

Strong Ramsey Numbers of All Sets of Three Graphs of Size 3

Gary Chartrand and Ping Zhang *

Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008-5248, USA

* Correspondence: ping.zhang@wmich.edu

Abstract

The strong Ramsey number $\bar{R}(F, H)$ of two graphs F and H without isolated vertices is the minimum positive integer n such that every red-blue (edge) coloring of K_n results in two edge-disjoint monochromatic subgraphs, one isomorphic to F and the other isomorphic to H . These numbers have been determined for all such pairs F, H of graphs of size 3. For a set S of $k \geq 2$ graphs without isolated vertices, the strong Ramsey number $\bar{R}(S)$ of S is the minimum positive integer n such that every red-blue coloring of K_n results in k pairwise edge-disjoint monochromatic subgraphs isomorphic to the graphs in S . The number $\bar{R}(S)$ is determined for all sets of three graphs of size 3. Other results and questions on strong Ramsey numbers are also presented.

Keywords: red-blue coloring; Ramsey number; strong Ramsey number

MSC: 05C15; 05C35; 05C55

1. Introduction

For two graphs F and H without isolated vertices, the *Ramsey number* $R(F, H)$ of F and H is the minimum positive integer n such that for every red-blue (edge) coloring of the complete graph K_n of order n , there exists a subgraph isomorphic to F all of whose edges are colored red (a *red* F) or a subgraph isomorphic to H all of whose edges are colored blue (a *blue* H). The (*diagonal*) *Ramsey number* $R(F)$ of a graph F without isolated vertices is the minimum positive integer n such that every red-blue coloring of K_n results in a subgraph of K_n isomorphic to F all of whose edges are colored the same (a monochromatic F). Due to the numerous intriguing and highly nontrivial problems occurring in Ramsey theory, the study of Ramsey numbers has attracted the attention of many researchers and has been one of most popular areas of research in graph theory (see refs. [1–6], for example). Many variations of Ramsey numbers have been introduced and studied which provide different approaches to the investigation of Ramsey numbers (see [7–12], for example).

The Ramsey number $R(F, H)$ of two graphs F and H without isolated vertices is the smallest positive integer n of a complete graph K_n such that every red-blue coloring of K_n yields a monochromatic subgraph that is *either* isomorphic to F *or* isomorphic to H (not necessarily both). This observation gave rise to a variation of Ramsey number $R(F, H)$ in [13] that requires a strong property. The *strong Ramsey number* $\bar{R}(F, H)$ of two non-isomorphic graphs F and H without isolated vertices is the minimum positive integer n such that every red-blue (edge) coloring of K_n results in two edge-disjoint monochromatic subgraphs, one isomorphic to F and the other isomorphic to H . Consequently, the strong Ramsey number $\bar{R}(F, H)$ requires that every red-blue coloring of $K_{\bar{R}(F, H)}$ results in two edge-disjoint monochromatic subgraphs either of different colors or the same color, one



Academic Editor: Fabio Caldarola

Received: 25 June 2026

Revised: 2 August 2026

Accepted: 12 August 2026

Published: 15 August 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

isomorphic to F and the other isomorphic to H . As mentioned in [13], since the Ramsey number $R(F, H)$ exists for every pair F, H of graphs by a result of Ramsey [14], the strong Ramsey number $\bar{R}(F, H)$ exists as well. In [13], the strong Ramsey number is determined for all pairs of graphs of size 2 or 3 without isolated vertices; while in [15], the strong Ramsey number is determined for all pairs of graphs of size 4 without isolated vertices.

The concept of the strong Ramsey number of two non-isomorphic graphs can be extended to sets of two or more non-isomorphic graphs. For a set $S = \{F_1, F_2, \dots, F_k\}$ of $k \geq 2$ pairwise non-isomorphic graphs without isolated vertices, the *strong Ramsey number* $\bar{R}(S)$ of S is the minimum positive integer n such that every red-blue coloring of K_n results in k pairwise edge-disjoint monochromatic subgraphs $F_1, F_2, \dots, F_k$. In particular, if $k = 2$, then $\bar{R}(S) = \bar{R}(F_1, F_2)$ is the strong Ramsey number of F_1 and F_2 . First, we present some preliminary information.

Proposition 1. *For every set S of pairwise non-isomorphic graphs without isolated vertices, the strong Ramsey number $\bar{R}(S)$ of S exists and $\bar{R}(S) \geq \max\{R(F) : F \in S\}$.*

Proof. Let $S = \{F_1, F_2, \dots, F_k\}$ be a set of $k \geq 2$ pairwise non-isomorphic graphs without isolated vertices. For $1 \leq i \leq k$, let $R(F_i) = n_i$ and let $n = \sum_{i=1}^k n_i$. Let there be given a red-blue coloring of K_n . Because $n \geq n_k$, there is a monochromatic subgraph F_k in K_n . Since deletion of the n_k vertices of F_k from K_n results in K_{n-n_k} , where $n - n_k \geq n_{k-1}$, there is a monochromatic subgraph F_{k-1} in K_{n-n_k} that is edge-disjoint from F_k . Continuing in this manner, we obtain pairwise edge-disjoint monochromatic subgraphs $F_1, F_2, \dots, F_k$. Thus, $\bar{R}(S)$ exists.

Next, let $\bar{R}(S) = n$. Since every red-blue coloring of K_n produces pairwise edge-disjoint monochromatic subgraphs $F_1, F_2, \dots, F_k$, it follows that $R(F_i) \leq n$ for $1 \leq i \leq k$ and so $\max\{R(F) : F \in S\} \leq n$. $\square$

Observation 1. *If S_1 and S_2 are two sets of pairwise non-isomorphic graphs without isolated vertices such that $S_1 \subseteq S_2$, then $\bar{R}(S_1) \leq \bar{R}(S_2)$.*

Since strong Ramsey numbers have been determined in [13] for all pairs of graphs of size 2 or 3 without isolated vertices, it follows that strong Ramsey numbers have been determined for all sets of two graphs of size 3. The goal here is to determine the strong Ramsey numbers of all sets of three graphs of size 3. There are five graphs of size 3 without isolated vertices, namely the triangle K_3 , the star $K_{1,3}$, the 4-path P_4 , the union $P_3 + K_2$ of the 3-path P_3 and the graph K_2 , and the matching $3K_2$, shown in Figure 1. Here, $R(P_4) = 5$, $R(K_{1,3}) = R(K_3) = R(P_3 + K_2) = 6$, and $R(3K_2) = 8$ (see [4,16]).

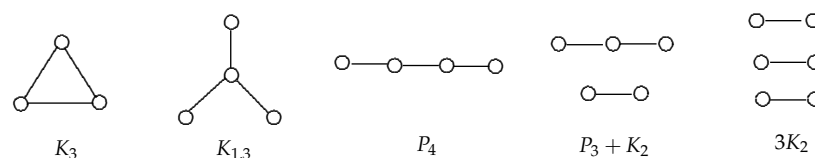


Figure 1. The five graphs of size 3 without isolated vertices.

We assume that all graphs in each set under consideration are pairwise non-isomorphic and have no isolated vertices. Notation and terminology not defined here may be found in the book [17] and additional information on Ramsey numbers can be found in [2,4,5].

2. Acyclic Graphs of Size 3

The first 3-element sets of graphs we consider consist of acyclic graphs. A graph is *acyclic* if it has no cycles. The four acyclic graphs of size 3 without isolated vertices

are P_4 , $P_3 + K_2$, $3K_2$, and $K_{1,3}$. It is useful to determine $\bar{R}(S)$ for the set $S = \{P_4, P_3 + K_2, 3K_2, K_{1,3}\}$ of all four of these graphs. First, we present a lemma.

Lemma 1. Every red-blue coloring of K_7 that results in the two edge-disjoint monochromatic subgraphs $3K_2$ and P_4 also results in the three pairwise edge-disjoint monochromatic subgraphs $3K_2$, P_4 , and $P_3 + K_2$.

Proof. Let there be given a red-blue coloring of $G = K_7$ with $V(G) = \{v_1, v_2, \dots, v_7\}$ that results in edge-disjoint monochromatic subgraphs $F_1 = 3K_2$ and $F_2 = P_4$, where say $E(F_1) = \{v_i v_{i+3} : i = 1, 2, 3\}$. Since $|V(F_1) \cup V(F_2)| \in \{6, 7\}$, there are two possibilities to consider.

Case 1. $|V(F_1) \cup V(F_2)| = 6$. Then $V(F_1) \cup V(F_2) = \{v_1, v_2, \dots, v_6\}$. We may assume that $F_2 = P_4$ is one of the four 4-paths (v_1, v_2, v_4, v_5) , (v_1, v_2, v_4, v_3) , (v_1, v_2, v_3, v_4) , (v_1, v_2, v_3, v_5) . We consider each of these four possibilities shown in Figure 2, where an edge in $F_1 = 3K_2$ is indicated by a dotted line and an edge in $F_2 = P_4$ is indicated by a dashed line.

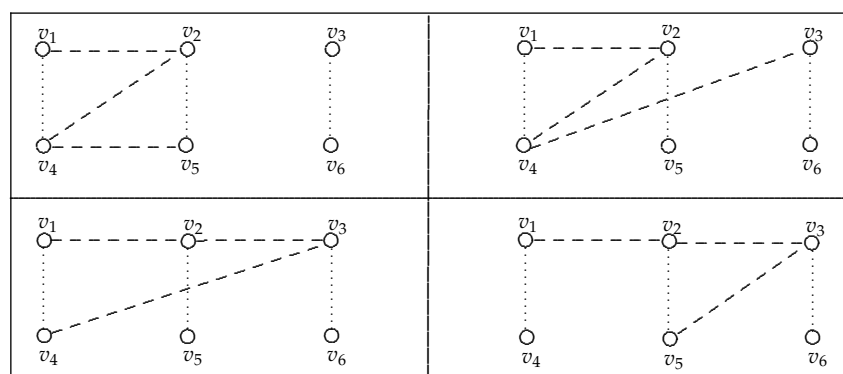


Figure 2. The four possible subgraphs $F_2 = P_4$ in Case 1.

- If $F_2 = (v_1, v_2, v_4, v_5)$ or $F_2 = (v_1, v_2, v_4, v_3)$, then at least two of the three edges v_7v_1, v_7v_2, v_7v_4 have the same color (by the Pigeonhole Principle), say v_7v_i and v_7v_j are red where $i, j \in \{1, 2, 4\}$.
- If $F_2 = (v_1, v_2, v_3, v_4)$ or $F_2 = (v_1, v_2, v_3, v_5)$, then at least two of the three edges v_7v_1, v_7v_2, v_7v_3 have the same color, say v_7v_i and v_7v_j are red where $i, j \in \{1, 2, 3\}$.

In either case, let $G' = G[\{v_1, v_2, \dots, v_7\} - \{v_i, v_j, v_7\}] - [E(F_1) \cup E(F_2)]$. If G' contains a red edge xy , then there is a red $P_3 + K_2 = (v_i, v_7, v_j) + (x, y)$ that is edge-disjoint from F_1 and F_2 . Thus, we may assume that G' is blue. First, suppose that $F_2 = (v_1, v_2, v_4, v_5)$. In this situation, at least two of the three edges v_7v_1, v_7v_2, v_7v_4 have the same color.

