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Qualitative Axioms Within Probability Calculus and Their Role in Comparing Previsions of Random Quantities

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Abstract

In this paper, the notion of probability is not undefined, so we extend to multilinear indices qualitative axioms which are not in conflict with those characterizing the development of modern probability theory: this is the main objective achieved by the current paper. The logical foundation of probability calculus is here extended by studying the mathematical expectation of random variables having two or more marginal variables as their components. Random variables, studied along with their probability distributions of a nonparametric nature, are geometric entities of which fundamental invariance properties are made explicit. The research gap addressed by this paper is the following: since the Cartesian product of two or more sets, where each of them is the image of a marginal random variable, is not commutative, noncommutative geometric objects coinciding with tensors come into play to make previsions of entities treating high-dimensional data. Empirical data given by time series of a finite length are handled. Time series of a finite length are formally seen as frequency distributions. They are also random variables. Hence, frequency distributions and random variables are shown to be the two sides of the same coin.

Keywords: tensor; utility; qualitative axioms; σ-algebra; time series; exchangeability

MSC: 15B33; 60B20; 62H05; 91B06; 91B16

1. Introduction

Statistical variables and frequency distributions defined with respect to statistical variables are geometric entities. Even random variables, studied along with their probability distributions of a nonparametric nature, are geometric entities. Since what can completely be observed by making a consequent classification is never an infinite number of elements belonging to a set, it is possible to pass from frequency distributions to random variables: frequency distributions and random variables are therefore the two sides of the same coin. Hence, they can be studied inside finite dimensional vector spaces over $\mathbb{R}$. On the other hand, every finite dimensional vector space over $\mathbb{R}$ can be seen as an affine space over itself, so it is true that it is only the affine properties which make sense.

First, we treat statistical variables. The potential set of values that an empirical quantity assumes in a field of observation gives rise to a statistical variable. Thus, the geometric representation of a simple statistical variable leads to the consideration of a vector [1]. In particular, if m is the positive integer identifying distinct values of the mathematical variable into account, then it is possible to consider the set of such values as that of the m components of a vector. This latter is an element of an m-dimensional vector space. Since the values of the simple statistical variable are real numbers, an m-dimensional vector space

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over $\mathbb{R}$ is considered. Hence, the m components of a vector are studied regarding a specific basis of an m-dimensional vector space over $\mathbb{R}$ [2,3]. Given a finite statistical population expressed by P and a simple statistical variable denoted by $X = (X_i)$, an item within P with respect to X is written as

$$U_i^X. \tag{1}$$

It is specifically that element of P where the value X_i of X is observed. In this paper, we focus on statistical populations whose elements are completely observed. They are accordingly finite statistical populations. The set of the values of X, where such values are not necessarily distinct, appearing in the items of P gives rise to a frequency distribution. If a subset of items within P shows the same value X_i characterizing each item of the subset under consideration, then the ordered pair given by (X_i, P_i), where P_i is a frequency, represents a simple frequency distribution as i varies from 1 to m. What has just been said will be extended later.

Second, we study previsions of random variables based on subjective probabilities [4]. Thus, in this paper, every probabilistic evaluation has exclusively a psychological value [5–7]. Our way of following the foundation of probability calculus is such that, in every estimation problem, it is convenient to separate what is logical from what has a purely empirical nature and value. One of our aims is therefore to put forward models that made explicit the set of formally admissible estimates at a first stage, without worrying whether there are reasons of some other order that could lead us to consider any of them more or less correct at a second stage. Those reasons go beyond the purely logical aspect of the estimation problem under consideration. Only mathematics can handle that aspect. A clear separation of the two phases or stages, connected with the two above aspects, is appropriate and necessary [8]. The first phase is of a formal nature. Here, coherent estimates, that is, previsions of a random variable, have to be characterized based on a generic state of information and knowledge associated with an individual. Such a phase has to be treated mathematically. The second phase consists of the choice of one of those possible estimates based on a more specific state of information and knowledge associated with the same individual. The second phase has to be left to practice, common sense, and the judgment of an individual. Our approach is advantageous because it includes, along with estimates that are not in themselves contradictory, also that particular evaluation which could be considered objectively correct at a second stage [9–11]. If a random variable has a uniquely determined prevision, then all individuals who align their psychological position to that specific prevision are judging correctly. The other individuals are simply wrong. However, apart from that, probability rules are the same for everyone [12–16].

Section 2 shows research works which are connected with what is said in this paper. Section 3 explains how statistical variables can geometrically be treated along with their frequency distributions. Qualitative axioms of an objective nature are shown in Section 4, where their psychological value is made explicit. Section 5 focuses on a specific scale of utility containing previsions of marginal random variables. Each prevision of them is a price expressed in terms of a monetary value. Finally, conclusions and future perspectives are contained in Section 6.

