \mathcal{F}(t_n+s,x^n_{s+t_n})ds.$$

Taking a limit on both sides induces

$$\bar{x}(t) \in \bar{x}(0) + \lim_{n\to\infty} \int_0^t \mathcal{F}(t_n+s,x^n_{s+t_n})ds.$$

Since $\mathcal{F}(t,\varphi)$ is upper semicontinuous in φ, uniformly in t, the following relation holds for any $t \in [0,T)$:

$$\lim_{n\to\infty} \mathcal{F}(t_n+t,x^n_{s+t_n}) \subset \lim_{n\to\infty} \mathcal{F}(t+t_n,\bar{x}_t).$$

Based on the Lebesgue's dominated convergence theorem and the definition of integral of set-valued function, we can derive

$$\lim_{n\to\infty} \int_0^t \mathcal{F}(t_n+s,x^n_{t_n+s})ds \in \int_0^t \lim_{n\to\infty} \mathcal{F}(t_n+s,\bar{x}_s)ds.$$

The proof is completed by combining the above relations. □

Lemma 2. *Consider the system (1). Suppose Assumption 1 holds; then the system (1) is WZSD.*

Proof of Lemma 2. Let $\{t_n\}_{n=1}^{\infty}$, and $\{x^n(t,t_n,\varphi_n)\}_{n=1}^{\infty}$ be, respectively, an unbounded time sequence, a bounded closed subset of D, and a sequence of solutions of system (1) satisfying $x^n_t \in Y$ for all $t \geq t_n$ and $n \in \mathbb{N}$, such that the limit output condition holds. Since $\{x^n(t,t_n,\varphi_n)\}_{n=1}^{\infty}$ is uniformly bounded and F maps $(0,\infty) \times Y$ into a bounded subset of $\mathbb{R}^n$, there exists a constant $B > 0$ such that, for almost all $t \in [0,\infty)$ and $n \in \mathbb{N}$,

$$\left|\dot{x}^n(t,t_n,\varphi_n)\right| \leq B.$$

Consequently, the sequence $\{\hat{x}\colon t \mapsto x(t+t_n+r, t_n, \varphi_n)\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous. By the Arzela–Ascoli theorem, there exists a subsequence $\{\hat{x}^{n_m}\}_{m=1}^{\infty}$ of $\{\hat{x}^n\}_{n=1}^{\infty}$ that converges uniformly to a continuous function $x\colon [-r, \infty) \to \mathbb{R}^n$. Moreover, the sequence $\{\hat{x}_t^{n_m}\}_{m=1}^{\infty}$ also converges uniformly to x_t. Since $h(t,\varphi)$ is continuous with respect to φ, uniformly in t, the sequence $\{h(t+t_{n_m}, \hat{x}_{t+t_{n_m}}^{n_m})\}_{m=1}^{\infty}$ converges uniformly to $\lim_{m\to\infty} h(t+t_{n_m}, x_t)$. Denote $\bar{t}_m = t_{n_m}$, $m \in \mathbb{N}$. From Assumption 1, for any $k \in \mathbb{N}$, there exists a $s_k > 0$ such that $\|x_{s_k}\|_r \le \frac{1}{k}$. We can further extract a subsequence $\{\bar{t}_{m_k}\}$ of $\{\bar{t}_m\}$ such that $m_k \ge s_k$. Then, by the triangle inequality, we have

$$\begin{aligned}\|x_{t_{m_k}+s_k}^{m_k}\|_r &= \left\|x_{t_{m_k}+s_k}^{m_k} - x_{\bar{t}_{m_k}} + x_{\bar{t}_{m_k}}\right\|_r,\\ &\le \left\|x_{t_{m_k}+s_k}^{m_k} - x_{\bar{t}_{m_k}}\right\|_r + \left\|x_{\bar{t}_{m_k}}\right\|_r.\end{aligned}$$