- ★ If v_7v_1 and v_7v_2 are red, then $G' = G[\{v_3, v_4, v_5, v_6\}] - [E(F_1) \cup E(F_2)]$ is blue. Thus, $F'_2 = (v_3, v_5, v_6, v_4)$ is a blue P_4 in G' that is edge-disjoint from F_1 and F_2 . If F_2 is red, then there is a red $P_3 + K_2 = (v_1, v_7, v_2) + (v_4, v_5)$ that is edge-disjoint from F_1 and F'_2 . If F_2 is blue, then there is a blue $P_3 + K_2 = (v_3, v_4, v_5) + (v_1, v_2)$ that is edge-disjoint from F_1 and F'_2 . This is illustrated in the first graph in Figure 3, where the solid vertices represent the four vertices of G' and a solid line represents a red edge.
- ★ If v_7v_1 and v_7v_4 are red, then $G' = G[\{v_2, v_3, v_5, v_6\}] - [E(F_1) \cup E(F_2)]$ is blue. By the argument above, we may assume that v_7v_2 is blue. Thus, $F'_2 = (v_3, v_2, v_6, v_5)$ is a blue P_4 in G' that is edge-disjoint from F_1 and F_2 . If F_2 is red, then there is a red $P_3 + K_2 = (v_2, v_4, v_5) + (v_1, v_7)$ that is edge-disjoint from F_1 and F'_2 . If F_2 is blue, then there is a blue $P_3 + K_2 = (v_1, v_2, v_7) + (v_4, v_5)$ that is edge-disjoint from F_1 and F'_2 .

This is illustrated in the second graph in Figure 3 where the solid vertices indicate the four vertices of G' .

- ★ If v_7v_2 and v_7v_4 are red, then $G' = G[\{v_1, v_3, v_5, v_6\}] - [E(F_1) \cup E(F_2)]$ is blue. We may assume that v_7v_1 is blue. Then $F'_2 = (v_3, v_1, v_6, v_5)$ is a blue P_4 in G' that is edge-disjoint from F_1 and F_2 . If F_2 is red, then there is a red $P_3 + K_2 = (v_1, v_2, v_7) + (v_4, v_5)$ that is edge-disjoint from F_1 and F'_2 . If F_2 is blue, then there is a blue $P_3 + K_2 = (v_7, v_1, v_2) + (v_4, v_5)$ that is edge-disjoint from F_1 and F'_2 . This is illustrated in the third graph in Figure 3 where the solid vertices indicate the four vertices of G' .

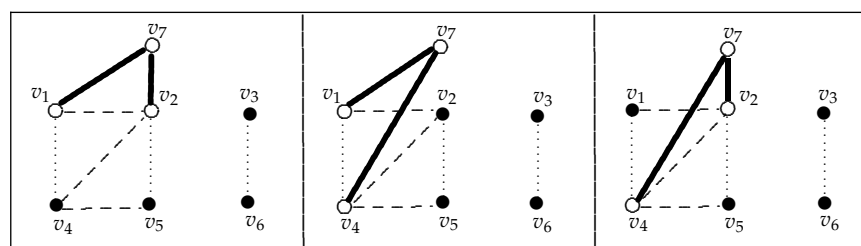


Figure 3. Illustrating steps in the proof of Case 1 where $F_2 = (v_1, v_2, v_4, v_5)$.

If F_2 is one of the other three 4-paths (v_1, v_2, v_4, v_3) , (v_1, v_2, v_3, v_4) , and (v_1, v_2, v_3, v_5) , then we apply the proof technique employed in the situation where $F_2 = (v_1, v_2, v_4, v_5)$ and use a similar argument to create pairwise edge-disjoint monochromatic subgraphs $3K_2$, P_4 and $P_3 + K_2$. For example, if $F_2 = P_4 = (v_1, v_2, v_3, v_4)$, then at least two of the three edges v_7v_1, v_7v_2, v_7v_3 have the same color, say red.

- ★ First, suppose that v_7v_1 and v_7v_2 are red. Let $G' = G[\{v_3, v_4, v_5, v_6\}] - [E(F_1) \cup E(F_2)]$. If G' contains a red edge xy , then there is a red $P_3 + K_2 = (v_1, v_7, v_2) + (x, y)$ that is edge-disjoint from F_1 and F_2 . If G' is blue, then $F'_2 = (v_3, v_5, v_6, v_4)$ is a blue P_4 in G' that is edge-disjoint from F_1 and F_2 . If F_2 is red, then there is a red $P_3 + K_2 = (v_1, v_7, v_2) + (v_4, v_3)$ that is edge-disjoint from F_1 and F'_2 . If F_2 is blue, then there is a blue $P_3 + K_2 = (v_3, v_4, v_5) + (v_1, v_2)$ that is edge-disjoint from F_1 and F'_2 .
- ★ Next, suppose that v_7v_1 and v_7v_3 are red. Let $G' = G[\{v_2, v_4, v_5, v_6\}] - [E(F_1) \cup E(F_2)]$. If G' contains a red edge xy , then there is a red $P_3 + K_2 = (v_1, v_7, v_3) + (x, y)$ that is edge-disjoint from F_1 and F_2 . If G' is blue, then $F'_2 = (v_2, v_4, v_5, v_6)$ is a blue P_4 in G' that is edge-disjoint from F_1 and F_2 . If F_2 is red, then there is a red $P_3 + K_2 = (v_7, v_1, v_2) + (v_3, v_4)$ that is edge-disjoint from F_1 and F'_2 . If F_2 is blue, then there is a blue $P_3 + K_2 = (v_3, v_4, v_6) + (v_1, v_2)$ that is edge-disjoint from F_1 and F'_2 .
- ★ Finally, suppose that v_7v_2 and v_7v_3 are red. Let $G' = G[\{v_1, v_4, v_5, v_6\}] - [E(F_1) \cup E(F_2)]$. If G' contains a red edge xy , then there is a red $P_3 + K_2 = (v_2, v_7, v_3) + (x, y)$ that is edge-disjoint from F_1 and F_2 . If G' is blue, then $F'_2 = (v_4, v_5, v_6, v_1)$ is a blue P_4 in G' that is edge-disjoint from F_1 and F_2 . If F_2 is red, then there is a red $P_3 + K_2 = (v_1, v_2, v_7) + (v_4, v_3)$ that is edge-disjoint from F_1 and F'_2 . If F_2 is blue, then there is a blue $P_3 + K_2 = (v_4, v_3, v_6) + (v_1, v_2)$ that is edge-disjoint from F_1 and F'_2 .

Case 2. $|V(F_1) \cup V(F_2)| = 7$. We may assume that $F_2 = P_4$ is one of the five 4-paths (v_7, v_1, v_2, v_4) , (v_7, v_1, v_2, v_3) , (v_1, v_7, v_4, v_2) , (v_1, v_7, v_2, v_4) , (v_1, v_7, v_2, v_3) . We consider each of these five possible 4-paths for F_2 shown in Figure 4.

- If $F_2 = (v_7, v_1, v_2, v_4)$ or $F_2 = (v_7, v_1, v_2, v_3)$, then at least two of the three edges v_7v_2, v_7v_4, v_7v_5 have the same color, say v_7v_i and v_7v_j are red where $i, j \in \{2, 4, 5\}$.
- If $F_2 = (v_1, v_7, v_4, v_2)$, then at least two of the three edges v_7v_2, v_7v_3, v_7v_6 have the same color, say v_7v_i and v_7v_j are red where $i, j \in \{2, 3, 6\}$.
- If $F_2 = (v_1, v_7, v_2, v_4)$, then at least two of the three edges v_7v_3, v_7v_4, v_7v_6 have the same color, say v_7v_i and v_7v_j are red where $i, j \in \{3, 4, 6\}$.

- If $F_2 = (v_1, v_7, v_2, v_3)$, then at least two of the three edges v_7v_3, v_7v_4, v_7v_5 have the same color, say v_7v_i and v_7v_j are red where $i, j \in \{3, 4, 5\}$.

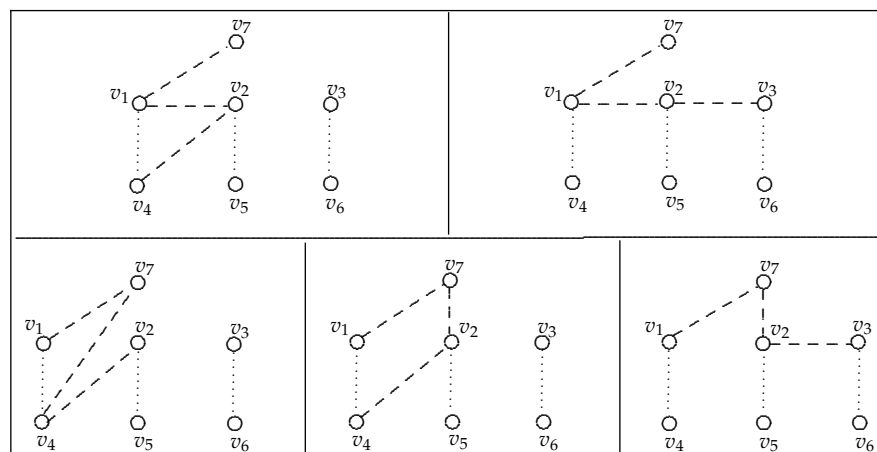


Figure 4. The five possible subgraphs $F_2 = P_4$ in Case 2.

In either case, let $G' = G[\{v_1, v_2, \dots, v_7\} - \{v_i, v_j, v_7\}] - [E(F_1) \cup E(F_2)]$. If G' contains a red edge xy , then there is a red $P_3 + K_2 = (v_i, v_7, v_j) + (x, y)$ that is edge-disjoint from F_1 and F_2 . Thus, we may assume that G' is blue. As in Case 1, an argument is given only for $F_2 = (v_1, v_7, v_4, v_2)$ since the proof technique and the arguments for other choices of F_2 are similar. In this situation, at least two of the three edges v_7v_2, v_7v_3, v_7v_6 have the same color.

- ★ If v_7v_2 and v_7v_3 are red or v_7v_2 and v_7v_6 are red, say the former, then $G' = G[\{v_1, v_4, v_5, v_6\}] - [E(F_1) \cup E(F_2)]$ is blue. Thus, $F'_2 = (v_4, v_5, v_1, v_6)$ is a blue P_4 in G' that is edge-disjoint from F_1 and F_2 . If F_2 is red, then there is a red $P_3 + K_2 = (v_1, v_7, v_3) + (v_2, v_4)$ that is edge-disjoint from F_1 and F'_2 . If F_2 is blue, then there is a blue $P_3 + K_2 = (v_5, v_6, v_4) + (v_1, v_7)$ that is edge-disjoint from F_1 and F'_2 . This is illustrated in the first graph in Figure 5 where the four solid vertices represent the vertices of G' .
- ★ If v_7v_3 and v_7v_6 are red, then $G' = G[\{v_1, v_2, v_4, v_5\}] - [E(F_1) \cup E(F_2)]$ is blue. We may assume that v_7v_2 is blue. Thus, $F'_2 = (v_2, v_1, v_5, v_4)$ is a blue P_4 in G' that is edge-disjoint from F_1 and F_2 . If F_2 is red, then there is a red $P_3 + K_2 = (v_1, v_7, v_3) + (v_2, v_4)$ that is edge-disjoint from F_1 and F'_2 . Thus, we may assume that F_2 is blue. If v_5v_6 is red, then there are edge-disjoint monochromatic subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_5, v_6, v_7, v_3)$ is red and $P_3 + K_2 = (v_7, v_1, v_2) + (v_4, v_5)$ is blue, that are edge-disjoint from F_1 . If v_5v_6 is blue, then there is a blue $P_3 + K_2 = (v_1, v_7, v_2) + (v_5, v_6)$, that is edge-disjoint from F_1 and F'_2 . This is illustrated in the second graph in Figure 5 where the four solid vertices represent the vertices of G' .

