

Article

The Concept of Soft Int-Modules: Structural Characterizations and Logistics Applications

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Abstract

Soft set theory provides useful mathematical tools for handling uncertainties. However, to address simultaneous multi-parameter constraints, a more dynamic intersectional approach is necessary. In this study, we introduce the concept of a soft intersectional module (soft int-module) and utilize level sets to characterize its underlying algebraic structure. Also, by defining soft int-submodules, we investigate whether they are closed under addition and multiplication operations with a soft int-ideal, and their behavior under module homomorphisms. Finally, using soft int-modules, a logistics-based mathematical model is presented to show how it solves simultaneous conditions in a supply chain network.

Keywords: soft int-ideal; soft int-module; soft int-submodule

MSC: 16D80; 13C13; 03E75

1. Introduction

The soft set concept, defined by Molodtsov [1] to model uncertainty and structures that depend on certain parameters, has been an important mathematical tool for years. Many studies have been conducted on operations in soft set theory [2–6]. In the light of Molodtsov's soft set definition, based on group, ring and module structures in classical algebra generalizations of soft groups [7–9], soft rings [10,11], soft semirings [5,12] and soft modules [13–15] have been proposed, and their properties have been widely studied. These generalizations offer new perspectives to handle uncertainty in abstract algebra.

Parallel to these studies, ref. [16] adopted a different approach to soft rings. In [16], by introducing the concept of soft intersectional ring concept, defined from R to $P(U)$, where R is a ring and U is a universal set, they examined soft int-ideals and their behavior under homomorphisms. In [17], the authors defined soft near-prime int-ideals and soft 1-absorbing prime int-ideals and examined their properties. They also examined the soft radical definition in these rings and their behaviors under homomorphism [17]. Also in [18,19], Ghosh et al. defined soft semiprimary int-ideals of ring and, by examining the soft prime and soft semiprime int-ideals of rings, they determined the basic properties and characterizations. All these studies present a strong generalization of classical ideals in a soft framework by soft int-ideals.

Soft module theory holds an important place as a natural extension of classical module theory. Existing approaches towards soft modules are based on Molodtsov's set-valued approach. That is, a soft set defined from a parameter set into the power set of a classical



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module is called a soft module if the image of each parameter is a submodule. Although important results have been obtained with this approach, it is quite similar to the classical structure, leading to the search for different foundations that can reveal deeper properties specific to soft algebraic structures. Despite the expanding literature on soft int-ideals in ring theory, their equivalent in soft module theory has not been developed yet. Although soft modules have been extensively studied, a module-theoretic framework directly based on the soft int-ideal approach has not yet been comprehensively investigated. To clearly demonstrate the mathematical necessity of the soft int-module concept we defined, it is necessary to examine the structural limitations of existing structures, such as fuzzy modules and soft modules. The main motivation for this study is the need for a structure that preserves the specific properties and structural identities of the elements by building a new bridge between the R module $\mathcal{M}$ and the $P(\mathcal{U})$ lattice. According to [20], a fuzzy module is not solely defined by the mapping $\mu : \mathcal{M} \rightarrow [0, 1]$; it satisfies the basic axioms $\mu(0_{\mathcal{M}}) = 1$, $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(rx) \geq \mu(x)$, for all $x, y \in \mathcal{M}, r \in R$. In this formulation, the image of the mapping represents the membership degrees of the algebraic elements. Furthermore, in the framework of soft set theory utilized in this paper, the terms parameters or constraints physically correspond to specific real-world operational conditions—such as delivery time windows, load capacities, or cost limits—that must be simultaneously satisfied in a logistics supply chain network. When elements with different properties are equated to the same numerical value, the properties and qualitative richness possessed by the algebraic elements are matched with a single number. To prevent this semantic information loss, there is an absolute need for a mathematically richer framework in which the image set consists not solely of scalars but rather of elements that preserve the specific properties and structural identities of the elements. As for the soft module, it is defined by the mapping $F : A \rightarrow P(\mathcal{M})$, where A is a set of parameters [15]. This approach assigns a subset of the module for each parameter. While this setup is certainly helpful for categorization, it falls short of directly modeling how parameters interact after the module's individual elements undergo algebraic operations (e.g., $x + y$). Because the mapping takes parameters as inputs rather than the elements themselves, it fails to build a dynamic link between the algebraic operations in $\mathcal{M}$ and the parameter sets. To minimize the issues we mentioned when defining soft int modules, we mapped the elements of the module to the elements of the power set. Thus, set-theoretic operations corresponded to the algebraic operations within the module. The additive behavior of the module is designed to mathematically govern how sets intersect, while the scalar product is designed to express the preservation and extension of subsets. In this way, by moving away from the scalar constraints of fuzzy modules and the parameter-focused approach of soft modules, we construct a different path for modeling that reveals mixed structural characterizations without incurring information loss.

In Section 3, we present an alternative perspective for soft module theory by adapting the properties of soft int-structures to the module environment. We carefully examine the soft int-module definition, soft int-submodules and their basic properties. More precisely, after defining the intersection, sum and product with soft int-ideals of soft int-submodules, we examine their behavior under module homomorphisms. Moreover, being inspired by the well-known relationship between level sets and soft submodules in the literature, we establish a similar approach between level sets and soft int-submodules. This provides a quite useful characterization of soft int-submodules in terms of level submodules.

Finally, in Section 4, as an application of soft int-modules, we model a logistics problem in the fast-moving consumer goods sector using a specialized soft int-module. Our soft int-module-based logistics model, which we propose to solve simultaneous constraints in supply chain networks, shows strong conceptual parallels with the operational dynamics

of tropical mathematics, a fundamental tool in optimization problems [21]. This structural similarity supports the interdisciplinary potential of the algebraic operations we have defined in applied logistics and their connection to mainstream optimization theories.

2. Preliminaries

Molodtsov [1] defined the soft set in the following manner: Let U be an initial universe set, E be a set of parameters, $\mathcal{P}(U)$ be the power set of U and $A \subseteq E$. A pair (F, A) is called a soft set over U , where F is a mapping given by $F : A \rightarrow \mathcal{P}(U)$.

Throughout this paper, R denotes a ring, $\mathcal{U}$ denotes an initial universe set and $\mathcal{P}(\mathcal{U})$ the power set of $\mathcal{U}$. A soft set over $\mathcal{U}$ is a mapping $\mathcal{F}_R : R \rightarrow \mathcal{P}(\mathcal{U})$. It is important to note that, unlike Molodtsov's standard definition where parameters are taken from a subset $A \subseteq E$, in this study, the soft set is defined directly over a ring R . Thus, the ring R intrinsically serves as the parameter set to adapt the soft structure to the algebraic framework. The set of all soft sets on ring R is denoted by $S(R, \mathcal{U})$. Let $\mathcal{F}_R, \mathcal{G}_R \in S(R, \mathcal{U})$. When $\mathcal{F}_R(r) \subseteq \mathcal{G}_R(r)$, for all $r \in R$, then $\mathcal{F}_R$ is called soft subset of $\mathcal{G}_R$ and denoted by $\mathcal{F}_R \subseteq \mathcal{G}_R$ [16].

Definition 1 ([16]). Let $\mathcal{F}_R \in S(R, \mathcal{U})$. Then, $\mathcal{F}_R$ is called a soft intersectional ring over $\mathcal{U}$ (briefly, soft int-ring) if $\mathcal{F}_R(x - y) \supseteq \mathcal{F}_R(x) \cap \mathcal{F}_R(y)$ and $\mathcal{F}_R(xy) \supseteq \mathcal{F}_R(x) \cap \mathcal{F}_R(y)$, for all $x, y \in R$.

The concept of soft int-ideals was originally introduced by Çitak et al. [16], who adopted the nomenclature “soft int-ideal over $\mathcal{U}$ ” to emphasize the role of the initial universe set. Subsequently, in [19], Ghosh and et al. [19] extensively investigated this structure under the name “soft int-ideal of R ”. Since both definitions are based on the same axiomatic conditions, we propose a unifying approach in our work by referring to it as the “soft int-ideal of R over $\mathcal{U}$.” This notation is also formally consistent with the notation used in fuzzy algebraic structures. We will use this notation throughout our work.

Definition 2 ([16]). Let $\mathcal{G}_R \in S(R, \mathcal{U})$. Then, $\mathcal{G}_R$ is called a soft int-ideal of R over $\mathcal{U}$, if it satisfies the following conditions, for all $x, y \in R$,

- (i) $\mathcal{G}_R(x - y) \supseteq \mathcal{G}_R(x) \cap \mathcal{G}_R(y)$,
- (ii) $\mathcal{G}_R(xy) \supseteq \mathcal{G}_R(x) \cup \mathcal{G}_R(y)$

Throughout our work, we denote the set of soft int-ideals of R over $\mathcal{U}$ by $SI_{int}(R, \mathcal{U})$.

Definition 3 ([16]). Let $\mathcal{G}_R \in S(R, \mathcal{U})$ and $\alpha \subseteq \mathcal{U}$. The set

$$\mathcal{G}_R^\alpha = \{x \in R \mid \mathcal{G}_R(x) \supseteq \alpha\}$$

is called the α -level set of $\mathcal{G}_R$.