Taking a limit on both sides derives

$$\lim_{k\to\infty} \left\|x_{t_{m_k}+s_k}^{m_k}\right\|_r = 0.$$

The proof is completed. □

Theorem 1. *Consider the system (1). Suppose that there exists a locally Lipschitz functional $V\colon \mathbb{R}_{\ge 0} \times \mathbb{R}^n \times C_r^n \to \mathbb{R}_{\ge 0}$ satisfying all the conditions of Proposition 1, $\alpha_i \in \mathcal{K}$ $(i = 1, 2)$ and a continuous function $\mu\colon [0,\infty) \to \mathbb{R}$ with $\mu(0) = 0$ and $\mu(s) > 0$ for all $s > 0$ such that*

1. *$\alpha_1(|\varphi(0)|) \le V(t, \varphi(0), \varphi) \le \alpha_2(\|\varphi\|_r)$ for all $(t,\varphi) \in \mathbb{R}_{\ge 0} \times D$;*
2. *$\dot{V} \le -\mu(h(t,\varphi))$ for all $(t,\varphi) \in \mathbb{R}_{\ge 0} \times D$.*

If Assumption 1 holds, then the origin is UAS. Furthermore, if $D = C_r^n$ and $\alpha_1 \in \mathcal{K}_\infty$, the origin is UGAS.

Proof of Theorem 1. US is obvious. By Proposition 1 and the conditions of Theorem 1, for any bounded closed subset Y of D and any solution $x(t, t_0, \varphi)$ satisfying $x_t \in Y$ for all $t \ge t_0$, there exists a constant $M(Y) > 0$ such that

$$\int_{t_0}^{\infty} \mu(h(t, x_t))dt \le V(t_0, \varphi) \le \alpha_2(\|\varphi\|_r) \le M(Y).$$

Using Lemma 2, the system is WZSD; then, applying Proposition 3, the origin is UGAS. □

4. Main Results

4.1. Limit Systems of Time-Varying Delay Differential Inclusions

In this section, a convergence concept for set-valued functional sequence is given, which is an extension of the set-valued function sequence [17].

Definition 4. *Let $Y \subset \mathbb{R}_{\ge 0} \times C_r^n$. A sequence $\{\mathcal{F}_i\colon Y \rightrightarrows \mathbb{R}^m\}_{i=1}^{\infty}$ of a set-valued functional is said to converge uniformly to $G\colon Y \rightrightarrows \mathbb{R}^m$ if, for every $\varepsilon > 0$ and $\rho > 0$, there exists a positive integer N such that, for any $i \ge N$,*

$$\begin{cases}\mathcal{F}_i(z) \cap \rho\mathbb{B} \subset G(z) + \varepsilon\mathbb{B} \\ G(z) \cap \rho\mathbb{B} \subset \mathcal{F}_i(z) + \varepsilon\mathbb{B}\end{cases} \quad \forall z \in Y. \tag{8}$$

For a closed bounded subset Y of C_r^n and a constant $\varrho > 0$, denote $S(\varrho, Y) = \{\varphi \in Y \mid \varphi$ absolutely continuous, $|\dot{\varphi}(s)| \le \varrho$, a.e.$\}$. Then, $S(\varrho, Y)$ is a compact set.

Definition 5. *Let $G\colon \mathbb{R}_{\geq 0} \times D \rightrightarrows \mathbb{R}^n$ be an upper semicontinuous set-valued functional and G take nonempty, convex and compact values on its domain. An unbounded sequence $\gamma = \{t_n \mid t_n \geq 0\}_{n=1}^{\infty}$ is said to be an admissible sequence associated with G if there exist a set-valued functional $G_\gamma\colon \mathbb{R}_{\geq 0} \times D \rightrightarrows \mathbb{R}^n$, which is upper semicontinuous and has compact and convex values on its domain, and a sequence of set-valued functionals $\{\hat{G}_n\colon (t,\varphi) \mapsto G(t+t_n,\varphi)\}_{n=1}^{\infty}$ such that $\{\hat{G}_n\colon (t,\varphi) \mapsto G(t+t_n,\varphi)\}_{n=1}^{\infty}$ converges uniformly to G_γ on $I \times S(\mu, Y)$ for every compact subset I of $[0,\infty)$, every closed bounded subset Y of D, and every constant $\mu > 0$.*

Denote the collection of all the admissible sequences associated with G by $\Lambda(G)$.