2. Literature Review

In this paper, empirical data are based on time series of a finite length. They are managed taking their basic properties into account [17–22]. Time series of a finite length are formally seen as frequency distributions. They are therefore ordered pairs of vectors defined with regard to statistical variables [23–28]. Time series of a finite length are also random variables. The notion of α-norm of a vector, introduced by Corrado Gini, is a particular scalar product of two vectors. Here, such a notion is accepted to be as the foundation of the

connection between the theory of vector spaces over $\mathbb{R}$ and that of statistical and random variables. The notion of α-norm of a tensor is considered whenever random variables, having two or more marginal variables as their components, are studied along with some of their indices [29–34]. If we compare what is developed in this paper with what appears elsewhere, then the logical foundation of probability calculus is here extended. Thus, the mathematical expectation of random variables, having two or more marginal variables as their components, is innovatively studied. Qualitative axioms connected with multilinear indices are accordingly treated.

3. Materials and Methods

The list of main symbols used in this paper is shown through the following table:

Symbol	Description
$x^i \dot{e}_i$	Einstein summation notation
x^i	contravariant component of $\dot{x}$
p_i	covariant component of $\dot{p}$
$p_{i_1 i_2}$	covariant component of the tensor p
$_{\dot{x}}\dot{d} = \dot{x} - \dot{\tilde{x}}$	vector that identifies deviations
$\|\dot{x}\|_\alpha^2$	α-norm of a vector
$\|_{(1)}\dot{d} - {}_{(2)}\dot{d}\|_\alpha^2$	linear metric
$\|{}_{12}f\|_\alpha^2$	α-norm of an antisymmetric tensor of order 2
${}_{12}f \odot {}_{34}f$	α-product of the tensors ${}_{12}f$ and ${}_{34}f$
$\|{}_{12}d\|_\alpha^2$	superficial metric
${}_{(1)}x_{i_1} = {}_{(1)}x^{i_2} p_{i_2 i_1}$	vector homography
${}_{(1)}\dot{x} \odot {}_{(2)}\dot{x}$	α-product between two vectors
r_{12}	the Bravais-Pearson correlation coefficient referred to X_{12}
r_{12}^2	the square of the Bravais-Pearson correlation coefficient referred to X_{12}
$R_{12,34}$	the Bravais-Pearson correlation coefficient referred to X_{34}^{12}

3.1. How a Simple Frequency Distribution Can Geometrically Be Studied

Let ${}_{(1)}S$ be a specific set of simple statistical variables having m values that are all distinct. Every element X belonging to ${}_{(1)}S$ is an ordered m-tuple of real numbers. Every element X belonging to ${}_{(1)}S$ is a vector of E_m, where E_m is an m-dimensional vector space over $\mathbb{R}$ having a Euclidean nature. The set of values of X is that of the components of $\dot{x}$, where one has $\dot{x} \in E_m$. Thus, ${}_{(1)}S$ is a subset of E_m. We write ${}_{(1)}S \subset E_m$. Here, ${}_{(1)}S \subset E_m$ does not exclude the possibility that one has ${}_{(1)}S = E_m$. A basis of E_m is orthonormal. The components of $\dot{x}$ regarding an orthonormal basis of E_m will always be contravariant within this context. Instead of referring to the value X_i of X, it is therefore possible to consider the component x^i of $\dot{x}$ regarding an orthonormal basis of E_m. Note the following:

Remark 1. *Given an orthonormal basis of E_m, we write*

$$\dot{x} = \sum_{i=1}^{m} x^i \dot{e}_i = x^i \dot{e}_i, \tag{2}$$

where $\{\dot{e}_i\}$, *with* $i = 1, 2, \ldots, m$, *is a specific set containing all the elements of an orthonormal basis of* E_m. *The components of* $\dot{x}$ *are the coefficients* x^i *of the linear combination given by* (2). *The vector* $\dot{x}$ *is understood as the set of its contravariant components regarding an orthonormal basis of* E_m. *Thus, we write* $\dot{x} = (x^i)$. *If another orthonormal basis of* E_m *is chosen, then the contravariant components of* $\dot{x}$ *remain unchanged, in the sense that one has*

$$\dot{x} = \sum_{i=1}^{m} x^i \dot{e}'_i = x^i \dot{e}'_i. \tag{3}$$

Hence, $\dot{x}$ *is said to be invariant.*

We need to introduce the vector representation of statistical weights to identify frequency distributions as elements of E_m. A covariant notation is chosen for them. Hence, we write $\dot{p} = (p_i)$. A frequency distribution is therefore given by