Theorem 1 ([19]). Let $\mathcal{G}_R \in S(R, \mathcal{U})$. Then, $\mathcal{G}_R$ is a soft int-ideal if and only if for every $A \subseteq \mathcal{U}$ with $A \subseteq \mathcal{G}_R(0_R)$, (where 0_R is the zero element of R), the set $\mathcal{G}_R^A$ is an ideal of R .

The above theorem shows that the concept of soft int-ideal is completely characterized by its level sets. Consequently, the theory of soft int-ideals can be regarded as a natural extension of the classical ideal theory.

Definition 4 ([16]). Let $\mathcal{G}_R, \mathcal{H}_R \in SI_{int}(R, \mathcal{U})$.

- (i) $\mathcal{G}_R \widetilde{\cap} \mathcal{H}_R$ is defined by $(\mathcal{G}_R \widetilde{\cap} \mathcal{H}_R)(r) = \mathcal{G}_R(r) \cap \mathcal{H}_R(r)$, for all $r \in R$.

(ii) The sum $\mathcal{G}_R + \mathcal{H}_R$ is defined by, for all $r \in R$,

$$(\mathcal{G}_R + \mathcal{H}_R)(r) = \bigcup_{r=a+b} (\mathcal{G}_R(a) \cap \mathcal{H}_R(b)).$$

(iii) The product $\mathcal{G}_R \cdot \mathcal{H}_R$ is defined by, for all $r \in R$,

$$(\mathcal{G}_R \cdot \mathcal{H}_R)(r) = \bigcup_{r=a \cdot b} (\mathcal{G}_R(a) \cap \mathcal{H}_R(b)).$$

If the element r cannot be written as the product of a and b , then it is defined as $(\mathcal{G}_R \cdot \mathcal{H}_R)(r) = \emptyset$.

Theorem 2 ([16]). Let $f : R \rightarrow S$ be a ring homomorphism and $\mathcal{G}_R$ be a soft int-ideal of R over $\mathcal{U}$ and $\mathcal{H}_S$ be a soft int-ideal of S over $\mathcal{U}$.

(i) If f is an epimorphism, then defined a soft set $f(\mathcal{G}_R)$ by, for $s \in S$,

$$f(\mathcal{G}_R)(s) = \bigcup_{s=f(r)} \mathcal{G}_R(r)$$

$f(\mathcal{G}_R)$ is a soft int-ideal of S over $\mathcal{U}$.

(ii) Defined a soft set $f^{-1}(\mathcal{H}_S)$ defined by, for all $r \in R$, $f^{-1}(\mathcal{H}_S)(r) = \mathcal{H}_S(f(r))$ and $f^{-1}(\mathcal{H}_S)$ is a soft int-ideal of R over $\mathcal{U}$.

3. Results

3.1. Intersectional Soft R -Modules

In this section, we adopt a new approach to soft modules and define soft intersecting modules based on the soft int-ring definition, examining their fundamental properties. Within the framework of soft set theory, the concept of a soft module holds an important place among widely studied algebraic structures. The basic structure of a soft module was first introduced by Sun et al. [15]. Let R be a ring, M an R -module and A a non-empty set of parameters. If $F(x)$ is a submodule of M for all $x \in A$, then $F : A \rightarrow P(M)$ is called a soft module over M [15]. In addition, while ref. [6] restricts a soft submodule (F, N) by mapping a known submodule N to $P(M)$, we map the entire module M directly to $P(\mathcal{U})$ for an arbitrary universe $\mathcal{U}$. Eliminating the requirement for a fundamental submodule and taking the universal set $\mathcal{U}$ instead of M naturally yields a more generalized soft algebraic structure.

Throughout this section, R is considered as a commutative ring with identity, $\mathcal{M}$ is a left R -module with the zero element of $0_{\mathcal{M}}$ and $\mathcal{U}$ the universal set. The set of all soft sets on R -module $\mathcal{M}$ is denoted by $S(\mathcal{M}, \mathcal{U})$.

Definition 5. Let $\mathcal{M}$ be an R -module and $\theta \in S(\mathcal{M}, \mathcal{U})$. Then, θ is called an intersectional soft R -module (briefly, soft int- R -module) over $\mathcal{U}$ if the following conditions are satisfied:

$$SM1. \quad \theta(0_{\mathcal{M}}) = \mathcal{U};$$

$$SM2. \quad \theta(m_1 + m_2) \supseteq \theta(m_1) \cap \theta(m_2), \text{ for all } m_1, m_2 \in \mathcal{M};$$

$$SM3. \quad \theta(rm) \supseteq \theta(m), \text{ for all } m \in \mathcal{M} \text{ and } r \in R.$$

The set of all intersectional soft R -modules over $\mathcal{U}$ is denoted by $S_{int}(R, \mathcal{M}, \mathcal{U})$. To simplify the notation, we will write intSRM instead of soft int- R -module. That is, $S_{int}(R, \mathcal{M}, \mathcal{U}) = \{\theta : \mathcal{M} \rightarrow P(\mathcal{U}) \mid \theta \text{ is an intSRM}\}$. In addition, if $\theta \in S_{int}(R, \mathcal{M}, \mathcal{U})$, then by the definition of intSRM, $\theta(-m) = \theta((-1_R) \cdot m) \supseteq \theta(m)$ and $\theta(m) = \theta((-1_R) \cdot (-m)) \supseteq \theta(-m)$, for all $m \in \mathcal{M}$. Thus, $\theta(-m) = \theta(m)$, for all $m \in \mathcal{M}$.

Example 1. Let $\mathcal{M} = \mathbb{Z}_4$ be the $\mathbb{Z}$ -module, $\mathcal{U} = \{x, y, z, t, w\}$ and $\theta : \mathcal{M} \rightarrow P(\mathcal{U})$ defined by, for all $m \in \mathcal{M}$, $\theta(m) = \begin{cases} \mathcal{U} & \text{if } m = \bar{0} \\ \{x, z\}, & \text{if } m \neq \bar{0} \end{cases}$.

One can easily show that θ is an intSRM. A demonstration of the satisfaction of the intSRM conditions, namely SM2 and SM3, can be readily substantiated. To facilitate a more profound comprehension of the subject, it is instructive to consider the specific case of $m_1 = \bar{1}, m_2 = \bar{3}, r = 6$.

- $\theta(\bar{1} + \bar{3}) = \mathcal{U} \supseteq \theta(m_1) \cap \theta(m_2) = \{x, z\}$
- $\theta(6.m) = \begin{cases} \mathcal{U} & \text{if } m = \bar{0}, \bar{2} \\ \{x, z\}, & \text{if } m = \bar{1}, \bar{3} \end{cases}$.

Thus, $\theta(r.m) \supseteq \theta(m)$, for all $r \in R, m \in \mathcal{M}$.

The concept of level sets is of great importance for a full understanding of the algebraic behavior of soft int-modules. Analyzing the behavior of level sets provides a powerful tool for systematically characterizing soft algebraic structures.

Definition 6. Let $\theta \in S_{int}(R\mathcal{M}, \mathcal{U})$ and arbitrary subset $\alpha \in P(\mathcal{U})$.

$$\theta^\alpha = \{m \in \mathcal{M} : \theta(m) \supseteq \alpha\}$$

is called α -level set of θ .

The following proposition establishes a connection between classical modules and soft int- R -modules by characterizing soft int- R -modules via their level sets, thereby providing a foundation for further study in this area.

Proposition 1. Let $\theta : \mathcal{M} \rightarrow P(\mathcal{U})$ be a soft set such that $\theta(0_{\mathcal{M}}) = \mathcal{U}$. Then, θ is an intSRM if and only if nonempty set θ^α is a submodule of $\mathcal{M}$, for all $\alpha \in P(\mathcal{U})$.

Proof. Let $\theta \in S_{int}(R\mathcal{M}, \mathcal{U})$ and α be an arbitrary subset of $\mathcal{U}$. Let us show that θ^α is a submodule of $\mathcal{M}$.

$0_{\mathcal{M}} \in \theta^\alpha$ since $\theta(0_{\mathcal{M}}) = \mathcal{U} \supseteq \alpha$, so $\theta^\alpha \neq \emptyset$, for all $\alpha \in P(\mathcal{U})$.

For all $m_1, m_2 \in \theta^\alpha$, $\theta(m_1 + m_2) \supseteq \theta(m_1) \cap \theta(m_2) \supseteq \alpha$, thus $m_1 + m_2 \in \theta^\alpha$.

For all $r \in R, m \in \theta^\alpha$, $\theta(r.m) \supseteq \theta(m) \supseteq \alpha$, thus $rm \in \theta^\alpha$. Therefore, θ^α is a submodule of $\mathcal{M}$.