Definition 6. *G is an asymptotically almost periodic functional (AAP functional) if there is an admissible sequence associated with G.*

An AAP functional has been introduced in [6]. Definition 6 is the corresponding version in functional space.

4.2. Example 1

Consider the following scalar delay differential inclusion motivated by an uncertain delay system:

$$\dot{x} \in \big[a(t), b(t)\big]x\big(t - d(t)\big),$$

where $a(t)$ and $b(t)$ are both uniformly continuous functions satisfying $\underline{\sigma} \leq a(t) < b(t) \leq \bar{\sigma}, \forall t \geq 0$ for two finite constants $\underline{\sigma}$ and $\bar{\sigma}$. $d(t)\colon [0,\infty) \to [0,r]$ is a uniform continuous function. The right-hand side of the delay differential inclusion above is $G(t,\varphi) = [a(t), b(t)]\varphi(-d(t))$.

For any unbounded sequence $\{t_n \mid t_n \geq 0\}_{n=1}^{\infty}$, there exists a subsequence $\gamma = \{t_{n_m}\}_{m=1}^{\infty}$ of $\{t_n\}_{n=1}^{\infty}$ such that the function sequences $\{a_{n_m}(t)\colon t \mapsto a(t+t_{n_m})\}_{m=1}^{\infty}$, $\{b_{n_m}(t)\colon t \mapsto b(t+t_{n_m})\}_{m=1}^{\infty}$ and $\{d_{n_m}(t)\colon t \mapsto d(t+t_{n_m})\}_{m=1}^{\infty}$ converge uniformly to a continuous function $a_\gamma(t)$, a continuous function $b_\gamma(t)$ and a continuous function $d_\gamma(t)$. Define G_γ as $G_\gamma(t,\varphi) = [a_\gamma(t), b_\gamma(t)]\varphi(-d_\gamma(t))$. The set-valued functional sequence $\{\hat{G}_n\colon (t,\varphi) \mapsto G(t+t_n,\varphi)\}_{n=1}^{\infty}$ converges uniformly to G_γ on $(0,\infty) \times S(\mu, Y)$ for each compact subset Y of C_r^n and constant $\mu > 0$. Therefore, G is an AAP functional.

Lemma 3. *Let $\mathcal{F}_i\colon [0,\infty) \times C_r^n \rightrightarrows \mathbb{R}^n$ be a sequence of set-valued functionals and for each $i \in \mathbb{N}$, let $\mathcal{F}_i$ be upper semicontinuous and take compact and convex values on its domain. Suppose that there exists a set-valued functional $G\colon [0,\infty) \times C_r^n \rightrightarrows \mathbb{R}^n$ that is upper semicontinuous and takes compact and convex values on its domain, such that $\mathcal{F}_i$ converges uniformly to G on $[0,\infty) \times Y$ for some compact subset Y of C_r^n. Let $\{x^n\colon [t_0 - r, \infty) \to \mathbb{R}^n\}_{n=1}^{\infty}$ be a solution sequence of the delay differential inclusions $\dot{x}(t) \in \mathcal{F}_n(t, x_t)$. If $\{x^n\}_{n=1}^{\infty}$ converges uniformly to y on some compact interval $[t_0 - r, T]$ and $x_t^n \in Y$ for all $n \in \mathbb{N}$ and all $t \in [t_0, T]$, then $y\colon [-r,\infty) \to \mathbb{R}^n$ is a solution of the delay differential inclusion $\dot{x} \in G(t, x_t)$.*