$$(\dot{x}, \dot{p}) \equiv (x^i, p_i). \tag{4}$$

We write $(\dot{x}, \dot{p}) \subset E_m$. It is completely inappropriate to use the term contravariant with respect to the components of the vectors that represent values of simple statistical variables. Similarly, it is completely inappropriate to use the term covariant with respect to the components of the vectors that represent the distributions of statistical weights. However, the usage we propose for such notations is not inappropriate. Since an orthonormal basis of E_m is always used, the distinction between contravariant and covariant components of vectors is not meaningful according to the theory of vector spaces over $\mathbb{R}$. Finally, if an orthonormal basis of E_m is used, then the contravariant and covariant components of any vector whatsoever coincide. Thus, the contravariant and covariant components of $\dot{x}$ coincide, as well as the ones of $\dot{p}$. The sum of the values of X pertaining to all the items within P is given by

$$\mathcal{I}(X) = \sum_{i=1}^{m} x^i p_i = x^i p_i. \tag{5}$$

If the size of P is equal to N, then the mean of X is expressed by

$$\overline{X} = \frac{1}{N} x^i p_i. \tag{6}$$

We use the vector $\dot{\bar{x}}$ to represent $\overline{X}$, whose components are given by $\bar{x}^i$. They are all equal. Another variable can be defined with respect to X. It is denoted by ${}_X d$. Its values represent deviations from $\dot{\bar{x}}$. We write

$${}_{\dot{x}}\dot{d} = \dot{x} - \dot{\bar{x}}, \tag{7}$$

whose contravariant components are expressed by

$${}_{\dot{x}}d^i = x^i - \bar{x}^i. \tag{8}$$

It is known that one has

$$\mathcal{I}({}_X d) = {}_{\dot{x}}d^i p_i = (x^i - \bar{x}^i) p_i = 0. \tag{9}$$

The α-norm of $\dot{x}$ is the following:

$$\|\dot{x}\|_\alpha^2 = (x^i)^2 p_i. \tag{10}$$

We write

$$\|\dot{x}\|_\alpha^2 \geq 0, \tag{11}$$

where $\|\dot{x}\|_\alpha^2 = 0$ means that the values of X are all equal to 0. Hence, it is possible to write

$$\|_{\dot{x}}\dot{d}\|_\alpha^2 = \sigma_X^2 \tag{12}$$

whenever the components of $\dot{p}$ represent relative frequencies summing to 1. Note the following:

Remark 2. *The Euclidean norm is given by*

$$\|\dot{x}\| = \sqrt{\dot{x} \cdot \dot{x}}. \tag{13}$$

Its square is expressed by

$$\|\dot{x}\|^2 = \dot{x} \cdot \dot{x}. \tag{14}$$

Hence, the α-norm of $\dot{x}$ denoted by $\|\dot{x}\|_\alpha^2$ derives from the α-criterion of concordance put forward by Corrado Gini. Here, unlike $\|\dot{x}\|^2$, statistical weights denoted by p_i are considered along with the contravariant components of $\dot{x}$ denoted by x^i.

Remark 3. *The notion of α-norm of a vector is accepted to be as the foundation of the connection between the theory of vector spaces over $\mathbb{R}$ and that of statistical variables. Such a notion identifies a metric of a quadratic nature. That is, given a vector written as*

$$_{(1)}\dot{d} - {}_{(2)}\dot{d}, \tag{15}$$

its α-norm is expressed by the following expression

$$\|_{(1)}\dot{d} - {}_{(2)}\dot{d}\|_\alpha^2 = \|_{(1)}\dot{d}\|_\alpha^2 + \|_{(2)}\dot{d}\|_\alpha^2 - 2({}_{(1)}\dot{d} \odot {}_{(2)}\dot{d}) \tag{16}$$

characterizing a distance. It is possible to write

$$\|_{(1)}\dot{d}\|_\alpha^2 = {}_{(1)}\dot{d} \odot {}_{(1)}\dot{d}, \tag{17}$$

and

$$\|_{(2)}\dot{d}\|_\alpha^2 = {}_{(2)}\dot{d} \odot {}_{(2)}\dot{d}. \tag{18}$$

The next subsection will clarify the meaning of ⊙. Anyway, (16) identifies a linear metric.