Conversely, θ^α be a submodule of $\mathcal{M}$ for all $\alpha \in P(\mathcal{U})$. By hypothesis, $\theta(0_{\mathcal{M}}) = \mathcal{U}$. Let $m_i \in \theta^{\theta(m_i)}$, for all $m_i \in \mathcal{M} (i = 1, 2)$ and since $\theta^{\theta(m_i)} \subseteq \theta^{\theta(m_1) \cap \theta(m_2)}$ thus $m_1, m_2 \in \theta^{\theta(m_1) \cap \theta(m_2)}$ and by the definition of submodule, $m_1 + m_2 \in \theta^{\theta(m_1) \cap \theta(m_2)}$ and $r.m \in \theta^{\theta(m)}$, for all $r \in R, m \in \mathcal{M}$. Hence, $\theta(m_1 + m_2) \supseteq \theta(m_1) \cap \theta(m_2)$ and $\theta(r.m) \supseteq \theta(m)$. Thus, $\theta \in S_{int}(R\mathcal{M}, \mathcal{U})$. $\square$

Definition 7. Let $\mathcal{M}$ be an R -module, $\mathcal{N}$ a submodule of $\mathcal{M}$, $\theta \in S_{int}(R\mathcal{M}, \mathcal{U})$, $\beta_{\mathcal{N}} : \mathcal{M} \rightarrow P(\mathcal{U})$ and $m \in \mathcal{M} - \mathcal{N}$ such that $\beta_{\mathcal{N}}(m) = \emptyset$. If the following conditions are satisfied, then $\beta_{\mathcal{N}}$ is called a soft int R -submodule of θ

- (i) $\beta_{\mathcal{N}} \subseteq \theta$,
- (ii) $\beta_{\mathcal{N}} \in S_{int}(R\mathcal{M}, \mathcal{U})$

It is denoted by $\beta_{\mathcal{N}} \leq_{s-int} \theta$. The set of all soft int R-submodules of θ will be denoted by $SS_{int}(\theta)$. That is, $SS_{int}(\theta) = \{\beta_{\mathcal{N}} | \beta_{\mathcal{N}} \leq_{s-int} \theta\}$.

Example 2. Let $\mathcal{M}$ be $\mathbb{Z}_{12}$ $\mathbb{Z}$ -module, $N = \langle \bar{3} \rangle = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$ be a submodule of $\mathcal{M}$ and $\mathcal{U} = \{x, y, z, t\}$. Let the soft set θ be defined by

$$\theta(m) = \begin{cases} \mathcal{U}, & \text{if } m \in \{\bar{0}, \bar{6}\} \\ \{x, y, z\}, & \text{if } m \in \{\bar{2}, \bar{4}, \bar{8}, \bar{10}\} \\ \{x, y\}, & \text{if } m \in \{\bar{1}, \bar{3}, \bar{5}, \bar{7}, \bar{9}, \bar{11}\} \end{cases}$$

It is straightforward to see that θ is a soft int-R-module.

Now, let us introduce a soft set $\beta_{\mathcal{N}}: \mathbb{Z}_{12} \rightarrow P(\mathcal{U})$ defined by

$$\beta_{\mathcal{N}}(m) = \begin{cases} \mathcal{U}, & \text{if } m = \bar{0} \\ \{x\}, & \text{if } m \in \{\bar{3}, \bar{6}, \bar{9}\} \end{cases}$$

From the definition, if $m \in \mathbb{Z}_{12} - N$, $\beta_{\mathcal{N}}(m) = \emptyset$. We can verify that $\beta_{\mathcal{N}}$ is a soft int-submodule of θ by observing the following conditions: Firstly, it is clear that $\beta_{\mathcal{N}}(m) \subseteq \theta(m)$ for all $m \in \mathbb{Z}_{12}$, which implies $\beta_{\mathcal{N}} \subseteq \theta$ and for the zero element of $\mathbb{Z}_{12}$, we have $\beta_{\mathcal{N}}(\bar{0}) = \mathcal{U}$. Finally, we examine the nonempty level sets $\beta_{\mathcal{N}}^A = \{m \in \mathbb{Z}_{12} | \beta_{\mathcal{N}}(m) \supseteq A\}$ for any subset $A \in P(\mathcal{U})$: For $A = \{x\}$,

$$\beta_{\mathcal{N}}^A = \{m \in \mathbb{Z}_{12} | \beta_{\mathcal{N}}(m) \supseteq \{x\}\} = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\} = \langle \bar{3} \rangle \leq \mathbb{Z}_{12}$$

For $A \neq \{x\}$,

$$\beta_{\mathcal{N}}^A = \{m \in \mathbb{Z}_{12} | \beta_{\mathcal{N}}(m) \supseteq A\} = \{\bar{0}\} \leq \mathbb{Z}_{12}$$

Since $\beta_{\mathcal{N}}^A$ is a submodule of $\mathbb{Z}_{12}$ for all $A \in P(\mathcal{U})$, thus $\beta_{\mathcal{N}} \leq_{s-int} \theta$.

3.2. Algebraic Operations of Soft Int-Submodules

To develop the theory of soft int-modules, it is natural to investigate how they behave under algebraic operations. Below, we define the intersection and sum of soft int-submodules, as well as the product and generalized product of a soft int-module and a soft int-ring. We will then show that the class of soft int-modules is closed under these operations and preserves its structural integrity.

Definition 8. Let $\mathcal{M}$ be an R-module, $\mathcal{N}$ and $\mathcal{K}$ two submodules of $\mathcal{M}$, $\theta \in S_{int}(R\mathcal{M}, \mathcal{U})$, $\beta_{\mathcal{N}}, \beta_{\mathcal{K}} \in SS_{int}(\theta)$ and f_R be a soft int-ring.

(i) The intersection of $\beta_{\mathcal{N}}$ and $\beta_{\mathcal{K}}$, denoted by $\beta_{\mathcal{N}} \tilde{\cap} \beta_{\mathcal{K}}$, is defined as, for all $m \in \mathcal{M}$

$$(\beta_{\mathcal{N}} \tilde{\cap} \beta_{\mathcal{K}})(m) = \beta_{\mathcal{N}}(m) \cap \beta_{\mathcal{K}}(m)$$

By the definition, when $m \notin \mathcal{N} \cap \mathcal{K}$ then $(\beta_{\mathcal{N}} \tilde{\cap} \beta_{\mathcal{K}})(m) = \emptyset$.

(ii) The sum of $\beta_{\mathcal{N}}$ and $\beta_{\mathcal{K}}$, denoted by $\beta_{\mathcal{N}} + \beta_{\mathcal{K}}: \mathcal{M} \rightarrow P(\mathcal{U})$, is defined by, for all $m \in \mathcal{M}$

$$(\beta_{\mathcal{N}} + \beta_{\mathcal{K}})(m) = \bigcup_{m=a+b} \{\beta_{\mathcal{N}}(a) \cap \beta_{\mathcal{K}}(b)\}$$

It is clear that, when $m \notin \mathcal{N} + \mathcal{K}$, $(\beta_{\mathcal{N}} + \beta_{\mathcal{K}})(m) = \emptyset$.

(iii) The product of f_R and β_N , denoted by $(f_R \circ \beta_N)$, defined by

$$(f_R \circ \beta_N)(m) = \bigcup_{m=r \cdot a} \{f_R(r) \cap \beta_N(a)\}$$

If the element m cannot be written as the product of $r \in R$ and $a \in N$, then it is defined as $(f_R \circ \beta_N)(m) = \emptyset$.

(iv) The generalized product of soft int-ring f_R and soft int-submodule β_N , denoted by $f_R \cdot \beta_N$, defined by, for all $m \in M$,

$$(f_R \cdot \beta_N)(m) = \bigcup_{m=\sum_{i=1}^n r_i \cdot a_i} \left\{ \bigcap_{i=1}^n (f_R(r_i) \cap \beta_N(a_i)) \right\}$$

where $r_i \in R$, $a_i \in M$.

Remark 1. It is clear that $(f_R \circ \beta_N) \neq (f_R \cdot \beta_N)$, and from this we can observe that the generalized product contains the product.

Proposition 2. Let $\theta \in S_{int}(R, M, \mathcal{U})$, $\beta_N, \beta_K \in SS_{int}(\theta)$ and f_R be a soft int-ring. Then, the following conditions holds:

- (i) $\beta_N + \beta_K \supseteq \beta_N$ and $\beta_N + \beta_K \supseteq \beta_K$
- (ii) $f_R \circ (\beta_N + \beta_K) \subseteq f_R \circ \beta_N + f_R \circ \beta_K$.

Proof.

- (i) For all $m \in M$, $m = m + 0_M$, then $\beta_N(m) = \beta_N(m) \cap \beta_K(0_M) \subseteq (\beta_N + \beta_K)(m)$. Thus, $\beta_N + \beta_K \supseteq \beta_N$. Similarly, we can show that $\beta_N + \beta_K \supseteq \beta_K$.
- (ii) Let $m \in M$. By the definition of the product of a soft int-ring and a soft int-module, we have:

$$\begin{aligned} (f_R \circ (\beta_N + \beta_K))(m) &= \bigcup_{m=r \cdot w} (f_R(r) \cap (\beta_N + \beta_K)(w)) \\ &= \bigcup_{m=r \cdot w} \left(f_R(r) \cap \bigcup_{w=n+k} (\beta_N(n) \cap \beta_K(k)) \right) \\ &= \bigcup_{m=r \cdot (n+k)} (f_R(r) \cap \beta_N(n) \cap \beta_K(k)) \\ &= \bigcup_{m=r \cdot n + r \cdot k} ((f_R(r) \cap \beta_N(n)) \cap (f_R(r) \cap \beta_K(k))) \end{aligned}$$

Now, let $a = r \cdot n$ and $b = r \cdot k$. If we take the union over all arbitrary elements $a, b \in M$ such that $m = a + b$, the resulting union will contain our specific case. Thus,

$$\begin{aligned} &\subseteq \bigcup_{m=a+b} \left(\bigcup_{a=r_1 \cdot n_1} (f_R(r_1) \cap \beta_N(n_1)) \cap \bigcup_{b=r_2 \cdot k_2} (f_R(r_2) \cap \beta_K(k_2)) \right) \\ &= \bigcup_{m=a+b} ((f_R \circ \beta_N)(a) \cap (f_R \circ \beta_K)(b)) \\ &= (f_R \circ \beta_N + f_R \circ \beta_K)(m) \end{aligned}$$

Therefore, for all $m \in M$, we conclude that $f_R \circ (\beta_N + \beta_K) \subseteq f_R \circ \beta_N + f_R \circ \beta_K$.