Proof of Lemma 3. Fix a point $\bar{t} \in [0,T]$. Since G is upper semicontinuous, for any constant $\varepsilon > 0$, there exists a constant $\eta > 0$ such that for any (t,φ) satisfying $|t - \bar{t}| \leq \eta$ and $\|\varphi - y_{\bar{t}}\|_r \leq 2\eta$, we have

$$G(t,\varphi) \subset G(\bar{t}, y_{\bar{t}}) + \varepsilon\mathbb{B}.$$

Since x^n converges uniformly to y, y is continuous and y_t is also continuous on $[t_0, T]$. As a result, there exists a constant $\bar{\eta} > 0$ with $\bar{\eta} \leq \eta$ such that for all $t \in \{s : |s - \bar{t}| \leq \bar{\eta}\}$, the following holds:

$$\|y_t - y_{\bar{t}}\|_r \leq \bar{\eta}.$$

Since $\mathcal{F}_n$ and x^n both converge uniformly, there exists an integer $N > 0$ such that for all $n > N$,

$$\|x_t^n - y_t\|_r \leq \bar{\eta}, \quad \mathcal{F}_n(t, \varphi) \subset G(t, \varphi) + \varepsilon\mathbb{B}.$$

Combining the above relations yields that for all $t \in \{s : |s - \bar{t}| \leq \bar{\eta}\}$ and all $n > N$,

$$\dot{x}^n(t) \in \mathcal{F}_n(t, x_t^n) \subset G(t, x_t^n) + \varepsilon\mathbb{B} \subset G(\bar{t}, y_{\bar{t}}) + 2\varepsilon\mathbb{B}.$$

By Lemma 15 in [18], in the open interval $\{s : |s - \bar{t}| < \bar{\eta}\}$, the following holds:

$$\dot{y}(t) \in G(\bar{t}, y_{\bar{t}}) + 2\varepsilon\mathbb{B}.$$

Taking all $\bar{t} \in [t_0, T]$, $[t_0, T]$ can be covered by a collection of such open intervals. Since $[t_0, T]$ is a compact set, there exists a finite open interval covering this interval $[t_0, T]$. Thus $y(t)$ is absolutely continuous in $[t_0, T]$ and differentiable almost everywhere. For any point $t \in [t_0, T]$ such that $y(t)$ is differentiable, $\dot{y}(t) \in G(t, y_t) + 2\varepsilon\mathbb{B}$. Since ε is arbitrary, $\dot{y}(t) \in G(t, y_t)$. □

4.3. Stability by Limit Systems of Delay Differential Inclusions

Let $\mathcal{F}$ and h in system (1) be AAP functionals. For any $\gamma \in \Lambda(\mathcal{F}) \cap \Lambda(h)$, a limit system for system (1) is given as follows:

$$\begin{aligned} \dot{x} &\in \mathcal{F}_\gamma(t, x_t), \\ y(t) &= h_\gamma(t, x_t). \end{aligned} \tag{9}$$

Assumption 2. *For any admissible sequence $\gamma \in \Lambda(\mathcal{F}) \cap \Lambda(h)$ and any bounded solution $\bar{x}\colon [0, \infty) \to \mathbb{R}^n$ of system (9) with $x_t \in Y$ for all $t \geq 0$, for some bounded closed subset Y of D such that $h_\gamma(t, \bar{x}_t) = 0$, $\forall t \geq 0$, it holds that either the origin is a ω-limit point of $\{\bar{x}_t \mid t \in [0, \infty)\}$ or $\bar{x}_{t_0} = 0$ for some $t_0 \geq 0$.*