3.2. How Geometrically to Study a Multiple Statistical Variable of Order 2 and Its Frequency Distribution

Let ${}_{(2)}S^{(2)}$ be a specific set of multiple statistical variables of order 2, whereas

$$X_{12} = \{{}_1X, {}_2X\} \tag{19}$$

is a generic variable belonging to ${}_{(2)}S^{(2)}$. Let $({}_1X, {}_2X)$ be the ordered pair of simple statistical variables, where such marginal variables are the components of X_{12}. Every element contained in ${}_{(2)}S^{(2)}$ is therefore a tensor belonging to

$$E_m \otimes E_m = E_m^{(2)}. \tag{20}$$

The set of the values of X_{12} is that of the components of a tensor of order 2 denoted by T. Hence, we write $T \in E_m \otimes E_m$. Here, m is both the dimension of E_m and the number of distinct values of ${}_1X$ and ${}_2X$, respectively. We also write ${}_{(2)}S^{(2)} \subset E_m \otimes E_m$, where the possibility that one has ${}_{(2)}S^{(2)} = E_m \otimes E_m$ is not here excluded. An orthonormal basis of E_m denoted by $\{\dot{e}_i\}$, with $i = 1, 2, \ldots, m$, is chosen to represent the values of X_{12}. Such values are therefore the components of T. Note the following:

Remark 4. *If the values of a simple statistical variable are the contravariant components of a vector, then the values of a multiple statistical variable are accordingly the contravariant components of a tensor.*

We write

$$T = {}_{(1)}\dot{x} \otimes {}_{(2)}\dot{x} = {}_{(1)}x^{i_1}{}_{(2)}x^{i_2}\dot{e}_{i_1} \otimes \dot{e}_{i_2}. \tag{21}$$

Nevertheless, a different writing of the classification scheme defined for X_{12} is possible. Instead of referring to the ordered pair $({}_1X, {}_2X)$, it is therefore possible to refer to the ordered pair $({}_2X, {}_1X)$. Thus, we write

$$T = {}_{(2)}\dot{x} \otimes {}_{(1)}\dot{x} = {}_{(2)}x^{i_2}{}_{(1)}x^{i_1}\dot{e}_{i_2} \otimes \dot{e}_{i_1}. \tag{22}$$

We have to write both (21) and (22) because T is intrinsically a noncommutative geometric entity. While the tensor T is the same, its components are not the same: the components of T which is written using (21) are not the same as the ones of the tensor T which is written using (22). We have to consider (21) and (22) simultaneously, so ${}_1X$ and ${}_2X$ are logically exchangeable [35–37]. It is possible to establish the following:

Definition 1. *The two marginal variables, being the components of a multiple statistical variable of order* 2*, are symmetric or invariant with respect to* 2! *permutations. They are therefore logically exchangeable. We write*

$$X_{12} = \{{}_1X, {}_2X\} = \{{}_2X, {}_1X\}.$$

Similarly, the n marginal variables, being the components of a multiple statistical variable of order n, are symmetric or invariant with respect to n! permutations. Hence, they are logically exchangeable.

Since we write

$$T = \sum_{i_1<i_2} \left({}_{(1)}x^{i_1}{}_{(2)}x^{i_2} - {}_{(1)}x^{i_2}{}_{(2)}x^{i_1} \right) \dot{e}_{i_1} \otimes \dot{e}_{i_2}, \tag{23}$$

(23) is an antisymmetric tensor of order 2 [38]. Note the following:

Remark 5. *The values of X_{12} have to be represented using the components of an antisymmetric tensor of order* 2*. That representation does not depend on how the classification scheme defined for X_{12} is written.*

There are accordingly antisymmetric tensors of order 2 that belong to $E_m^{(2)\wedge}$. A specific set of multiple statistical variables of order 2 is therefore denoted by ${}_{(2)}S^{(2)\wedge}$. Thus, we write ${}_{(2)}S^{(2)\wedge} \subset E_m^{(2)\wedge}$. Accordingly, we pass from

$$\dim E_m^{(2)} = m^2 \tag{24}$$

to

$$\dim E_m^{(2)\wedge} = \binom{m}{2}. \tag{25}$$

We wish to make more understandable what has just been said. Check the following:

Example 1. *Let*

$${}_{(1)}\dot{x} = {}_{(1)}x^1\dot{e}_1 + {}_{(1)}x^2\dot{e}_2 + {}_{(1)}x^3\dot{e}_3 = 3\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 4\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\2\\4\end{pmatrix}$$

and

$$_{(2)}\dot{x} = {}_{(2)}x^1\dot{e}_1 + {}_{(2)}x^2\dot{e}_2 + {}_{(2)}x^3\dot{e}_3 = 5\begin{pmatrix}1\\0\\0\end{pmatrix} + 6\begin{pmatrix}0\\1\\0\end{pmatrix} + 7\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}5\\6\\7\end{pmatrix}$$

be the two vectors representing the distinct values for the two simple statistical variables of X_{12}*. Accordingly, we write*

$$T = {}_{(1)}x^1{}_{(2)}x^1\dot{e}_1 \otimes \dot{e}_1 + {}_{(1)}x^1{}_{(2)}x^2\dot{e}_1 \otimes \dot{e}_2 + {}_{(1)}x^1{}_{(2)}x^3\dot{e}_1 \otimes \dot{e}_3 + \ldots + {}_{(1)}x^3{}_{(2)}x^3\dot{e}_3 \otimes \dot{e}_3,$$

that is,

$$T = 15\dot{e}_1 \otimes \dot{e}_1 + 18\dot{e}_1 \otimes \dot{e}_2 + 21\dot{e}_1 \otimes \dot{e}_3 + \ldots + 28\dot{e}_3 \otimes \dot{e}_3.$$