□

Below, we present a counterexample showing the reverse inclusion $f_R \circ \beta_N + f_R \circ \beta_K \subseteq f_R \circ (\beta_N + \beta_K)$ does not hold in general.

Example 3. Let $R = \mathbb{Z}[y, t]$ be the polynomial ring, $\mathcal{M} = R \oplus R$ be a free R -module and $\mathcal{N}, \mathcal{K}$ are submodules of $\mathcal{M}$. Let the universal set be $\mathcal{U} = \{u\}$.

We define a soft int-ring $f_R: R \rightarrow P(\mathcal{U})$ based on the ideal I generated by y and t ,

$$f_R(p) = \begin{cases} \{u\}, & \text{if } p \in I \\ \emptyset, & \text{if } p \notin I \end{cases}$$

and we define two soft int-submodules $\beta_N, \beta_K: \mathcal{M} \rightarrow P(\mathcal{U})$ as follows:

$$\beta_N(a, b) = \begin{cases} \{u\}, & \text{if } b = 0 \\ \emptyset, & \text{if } b \neq 0, \end{cases} \quad \text{and} \quad \beta_K(a, b) = \begin{cases} \{u\}, & \text{if } a = 0 \\ \emptyset, & \text{if } a \neq 0. \end{cases}$$

Let $m = (y, t) \in \mathcal{M}$, we can express $(y, t) = (y, 0) + (0, t) = y \cdot (1, 0) + t \cdot (0, 1)$. Therefore,

$$(f_R \circ \beta_N + f_R \circ \beta_K)(y, t) = \{u\}.$$

On the otherside, by definition, $(f_R \circ (\beta_N + \beta_K))(y, t) = \bigcup_{(y,t)=r \cdot (z_1, z_2)} (f_R(r) \cap (\beta_N + \beta_K)(z_1, z_2)) = \emptyset$, since for the $(y, t) = (r \cdot z_1, r \cdot z_2) \in \mathbb{Z}[y, t]$, the scalar $r \in R$ must divide both y and t . Then, $r = 1$ and $r = -1$. However, $1, -1 \notin I$, therefore $f_R(1) = \emptyset$ and $f_R(-1) = \emptyset$. Thus, this clearly demonstrates that $f_R \circ \beta_N + f_R \circ \beta_K \not\subseteq f_R \circ (\beta_N + \beta_K)$.

As demonstrated by the preceding counterexample, the single-term product of a soft int-ring and a soft int-module only satisfies a one-sided distributive property. In the next proposition, we will prove that the distributive equality property given above holds for the generalized product.

Proposition 3. Let f_R, g_R be soft int-rings with $f_R(0) = \mathcal{U}$ and $\theta \in S_{int}(R\mathcal{M}, \mathcal{U})$, $\beta_N, \beta_K \in SS_{int}(\theta)$; then, the following distributive properties hold:

- (i) $f_R \cdot (\beta_N + \beta_K) = f_R \cdot \beta_N + f_R \cdot \beta_K$
- (ii) $(f_R + g_R) \cdot \beta_N = f_R \cdot \beta_N + g_R \cdot \beta_N$

Proof.

- (i) We first show that $f_R \cdot (\beta_N + \beta_K) \subseteq f_R \cdot \beta_N + f_R \cdot \beta_K$. Let $m \in \mathcal{M}$. By the definition of the generalized product, $(f_R \cdot (\beta_N + \beta_K))(m) = *$

$$\begin{aligned} * &= \bigcup_{m=\sum_{i=1}^n r_i a_i} \left\{ \bigcap_{i=1}^n (f_R(r_i) \cap (\beta_N + \beta_K)(a_i)) \right\} \\ &= \bigcup_{\substack{m=\sum_{i=1}^n r_i (y_i + z_i) \\ a_i = y_i + z_i}} \left\{ \bigcap_{i=1}^n (f_R(r_i) \cap \beta_N(y_i) \cap \beta_K(z_i)) \right\} \\ &\subseteq \bigcup_m \left\{ \left(\bigcap_{i=1}^n (f_R(r_i) \cap \beta_N(y_i)) \right) \cap \left(\bigcap_{i=1}^n (f_R(r_i) \cap \beta_K(z_i)) \right) \right\} \end{aligned}$$

where $m = \underbrace{\sum_{i=1}^n r_i y_i}_u + \underbrace{\sum_{i=1}^n r_i z_i}_v$. This resulting union is exactly a subset of the definition of $(f_R \cdot \beta_N)(u) \cap (f_R \cdot \beta_K)(v)$ over all possible $m = u + v$. Thus,

$$(f_R \cdot (\beta_N + \beta_K))(m) \subseteq (f_R \cdot \beta_N + f_R \cdot \beta_K)(m).$$

Conversely, we show that

$$(f_R \cdot \beta_N + f_R \cdot \beta_K)(m) \subseteq (f_R \cdot (\beta_N + \beta_K))(m).$$

By Definition 8,

$$(f_R \cdot \beta_N + f_R \cdot \beta_K)(m) = \bigcup_{m=u+v} \{(f_R \cdot \beta_N)(u) \cap (f_R \cdot \beta_K)(v)\}$$

Let $u = \sum_{j=1}^k r'_j y_j$ and $v = \sum_{l=1}^p r''_l z_l$. We expand the product definitions for specific representations of u and v . Then,

$$\begin{aligned} & (f_R \cdot \beta_N)(u) \cap (f_R \cdot \beta_K)(v) \\ &= \bigcup_{\substack{u=\sum_{j=1}^k r'_j y_j \\ v=\sum_{l=1}^p r''_l z_l}} \left\{ \left(\bigcap_{j=1}^k (f_R(r'_j) \cap \beta_N(y_j)) \right) \cap \left(\bigcap_{l=1}^p (f_R(r''_l) \cap \beta_K(z_l)) \right) \right\} \end{aligned}$$

Now, consider the sum $m = u + v = \sum_{j=1}^k r'_j y_j + \sum_{l=1}^p r''_l z_l$. This is a single finite sum with $k + p$ terms. We know that $y_j = y_j + 0_M$. We also know $\beta_K(0_M) = U$. We obtain

$$(\beta_N + \beta_K)(y_j) \supseteq \beta_N(y_j) \cap \beta_K(0_M) = \beta_N(y_j) \cap U = \beta_N(y_j)$$

Similarly, we write $z_l = 0_M + z_l$. Since $\beta_N(0_M) = U$, we obtain

$$(\beta_N + \beta_K)(z_l) \supseteq \beta_N(0_M) \cap \beta_K(z_l) = U \cap \beta_K(z_l) = \beta_K(z_l)$$

We substitute these two subset relations back into our intersection expression,

$$\begin{aligned} & \left(\bigcap_{j=1}^k (f_R(r'_j) \cap \beta_N(y_j)) \right) \cap \left(\bigcap_{l=1}^p (f_R(r''_l) \cap \beta_K(z_l)) \right) \\ & \subseteq \left(\bigcap_{j=1}^k (f_R(r'_j) \cap (\beta_N + \beta_K)(y_j)) \right) \cap \left(\bigcap_{l=1}^p (f_R(r''_l) \cap (\beta_N + \beta_K)(z_l)) \right) \end{aligned}$$

This resulting expression is just one specific intersection term for the element m . The definition of $(f_R \cdot (\beta_N + \beta_K))(m)$ takes the union over all possible finite sum representations. Thus, it naturally contains this specific $(k + p)$ term intersection. Taking the union over all u and v proves the inclusion

$$(f_R \cdot \beta_N + f_R \cdot \beta_K)(m) \subseteq (f_R \cdot (\beta_N + \beta_K))(m)$$