Lemma 4. *Suppose Assumption 2 holds; then the system (1) is WZSD.*

Proof of Lemma 4. Let Y be a bounded closed subset of D and $\bar{x}\colon [-r, \infty) \to \mathbb{R}^n$ be a limit solution of system (1) with respect to an unbounded sequence $\gamma = \{t_n \mid t_n \geq 0\}_{n=1}^\infty$ and a solution sequence $\{x^n(t, t_n, \varphi_n)\}_{n=1}^\infty$ with $x_t^n \in Y$ for all $t \in [0, \infty)$ and all $n \in \mathbb{N}$. Then $\tilde{x}\colon t \mapsto \bar{x}(t + r)$ is also a limit solution of system (1) with respect to an unbounded sequence $\bar{\gamma} = \{\bar{t}_n\}_{n=1}^\infty$ satisfying $\bar{t}_n = t_n + r$ and a solution sequence $\{\tilde{x}^n\colon t \mapsto x^n(t + r)\}_{n=1}^\infty$. Since $\mathcal{F}$ maps $[0, \infty) \times Y$ into a bounded subset of $\mathbb{R}^n$, there exists a $B > 0$ such that $|\dot{\tilde{x}}^n(t, t_n, \varphi_n)| \leq B$, for almost all $t \in [-r, \infty)$ and all $n \in \mathbb{N}$. Since $\mathcal{F}$ and h are both AAP functionals, there exist a subsequence $\gamma = \{t_{n_k}\}$ and $\mathcal{F}_\gamma$ and h_γ such that $\{\hat{\mathcal{F}}\colon (t, \varphi) \mapsto \mathcal{F}(t + t_{n_k}, \varphi)\}$ and $\{\hat{h}\colon (t, \varphi) \mapsto h(t + t_{n_k}, \varphi)\}$ converge uniformly to $\mathcal{F}_\gamma$ and h_γ on $[0, \infty) \times S(B, Y)$. By Lemma 3, $\tilde{x}$ is a solution of the following limit system:

$$\dot{x} \in \mathcal{F}_{\bar{\gamma}}(t, x_t).$$

Therefore, Assumption 2 implies that Assumption 1 holds. By Lemma 2, the system (1) is WZSD. □

Remark 4. *The WZSD is a key property to guarantee the uniform attractivity. However, it is very difficult to check the WZSD of the time-varying delay differential inclusion directly. By Lemma 4, the WZSD of the original delay differential inclusion (1) can be checked by the limit system (9) of (1).*

Theorem 2. *Consider the system (1). Suppose that there exists a locally Lipschitz functional $V\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times C_r^n \to \mathbb{R}_{\geq 0}$ satisfying all the conditions of Proposition 1, $\alpha_i \in \mathcal{K}$ $(i = 1, 2)$ and a continuous function $\mu\colon [0, \infty) \to \mathbb{R}$ with $\mu(0) = 0$ and $\mu(s) > 0$ for all $s > 0$ such that*

1. $\alpha_1\big(|\varphi(0)|\big) \leq V(t, \varphi(0), \varphi) \leq \alpha_2\big(\|\varphi\|_r\big)$ *for all* $(t, \varphi) \in \mathbb{R}_{\geq 0} \times D$;
2. $\dot{\bar{V}} \leq -\mu\big(h(t, \varphi)\big)$ *for all* $(t, \varphi) \in \mathbb{R}_{\geq 0} \times D$.

If $\mathcal{F}$ and h are AAP functionals and Assumption 2 holds, then the origin is UAS. Moreover, if $D = C_r^n$, the origin is UGAS.

Proof of Theorem 2. The US of the origin is straightforward because V is a weak Lyapunov functional. From Proposition 1 and the conditions of Theorem 2, for any bounded closed subset Y of D and any solution $x(t, t_0, \varphi)$ with $x_t \in Y$, there exists $M(Y) > 0$ such that

$$\int_{t_0}^{\infty} \mu\big(h(t, x_t)\big)dt \leq V(t_0, \varphi) \leq \alpha_2\big(\|\varphi\|_r\big) \leq M(Y).$$

Because $\mathcal{F}$ and h are AAP functionals and Assumption 2 holds based on the conditions of Theorem 2, the system (1) is WZSD from Lemma 4. Therefore, the origin is UAS by Proposition 3. □

Assumption 3. $\lim_{t \to \infty} \bar{x}(t) = 0$ *if $\bar{x}$ is a limit solution of (1) associated with an unbounded time sequence $\{t_n\}_{n=1}^{\infty}$.*

Remark 5. *Theorem 2 offers us sufficient conditions to check the UAS of time-varying delay differential inclusions. The output-PE condition of system (1) can be checked by the limit systems (9).*

Theorem 3. *Consider system (1) and let $D = C_r^n$. The following statements are equivalent:*

1. *The origin of system (1) is UGAS;*
2. *The origin of system (1) is UGS and Hypothesis 1 and Assumption 3 hold.*