We always focus on the components of T*. The same is true for any vector whatsoever which is written as a linear combination of basis vectors. It is possible to change the order of the two vectors representing the distinct values for the two simple statistical variables denoted by* ${}_1X$ *and* ${}_2X$*, respectively. Hence, we write*

$$_{(2)}\dot{x} = {}_{(2)}x^1\dot{e}_1 + {}_{(2)}x^2\dot{e}_2 + {}_{(2)}x^3\dot{e}_3 = 5\begin{pmatrix}1\\0\\0\end{pmatrix} + 6\begin{pmatrix}0\\1\\0\end{pmatrix} + 7\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}5\\6\\7\end{pmatrix}$$

and

$$_{(1)}\dot{x} = {}_{(1)}x^1\dot{e}_1 + {}_{(1)}x^2\dot{e}_2 + {}_{(1)}x^3\dot{e}_3 = 3\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 4\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\2\\4\end{pmatrix}.$$

Accordingly, we obtain

$$T = {}_{(2)}x^1{}_{(1)}x^1\dot{e}_1 \otimes \dot{e}_1 + {}_{(2)}x^1{}_{(1)}x^2\dot{e}_1 \otimes \dot{e}_2 + {}_{(2)}x^1{}_{(1)}x^3\dot{e}_1 \otimes \dot{e}_3 + \ldots + {}_{(2)}x^3{}_{(1)}x^3\dot{e}_3 \otimes \dot{e}_3,$$

that is,

$$T = 15\dot{e}_1 \otimes \dot{e}_1 + 10\dot{e}_1 \otimes \dot{e}_2 + 20\dot{e}_1 \otimes \dot{e}_3 + \ldots + 28\dot{e}_3 \otimes \dot{e}_3.$$

We have to consider both classification schemes. For this reason, we observe

$$T_{ij} = -T_{ji},$$

so T *is an antisymmetric tensor of order* 2*. This means that its components, whose two indices are equal, are zero. In addition, the number of independent components is drastically reduced. Finally, we write*

$$T = \begin{vmatrix}3 & 2\\5 & 6\end{vmatrix}\dot{e}_1 \otimes \dot{e}_2 + \begin{vmatrix}3 & 4\\5 & 7\end{vmatrix}\dot{e}_1 \otimes \dot{e}_3 + \begin{vmatrix}2 & 4\\6 & 7\end{vmatrix}\dot{e}_2 \otimes \dot{e}_3.$$

Here, there are three independent components of T*. The generic contravariant component of the following tensor of order* 2 *denoted by* ${}_{12}f$ *is written as*

$${}_{12}f^{(i_1\,i_2)} = \begin{vmatrix}{}_{(1)}x^{i_1} & {}_{(1)}x^{i_2}\\{}_{(2)}x^{i_1} & {}_{(2)}x^{i_2}\end{vmatrix}.$$

The tensor ${}_{12}f$ *is called the tensor of the values of* X_{12}*. More explicitly, one has*

$${}_{12}f^{(12)} = \begin{vmatrix}{}_{(1)}x^1 & {}_{(1)}x^2\\{}_{(2)}x^1 & {}_{(2)}x^2\end{vmatrix} = \begin{vmatrix}3 & 2\\5 & 6\end{vmatrix},$$

$$_{12}f^{(13)} = \begin{vmatrix} {}_{(1)}x^1 & {}_{(1)}x^3 \\ {}_{(2)}x^1 & {}_{(2)}x^3 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix},$$

and

$$_{12}f^{(23)} = \begin{vmatrix} {}_{(1)}x^2 & {}_{(1)}x^3 \\ {}_{(2)}x^2 & {}_{(2)}x^3 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 6 & 7 \end{vmatrix}.$$

Each determinant written above expresses the same value of X_{12} *with regard to both classification schemes based on* $({}_1\mathrm{X}, {}_2\mathrm{X})$ *and* $({}_2\mathrm{X}, {}_1\mathrm{X})$*, respectively.*

Note the following:

Remark 6. *After writing*

$$T = \begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} \dot{e}_1 \otimes \dot{e}_2 + \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} \dot{e}_1 \otimes \dot{e}_3 + \begin{vmatrix} 2 & 4 \\ 6 & 7 \end{vmatrix} \dot{e}_2 \otimes \dot{e}_3,$$

it follows that one has

$$T = (18 - 10)\dot{e}_1 \otimes \dot{e}_2 + (21 - 20)\dot{e}_1 \otimes \dot{e}_3 + (14 - 24)\dot{e}_2 \otimes \dot{e}_3,$$

that is,

$$T = 18\dot{e}_1 \otimes \dot{e}_2 - 10e_1 \otimes \dot{e}_2 + \ldots + 14\dot{e}_2 \otimes \dot{e}_3 - 24\dot{e}_2 \otimes \dot{e}_3.$$

The other components of T are not there because they are equal to zero. We observe, for instance,

$$(15 - 15)\dot{e}_1 \otimes \dot{e}_1.$$

If a tensor is used because it extends the notion of vector in a natural way, then an antisymmetric tensor of order 2 *is more specifically used to express the mathematical writing of* X_{12} *in a more compact way.*

If another orthonormal basis of E_3 is chosen, then the contravariant components of the following vectors

$${}_{(1)}\dot{x} = 3\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} + 2\begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \\ 4 \end{pmatrix}$$

and

$${}_{(2)}\dot{x} = 5\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} + 6\begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} + 7\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{11}{\sqrt{2}} \\ 7 \end{pmatrix}$$

are the same. Note the following:

Remark 7. *A tensor of order* 2 *is said to be of an antisymmetric nature because if we consider all its components, then an antisymmetric matrix arises. In particular, from*

$$T = \begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} \dot{e}_1 \otimes \dot{e}_2 + \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} \dot{e}_1 \otimes \dot{e}_3 + \begin{vmatrix} 2 & 4 \\ 6 & 7 \end{vmatrix} \dot{e}_2 \otimes \dot{e}_3,$$

it follows that the square matrix of order 3 *given by*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 8 & 1 \\ -8 & 0 & -10 \\ -1 & 10 & 0 \end{bmatrix}$$

is of an antisymmetric nature.

Here, ${}_1X$ and ${}_2X$ are thought to be homogeneous and expressed in the same unit of measurement. However, that limitation can easily be overcome. Statistical weights associated with all the values of ${}_1X$ and ${}_2X$ are frequencies. In particular, such weights can be relative frequencies. They are the covariant components of a tensor of order 2. This latter is called the association tensor. It is denoted by $p = (p_{i_1 i_2})$. It is not a tensor of an antisymmetric nature. We therefore write $p \in E_m^{(2)}$. Accordingly, it is possible to consider the following table:

vector 1 \ vector 2	${}_{(2)}x^1$	${}_{(2)}x^2$	$\cdots$	${}_{(2)}x^m$	Sum
${}_{(1)}x^1$	p_{11}	p_{12}	$\cdots$	p_{1m}	${}_{(2)}x_1$
${}_{(1)}x^2$	p_{21}	p_{22}	$\cdots$	p_{2m}	${}_{(2)}x_2$
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$
${}_{(1)}x^m$	p_{m1}	p_{m2}	$\cdots$	p_{mm}	${}_{(2)}x_m$
Sum	${}_{(1)}x_1$	${}_{(1)}x_2$	$\cdots$	${}_{(1)}x_m$	1

The sum of the values with respect to ${}_1X$ of the items within P that have different values related to ${}_1X$—but they all have the same value ${}_2X_{i_2}$ with regard to ${}_2X$—is denoted by

$$\mathfrak{I}({}_1X|{}_2X_{i_2}) = {}_{(1)}x^{i_1} p_{i_1 i_2} = {}_{(1)}x_{i_2}. \tag{26}$$

Similarly, we write

$$\mathfrak{I}({}_2X|{}_1X_{i_1}) = {}_{(2)}x^{i_2} p_{i_2 i_1} = {}_{(2)}x_{i_1}, \tag{27}$$

where both (26) and (27) are vector homographies. Hence, the sequence given by

$$({}_{(1)}x_1, {}_{(1)}x_2, \ldots, {}_{(1)}x_m) \tag{28}$$

is called the marginal frequency distribution of ${}_1X$ with respect to ${}_2X$. It is decomposed. Conversely, the sequence given by

$$({}_{(2)}x_1, {}_{(2)}x_2, \ldots, {}_{(2)}x_m) \tag{29}$$

is called the marginal frequency distribution of ${}_2X$ with respect to ${}_1X$. It is decomposed. The components of the antisymmetric tensor of order 2 that simultaneously represents the marginal frequency distributions related to X_{12} are written as