- (ii) We show that $(f_R + g_R) \cdot \beta_N = f_R \cdot \beta_N + g_R \cdot \beta_N$. Firstly, we establish that $((f_R + g_R) \cdot \beta_N)(m) \subseteq (f_R \cdot \beta_N + g_R \cdot \beta_N)(m)$, for all $m \in \mathcal{M}$. Let $m \in \mathcal{M}$. By Definition 8,

$$\begin{aligned} ((f_R + g_R) \cdot \beta_N)(m) &= \bigcup_{m = \sum_{i=1}^n r_i a_i} \left\{ \bigcap_{i=1}^n ((f_R + g_R)(r_i) \cap \beta_N(a_i)) \right\} \\ &= \bigcup_{m = \sum_{i=1}^n (s_i + t_i) a_i} \left\{ \bigcap_{i=1}^n \left(\bigcup_{r_i = s_i + t_i} (f_R(s_i) \cap g_R(t_i)) \cap \beta_N(a_i) \right) \right\} \end{aligned}$$

where $(f_R + g_R)(r_i) = \bigcup_{r_i = s_i + t_i} (f_R(s_i) \cap g_R(t_i))$.

$$\begin{aligned} &= \bigcup_{\substack{m = \sum_{i=1}^n s_i a_i + \sum_{i=1}^n t_i a_i \\ r_i = s_i + t_i}} \left\{ \bigcap_{i=1}^n (f_R(s_i) \cap \beta_N(a_i) \cap g_R(t_i) \cap \beta_N(a_i)) \right\} \\ &\subseteq \bigcup_{m = u + v} \left\{ \left(\bigcap_{i=1}^n (f_R(s_i) \cap \beta_N(a_i)) \right) \cap \left(\bigcap_{i=1}^n (g_R(t_i) \cap \beta_N(a_i)) \right) \right\} \end{aligned}$$

(By setting $u = \sum_{i=1}^n s_i a_i$ and $v = \sum_{i=1}^n t_i a_i$, we get $m = u + v$.) Thus, $((f_R + g_R) \cdot \beta_N)(m) \subseteq (f_R \cdot \beta_N + g_R \cdot \beta_N)(m)$.

Conversely, we show

$$(f_R \cdot \beta_N + g_R \cdot \beta_N) \subseteq ((f_R + g_R) \cdot \beta_N).$$

Let $m \in \mathcal{M}$ and by Definition 8

$$(f_R \cdot \beta_N + g_R \cdot \beta_N)(m) = \bigcup_{m = u + v} \{ (f_R \cdot \beta_N)(u) \cap (g_R \cdot \beta_N)(v) \}$$

Let $u = \sum_{j=1}^k s_j y_j$ and $v = \sum_{l=1}^p t_l z_l$. For $n = k + p$ let s_i be as given, $t_i = 0$, and $a_i = y_i$. For $1 \leq i \leq k$, $r_i = s_i + 0$ and for $k + 1 \leq i \leq n$, $r_i = 0 + t_i$.

$$\bigcap_{i=1}^n ((f_R(s_i) \cap g_R(t_i)) \cap \beta_N(a_i)) \subseteq ((f_R + g_R) \cdot \beta_N)(m)$$

This proves the equality in (ii).

□

Proposition 4. Let $\theta \in S_{int}(R\mathcal{M}, \mathcal{U})$, $\beta_N, \beta_K \in SS_{int}(\theta)$ and f_R be a soft int-ring such that $f_R(0) = \mathcal{U}$. Then, $\beta_N \tilde{\cap} \beta_K$, $\beta_N + \beta_K$, $f_R \circ \beta_N$ and $f_R \cdot \beta_N$ are soft int R -submodules of θ .

Proof. For all $x \in \mathcal{M}$, we have $(\beta_N \tilde{\cap} \beta_K)(x) = \beta_N(x) \cap \beta_K(x) \subseteq \theta(x)$, and $(\beta_N \tilde{\cap} \beta_K)(0_M) = \beta_N(0_M) \cap \beta_K(0_M) = \mathcal{U}$. Thus, for all $x, y \in M$ and $r \in R$

$$\begin{aligned} (\beta_N \tilde{\cap} \beta_K)(x + y) &= \beta_N(x + y) \cap \beta_K(x + y) \\ &\supseteq (\beta_N(x) \cap \beta_N(y)) \cap (\beta_K(x) \cap \beta_K(y)) \\ &= \beta_N(x) \cap \beta_K(x) \cap \beta_N(y) \cap \beta_K(y) \\ &= (\beta_N \tilde{\cap} \beta_K)(x) \cap (\beta_N \tilde{\cap} \beta_K)(y). \end{aligned}$$

and

$$\begin{aligned}(\beta_{\mathcal{N}} \widetilde{\cap} \beta_{\mathcal{K}})(r \cdot x) &= \beta_{\mathcal{N}}(r \cdot x) \cap \beta_{\mathcal{K}}(r \cdot x) \\ &\supseteq \beta_{\mathcal{N}}(x) \cap \beta_{\mathcal{K}}(x) \\ &= (\beta_{\mathcal{N}} \widetilde{\cap} \beta_{\mathcal{K}})(x).\end{aligned}$$

Hence, $\beta_{\mathcal{N}} \widetilde{\cap} \beta_{\mathcal{K}}$ is a soft int-submodule of θ .

Now, we show that $\beta_{\mathcal{N}} + \beta_{\mathcal{K}} \in SS_{int}(\theta)$.

$$\begin{aligned}(\beta_{\mathcal{N}} + \beta_{\mathcal{K}})(m) &= \bigcup_{m=a+b} (\beta_{\mathcal{N}}(a) \cap \beta_{\mathcal{K}}(b)) \\ &\subseteq \bigcup_{m=a+b} (\theta(a) \cap \theta(b)) \\ &\subseteq \bigcup_{m=a+b} \theta(a+b) \subseteq \theta(m).\end{aligned}$$

Thus, $\beta_{\mathcal{N}} + \beta_{\mathcal{K}} \subseteq \theta$ and

Now for all $r \in R$ and $m, m_1, m_2 \in \mathcal{M}$, let

$$(\beta_{\mathcal{N}} + \beta_{\mathcal{K}})(m_1) \cap (\beta_{\mathcal{N}} + \beta_{\mathcal{K}})(m_2) = *$$

$$\begin{aligned}*&= \left(\bigcup_{m_1=n_1+k_1} \{\beta_{\mathcal{N}}(n_1) \cap \beta_{\mathcal{K}}(k_1)\} \right) \cap \left(\bigcup_{m_2=n_2+k_2} \{\beta_{\mathcal{N}}(n_2) \cap \beta_{\mathcal{K}}(k_2)\} \right) \\ &= \bigcup_{\substack{m_1=n_1+k_1 \\ m_2=n_2+k_2}} \{\beta_{\mathcal{N}}(n_1) \cap \beta_{\mathcal{N}}(n_2) \cap \beta_{\mathcal{K}}(k_1) \cap \beta_{\mathcal{K}}(k_2)\} \\ &\subseteq \bigcup_{m_1+m_2=(n_1+n_2)+(k_1+k_2)} \{\beta_{\mathcal{N}}(n_1+n_2) \cap \beta_{\mathcal{K}}(k_1+k_2)\} \\ &\subseteq \bigcup_{m_1+m_2=u+v} \{(\beta_{\mathcal{N}}(u) \cap \beta_{\mathcal{K}}(v))\} = (\beta_{\mathcal{N}} + \beta_{\mathcal{K}})(m_1+m_2).\end{aligned}$$

$$\begin{aligned}(\beta_{\mathcal{N}} + \beta_{\mathcal{K}})(m) &= \bigcup_{m=a+b} (\beta_{\mathcal{N}}(a) \cap \beta_{\mathcal{K}}(b)) \\ &\subseteq \bigcup_{r \cdot a + r \cdot b = r \cdot (a+b) = r \cdot m} \{\beta_{\mathcal{N}}(r \cdot a) \cap \beta_{\mathcal{K}}(r \cdot b)\} \\ &\subseteq (\beta_{\mathcal{N}} + \beta_{\mathcal{K}})(r \cdot m).\end{aligned}$$

Thus, $\beta_{\mathcal{N}} + \beta_{\mathcal{K}} \in SS_{int}(\theta)$.

Finally, we will show that $f_R \circ \beta_{\mathcal{N}} \in SS_{int}(\theta)$. For all $n, m \in \mathcal{M}$ and $r \in R$, we have:

$$\begin{aligned}(f_R \circ \beta_{\mathcal{N}})(m) &= \bigcup_{m=r \cdot n} (f_R(r) \cap \beta_{\mathcal{N}}(n)) \\ &\subseteq \bigcup_{m=r \cdot n} (f_R(r) \cap \theta(n)) \\ &\subseteq \bigcup_{m=r \cdot n} \theta(r \cdot n) = \theta(m).\end{aligned}$$

For all $n_1, n_2, m_1, m_2 \in M$ and $r_1, r_2 \in R$,

$$\begin{aligned} & (f_R \circ \beta_N)(m_1) \cap (f_R \circ \beta_N)(m_2) = * \\ & * = \left(\bigcup_{m_1=r_1 \cdot n_1} \{f_R(r_1) \cap \beta_N(n_1)\} \right) \cap \left(\bigcup_{m_2=r_2 \cdot n_2} \{f_R(r_2) \cap \beta_N(n_2)\} \right) \\ & = \bigcup_{\substack{m_1=r_1 \cdot n_1 \\ m_2=r_2 \cdot n_2}} \{f_R(r_1) \cap \beta_N(n_1) \cap f_R(r_2) \cap \beta_N(n_2)\} \\ & \subseteq \bigcup_{\substack{m_1=r_1 \cdot n_1 \\ m_2=r_2 \cdot n_2}} \{(f_R \circ \beta_N)(r_1 \cdot n_1) \cap (f_R \circ \beta_N)(r_2 \cdot n_2)\} \\ & \subseteq \bigcup_{r_1 \cdot n_1 + r_2 \cdot n_2} \{f_R \circ \beta_N + f_R \circ \beta_N\}(r_1 \cdot n_1 + r_2 \cdot n_2) \\ & \subseteq (f_R \circ \beta_N + f_R \circ \beta_N)(m_1 + m_2) \\ & \subseteq (f_R \circ \beta_N)(m_1 + m_2) \end{aligned}$$

and for all $n, m \in M$ and $r, r_1 \in R$, then

$$\begin{aligned} (f_R \circ \beta_N)(m) &= \bigcup_{m=r \cdot n} \{f_R(r) \cap \beta_N(n)\} \\ &\subseteq \bigcup_{m=r \cdot n} \{f_R(r) \cap \beta_N(r_1 \cdot n)\} \\ &\subseteq \bigcup_{r_1 \cdot m = r_1 \cdot r \cdot n} (f_R \circ \beta_N)(r_1 \cdot r \cdot n) \\ &= (f_R \circ \beta_N)(r_1 \cdot m). \end{aligned}$$

By using the same steps above, it can be demonstrated $f_R \cdot \beta_N$ is also a soft int-module.