□

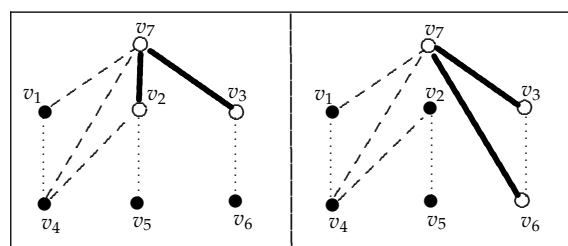


Figure 5. Illustrating steps in the proof of Case 2 when $F_2 = (v_1, v_7, v_4, v_2)$.

We are now prepared to determine the strong Ramsey number of the set of the four acyclic graphs of size 3 without isolated vertices.

Theorem 1. If $S = \{P_4, P_3 + K_2, 3K_2, K_{1,3}\}$, then $\bar{R}(S) = 8$.

Proof. Since $\max\{R(F) : F \in S\} = R(3K_2) = 8$, it follows by Proposition 1 that $\bar{R}(S) \geq 8$. It remains to show that $\bar{R}(S) \leq 8$. Let there be given a red-blue coloring of $G = K_8$ with $V(G) = \{v_1, v_2, \dots, v_8\}$. Since $\bar{R}(P_4, 3K_2) = 8$ by Theorem 2, there are edge-disjoint monochromatic subgraphs $F_1 = 3K_2$ and $F_2 = P_4$ in G , where say $E(F_1) = \{v_i v_{i+3} : i = 1, 2, 3\}$. If $|V(F_1) \cup V(F_2)| \leq 7$, where say $v_8 \notin V(F_1) \cup V(F_2)$, then the red-blue coloring of K_7 with $V(K_7) = \{v_1, v_2, \dots, v_7\}$ contains edge-disjoint monochromatic subgraphs $3K_2$ and P_4 . By Lemma 1, there are pairwise edge-disjoint monochromatic subgraphs $3K_2, P_4$, and $P_3 + K_2$ in G . Since at least three of the seven edges incident with v_8 in G have the same color, there is a monochromatic $K_{1,3}$ centered at v_8 that is edge-disjoint from $3K_2, P_4$, and $P_3 + K_2$ and so $\bar{R}(S) \leq 8$. Thus, we may assume that $|V(F_1) \cup V(F_2)| = 8$. First, we verify the following claim.

Claim: There are pairwise edge-disjoint monochromatic subgraphs $3K_2, P_4$, and $P_3 + K_2$ in G such that some vertex v of G is incident with at most two edges in $E(3K_2) \cup E(P_4) \cup E(P_3 + K_2)$.

To verify this claim, we may assume that $F_2 = P_4$ is one of the six 4-paths (v_7, v_8, v_1, v_2) , (v_7, v_1, v_8, v_2) , (v_7, v_1, v_8, v_4) , (v_7, v_1, v_2, v_8) , (v_1, v_7, v_8, v_4) , (v_1, v_7, v_8, v_2) shown in Figure 6.

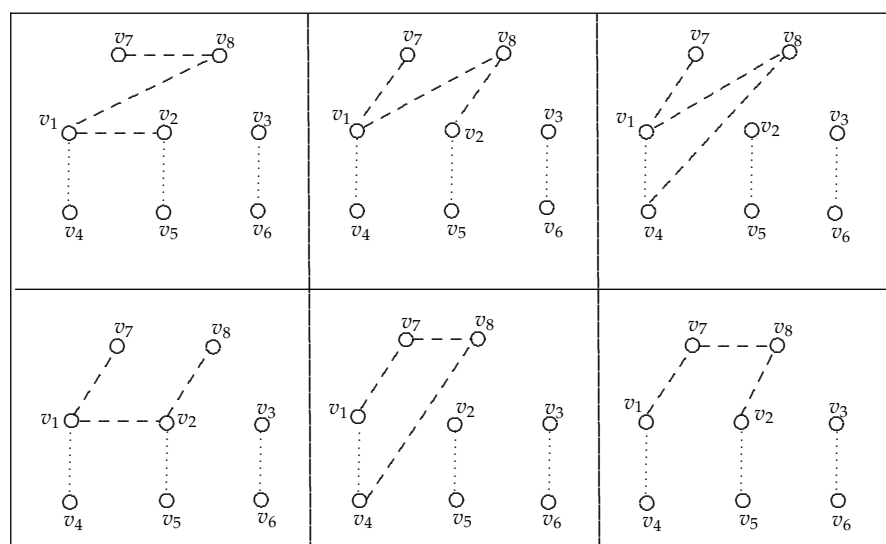


Figure 6. The six possible subgraphs $F_2 = P_4$ in G .

We now consider these six possible 4-paths for F_2 . In each of these cases, at least two of the three edges $v_7 v_2, v_7 v_4, v_7 v_5$ have the same color, say $v_7 v_i$ and $v_7 v_j$ are red where $i, j \in \{2, 4, 5\}$. Let $G' = G[\{v_1, v_2, \dots, v_8\} - \{v_i, v_j, v_7\}] - [E(F_1) \cup E(F_2)]$. If G' contains a red edge xy , then there is a red $P_3 + K_2 = (v_i, v_7, v_j) + (x, y)$ that is edge-disjoint from F_1 and F_2 and $v \in \{v_6, v_7\}$. Thus, we may assume that G' is blue. First, suppose that $F_2 = P_4 = (v_7, v_8, v_1, v_2)$. At least two of the three edges $v_7 v_2, v_7 v_4, v_7 v_5$ have the same color, say red.

★ If $v_7 v_2$ and $v_7 v_4$ are red, then $G' = G[\{v_1, v_3, v_5, v_6, v_8\}] - [E(F_1) \cup E(F_2)]$ is blue. If F_2 is red, then there are edge-disjoint monochromatic subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_8, v_3, v_5, v_1)$ is blue and $P_3 + K_2 = (v_4, v_7, v_2) + (v_1, v_8)$ is red, that are edge-disjoint from F_1 . If F_2 is blue, then there are edge-disjoint blue subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_8, v_3, v_1, v_5)$ and $P_3 + K_2 = (v_8, v_1, v_2) + (v_3, v_5)$, that are edge-disjoint from F_1 . In either situation, let $v = v_6$. The first graph in Figure 7 illustrates this situation where the four solid vertices represent the vertices of G' .

- ★ If v_7v_2 and v_7v_5 are red, then $G' = G[\{v_1, v_3, v_4, v_6, v_8\}] - [E(F_1) \cup E(F_2)]$ is blue. If F_2 is red, then there are edge-disjoint monochromatic subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_4, v_8, v_3, v_1)$ is blue and $P_3 + K_2 = (v_2, v_7, v_5) + (v_1, v_8)$ is red, that are edge-disjoint from F_1 . If F_2 is blue, then there are edge-disjoint blue subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_8, v_3, v_1, v_6)$ and $P_3 + K_2 = (v_7, v_8, v_4) + (v_1, v_2)$, that are edge-disjoint from F_1 . In either situation, let $v = v_6$. The second graph in Figure 7 illustrates this situation where the four solid vertices represent the vertices of G' .
- ★ If v_7v_4 and v_7v_5 are red, then $G' = G[\{v_1, v_2, v_3, v_6, v_8\}] - [E(F_1) \cup E(F_2)]$ is blue. If F_2 is red, then there are edge-disjoint monochromatic subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_1, v_3, v_2, v_8)$ is blue and $P_3 + K_2 = (v_4, v_7, v_5) + (v_1, v_2)$ is red, that are edge-disjoint from F_1 . Let $v = v_6$. If F_2 is blue, then there are edge-disjoint blue subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_8, v_2, v_6, v_1)$ and $P_3 + K_2 = (v_7, v_8, v_1) + (v_2, v_3)$, that are edge-disjoint from F_1 . Let $v = v_7$. The third graph in Figure 7 illustrates this situation where the four solid vertices represent the vertices of G' .

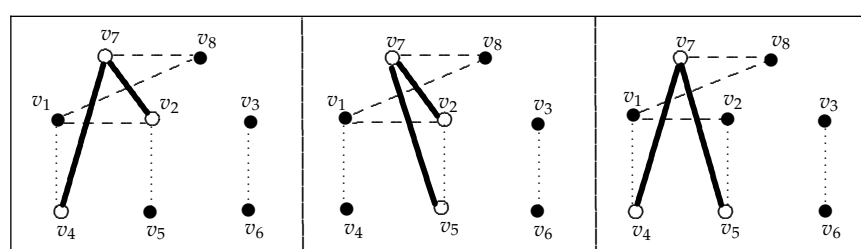


Figure 7. Illustrating the steps in the proof of Theorem 1.

If F_2 is one of the other five 4-paths (v_7, v_1, v_8, v_2) , (v_7, v_1, v_8, v_4) , (v_7, v_1, v_2, v_8) , (v_1, v_7, v_8, v_4) , (v_1, v_7, v_8, v_2) , then we apply the proof technique employed in the situation when $F_2 = P_4 = (v_7, v_8, v_1, v_2)$ and use a similar argument to create pairwise edge-disjoint monochromatic subgraphs $3K_2$, P_4 and $P_3 + K_2$ such that some vertex v of G is incident with at most two edges in $E(3K_2) \cup E(P_4) \cup E(P_3 + K_2)$. For example, if $F_2 = P_4 = (v_7, v_1, v_8, v_2)$, then at least two of the three edges v_7v_2, v_7v_4, v_7v_5 have the same color, say red.