Proof of Theorem 3. The proof is that 2. $\Rightarrow$ 1. is the same as that in Theorem 2. The focus is located on 1. $\Rightarrow$ 2. We first show Hypothesis 1 holds. Let the closed bounded subset Y of C_r^n and the positive constant σ be arbitrary. Since $h(t, 0) = 0$ for all $t \geq 0$ and $h(t, \varphi)$ is continuous in φ, uniformly in t, there exists a $\varepsilon(\sigma) > 0$ such that $|h(t, \varphi)| \leq \sigma$ for all $t \geq 0$ and all $\varphi \in C_r^n$ satisfying $\|\varphi\|_r \leq \varepsilon(\sigma)$. From UGAS, there exist $T(\varepsilon(\sigma), Y) > 0$ and $\bar{M}(Y) > 0$ such that for any solution of system (1), the following holds:

$$\begin{cases} |h(t, x_t)| \leq \bar{M}(Y), \forall t \geq 0 \\ \|x_t\|_r \leq \varepsilon, \forall t \geq T(\varepsilon(\sigma), Y). \end{cases} \quad \forall z \in Y. \tag{10}$$

Therefore, the following inequality holds:

$$\int_{t_0}^{\infty} \big(|h(t, x_t)| - \sigma\big)dt \leq \bar{M}T(\varepsilon(\sigma), Y).$$

The proof of Hypothesis 1 is completed with $\mu(y) = |y|$ and $M(\sigma, Y) = \bar{M}(Y)T(\varepsilon(\sigma), Y)$.

We next prove Assumption 3. Let Y be a bounded closed subset of C_r^n and $\bar{x}\colon [-r, \infty) \to \mathbb{R}^n$ be a limiting solution of system (1) associated with an unbounded time sequence $\{t_n\}_{n=1}^{\infty}$ satisfying $\bar{x}_t \in Y$ for all $t \geq 0$. Then, there exists a sequence of solutions $x^n(t, t_n, \varphi_n)$ of system (1) satisfying $x_t^n \in Y$ for all $n \in \mathbb{N}$ and all $t \geq t_n$, which converges uniformly to $\bar{x}$ on every compact subset of $[0, \infty)$. Since the origin is UGAS, it follows immediately that $\lim_{t\to\infty} \bar{x}_t = 0$. $\square$

5. Applications to Adaptive Control

Consider the following systems with state delays:

$$\dot{y}_p = a_p y_p(t) + b_p y_p(t - r) + k_p u(t). \tag{11}$$

The control goal is such that the system above tracks the following reference model

$$\dot{y}_m = a_m y_m(t) + b_m y_m(t - r) + k_m r(t). \tag{12}$$

The following controller is adopted:

$$\begin{aligned} u(t) &= \theta_1 r(t) + \theta_2 y_p(t) + \theta_3 y_p(t - r) \\ \dot{\theta}_1 &= -\gamma\big(y_p(t) - y_m(t)\big) r(t) \\ \dot{\theta}_2 &= -\gamma\big(y_p(t) - y_m(t)\big) y_p(t) \\ \dot{\theta}_3 &= -\gamma\big(y_p(t) - y_m(t)\big) y_p(t - r). \end{aligned} \tag{13}$$

In (11)–(13), the parameters a_p, b_p and k_p are unknown. Denoting $\hat{\theta}_1 = \frac{k_m}{k_p}$, $\hat{\theta}_2 = \frac{a_m - a_p}{k_p}$ and $\hat{\theta}_3 = \frac{b_m - b_p}{k_p}$, define $e_o = y_p - y_m$, $\zeta_1 = \theta_1 - \hat{\theta}_1$, $\zeta_2 = \theta_2 - \hat{\theta}_2$, and $\zeta_3 = \theta_3 - \hat{\theta}_3$. Then, the reference model can be reformulated as

$$\dot{y}_m = a_p y_m(t) + b_p y_m(t - r) + k_p\big(\hat{\theta}_1 r + \hat{\theta}_2 y_m(t) + \hat{\theta}_3 y_m(t - r)\big).$$