□

Example 4. Let $M = \mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ be a $\mathbb{Z}$ -module and $\mathcal{U} = \{x, y, z\}$ be the universal set. Firstly, we define a soft int-ring $f_{\mathbb{Z}}: \mathbb{Z} \rightarrow P(\mathcal{U})$ over $\mathbb{Z}$ and soft int-module θ as follows:

$$f_{\mathbb{Z}}(r) = \begin{cases} \mathcal{U}, & \text{if } r = 0 \\ \{x, y\}, & \text{if } r \in 2\mathbb{Z} - \{0\} \\ \{x\}, & \text{if } r \in \mathbb{Z} - 2\mathbb{Z}. \end{cases}$$

and

$$\theta(m) = \begin{cases} \mathcal{U}, & \text{if } m \in \{\bar{0}, \bar{3}\} \\ \{x, y\}, & \text{if } m \in \{\bar{1}, \bar{2}, \bar{4}, \bar{5}\}. \end{cases}$$

Next, we define a soft int-submodule $\beta_N: \mathbb{Z}_6 \rightarrow P(\mathcal{U})$ of θ where the submodule $N = \langle \bar{3} \rangle$ of $\mathcal{M}$ as follows:

$$\beta_N(m) = \begin{cases} \mathcal{U}, & \text{if } m = \bar{0} \\ \{x, z\}, & \text{if } m = \bar{3} \end{cases}$$

Now, we compute the product $f_{\mathbb{Z}} \circ \beta_N$ by using the definition, for $m \in \mathbb{Z}_6$, $(f_{\mathbb{Z}} \circ \beta_N)(m) = \bigcup_{m=r \cdot n} (f_{\mathbb{Z}}(r) \cap \beta_N(n))$

- If $m = \bar{0}$, since $\bar{0} = 0 \cdot \bar{0}$ clearly, $(f_{\mathbb{Z}} \circ \beta_N)(\bar{0}) = \mathcal{U}$.
- If $m = \bar{3}$, the only way to obtain $\bar{3} = r \cdot n$ such that $\beta_N(n) \neq \emptyset$ is when $n = \bar{3}$. The equation $r \cdot \bar{3} = \bar{3}$ in $\mathbb{Z}_6$ implies $r \in \mathbb{Z} \setminus 2\mathbb{Z}$. For any odd integer r , we have $f_{\mathbb{Z}}(r) = \{x\}$. Thus, $f_{\mathbb{Z}}(r) \cap \beta_N(\bar{3}) = \{x\} \cap \{x, z\} = \{x\}$. Therefore, $(f_{\mathbb{Z}} \circ \beta_N)(\bar{3}) = \{x\}$.

- For $m \notin \langle 3 \rangle$, writing it is not possible as any $m \in \{\bar{1}, \bar{2}, \bar{4}, \bar{5}\}$ as $r.n$ where $n \in \mathcal{N}$. Since $\beta_{\mathcal{N}}(n) = \emptyset$, $(f_{\mathbb{Z}} \circ \beta_{\mathcal{N}})(m) = \emptyset$.

Result of the product $f_{\mathbb{Z}} \circ \beta_{\mathcal{N}}$ is given by:

$$(f_{\mathbb{Z}} \circ \beta_{\mathcal{N}})(m) = \begin{cases} \mathcal{U}, & \text{if } m = \bar{0} \\ \{x\}, & \text{if } m = \bar{3} \\ \emptyset, & \text{if } m \notin \langle \bar{3} \rangle. \end{cases}$$

It is easy to verify that this resulting soft set is a soft int-submodule of θ .

Definition 9. Let $\theta \in S_{\text{int}}(M, \mathcal{U})$ and $\beta_{\mathcal{N}} \leq_{S-\text{int}} \theta$. For any scalar $r \in R$, we define the scalar multiplication $r.\beta_{\mathcal{N}}$ such that for all $m \in M$,

$$(r.\beta_{\mathcal{N}})(m) = \bigcup_{m=r.a} \beta_{\mathcal{N}}(a)$$

Proposition 5. Let $\beta_{\mathcal{N}}$ be a soft int-submodule of θ and let $r, r_1, r_2 \in R$. Then, the following properties hold:

- $r.\beta_{\mathcal{N}} \subseteq \beta_{\mathcal{N}}$
- $r_1.(r_2.\beta_{\mathcal{N}}) = (r_1.r_2).\beta_{\mathcal{N}}$

Proof.

- Let $m \in M$. Since $\beta_{\mathcal{N}}$ is a soft int-submodule, we have

$$\begin{aligned} (r.\beta_{\mathcal{N}})(m) &= \bigcup_{m=r.a} \beta_{\mathcal{N}}(a) \subseteq \bigcup_{m=r.a} \beta_{\mathcal{N}}(r.a) \\ &= \beta_{\mathcal{N}}(m) \end{aligned}$$

Thus, $(r.\beta_{\mathcal{N}})(m) \subseteq \beta_{\mathcal{N}}(m)$, which concludes the proof of (i).

- Let $m \in M$. Then,

$$\begin{aligned} (r_1.(r_2.\beta_{\mathcal{N}}))(m) &= \bigcup_{m=r_1.x} (r_2.\beta_{\mathcal{N}})(x) = \bigcup_{m=r_1.x} \left(\bigcup_{x=r_2.b} \beta_{\mathcal{N}}(b) \right) \\ &= \bigcup_{m=r_1.(r_2.b)=(r_1.r_2).b} \beta_{\mathcal{N}}(b) = ((r_1.r_2).\beta_{\mathcal{N}})(m) \end{aligned}$$

This completes the proof of (ii).

□

3.3. Behavior of Soft Int-Modules Under Homomorphisms

Homomorphisms are mappings that preserve algebraic structures. In classical module theory, the homomorphic image and preimage of a submodule are also submodules. In this section, we examine the images and preimages of soft int-submodules under module homomorphisms.

Definition 10. Let $\mathcal{M}$ and $\mathcal{N}$ be two R -modules, $f : \mathcal{M} \rightarrow \mathcal{N}$ be an R module homomorphism, and let $\theta \in S_{\text{int}}(\mathcal{M}, \mathcal{U})$ and $\gamma \in S_{\text{int}}(\mathcal{N}, \mathcal{U})$. Then, $f(\theta) : \mathcal{N} \rightarrow P(\mathcal{U})$ defined by, for all $n \in \mathcal{N}$,

$$f(\theta)(n) = \begin{cases} \bigcup_{m \in f^{-1}(n)} \theta(m), & \text{if } m = f^{-1}(n) \neq \emptyset \\ \emptyset, & \text{if } m = f^{-1}(n) = \emptyset \end{cases}$$

and, $f^{-1}(\gamma) : \mathcal{M} \rightarrow P(\mathcal{U})$ defined by, for all $m \in \mathcal{M}$,

$$f^{-1}(\gamma)(m) = \gamma(f(m)).$$

Proposition 6. Let $\mathcal{M}$ and $\mathcal{N}$ be two R -modules, $f : \mathcal{M} \rightarrow \mathcal{N}$ be an R module homomorphism, and let $\theta \in S_{\text{int}}(\mathcal{M}, \mathcal{U})$ and $\gamma \in S_{\text{int}}(\mathcal{N}, \mathcal{U})$; then, $f(\theta) \in S_{\text{int}}(\mathcal{N}, \mathcal{U})$ and $f^{-1}(\gamma) \in S_{\text{int}}(\mathcal{M}, \mathcal{U})$.

Proof. First of all, we show that $f(\theta)(0_{\mathcal{N}}) = \mathcal{U}$. Since $f(0_{\mathcal{M}}) = 0_{\mathcal{N}}$, $f^{-1}(0_{\mathcal{N}}) \neq \emptyset$. Hence, $f(\theta)(0_{\mathcal{N}}) = \bigcup_{m \in f^{-1}(0_{\mathcal{N}})} \theta(m) = \theta(0_{\mathcal{M}}) = \mathcal{U}$.