- ★ If v_7v_2 and v_7v_4 are red, then $G' = G[\{v_1, v_3, v_5, v_6, v_8\}] - [E(F_1) \cup E(F_2)]$ is blue. If F_2 is red, then there are edge-disjoint monochromatic subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_8, v_3, v_1, v_5)$ is blue and $P_3 + K_2 = (v_4, v_7, v_2) + (v_1, v_8)$ is red, that are edge-disjoint from F_1 . If F_2 is blue, then there are edge-disjoint blue subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_8, v_3, v_1, v_5)$ and $P_3 + K_2 = (v_1, v_8, v_2) + (v_3, v_5)$, that are edge-disjoint from F_1 . In either situation, let $v = v_6$.
- ★ If v_7v_2 and v_7v_5 are red, then $G' = G[\{v_1, v_3, v_4, v_6, v_8\}] - [E(F_1) \cup E(F_2)]$ is blue. If F_2 is red, then there are edge-disjoint monochromatic subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_4, v_8, v_3, v_1)$ is blue and $P_3 + K_2 = (v_2, v_7, v_5) + (v_1, v_8)$ is red, that are edge-disjoint from F_1 . If F_2 is blue, then there are edge-disjoint blue subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_4, v_8, v_3, v_1)$ and $P_3 + K_2 = (v_7, v_1, v_8) + (v_3, v_4)$, that are edge-disjoint from F_1 . In either situation, let $v = v_6$.
- ★ If v_7v_4 and v_7v_5 are red, then $G' = G[\{v_1, v_2, v_3, v_6, v_8\}] - [E(F_1) \cup E(F_2)]$ is blue. If F_2 is red, then there are edge-disjoint monochromatic subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_2, v_1, v_3, v_8)$ is blue and $P_3 + K_2 = (v_4, v_7, v_5) + (v_2, v_8)$ is red, that are edge-disjoint from F_1 . Let $v = v_6$. If F_2 is blue, then there are edge-disjoint blue subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_8, v_2, v_6, v_1)$ and $P_3 + K_2 = (v_7, v_1, v_8) + (v_2, v_3)$, that are edge-disjoint from F_1 . Let $v = v_7$.

This completes the verification of the claim. By the claim, there is a vertex v that is incident with at least five edges in $E(G) - [E(3K_2) \cup E(P_4) \cup E(P_3 + K_2)]$. Since three of

these five edges have the same color, there is a monochromatic $K_{1,3}$ that is edge-disjoint from $3K_2$, P_4 , and $P_3 + K_2$. Hence, there are pairwise edge-disjoint monochromatic subgraphs $3K_2$, P_4 , $P_3 + K_2$, and $K_{1,3}$ in G . Therefore, $\bar{R}(S) \leq 8$ and so $\bar{R}(S) = 8$. $\square$

3. The 3-Element Sets of Graphs of Size 3

There are five graphs of size 3 containing no isolated vertices, namely K_3 , $K_{1,3}$, P_4 , $P_3 + K_2$, and $3K_2$ and so ten 3-element sets S of graphs of size 3. If $3K_2 \in S$, then $\max\{R(F) : F \in S\} = R(3K_2) = 8$, while if $3K_2 \notin S$, then $\max\{R(F) : F \in S\} = 6$. We show that $\bar{R}(S) = \max\{R(F) : F \in S\}$ for every 3-element set S of graphs of size 3 without isolated vertices. It is useful to know the strong Ramsey numbers of all pairs of graphs of size 3 without isolated vertices.

Theorem 2 ([13]). *If $F, H \in \{K_{1,3}, P_4, K_3, P_3 + K_2, 3K_2\}$, then*

$$\bar{R}(F, H) = \begin{cases} 6 & \text{if } F, H \in \{K_3, K_{1,3}, P_4, P_3 + K_2\} \\ 8 & \text{if } F \in \{K_3, K_{1,3}, P_4, P_3 + K_2\} \text{ and } H = 3K_2. \end{cases}$$

We begin with those sets S containing $3K_2$.

3.1. The 3-Element Sets of Graphs of Size 3 Containing $3K_2$

There are six sets S of graphs of size 3 containing no isolated vertices such that $3K_2 \in S$, three of which consist of the acyclic graphs of size 3 and the remaining three contain the triangle K_3 . The following is a consequence of Observation 1, Proposition 1, and Theorem 1.

Corollary 1. *If $S = \{F_1, F_2, 3K_2\}$ where $F_1, F_2 \in \{P_4, P_3 + K_2, K_{1,3}\}$, then $\bar{R}(S) = 8$.*

Theorem 3. *If $S = \{F, K_3, 3K_2\}$ where $F \in \{K_{1,3}, P_4, P_3 + K_2\}$, then $\bar{R}(S) = 8$.*

Proof. Since $\bar{R}(S) \geq R(3K_2) = 8$ by Proposition 1, it remains to show that $\bar{R}(S) \leq 8$. Let there be given a red-blue coloring of $G = K_8$ with $V(G) = \{v_1, v_2, \dots, v_8\}$. Since $\bar{R}(K_3, 3K_2) = 8$ by Theorem 2, G contains edge-disjoint monochromatic subgraphs $F_1 = K_3$ and $F_2 = 3K_2$. We may assume that $E(F_2) = \{v_i v_{i+3} : i = 1, 2, 3\}$ and $F_1 = K_3$ is one of the three triangles (v_1, v_2, v_3, v_1) , (v_1, v_2, v_7, v_1) , and (v_1, v_7, v_8, v_1) shown in Figure 8. Since $F \in \{K_{1,3}, P_4, P_3 + K_2\}$, we consider these three cases.

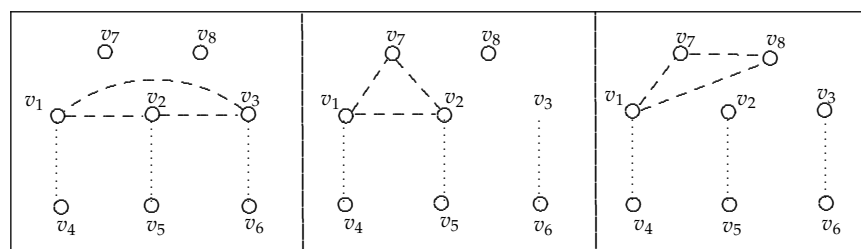


Figure 8. The three possible subgraphs $F_1 = K_3$ in G .

Case 1. $F = K_{1,3}$. Let $G_1 = G[E(F_1) \cup E(F_2)]$. Since $\delta(G_1) \leq 1$, there is a vertex $v \in V(G)$ that is incident with at least five edges in $E(G) - [E(F_1) \cup E(F_2)]$ and at least three of these five edges have the same color. Hence, there is a monochromatic $K_{1,3}$ that is edge-disjoint from F_1 and F_2 .

Case 2. $F = P_4$. At least two of the three edges $v_8 v_4, v_8 v_5, v_8 v_6$ have the same color, say $v_8 v_4$ and $v_8 v_5$ are red (since the arguments are similar for other choices of two red edges). If $v_4 v_2$ or $v_4 v_3$ is red, then there is a red P_4 that is edge-disjoint from F_1 and F_2 . Thus, we

may assume that v_4v_2 and v_4v_3 are blue. Regardless of the color of the edge v_3v_5 , there is a monochromatic P_4 that is edge-disjoint from F_1 and F_2 .

Case 3. $F = P_3 + K_2$. First, suppose that $K_3 = (v_1, v_2, v_3, v_1)$ or $K_3 = (v_1, v_2, v_7, v_1)$. At least two of the three edges v_8v_1, v_8v_2, v_8v_3 have the same color, say v_8v_i and v_8v_j are red where $i, j \in \{1, 2, 3\}$. If one of the edges in $G' = G[\{v_4, v_5, v_6\}] = K_3$ is red, then there is a red $P_3 + K_2$ in G that is edge-disjoint from F_1 and F_2 . Thus, we may assume that G' is a blue K_3 . Let $k \in \{1, 2, 3\} - \{i, j\}$. If v_8v_k is blue, then there is a blue $P_3 + K_2 = (v_4, v_5, v_6) + (v_k, v_8)$ that is edge-disjoint from F_1 and F_2 . Hence, we may assume that v_8v_k is red and so all of v_8v_1, v_8v_2, v_8v_3 are red. If one of v_1v_5 and v_3v_5 is red, say v_1v_5 , then $P_3 + K_2 = (v_2, v_8, v_3) + (v_1, v_5)$ is red that is edge-disjoint from F_1 and F_2 . Thus, we may assume that v_1v_5 and v_3v_5 are blue. Then $P_3 + K_2 = (v_1, v_5, v_3) + (v_4, v_6)$ is blue that is edge-disjoint from F_1 and F_2 .

Next, suppose that $K_3 = (v_1, v_7, v_8, v_1)$. At least two of the three edges v_8v_4, v_8v_5, v_8v_6 have the same color, say v_8v_4 and v_8v_5 are red (since the arguments are similar for other choices of two red edges). If one of the edges in $G' = G[\{v_1, v_2, v_3\}] = K_3$ is red, then there is a red $P_3 + K_2$ in G that is edge-disjoint from F_1 and F_2 . Thus, we may assume that G' is a blue K_3 . If one of the edges in $G'' = G[\{v_4, v_5, v_6\}] = K_3$ is blue, then there is a blue $P_3 + K_2$ in G that is edge-disjoint from F_1 and F_2 . Thus, we may assume that G'' is a red K_3 . If v_8v_3 is red, then there is a red $P_3 + K_2 = (v_4, v_5, v_6) + (v_3, v_8)$ that is edge-disjoint from F_1 and F_2 . Hence, we may assume that v_3v_8 is blue. For $\{i, j\} = \{4, 5\}$, this is illustrated in Figure 9, where a red edge is indicated by a solid line and a blue edge is indicated by a thin line. Regardless of the color of v_1v_5 , there is a monochromatic $P_3 + K_2$ that is edge-disjoint from F_1 and F_2 .

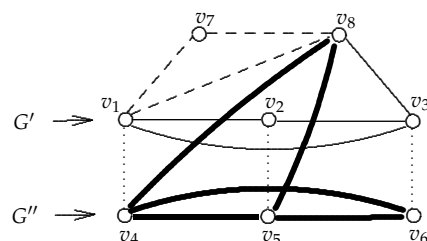


Figure 9. A step in the proof in Case 3 of Theorem 3.

Hence, for each $F \in \{K_{1,3}, P_4, P_3 + K_2\}$, there are pairwise edge-disjoint monochromatic subgraphs $F, K_3, 3K_2$ in G . Therefore, $\bar{R}(S) \leq 8$ and so $\bar{R}(S) = 8$. $\square$

In summary, we have the following.

Theorem 4. If S is a set of three graphs of size 3 without isolated vertices containing $3K_2$, then $\bar{R}(S) = \max\{R(F) : F \in S\} = 8$.

3.2. The 3-Element Sets of Graphs of Size 3 Not Containing $3K_2$

We now turn to the four 3-element sets S of graphs of size 3 without isolated vertices such that $3K_2 \notin S$. In this case, $\max\{R(F) : F \in S\} = 6$. We show that $\bar{R}(S) = 6$ for each such set S . We begin with $S = \{K_3, K_{1,3}, P_4\}$ consisting of the three connected graphs of size 3.

Theorem 5. If $S = \{K_3, K_{1,3}, P_4\}$, then $\bar{R}(S) = 6$.

Proof. Since $\max\{R(F) : F \in S\} = R(K_3) = 6$, it follows by Proposition 1 that $\bar{R}(S) \geq 6$. It remains to show that $\bar{R}(S) \leq 6$. Let there be given a red-blue coloring of $G = K_6$ with $V(G) = \{v_1, v_2, \dots, v_6\}$. Since $\bar{R}(K_3, P_4) = 6$ by Theorem 2, there are edge-disjoint

monochromatic subgraphs $F_1 = K_3$ and $F_2 = P_4$ in G . If $|V(F_1) \cup V(F_2)| \leq 5$, where say $v_6 \notin V(F_1) \cup V(F_2)$, then at least three of the five edges v_6v_i ($1 \leq i \leq 5$) have the same color and so there is a monochromatic subgraph $F_3 = K_{1,3}$ that is edge-disjoint from F_1 and F_2 . We may therefore assume that $V(F_1) \cup V(F_2) = V(G)$. Further, we may assume that $F_1 = K_3 = (v_1, v_2, v_3, v_1)$ and $F_2 = P_4$ is either (v_3, v_4, v_5, v_6) or (v_6, v_3, v_4, v_5) , as shown in Figure 10. We consider these two cases.