$${}_{12}f_{(i_1 i_2)} = \begin{vmatrix} {}_{(1)}x_{i_1} & {}_{(1)}x_{i_2} \\ {}_{(2)}x_{i_1} & {}_{(2)}x_{i_2} \end{vmatrix} = \begin{vmatrix} {}_{(1)}x^{i_2} p_{i_2 i_1} & {}_{(1)}x^{i_1} p_{i_1 i_2} \\ {}_{(2)}x^{i_2} p_{i_2 i_1} & {}_{(2)}x^{i_1} p_{i_1 i_2} \end{vmatrix}. \tag{30}$$

If we consider the following ordered triplet

$$({}_{(1)}\dot{x}, {}_{(2)}\dot{x}, p), \tag{31}$$

then one has ${}_{(2)}S^{(1)} \subset E_m$, where ${}_{(2)}S^{(1)}$ is the vector space of the simple statistical variables that are the components of X_{12}. Here, the possibility that one has ${}_{(2)}S^{(1)} = E_m$ is not excluded. One has ${}_{(1)}\dot{x}, {}_{(2)}\dot{x} \in E_m$, whereas p is a tensor of order 2. From the scalar product of two vectors which is written as

$$ {}_{(1)}\dot{x} \cdot {}_{(2)}\dot{x} = \sum_{i_1=1}^{m} {}_{(1)}x^{i_1} {}_{(2)}x^{i_1}, \tag{32} $$

it follows that their α-product denoted by

$$ ({}_{(1)}\dot{x} \odot {}_{(2)}\dot{x})_p = {}_{(1)}\dot{x} \odot {}_{(2)}\dot{x} \tag{33} $$

intrinsically depends on p. Since a vector homography, via the tensor p, transforms the vector ${}_{(2)}\dot{x}$ into the vector ${}_{(2)}\dot{x}'$, it is also possible to write

$$ {}_{(1)}\dot{x} \odot {}_{(2)}\dot{x} = {}_{(1)}x^{i_1} {}_{(2)}x^{i_2} p_{i_1 i_2} = {}_{(1)}\dot{x} \cdot {}_{(2)}\dot{x}'. \tag{34} $$

A particular α-product is the following

$$ \|{}_{(1)}\dot{x}\|_\alpha^2 = {}_{(1)}\dot{x} \odot {}_{(1)}\dot{x} = {}_{(1)}x^{i_1} {}_{(1)}x^{i_1} p_{i_1 i_1}. \tag{35} $$

Note the following:

Remark 8. *Every α-product between two vectors depends on p. Since vector homographies always take place, every α-product between two vectors is a particular scalar product.*

Check the following:

Example 2. *Given the following table:*

vector 1 \ *vector* 2	*5*	*6*	*7*	*Sum*
3	*0.05*	*0.05*	*0.1*	*0.2*
2	*0.05*	*0.05*	*0.1*	*0.2*
4	*0.2*	*0.2*	*0.2*	*0.6*
Sum	*0.3*	*0.3*	*0.4*	*1*

a vector homography transforms the vector given by

$$ {}_{(2)}\dot{x} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} $$

into the vector expressed by

$$ {}_{(2)}\dot{x}' = \begin{pmatrix} 1.25 \\ 1.25 \\ 3.6 \end{pmatrix}. $$

One has

$$ {}_{(1)}\dot{x} \odot {}_{(2)}\dot{x} = {}_{(1)}\dot{x} \cdot {}_{(2)}\dot{x}' = 20.65. $$

If we consider the following table:

vector 2 / *vector* 1	*3*	*2*	*4*	*Sum*
3	*0.2*	*0*	*0*	*0.2*
2	*0*	*0.2*	*0*	*0.2*
4	*0*	*0*	*0.6*	*0.6*
Sum	*0.2*	*0.2*	*0.6*	*1*

then the absolute maximum of concordance among the values that appear in this pattern is achieved. We therefore write

$$\|_{(1)}\dot{x}\|_{\alpha}^{2} = {}_{(1)}\dot{x} \odot {}_{(1)}\dot{x} = 12.2.$$

Whenever we pass from an α-product to an α-norm, then the α-criterion of concordance put forward by Corrado Gini is considered.