Now, we prove that $f(\theta)(n_1 + n_2) \supseteq f(\theta)(n_1) \cap f(\theta)(n_2)$, for all $n_1, n_2 \in \mathcal{N}$.

Case 1. If $f^{-1}(n_1) = \emptyset$ or $f^{-1}(n_2) = \emptyset$, then it is clear that $f(\theta)(n_1 + n_2) \supseteq f(\theta)(n_1) \cap f(\theta)(n_2)$.

Case 2. If $f^{-1}(n_1) \neq \emptyset$ and $f^{-1}(n_2) \neq \emptyset$ then

$$\begin{aligned} f(\theta)(n_1) \cap f(\theta)(n_2) &= \left[\bigcup_{m_1 \in f^{-1}(n_1)} \theta(m_1) \right] \cap \left[\bigcup_{m_2 \in f^{-1}(n_2)} \theta(m_2) \right] \\ &= \bigcup_{m_1 \in f^{-1}(n_1), m_2 \in f^{-1}(n_2)} \theta(m_1) \cap \theta(m_2) \\ &\subseteq \bigcup_{m_1 \in f^{-1}(n_1), m_2 \in f^{-1}(n_2)} \theta(m_1 + m_2) \\ &\subseteq \bigcup_{m_1 + m_2 \in f^{-1}(n_1 + n_2)} \theta(m_1 + m_2) \\ &= f(\theta)(n_1 + n_2) \end{aligned}$$

Finally, we prove that $f(\theta)(r.n) \supseteq f(\theta)(n)$, for all $n \in \mathcal{N}$, $r \in R$. If $f^{-1}(n) = \emptyset$, then $f(\theta)(n) = \emptyset \subseteq f(\theta)(r.n)$. If $f^{-1}(n) \neq \emptyset$, then

$$f(\theta)(n) = \left(\bigcup_{m \in f^{-1}(n)} \theta(m) \right) \subseteq \left(\bigcup_{r.m \in f^{-1}(r.n)} \theta(r.m) \right) = f(\theta)(r.n)$$

Thus, $f(\theta) \in S_{\text{int}}(\mathcal{N}, \mathcal{U})$. Now, we prove that $f^{-1}(\gamma) \in S_{\text{int}}(\mathcal{M}, \mathcal{U})$.

$f^{-1}(\gamma)(0_{\mathcal{M}}) = \gamma(f(0_{\mathcal{M}})) = \gamma(0_{\mathcal{N}}) = \mathcal{U}$, and for all $m_1, m_2 \in \mathcal{M}$ $r \in R$,

$$\begin{aligned} f^{-1}(\gamma)(m_1 + m_2) &= \gamma(f(m_1 + m_2)) \\ &= \gamma(f(m_1) + f(m_2)) \\ &\supseteq \gamma(f(m_1)) + \gamma(f(m_2)) \\ &= f^{-1}(\gamma)(m_1) + f^{-1}(\gamma)(m_2) \\ f^{-1}(\gamma)(r.m) &= \gamma(f(r.m)) = \gamma(r.f(m)) \\ &\supseteq \gamma(f(m)) = f^{-1}(\gamma)(m) \end{aligned}$$

□

Example 5. $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be a module homomorphism defined by $f(x) = 2.x$ and $\mathcal{U} = \{x, y, z, t\}$.

$$\theta : \mathbb{Z} \rightarrow P(\mathcal{U}) \text{ defined by } \theta(n) = \begin{cases} \mathcal{U}, & \text{if } n \in 6\mathbb{Z} \\ \{x, y\}, & \text{if } n \in 3\mathbb{Z} - 6\mathbb{Z} \\ \emptyset, & \text{if } n \in \mathbb{Z} - 3\mathbb{Z} \end{cases}$$

$\theta \in S_{int}(\mathbb{Z}, \mathcal{U})$.

- $f(\theta)(0) = \bigcup_{m \in f^{-1}(0)} \theta(m) = \theta(0) = \mathcal{U}$
- If $n = 2k + 1$, $f^{-1}(n) = \emptyset$. From this $f(\theta)(n) = \emptyset$.
- If $n \in 2\mathbb{Z} - 3\mathbb{Z}$, then $f(\theta)(n) = \bigcup_{m \in f^{-1}(n)} \theta(m) = \emptyset$.
- If $n \in 6\mathbb{Z} - 12\mathbb{Z}$ then $f(\theta)(n) = \bigcup_{m \in f^{-1}(n)} \theta(m) = \{x, y\}$.
- If $n \in 12\mathbb{Z}$ then $f(\theta)(n) = \bigcup_{m \in f^{-1}(n)} \theta(m) = \mathcal{U}$.

$$\text{Hence, } f(\theta)(n) = \begin{cases} \mathcal{U}, & \text{if } n \in 12\mathbb{Z} \\ \{x, y\}, & \text{if } n \in 6\mathbb{Z} - 12\mathbb{Z} \\ \emptyset, & \text{if } n \in \mathbb{Z} - 6\mathbb{Z} \end{cases}$$

One can easily show that $f(\theta)$ is a soft int-R-module.

Example 6. Let $f : \mathbb{Z} \rightarrow \mathbb{Z}_6$ defined by $f(x) = \bar{x}$ be a module homomorphism and $\mathcal{U} = \{x, y, z, t\}$. $\gamma : \mathbb{Z}_6 \rightarrow P(\mathcal{U})$ is a soft int-R-module defined by

$$\gamma(\bar{x}) = \begin{cases} \mathcal{U}, & \text{if } \bar{x} = \bar{0} \\ \{y\}, & \text{if } \bar{x} \in \{\bar{2}, \bar{4}\} \\ \emptyset, & \text{if } \bar{x} \in \{\bar{1}, \bar{3}, \bar{5}\} \end{cases}$$

By the division algorithm, any $x \in \mathbb{Z}$, $x = 6k + r$ where $r = 0, 1, 2, 3, 4, 5$. $f^{-1}(\gamma)(x) = \gamma(f(x)) = \gamma(\bar{r})$. For example, $f^{-1}(\gamma)(-2) = \gamma(f(-2)) = \gamma(\bar{-2}) = \gamma(\bar{4}) = \{y\}$. Thus,

$$f^{-1}(\gamma) = \begin{cases} \mathcal{U}, & \text{if } x \in 6\mathbb{Z} \\ \{y\}, & \text{if } x \in 2\mathbb{Z} - 6\mathbb{Z} \\ \emptyset, & \text{if } x \in \mathbb{Z} - 2\mathbb{Z} \end{cases}$$

One can easily show that $f^{-1}(\gamma)$ is a soft int-R-module.

4. Applications

4.1. Model for Supply Chain Logistics over the $\mathbb{Z}_6$ -Module $\mathbb{Z}_6$

In the Fast-Moving Consumer Goods sector, especially for high-tonnage products like water, the management of the supply chain plays an important role in the optimization of logistic operations. In order to reduce costs in shipments made from production facilities to the regional warehouses of dealers, it is desired that the orders comply with the “Full Pallet” and “Full Truckload” (FTL) standards. Full Truckload capacity is the process of filling a transportation vehicle with standard pallet groups without leaving any empty space, so as to reach its maximum volume limit. “Fractional” or “partial” additional orders that do not comply with this rule disrupt the standardized assembly line flow and operational integrity of the factories, and at the same time lead to waste in transportation costs. For this reason, while large-scale factories generally accept only orders in the form of Full Truckload multiples, fractional additional demands are directed to higher-cost boutique facilities. Finding a model that will provide the ability to make quick decisions in order to save both

time and cost in moments of sudden water need occurring in situations such as seasonal changes, earthquakes, and pandemics will greatly facilitate the process.

In this section, using the proposed soft int-module structure, the capacity limits of a regional warehouse and “Full Truckload” logistic rules are modeled algebraically over the module $\mathbb{Z}_6$.