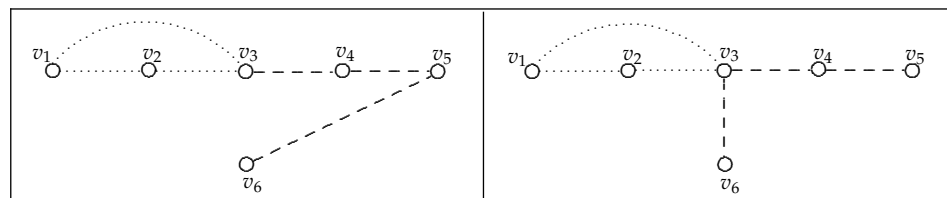


Figure 10. The two possible subgraphs $F_2 = P_4$ in G .

Case 1. $F_2 = P_4 = (v_3, v_4, v_5, v_6)$. Let $X = \{v_6v_i : i = 1, 2, 3, 4\}$. If three edges in X have the same color, then there is a monochromatic $K_{1,3}$ in G that is edge-disjoint from F_1 and F_2 . Thus, we may assume that exactly two edges in X are red, say v_6v_1 and v_6v_i are red where $i \in \{2, 3, 4\}$. Hence, there are three possibilities. An argument is given only for $i = 2$ since the proof technique is the same and the arguments for the choices $i = 3$ and $i = 4$ are similar.

Suppose that v_6v_1 and v_6v_2 are red. If both v_1v_4 and v_1v_5 are red, then there is a red $F_3 = K_{1,3}$ with $E(F_3) = \{v_1v_i : i = 4, 5, 6\}$ in G that is edge-disjoint from F_1 and F_2 . Thus, we may assume that at least one of v_1v_4 and v_1v_5 is blue, which results in three possibilities.

- ★ First, suppose that v_1v_4 is blue and v_1v_5 is red. Then (v_5, v_1, v_6, v_2) is a red P_4 and (v_1, v_4, v_6, v_3) is a blue P_4 . If F_2 is red, then there are edge-disjoint monochromatic subgraphs P_4 and $K_{1,3}$ where $P_4 = (v_1, v_4, v_6, v_3)$ is blue and $K_{1,3}$ is red with $E(K_{1,3}) = \{v_6v_i : i = 1, 2, 5\}$ that are edge-disjoint from F_1 . If F_2 is blue, then there are edge-disjoint monochromatic subgraphs P_4 and $K_{1,3}$ where $P_4 = (v_5, v_1, v_6, v_2)$ is red and $K_{1,3}$ is blue with $E(K_{1,3}) = \{v_6v_i : i = 3, 4, 5\}$ that are edge-disjoint from F_1 .
- ★ Next, suppose that v_1v_4 is red and v_1v_5 is blue. Then (v_4, v_1, v_6, v_2) is a red P_4 . If F_2 is red, then regardless of the color of v_3v_5 , there are edge-disjoint monochromatic subgraphs P_4 and $K_{1,3}$ that are edge-disjoint from F_1 . For example, if v_3v_5 is red, then $P_4 = (v_1, v_4, v_3, v_5)$ is red and $K_{1,3}$ is red with $E(K_{1,3}) = \{v_6v_i : i = 1, 2, 5\}$. If v_3v_5 is blue, then $P_4 = (v_4, v_6, v_3, v_5)$ is blue and $K_{1,3}$ is red with $E(K_{1,3}) = \{v_6v_i : i = 1, 2, 5\}$. If F_2 is blue, then there are edge-disjoint monochromatic subgraphs P_4 and $K_{1,3}$ where $P_4 = (v_4, v_1, v_6, v_2)$ is red and $K_{1,3}$ is blue with $E(K_{1,3}) = \{v_6v_i : i = 3, 4, 5\}$ that are edge-disjoint from F_1 .
- ★ Finally, suppose that both v_1v_4 and v_1v_5 are blue. Then $F'_2 = (v_1, v_4, v_6, v_3)$ is a blue P_4 . If F_2 is red, then $K_{1,3}$ is red with $E(K_{1,3}) = \{v_6v_i : i = 1, 2, 5\}$ is edge-disjoint from F'_2 and F_1 . If F_2 is blue, then $K_{1,3}$ is blue with $E(K_{1,3}) = \{v_5v_i : i = 1, 4, 6\}$ is edge-disjoint from F'_2 and F_1 .

Case 2. $F_2 = P_4 = (v_6, v_3, v_4, v_5)$. Let $Y = \{v_6v_i : i = 1, 2, 4, 5\}$. If three edges in Y have the same color, then there is a monochromatic $K_{1,3}$ in G that is edge-disjoint from F_1 and F_2 . Thus, we may assume that exactly two edges in Y are red, say v_6v_1 and v_6v_i are red, where $i \in \{2, 4, 5\}$, which results in three possibilities. An argument is given only for $i = 2$ since the proof technique is the same and the arguments for the choices $i = 4$ and $i = 5$ are similar.

Suppose that v_6v_1 and v_6v_2 are red. If both v_1v_4 and v_1v_5 are red, then there is a red $F_3 = K_{1,3}$ with $E(F_3) = \{v_1v_i : i = 4, 5, 6\}$ in G that is edge-disjoint from F_1 and F_2 , for otherwise at least one of v_1v_4 and v_1v_5 is blue, which results in three possibilities.

- ★ First, suppose that v_1v_4 is blue and v_1v_5 is red. Then (v_5, v_1, v_6, v_2) is a red P_4 and (v_1, v_4, v_6, v_5) is a blue P_4 . Regardless of the color of F_2 , there are edge-disjoint monochromatic subgraphs P_4 and $K_{1,3}$ that are edge-disjoint from F_1 .
- ★ Next, suppose that v_1v_4 is red and v_1v_5 is blue. Then (v_4, v_1, v_6, v_2) is a red P_4 and (v_1, v_5, v_6, v_4) is a blue P_4 . Regardless of the color of F_2 , there are edge-disjoint monochromatic subgraphs P_4 and $K_{1,3}$ that are edge-disjoint from F_1 .
- ★ Finally, suppose that both v_1v_4 and v_1v_5 are blue. Then (v_1, v_4, v_6, v_5) is a blue P_4 . If F_2 is red, then there are edge-disjoint monochromatic subgraphs P_4 and $K_{1,3}$ where $P_4 = (v_1, v_4, v_6, v_5)$ is blue and $K_{1,3}$ is red with $E(K_{1,3}) = \{v_6v_i : i = 1, 2, 3\}$ that are edge-disjoint from F_1 . If F_2 is blue, then there are edge-disjoint blue subgraphs P_4 and $K_{1,3}$ where $P_4 = (v_1, v_5, v_4, v_3)$ and $E(K_{1,3}) = \{v_6v_i : i = 3, 4, 5\}$ that are edge-disjoint from F_1 .

Hence, G contains pairwise edge-disjoint monochromatic subgraphs $K_3, K_{1,3}, P_4$. Therefore, $\bar{R}(S) \leq 6$ and so $\bar{R}(S) = 6$. $\square$

Theorem 6. If $S = \{K_3, K_{1,3}, P_3 + K_2\}$, then $\bar{R}(S) = 6$.

Proof. Since $\max\{R(F) : F \in S\} = R(K_3) = 6$, it follows by Proposition 1 that $\bar{R}(S) \geq 6$. Thus, it remains to show that $\bar{R}(S) \leq 6$. Let there be given a red-blue coloring of $G = K_6$ with $V(G) = \{v_1, v_2, \dots, v_6\}$. Since $\bar{R}(K_3, P_3 + K_2) = 6$ by Theorem 2, there are edge-disjoint monochromatic subgraphs $F_1 = K_3$ and $F_2 = P_3 + K_2$ in G . We may assume that $V(F_1) = \{v_1, v_2, v_3\}$ and $F_2 = P_3 + K_2$ is one of the following three subgraphs $(v_3, v_4, v_5) + (v_1, v_6)$, $(v_4, v_3, v_5) + (v_1, v_6)$, $(v_1, v_6, v_2) + (v_4, v_5)$ in G shown in Figure 11. We consider these three possibilities.

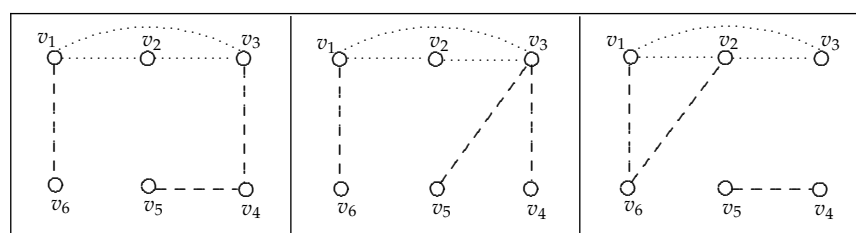


Figure 11. The three possible subgraphs $F_2 = P_3 + K_2$ in G .

In each situation, at least two of the three edges v_5v_1, v_5v_2, v_5v_6 have the same color, say (1) v_5v_1 and v_5v_6 are red, (2) v_5v_1 and v_5v_2 are red, or (3) v_5v_2 and v_5v_6 are red. We provide an argument only for (1) since the proof technique is the same and the arguments for (2) and (3) are similar.

First, suppose that $F_2 = P_3 + K_2 = (v_3, v_4, v_5) + (v_1, v_6)$. If v_5v_1 and v_5v_6 are red, then v_5v_2 and v_5v_3 are blue; for otherwise there is a red $K_{1,3}$ that is edge-disjoint from F_1 and F_2 .

- ★ If F_2 is red, then there is a red $F_3 = K_{1,3}$ with $E(F_3) = \{v_5v_i : i = 1, 4, 6\}$. If one of v_2v_6 and v_2v_4 is red, say v_2v_6 , then there is a red $P_3 + K_2 = (v_1, v_6, v_2) + (v_3, v_4)$ that is edge-disjoint from F_1 and F_3 . If v_2v_6 and v_2v_4 are blue, then there is a blue $P_3 + K_2 = (v_6, v_2, v_4) + (v_3, v_5)$ that is edge-disjoint from F_1 and F_3 .
- ★ If F_2 is blue, then there is a blue $F_3 = K_{1,3}$ with $E(F_3) = \{v_5v_i : i = 2, 3, 4\}$. If one of v_2v_6 and v_2v_4 is blue, say v_2v_6 , then there is a blue $P_3 + K_2 = (v_1, v_6, v_2) + (v_3, v_4)$ that is edge-disjoint from F_1 and F_3 . If v_2v_6 and v_2v_4 are red, then there is a red $P_3 + K_2 = (v_6, v_2, v_4) + (v_1, v_5)$ that is edge-disjoint from F_1 and F_3 .