The plant can also be reformulated as

$$\dot{y}_p = a_p y_p(t) + b_p y_p(t - r) + k_p\big(\theta_1 r(t) + \theta_2 y_p(t) + \theta_3 y_p(t - r)\big).$$

We can derive

$$\begin{aligned} \dot{e}_o &= a_p e_o(t) + b_p e_o(t - r) + k_p\big(\theta_1 - \hat{\theta}_1\big) r(t) \\ &\quad + k_p\big(\theta_2 y_p(t) - \hat{\theta}_2 y_m(t)\big) + k_p\big(\theta_3 y_p(t - r) - \hat{\theta}_3 y_m(t - r)\big) \\ &= a_p e_o(t) + b_p e_o(t - r) + k_p \zeta_1 r(t) + k_p \zeta_2 y_p(t) + k_p \zeta_3 y_p(t - r) \\ &= a_m e_o(t) + b_m e_o(t - r) + k_p \zeta_1 r(t) + k_p \zeta_2 y_p(t) + k_p \zeta_3 y_p(t - r). \end{aligned}$$

Then the whole closed-loop system can be rewritten as

$$\begin{aligned} \dot{e}_o &= a_m e_o(t) + b_m e_o(t - r) + k_p \zeta_1 r(t) + k_p \zeta_2 y_p(t) + k_p \zeta_3 y_p(t - r) \\ \dot{\zeta}_1 &= -\gamma e_o(t) r(t) \\ \dot{\zeta}_2 &= -\gamma e_o(t) y_p(t) \\ \dot{\zeta}_3 &= -\gamma e_o(t) y_p(t - r). \end{aligned}$$

Denote $x = [e_o, \zeta_1, \zeta_2, \zeta_3]^T \in \mathbb{R}^4$ and $\phi = [\phi_1, \phi_2, \phi_3, \phi_4]^T \in C_r^4$. Choose functional

$$V(x,\phi) = \frac{1}{2}e_o^2 + \frac{k_p}{\gamma}\zeta_1^2 + \frac{k_p}{\gamma}\zeta_2^2 + \frac{k_p}{\gamma}\zeta_3^2 + \frac{|b_m|}{2}\int_{t-r}^{t}\phi_1^2(s)ds.$$

Computing the derivative of V yields that

$$\begin{aligned}\dot{V}(\phi(0),\phi) &\leq a_m\phi_1^2(0) + b_m\phi_1(0)\phi_1(-r) + \frac{|b_m|}{2}\phi_1^2(0) - \frac{|b_m|}{2}\phi_1^2(-r) \\ &\leq \frac{a_m + |b_m|}{2}\phi_1^2(0).\end{aligned}$$

Thus, the closed-loop system is uniformly stable if $a_m + |b_m| < 0$ and can be rewritten as

$$\dot{x} = \begin{bmatrix} a_m & k_pB(t) \\ B^T(t)\gamma & 0 \end{bmatrix} x(t) + \begin{bmatrix} b_m & 0 \\ 0 & 0 \end{bmatrix} x(t-r)$$

where

$$B(t) = \begin{bmatrix} r(t) & y_p(t) & y_p(t-r) \end{bmatrix}.$$

Define $h(\phi) = \phi_1(0)$ and

$$f(t,\phi) = \begin{bmatrix} a_m & k_pB(t) \\ B^T(t)\gamma & 0 \end{bmatrix} \phi(0) + \begin{bmatrix} b_m & 0 \\ 0 & 0 \end{bmatrix} \phi(-r).$$

Suppose that $B(t)$ is uniformly bounded; then the functional $f(t,\phi)$ is uniformly continuous in ϕ, uniformly in t. Let $\bar{x}$ be a limit solution of the closed-loop system satisfying $h(\bar{x}_t) = 0$, $\forall t \geq 0$, then $e_o(t) = 0$ for all $t \geq 0$. Based on Proposition 4, for all $t \in [0,\infty)$, we have

$$\begin{gathered} 0 = \int_s^{s+\delta} k_pB(s+t_n)[\zeta_1(s), \zeta_2(s), \zeta_3(s)]^T ds, \\ \zeta_1(t) = \zeta_1(0), \quad \zeta_2(t) = \zeta_2(0), \quad \zeta_3(t) = \zeta_3(0). \end{gathered}$$