4.1.1. Construction of the Algebraic Model

- **Universal Set U :** Let the set of factories that the chain market has an agreement with to supply water to the regional warehouse be $U = \{f_1, f_2, f_3, f_4, f_5\}$. Here, let f_1, f_2 , and f_3 be large-scale factories operating only at full capacity (Full Truckload), and let f_4 and f_5 be flexible but costly boutique factories that can also produce fractional orders.
- **Ring and Module ($R = M = \mathbb{Z}_6$):** Suppose that the regional warehouse has a maximum cyclic capacity of six pallets. The “additional order” demanded by the market in addition to the planned main shipments is defined as $x \in \mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$.
- **Logistic Rule:** Let us assume that the full capacity of a truck is three pallets. In this case, the $\{\bar{0}, \bar{3}\}$ subset within the $\mathbb{Z}_6$ module represents the “Full Truckload” orders that do not disrupt the logistic integrity and is a submodule of $\mathbb{Z}_6$.
- **Soft Int-Module (θ):** The θ function, which assigns the subset of factories that can produce that demand without getting stuck in logistic constraints to each $\bar{x} \in \mathbb{Z}_6$ additional order vector, is defined as follows:

$$\theta(x) = \begin{cases} U, & \text{if } x \in \{\bar{0}, \bar{3}\} \text{ (Full Truckload order)} \\ \{f_4, f_5\}, & \text{if } x \in \{\bar{1}, \bar{2}, \bar{4}, \bar{5}\} \text{ (Fractional order)} \end{cases}$$

4.1.2. Proof of Soft Int-Module Axioms and Logistic Interpretations

- Condition SM1** The case of no addition to the initial order ($\theta(\bar{0}) = U$)
Algebraic Interpretation: Having no additional orders (the $\bar{0}$ state) does not change the logistic system in any way.
Logistic Equivalent: If the chain market does not demand any additional orders or revisions, no external intervention is made to the routine shipment cycles of the factories. Since the “Full Truckload” constraint is not violated, theoretically all factories in the universal set (U) support this reference state.
- Condition SM2: Preservation of Intersection under Operations** ($\theta(x + y) \supseteq \theta(x) \cap \theta(y)$), for all $x, y \in \mathbb{Z}_6$.
Algebraic Interpretation: We can test this situation with two different extreme scenarios:
 - Sum of Two Full Truckloads;** Let $x = \bar{3}$ and $y = \bar{3}$. Then, $x + y \equiv \bar{0} \pmod{6}$. In this case, $\theta(\bar{0}) \supseteq \theta(\bar{3}) \cap \theta(\bar{3})$, that is, $U \supseteq U \cap U$ is satisfied.
 - Sum of Two Fractional Orders:** Suppose the market has placed fractional orders of x and y for two different regions. Only flexible factories can meet these two orders: $\theta(x) = \theta(y) = \{f_4, f_5\}$.
Case1. $x + y = \bar{0}, \bar{3} \pmod{6}$ (Full Truckload). In this case, $\theta(\bar{3}) = U$ is obtained. Algebraically, the inclusion $U \supseteq \{f_4, f_5\} \cap \{f_4, f_5\}$ is perfectly satisfied.
Case2. $x + y \neq \bar{0}, \bar{3} \pmod{6}$ (Fractional order). In this case, $\theta(x + y) = \{f_4, f_5\} = \theta(x) \cap \theta(y)$.

Logistic Equivalent: If the canceled and added fractional demands ($\bar{5}$ and $\bar{2}$) in the logistic network balance each other and the difference modularly settles into the “Full Truckload” rule (multiples of $\bar{3}$), the system suddenly returns to its most efficient state (optimum point) where all factories (U) can serve. The intersection is strictly preserved and the system stability expands.

(iii) Condition SM3: Expansion under Scalar Multiplication ($\theta(r \cdot x) \supseteq \theta(x)$)

Algebraic Interpretation: Let us take the $x = \bar{3}$ additional order representing the Full Truckload capacity. As is known, $\theta(\bar{3}) = U$. When we multiply this order by any $r \in \mathbb{Z}_6$ demand coefficient; $r \cdot \bar{3} \equiv \bar{0} \pmod{6}$ or $r \cdot \bar{3} \equiv \bar{3} \pmod{6}$. Thus, $\theta(r \cdot x) = U \supseteq \theta(x)$ for all $r \in \mathbb{Z}_6$ and $x = \bar{3}$. If $x \in \mathbb{Z}_6 - \{\bar{0}, \bar{3}\}$, it is easy check that $\theta(r \cdot x) \supseteq \theta(x)$ for all $r \in \mathbb{Z}_6$.

Logistic Equivalent: No matter how much you multiply a legitimate order of 3s (Full Truckload), the logistic integrity is never broken. Even when the scale grows, the supplier pool (U) is preserved and no order falls into a fractional state.

We can show the practical need for this structure by adding a second constraint to our logistics problem. Real-world situations, like sudden regional demands or disasters, actually force us to filter multiple criteria at once. So, we bring in a new logistics constraint: “Delivery Time/Urgency”.

Imagine a case where the regional warehouse suddenly needs to ship an extra order very fast. If there is no extra demand ($x = \bar{0}$), the normal logistics flow just keeps going. However, if a demand is active ($x \neq \bar{0}$), we face a geographic problem. Out of our factory set $U = \{f_1, f_2, f_3, f_4, f_5\}$, only f_2 (a large-scale facility) and f_4 (a boutique facility) are close enough to make a same-day delivery.

We define our time-sensitive capacity soft int-module $\alpha : \mathbb{Z}_6 \rightarrow P(U)$ by, for all $x \in \mathbb{Z}_6$,

$$\alpha(x) = \begin{cases} U, & \text{if } x = \bar{0} \\ \{f_2, f_4\}, & \text{if } x \in \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\} \end{cases}$$

Now, the supply chain faces a challenge. It has to handle a sudden fractional order like $x = \bar{2}$ and meet both the capacity rule (θ) and the time constraint (α) at the same time. Instead of using step-by-step filtering algorithms, we solve this algebraically by intersecting the soft int-modules. Let $\mu = \theta \tilde{\cap} \alpha$ be our optimum decision module. For the fractional order $x = \bar{2}$, the calculation is straightforward:

$$\begin{aligned} \mu(\bar{2}) &= \theta(\bar{2}) \cap \alpha(\bar{2}) \\ \mu(\bar{2}) &= \{f_4, f_5\} \cap \{f_2, f_4\} = \{f_4\} \end{aligned}$$

Why this beats fuzzy modules? If we tried modeling this problem with a fuzzy module, a fractional order would just turn into a plain number. Take a fuzzy module ν for example; we would obtain a value like $\nu(\bar{2}) = t$, where $t \in [0, 1]$. That scalar number might indicate that a constraint exists, but it gives absolutely no clue which factory the system actually needs to contact. The soft int-module structure fixes this algebraically. As the math directly shows, the result $\mu(\bar{2}) = \{f_4\}$ identifies f_4 as the unique facility capable of processing a fractional order under emergency time limits.

To explicitly demonstrate the superiority of the proposed structure in handling simultaneous constraints without information loss, a multidimensional comparison of the algebraic modeling approaches for a partial order scenario is summarized in Table 1.

Table 1. Comparison of algebraic modeling approaches for partial orders ($x = \bar{2}$) under multi-criteria constraints.

Modeling Approach	Image Set	Output for $x = \bar{2}$	Information Loss	Decision Status
Standard IF-THEN Logic	Boolean $\{0, 1\}$	Error/Reject	High (Rejects the order)	Requires manual intervention
Fuzzy Module	Scalar $[0, 1]$	$\mu(\bar{2}) = 0.4$ (Low suitability)	High (Factory IDs are lost)	Cannot route to a specific factory
Proposed Soft Int-Module	Power Set $P(U)$	$\Psi(\bar{2}) = \{f_4\}$	None	Automatic routing to f_4

As a result, this established model proves that soft int-modules are not just a theoretical abstraction, but on the contrary, they can work as a fast and automated decision-making algorithm that reduces costs and adapts to emergency situations in modern supply chain optimization.

5. Conclusions

In this study, using the int-structure approach, which provides a new perspective on soft algebraic structures, we defined a new algebraic structure over a module, which we called “soft intersectional module”. By giving the basic properties of soft int-modules and the definition of soft int-submodules, we showed that soft int-submodules preserve closure under scalar multiplication and module addition, and we provided examples. Also, we examined the behaviors of soft int-modules under module homomorphisms. To show the practical applicability of this structure, we gave a logistics-based modeling example. To summarize the operational and computational advantages of the developed soft int-module theory over classical and fuzzy approaches in complex logistics decision processes, a general performance comparison is presented in Table 2. Furthermore, defining and investigating the notion of soft Hilbert int-modules stands out as a highly promising direction for our future research endeavors. Beyond the algebraic results we obtained, this concept is evaluated to have potential for application across different disciplines as well. To make sense of complex datasets, machine learning algorithms often convert them into vector spaces where linear operations such as scalar multiplication and addition can be performed. Since vector spaces are also special modules, the properties of preservation of intersection under summation ($F(x + y) \supseteq F(x) \cap F(y)$) and expansion under scalar multiplication ($F(rx) \supseteq F(x)$), which are the basic axioms of soft int-modules, exhibit a theoretical harmony with the concepts of scale invariance and preservation of semantic integrity sought in data representation. Considering this theoretical similarity, it is anticipated that the defined soft int-module structure can serve as an alternative mathematical tool for data analysis and modeling in machine learning research in future studies.

Table 2. Performance and output comparison of different module structures within the logistics model.

Logistics Model Criteria	Classical Soft Module	Fuzzy Module	Proposed Soft Int-Module
Multi-Constraint Processing	Sequential (Step-by-step)	Membership degree based	Simultaneous (Dynamic Intersection)
Capacity & Time Limits	Evaluated separately	Handled via min-max logic	Handled concurrently via level sets
Decision Output	Single optimal parameter set	Continuous truth values in $[0, 1]$	Intersectional exact submodule
Computational Overhead	High (Iterative filtering needed)	Moderate	Reduced (Direct intersection logic)
Applicability to Supply Chains	Limited for complex networks	Suitable for vague conditions	Highly effective for exact multi-parameters

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Abbreviations

The following abbreviations are used in this manuscript:

int-module	Intersectional module
int-ring	Intersectional ring
int-submodule	Intersectional submodule

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