Next, suppose that $F_2 = P_3 + K_2 = (v_4, v_3, v_5) + (v_1, v_6)$. If v_5v_1 and v_5v_6 are red, then v_5v_2 and v_5v_4 are blue; for otherwise there is a red $K_{1,3}$ that is edge-disjoint from F_1 and F_2 .

- ★ If F_2 is red, then there is a red $F_3 = K_{1,3}$ with $E(F_3) = \{v_5v_i : i = 1, 3, 6\}$. If one of v_2v_6 and v_2v_4 is red, say v_2v_6 , then there is a red $P_3 + K_2 = (v_1, v_6, v_2) + (v_3, v_4)$ that is edge-disjoint from F_1 and F_3 . If v_2v_6 and v_2v_4 are blue, then there are edge-disjoint monochromatic subgraphs $K_{1,3}$ and $P_3 + K_2$, where $K_{1,3}$ is blue with $E(K_{1,3}) = \{v_2v_i : i = 4, 5, 6\}$ and $P_3 + K_2 = (v_6, v_1, v_5) + (v_3, v_4)$ is red, that are edge-disjoint from F_1 .
- ★ If F_2 is blue, then there is a blue $F_3 = K_{1,3}$ with $E(F_3) = \{v_5v_i : i = 2, 3, 4\}$. If one of v_2v_6 and v_2v_4 is blue, say v_2v_6 , then there is a blue $P_3 + K_2 = (v_1, v_6, v_2) + (v_3, v_4)$ that is edge-disjoint from F_1 and F_3 . If v_2v_6 and v_2v_4 are red, then there is a red $P_3 + K_2 = (v_6, v_2, v_4) + (v_1, v_5)$ that is edge-disjoint from F_1 and F_3 .

Finally, suppose that $F_2 = P_3 + K_2 = (v_1, v_6, v_2) + (v_4, v_5)$. If v_5v_1 and v_5v_6 are red, then v_5v_2 and v_5v_3 are blue; for otherwise there is a red $K_{1,3}$ that is edge-disjoint from F_1 and F_2 .

- ★ If F_2 is red, then there is a red $F_3 = K_{1,3}$ with $E(F_3) = \{v_5v_i : i = 1, 4, 6\}$. Then v_4v_6 is red, for otherwise there is a blue $P_3 + K_2 = (v_2, v_5, v_3) + (v_4, v_6)$ that is edge-disjoint from F_1 and F_3 . If v_3v_4 is red, then there is a red $P_3 + K_2 = (v_1, v_6, v_2) + (v_3, v_4)$ that is edge-disjoint from F_1 and F_3 . Thus, we may assume that v_3v_4 is blue. If v_3v_6 is blue, then there are edge-disjoint monochromatic subgraphs $K_{1,3}$ and $P_3 + K_2$, where $K_{1,3}$ is blue with $E(K_{1,3}) = \{v_3v_i : i = 4, 5, 6\}$ and $P_3 + K_2 = (v_1, v_6, v_2) + (v_4, v_5)$ is red, that are edge-disjoint from F_1 . If v_3v_6 is red, then there are edge-disjoint red subgraphs $K_{1,3}$ with $E(K_{1,3}) = \{v_6v_i : i = 3, 4, 5\}$ and $P_3 + K_2 = (v_1, v_6, v_2) + (v_4, v_5)$ that are edge-disjoint from F_1 . This procedure is illustrated in Figure 12.

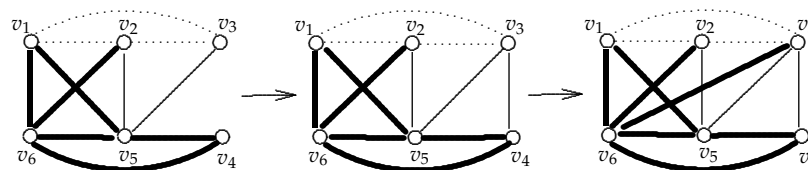


Figure 12. Illustrating the arguments when F_2 is red.

- ★ If F_2 is blue, then there is a blue $F_3 = K_{1,3}$ with $E(F_3) = \{v_5v_i : i = 2, 3, 4\}$. Regardless of the color of v_3v_4 , there is monochromatic $P_3 + K_2$ that is edge-disjoint from F_1 and F_3 .

Thus, there are pairwise edge-disjoint subgraphs $K_3, K_{1,3}, P_3 + K_2$ in G . Therefore, $\bar{R}(S) \leq 6$ and so $\bar{R}(S) = 6$. $\square$

Theorem 7. If $S = \{K_3, P_4, P_3 + K_2\}$, then $\bar{R}(S) = 6$.

Proof. Since $\max\{R(F) : F \in S\} = R(K_3) = 6$, it follows by Proposition 1 that $\bar{R}(S) \geq 6$. Thus, it remains to show that $\bar{R}(S) \leq 6$. Let there be given a red-blue coloring of $G = K_6$ with $V(G) = \{v_1, v_2, \dots, v_6\}$. Since $\bar{R}(K_3, P_4) = 6$ by Theorem 2, there are edge-disjoint monochromatic subgraphs $F_1 = K_3$ and $F_2 = P_4$ in G . We may assume that $V(F_1) = \{v_1, v_2, v_3\}$ and $F_2 = P_4$ is one of the following four subgraphs (v_1, v_4, v_5, v_2) , (v_4, v_1, v_5, v_2) , (v_1, v_4, v_5, v_6) , (v_4, v_1, v_5, v_6) in G shown in Figure 13. We consider these four possibilities.

In each situation, at least two of the three edges v_6v_1, v_6v_2, v_6v_3 have the same color, namely either (1) v_6v_1 and v_6v_2 are red, (2) v_6v_1 and v_6v_3 are red, or (3) v_6v_2 and v_6v_3 are red. We provide an argument only for (1) since the proof technique is the same and the arguments for (2) and (3) are similar.

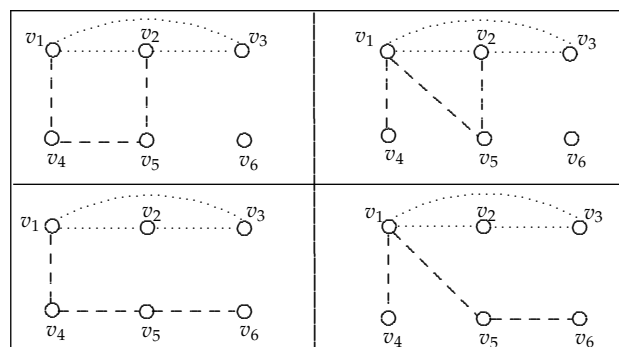


Figure 13. The four possible subgraphs $F_2 = P_4$ in G .

First, suppose that $F_2 = P_4 = (v_1, v_4, v_5, v_2)$. If v_6v_1 and v_6v_2 are red, then v_3v_4 and v_3v_5 are blue; for otherwise, say v_3v_4 is red, there is a red $P_3 + K_2 = (v_1, v_6, v_2) + (v_3, v_4)$ that is edge-disjoint from F_1 and F_2 .

- ★ First, suppose that F_2 is red. If one of $v_4v_2, v_4v_6, v_5v_1, v_5v_6$ is blue, say v_4v_2 , then there are edge-disjoint monochromatic subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_2, v_4, v_3, v_5)$ is blue and $P_3 + K_2 = (v_1, v_4, v_5) + (v_2, v_6)$ is red, that are edge-disjoint from F_1 . If $v_4v_2, v_4v_6, v_5v_1, v_5v_6$ are all red, then there are edge-disjoint red subgraphs $P_4 = (v_4, v_2, v_5, v_6)$ and $P_3 + K_2 = (v_1, v_4, v_5) + (v_2, v_6)$ that are edge-disjoint from F_1 .
- ★ Next, suppose that F_2 is blue. If v_1v_5 or v_2v_4 is red, say v_1v_5 , then there are edge-disjoint monochromatic subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_5, v_1, v_6, v_2)$ is red and $P_3 + K_2 = (v_1, v_4, v_3) + (v_2, v_5)$ is blue, that are edge-disjoint from F_1 . If v_1v_5 and v_2v_4 are blue, there are edge-disjoint blue subgraphs $P_4 = (v_1, v_5, v_4, v_2)$ and $P_3 + K_2 = (v_1, v_4, v_3) + (v_2, v_5)$ that are edge-disjoint from F_1 .

Next, suppose that $F_2 = P_4 = (v_4, v_1, v_5, v_2)$. If one of v_3v_4, v_3v_5, v_4v_5 is red, say v_3v_4 , then there is a red $P_3 + K_2 = (v_1, v_6, v_2) + (v_3, v_4)$ that is edge-disjoint from F_1 and F_3 . Thus, we may assume that v_3v_4, v_3v_5, v_4v_5 are blue.

- ★ First, suppose that F_2 is red. If v_5v_6 or v_3v_6 is blue, say the former, then there are edge-disjoint monochromatic subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_6, v_5, v_4, v_3)$ is blue and $P_3 + K_2 = (v_4, v_1, v_5) + (v_2, v_6)$ is red, that are edge-disjoint from F_1 . If v_5v_6 and v_3v_6 are red, then there are edge-disjoint red subgraphs $P_4 = (v_1, v_6, v_2, v_5)$ and $P_3 + K_2 = (v_4, v_1, v_5) + (v_3, v_6)$ that are edge-disjoint from F_1 .
- ★ Next, suppose that F_2 is blue. If v_2v_4 is red, then there are edge-disjoint monochromatic subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_4, v_2, v_6, v_1)$ is red and $P_3 + K_2 = (v_1, v_4, v_3) + (v_2, v_5)$ is blue that are edge-disjoint from F_1 . If v_2v_4 is blue, then there are edge-disjoint blue subgraphs $P_4 = (v_2, v_4, v_5, v_3)$ and $P_3 + K_2 = (v_1, v_4, v_3) + (v_2, v_5)$ that are edge-disjoint from F_1 .

We now assume that $F_2 = P_4 = (v_1, v_4, v_5, v_6)$. If v_3v_4 or v_3v_5 is red, say the former, then there is a red $P_3 + K_2 = (v_1, v_6, v_2) + (v_3, v_4)$ that is edge-disjoint from F_1 and F_3 . Thus, we may assume that v_3v_4 and v_3v_5 are blue.

- ★ First, suppose that F_2 is red. If one of $v_4v_2, v_4v_6, v_5v_1, v_5v_2$ is blue, say v_4v_2 , then there are edge-disjoint monochromatic subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_2, v_4, v_3, v_5)$ is blue and $P_3 + K_2 = (v_1, v_4, v_5) + (v_2, v_6)$ is red, that are edge-disjoint from F_1 . If $v_4v_2, v_4v_6, v_5v_1, v_5v_2$ are all red, then there are edge-disjoint red subgraphs $P_4 = (v_1, v_5, v_2, v_4)$ and $P_3 + K_2 = (v_1, v_4, v_5) + (v_2, v_6)$ that are edge-disjoint from F_1 .
- ★ Next, suppose that F_2 is blue. If v_2v_4 is red, then there are edge-disjoint monochromatic subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_4, v_2, v_6, v_1)$ is red and $P_3 + K_2 = (v_1, v_4, v_3) + (v_5, v_6)$ is blue, that are edge-disjoint from F_1 . If v_2v_4 is blue, then there are

edge-disjoint blue subgraphs $P_4 = (v_2, v_4, v_5, v_3)$ and $P_3 + K_2 = (v_1, v_4, v_3) + (v_5, v_6)$ that are edge-disjoint from F_1 .