Assumption 1 reduces to the following implication:

$$\lim_{n\to\infty}\int_t^{t+\delta} k_pB(t+t_n)\zeta dt = 0, \forall t \geq t_n \implies \zeta = 0.$$

Therefore, a sufficient condition such that the origin of closed-loop system is uniformly asymptotically stable is concluded in the following condition which is often encountered in adaptive control [19]. Specifically, there exist $\delta > 0$ and $\epsilon^* > 0$ such that for each unit vector ζ, the following holds:

$$\lim_{n\to\infty}|\int_t^{t+\delta} B(t+t_n)\zeta dt| \geq \epsilon^*, \forall t \geq t_n.$$

In the simulation, we let the parameters be given as follows: $a_m = -2$, $b_m = -1$, $k_m = 1$, $a_p = 1$, $b_p = 2$, $k_p = 3$, $r = 1$ and $\gamma = 2.5$. The initial states are given as $y_p = y_m = \theta_1 = \theta_2 = \theta_3 = 0$. By using the above parameters, the trajectories of system y_p and the reference model y_m are shown in Figure 1, and the corresponding error is also given in Figure 1. Moreover, the evolutions of unknown parameters and estimated parameters are presented in Figure 2. The simulation results demonstrate that the considered system has desirable tracking performance. Furthermore, it verifies the effectiveness of the theoretical results developed in this paper.

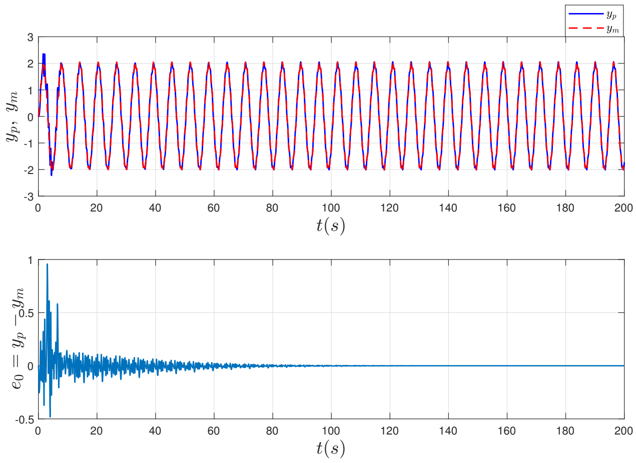

Figure 1. Tracking trajectory and tracking error e_0.

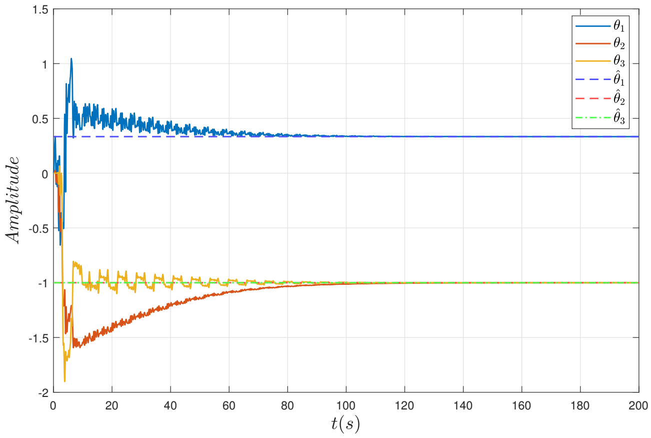

Figure 2. Evolutions of unknown parameters and estimated parameters.

6. Conclusions

In this paper, several new stability criteria for the UAS of time-varying delay differential inclusions are established. An infinite-dimensional version of the WZSD condition is proposed, and under certain integral conditions, the UAS of time-varying delay differential inclusions can be guaranteed. Furthermore, the limiting systems associated with the time-varying delay differential inclusions are introduced to verify the WZSD condition. Finally, a numerical tracking control example is presented to demonstrate the effectiveness of the proposed results.

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