Finally, suppose that $F_2 = P_4 = (v_4, v_1, v_5, v_6)$. If one of v_3v_4, v_3v_5, v_4v_5 is red, say v_3v_4 , then there is a red $P_3 + K_2 = (v_1, v_6, v_2) + (v_3, v_4)$ that is edge-disjoint from F_1 and F_3 . Thus, we may assume that v_3v_4, v_3v_5, v_4v_5 are blue.

- ★ First, suppose that F_2 is red. If v_3v_6 is blue, then there are edge-disjoint monochromatic subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_5, v_4, v_3, v_6)$ is blue and $P_3 + K_2 = (v_4, v_1, v_5) + (v_2, v_6)$ is red, that are edge-disjoint from F_1 . If v_3v_6 is red, then there are edge-disjoint red subgraphs $P_4 = (v_5, v_1, v_6, v_2)$ and $P_3 + K_2 = (v_5, v_6, v_3) + (v_1, v_4)$ that are edge-disjoint from F_1 .
- ★ Next, suppose that F_2 is blue. If v_2v_5 or v_2v_4 is red, say the former, then there are edge-disjoint monochromatic subgraphs P_4 and $P_3 + K_2$, where $P_4 = (v_5, v_2, v_6, v_1)$ is red and $P_3 + K_2 = (v_1, v_4, v_3) + (v_5, v_6)$ is blue, that are edge-disjoint from F_1 . If v_2v_5 and v_2v_4 are blue, then there are edge-disjoint blue subgraphs $P_4 = (v_1, v_5, v_4, v_2)$ and $P_3 + K_2 = (v_1, v_4, v_3) + (v_5, v_6)$ that are edge-disjoint from F_1 .

Hence, there are pairwise edge-disjoint monochromatic subgraphs $K_3, P_4, P_3 + K_2$ in G . Therefore, $\bar{R}(S) \leq 6$ and so $\bar{R}(S) = 6$. $\square$

Theorem 8. If $S = \{P_4, P_3 + K_2, K_{1,3}\}$, then $\bar{R}(S) = 6$.

Proof. Since $\max\{R(F) : F \in S\} = R(K_{1,3}) = 6$, it follows by Proposition 1 that $\bar{R}(S) \geq 6$. Thus, it remains to show that $\bar{R}(S) \leq 6$. Let there be given a red-blue coloring of $G = K_6$ with $V(G) = \{v_1, v_2, \dots, v_6\}$. Since $\bar{R}(P_4, K_{1,3}) = 6$ by Theorem 2, there are edge-disjoint monochromatic subgraphs $F_1 = P_4$ and $F_2 = K_{1,3}$ in G . We may assume that $F_1 = P_4 = (v_1, v_2, v_3, v_4)$. For the location of F_2 in G , we consider two cases, according to the location of the center of F_2 .

Case 1. The center of $K_{1,3}$ is either v_5 or v_6 , say the former. By symmetry, we may assume that $E(F_2)$ is one of the following six sets $\{v_5v_i : i = 1, 2, 3\}$, $\{v_5v_i : i = 1, 2, 4\}$, $\{v_5v_i : i = 1, 2, 6\}$, $\{v_5v_i : i = 1, 3, 6\}$, $\{v_5v_i : i = 1, 4, 6\}$, $\{v_5v_i : i = 2, 3, 6\}$ shown in Figure 14.

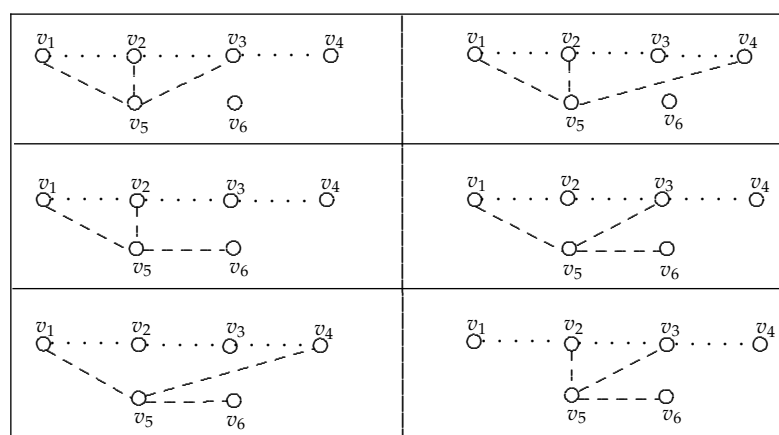


Figure 14. The six possible subgraphs for $F_2 = K_{1,3}$ in Case 1.

We provide an argument only for $E(F_2) = \{v_5v_i : i = 1, 2, 3\}$ since the proof technique is the same and the arguments for other choices of F_2 are similar. At least two of v_5v_i ($i = 1, 2, 3$) have the same color, say red. There are three possibilities.

- First, suppose that v_5v_1 and v_5v_2 are red. Then v_4v_5 is blue, for otherwise there is a red $P_3 + K_2 = (v_1, v_6, v_2) + (v_4, v_5)$ that is edge-disjoint from F_1 and F_2 .

- ★ If F_2 is red, then v_5v_6 is blue, for otherwise there are edge-disjoint red subgraphs $K_{1,3}$ with $E(K_{1,3}) = \{v_5v_i : i = 1, 2, 6\}$ and $P_3 + K_2 = (v_1, v_6, v_2) + (v_3, v_5)$ that are edge-disjoint from F_1 . Then v_1v_3 is red, for otherwise there is a blue $P_3 + K_2 = (v_4, v_5, v_6) + (v_1, v_3)$ that is edge-disjoint from F_1 and F_2 . Then v_1v_4 and v_2v_4 are blue, for otherwise, say v_1v_4 is red, then there are edge-disjoint red subgraphs $K_{1,3}$ with $E(K_{1,3}) = \{v_1v_i : i = 3, 4, 5\}$ and $P_3 + K_2 = (v_1, v_6, v_2) + (v_3, v_5)$ that are edge-disjoint from F_1 . Then $P_3 + K_2 = (v_1, v_4, v_2) + (v_5, v_6)$ that is edge-disjoint from F_1 and F_2 .
- ★ We now assume that F_2 is blue. First, suppose that v_5v_6 is red. Then v_1v_4 and v_2v_4 are blue, for otherwise, say v_1v_4 is red, then $P_3 + K_2 = (v_2, v_6, v_5) + (v_1, v_4)$ is edge-disjoint from F_1 and F_2 . Then there are edge-disjoint monochromatic subgraphs $K_{1,3}$ and $P_3 + K_2$, where $K_{1,3}$ is red with $E(K_{1,3}) = \{v_6v_i : i = 1, 2, 5\}$ and $P_3 + K_2 = (v_1, v_5, v_3) + (v_2, v_4)$ is blue, that are edge-disjoint from F_1 . Next, suppose that v_5v_6 is blue. Then v_3v_6 and v_4v_6 are red, for otherwise, say v_3v_6 is blue, then there are edge-disjoint blue subgraphs $K_{1,3}$ with $E(K_{1,3}) = \{v_5v_i : i = 3, 4, 6\}$ and $P_3 + K_2 = (v_1, v_5, v_2) + (v_3, v_6)$ that are edge-disjoint from F_1 . If v_1v_4 is red, there is a red $P_3 + K_2 = (v_2, v_6, v_3) + (v_1, v_4)$ that is edge-disjoint from F_1 and F_2 . If v_1v_4 is blue, then there are blue subgraphs $K_{1,3}$ with $E(K_{1,3}) = \{v_5v_i : i = 1, 4, 6\}$ and $P_3 + K_2 = (v_2, v_5, v_3) + (v_1, v_4)$ that are edge-disjoint from F_1 .
- Next, suppose that v_6v_1 and v_6v_3 are red. By the argument above, we may assume that v_6v_2 is blue. Then v_2v_4 and v_5v_4 are blue, for otherwise, say v_2v_4 is red, there is a red $P_3 + K_2 = (v_1, v_6, v_3) + (v_2, v_4)$ that is edge-disjoint from F_1 and F_2 .
 - ★ If F_2 is red, then v_1v_3 and v_1v_4 are red, for otherwise, say v_1v_3 is blue, there is a blue $P_3 + K_2 = (v_2, v_4, v_5) + (v_1, v_3)$ that is edge-disjoint from F_1 and F_2 . Then there are edge-disjoint red subgraphs $K_{1,3}$ with $E(K_{1,3}) = \{v_1v_i : i = 3, 4, 6\}$ and $P_3 + K_2 = (v_1, v_5, v_2) + (v_3, v_6)$ that are edge-disjoint from F_1 .
 - ★ If F_2 is blue, then there are edge-disjoint blue subgraphs $K_{1,3}$ with $E(K_{1,3}) = \{v_5v_i : i = 2, 3, 4\}$ and $P_3 + K_2 = (v_4, v_2, v_6) + (v_1, v_5)$ that are edge-disjoint from F_1 .
- Finally, suppose that v_6v_2 and v_6v_3 are red. We may assume that v_6v_1 is blue. Then v_1v_4 and v_5v_4 are blue, for otherwise, say v_1v_4 is red, there is a red $P_3 + K_2 = (v_2, v_6, v_3) + (v_1, v_4)$ that is edge-disjoint from F_1 and F_2 .
 - ★ First, suppose that F_2 is red. If one of v_4v_6, v_4v_2, v_1v_3 is blue, say v_4v_6 , then there are edge-disjoint monochromatic subgraphs $K_{1,3}$ and $P_3 + K_2$, where $K_{1,3}$ is blue with $E(K_{1,3}) = \{v_4v_i : i = 1, 5, 6\}$ and $P_3 + K_2 = (v_1, v_5, v_2) + (v_3, v_6)$ is red, that are edge-disjoint from F_1 . If all v_4v_6, v_4v_2, v_1v_3 are red, then there is a red $P_3 + K_2 = (v_2, v_4, v_6) + (v_1, v_3)$ that is edge-disjoint from F_1 and F_2 .
 - ★ Next, suppose that F_2 is blue. If v_5v_6 is blue, then there are edge-disjoint blue subgraphs $K_{1,3}$ with $E(K_{1,3}) = \{v_5v_i : i = 1, 4, 6\}$ and $P_3 + K_2 = (v_2, v_5, v_3) + (v_1, v_4)$ that are edge-disjoint from F_1 . If v_5v_6 is red, then there are edge-disjoint monochromatic subgraphs $K_{1,3}$ and $P_3 + K_2$, where $K_{1,3}$ is red with $E(K_{1,3}) = \{v_6v_i : i = 2, 3, 5\}$ and $P_3 + K_2 = (v_2, v_5, v_3) + (v_1, v_4)$ is blue, that are edge-disjoint from F_1 .

Case 2. The center of $K_{1,3}$ belongs to $\{v_1, v_2, v_3, v_4\}$. By symmetry, we may assume that $E(F_2)$ is one of the following four sets $\{v_1v_i : i = 3, 5, 6\}$, $\{v_1v_i : i = 4, 5, 6\}$, $\{v_1v_i : i = 3, 4, 5\}$, $\{v_2v_i : i = 4, 5, 6\}$, shown in Figure 15.

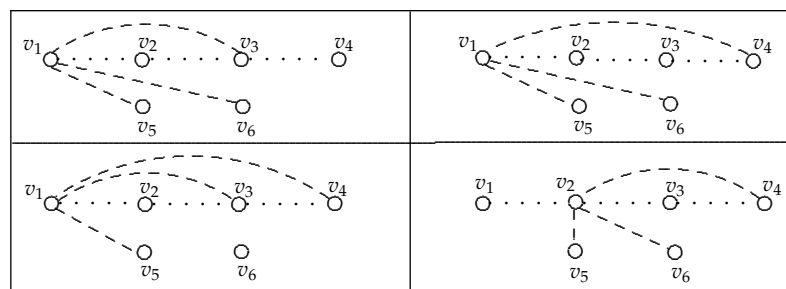


Figure 15. The four possible subgraphs for $F_2 = K_{1,3}$ in Case 2.

We provide an argument only for $E(F_2) = \{v_1v_i : i = 3, 5, 6\}$ since the proof technique is the same and the arguments for other choices of F_2 are similar. At least two edges in $\{v_6v_i : i = 2, 3, 4\}$ have the same color, say red. There are three possibilities.

- First, suppose that v_6v_2 and v_6v_3 are red. Then v_4v_1 and v_4v_5 are blue, for otherwise, say v_4v_1 is red, there is a red $P_3 + K_2 = (v_2, v_6, v_3) + (v_1, v_4)$ that is edge-disjoint from F_1 and F_2 .
 - ★ If F_2 is red, then there is a red $F'_2 = K_{1,3}$ with $E(F'_2) = \{v_6v_i : i = 1, 2, 3\}$. If v_2v_4 is red, then there is a red $P_3 + K_2 = (v_3, v_1, v_5) + (v_2, v_4)$ that are edge-disjoint from F_1 and F'_2 . If v_2v_4 is blue, then there are edge-disjoint monochromatic subgraphs $K_{1,3}$ and $P_3 + K_2$, where $K_{1,3}$ is blue with $E(K_{1,3}) = \{v_4v_i : i = 1, 2, 5\}$ and $P_3 + K_2 = (v_2, v_6, v_3) + (v_1, v_5)$ is red, that are edge-disjoint from F_1 .
 - ★ If F_2 is blue, then there is blue $F_3 = P_3 + K_2 = (v_3, v_1, v_6) + (v_4, v_5)$ that is edge-disjoint from F_1 . If v_4v_6 or v_5v_6 is red, say the former, then there is a red $K_{1,3}$ with $E(K_{1,3}) = \{v_6v_i : i = 2, 3, 4\}$ that is edge-disjoint from F_1 and F_3 . Thus, we may assume that v_4v_6 and v_5v_6 are blue. If v_2v_4 or v_2v_5 is blue, say the former, then there is a blue $K_{1,3}$ with $E(K_{1,3}) = \{v_4v_i : i = 1, 2, 6\}$ that is edge-disjoint from F_1 and F_3 . If v_2v_4 and v_2v_5 are red, then there is a red $K_{1,3}$ with $E(K_{1,3}) = \{v_2v_i : i = 4, 5, 6\}$ that is edge-disjoint from F_1 and F_3 .
- Next, suppose that v_6v_2 and v_6v_4 are red. By the argument above, we may assume that v_6v_3 is blue. Then v_3v_5 is blue, for otherwise there is a red $P_3 + K_2 = (v_2, v_6, v_4) + (v_3, v_5)$ that is edge-disjoint from F_1 and F_2 . This implies that v_1v_4 and v_2v_4 are red, for otherwise, say v_1v_4 is blue, there is a blue $P_3 + K_2 = (v_5, v_3, v_6) + (v_1, v_4)$ that is edge-disjoint from F_1 and F_2 .
 - ★ If F_2 is red, then there are edge-disjoint red subgraphs $K_{1,3}$ with $E(K_{1,3}) = \{v_1v_i : i = 3, 4, 6\}$ and $P_3 + K_2 = (v_4, v_2, v_6) + (v_1, v_5)$ that are edge-disjoint from F_1 .
 - ★ We now assume that F_2 is blue. If v_2v_5 is red, there is a red $P_3 + K_2 = (v_1, v_4, v_6) + (v_2, v_5)$ is edge-disjoint from F_1 and F_2 . If v_2v_5 is blue, then there are edge-disjoint monochromatic subgraphs $K_{1,3}$ and $P_3 + K_2$, where $K_{1,3}$ is red with $E(K_{1,3}) = \{v_4v_i : i = 1, 2, 6\}$ and $P_3 + K_2 = (v_1, v_3, v_6) + (v_2, v_5)$ is blue, that are edge-disjoint from F_1 .
- Finally, suppose that v_6v_3 and v_6v_4 are red. We may assume that v_6v_2 is blue. Then v_2v_5 is blue, for otherwise there is a red $P_3 + K_2 = (v_3, v_6, v_4) + (v_2, v_5)$ that is edge-disjoint from F_1 and F_2 . This implies that v_1v_4 is red, for otherwise there is a blue $P_3 + K_2 = (v_5, v_2, v_6) + (v_1, v_4)$ that is edge-disjoint from F_1 and F_2 .
 - ★ If F_2 is red, then there are edge-disjoint red subgraphs $K_{1,3}$ with $E(K_{1,3}) = \{v_1v_i : i = 3, 4, 6\}$ and $P_3 + K_2 = (v_3, v_6, v_4) + (v_1, v_5)$ that are edge-disjoint from F_1 .
 - ★ We now assume that F_2 is blue. If v_5v_6 or v_2v_4 is red, say the former, then there are edge-disjoint monochromatic subgraphs $K_{1,3}$ and $P_3 + K_2$, where $K_{1,3}$ is red with $E(K_{1,3}) = \{v_6v_i : i = 3, 4, 5\}$ and $P_3 + K_2 = (v_5, v_1, v_3) + (v_2, v_6)$

is blue, that are edge-disjoint from F_1 . If v_5v_6 and v_2v_4 are blue, then there are edge-disjoint blue subgraphs $K_{1,3}$ with $E(K_{1,3}) = \{v_5v_i : i = 1, 2, 6\}$ and $P_3 + K_2 = (v_4, v_2, v_6) + (v_1, v_3)$ that are edge-disjoint from F_1 .

Hence, there are pairwise edge-disjoint monochromatic subgraphs $P_4, P_3 + K_2, K_{1,3}$ in G . Therefore, $\bar{R}(S) \leq 6$ and so $\bar{R}(S) = 6$. $\square$

In summary, we have the following.

Theorem 9. *If S is a set of three graphs of size 3 without isolated vertices not containing $3K_2$, then $\bar{R}(S) = \max\{R(F) : F \in S\} = 6$.*

4. Closing Remarks

By Theorems 2, 4, and 9, we have the following result.

Theorem 10. *For every set S of two or three graphs of size 3 without isolated vertices,*

$$\bar{R}(S) = \max\{R(F) : F \in S\}.$$

By Theorem 1, if S is the set of the four acyclic graphs of size 3 without isolated vertices, then $\bar{R}(S) = \max\{R(F) : F \in S\}$. This brings up the question as to whether this fact holds for the remaining 4-element subsets or the entire set $\{K_3, K_{1,3}, P_4, P_3 + K_2, 3K_2\}$ of the five graphs of size 3 without isolated vertices. Furthermore, there is the more general question as to whether this fact can be extended to broader sets of graphs.

Author Contributions: Conceptualization, G.C. and P.Z.; methodology, G.C. and P.Z.; software, P.Z.; validation, G.C. and P.Z.; formal analysis, G.C. and P.Z.; investigation, G.C. and P.Z.; resources, G.C. and P.Z.; data curation, G.C. and P.Z.; writing—original draft preparation, G.C. and P.Z.; writing—review and editing, G.C. and P.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: We thank the anonymous referees whose valuable suggestions resulted in an improved paper.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Chvátal, V. Tree-complete graph Ramsey numbers. *J. Graph Theory* **1977**, *1*, 93. [CrossRef]
2. Graham, R.L.; Rothschild, B.L.; Spencer, J.H. *Ramsey Theory*, 2nd ed.; Wiley: New York, NY, USA, 2013.
3. Greenwood, R.E.; Gleason, A.M. Combinatorial relations and chromatic graphs. *Canad. J. Math.* **1955**, *7*, 1–7. [CrossRef]
4. Radziszowski, S.P. Small Ramsey numbers. *Electron. J. Combin.* **2026**. Available online: <https://www.cs.rit.edu/~spr/EJC/sur.pdf> (accessed on 24 June 2026).
5. Robertson, A. *Fundamentals of Ramsey Theory*; Chapman and Hall/CRC: New York, NY, USA, 2021.
6. Soifer, A. *The New Mathematical Coloring Book*, 2nd ed.; Springer: New York, NY, USA, 2024.
7. Beineke, L.W.; Schwenk, A.J. On a bipartite form of the Ramsey problem. In Proceedings of the Fifth British Combinatorial Conference, Aberdeen, UK, 14–18 July 1975; pp. 17–22.
8. Bialostocki, A.; Voxman, W. Generalizations of some Ramsey-type theorems for matchings. *Discret. Math.* **2001**, *239*, 101–107. [CrossRef]
9. Conlon, D.; Nenadov, R.; Trujić, M. The size-Ramsey number of cubic graphs. *Bull. Lond. Math. Soc.* **2022**, *54*, 2135–2150. [CrossRef]
10. Eroh, L. Rainbow Ramsey Numbers. Ph.D. Thesis, Western Michigan University, Kalamazoo, MI, USA, 2000.
11. Jent, E. Multiple Monochromatic Subgraphs in Edge-Colored Graphs. Ph.D. Thesis, Western Michigan University, Kalamazoo, MI, USA, 2025.

12. Olejniczak, D. Variations in Ramsey Theory. Ph.D. Thesis, Western Michigan University, Kalamazoo, MI, USA, 2018.
13. Chartrand, G.; Zhang, P. Strong Ramsey numbers of graphs. *Electron. J. Math.* **2026**, *11*, 141–153. [[CrossRef](#)]
14. Ramsey, F.P. On a problem of formal logic. *Proc. Lond. Math. Soc.* **1930**, *30*, 264–286. [[CrossRef](#)]
15. Chartrand, G.; Zhang, P. Strong Ramsey numbers of graphs of size 4. *Mathematics* **2026**, *14*, 2158. [[CrossRef](#)]
16. Burr, S.A. Diagonal Ramsey numbers for small graphs. *J. Graph Theory* **1983**, *7*, 57–67. [[CrossRef](#)]
17. Chartrand, G.; Zhang, P. *Chromatic Graph Theory*, 2nd ed.; Chapman & Hall/CRC Press: Boca Raton, FL, USA, 2020.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.