Given ${}_{(1)}\dot{x}$ and ${}_{(2)}\dot{x}$, it is possible to consider the following linear combination

$$\dot{y} = {}_{(1)}\dot{x} + \lambda {}_{(2)}\dot{x}, \tag{36}$$

with $\lambda \in \mathbb{R}$. From the α-norm of $\dot{y}$ denoted by $\|\dot{y}\|_{\alpha}^{2}$, it follows that the inequality expressed by

$$|{}_{(1)}\dot{x} \odot {}_{(2)}\dot{x}| \leq \|_{(1)}\dot{x}\|_{\alpha}\|_{(2)}\dot{x}\|_{\alpha} \tag{37}$$

holds. It is called the α-generalized Cauchy–Schwarz inequality. We also write

$$\|_{(1)}\dot{x} + {}_{(2)}\dot{x}\|_{\alpha} \leq \|_{(1)}\dot{x}\|_{\alpha} + \|_{(2)}\dot{x}\|_{\alpha}. \tag{38}$$

This latter is called the α-generalized triangle inequality. Moreover, the following expression

$$\cos\gamma = \frac{{}_{(1)}\dot{x} \odot {}_{(2)}\dot{x}}{\|_{(1)}\dot{x}\|_{\alpha}\|_{(2)}\dot{x}\|_{\alpha}} \tag{39}$$

also holds. In particular, using deviations, (39) becomes

$$\cos\gamma = \frac{{}_{(1)}\dot{d} \odot {}_{(2)}\dot{d}}{\|_{(1)}\dot{d}\|_{\alpha}\|_{(2)}\dot{d}\|_{\alpha}}. \tag{40}$$

That is, (40) expresses the Bravais-Pearson correlation coefficient. From (36), it follows that the α-generalized Cauchy–Schwarz inequality, the α-generalized triangle inequality, and the cosine of the angle γ between two vectors are mathematical expressions characterizing ${}_{(2)}S^{(1)}$ metrically. The α-norm of the tensor denoted by ${}_{12}f$, and associated with X_{12}, is given by

$$\|_{12}f\|_{\alpha}^{2} = \begin{vmatrix} \|_{(1)}\dot{x}\|_{\alpha}^{2} & {}_{(1)}\dot{x} \odot {}_{(2)}\dot{x} \\ {}_{(2)}\dot{x} \odot {}_{(1)}\dot{x} & \|_{(2)}\dot{x}\|_{\alpha}^{2} \end{vmatrix}, \tag{41}$$

whereas one has

$$\|_{12}d\|_{\alpha}^{2} = \begin{vmatrix} \|_{(1)}\dot{d}\|_{\alpha}^{2} & {}_{(1)}\dot{d} \odot {}_{(2)}\dot{d} \\ {}_{(2)}\dot{d} \odot {}_{(1)}\dot{d} & \|_{(2)}\dot{d}\|_{\alpha}^{2} \end{vmatrix} \tag{42}$$

if deviations characterizing the tensor denoted by ${}_{12}d$ are used. A frequency distribution of X_{12} is expressed via two marginal distributions characterizing ${}_{1}X$ and ${}_{2}X$, respectively. Such distributions are summarized. Furthermore, they are combined to obtain a square

matrix of order 2, whose determinant is a multilinear index referred to X_{12}. Hence, a frequency distribution of X_{12} operationally identifies a square matrix of order 2 given by

$$\begin{bmatrix} \|_{(1)}\dot{x}\|_{\alpha}^{2} & {}_{(1)}\dot{x} \odot {}_{(2)}\dot{x} \\ {}_{(2)}\dot{x} \odot {}_{(1)}\dot{x} & \|_{(2)}\dot{x}\|_{\alpha}^{2} \end{bmatrix}, \tag{43}$$

or by

$$\begin{bmatrix} \|_{(1)}\dot{d}\|_{\alpha}^{2} & {}_{(1)}\dot{d} \odot {}_{(2)}\dot{d} \\ {}_{(2)}\dot{d} \odot {}_{(1)}\dot{d} & \|_{(2)}\dot{d}\|_{\alpha}^{2} \end{bmatrix}, \tag{44}$$

where each entry of the two above matrices, given by (43) and (44), depends on an association tensor. Entries running from the top-left corner to the bottom-right corner of the two above matrices have association tensors coinciding with the masses of one of the two marginal distributions. Conversely, entries running from the top-right corner to the bottom-left corner of the same matrices have association tensors based on the masses of two marginal distributions characterizing ${}_{1}X$ and ${}_{2}X$, respectively.

3.3. How Geometrically to Study a Multiple Statistical Variable of Order 4 *and Its Frequency Distribution*

Later in this research paper, we will consider multiple statistical variables of order 4 too. In particular, we will deal with divided quadruple statistical variables denoted by

$$X_{34}^{12} \equiv \{X_{12}, X_{34}\}. \tag{45}$$

The components of a divided quadruple statistical variable are therefore multiple statistical variables of order 2, X_{12} and X_{34}, each of which is studied through an antisymmetric tensor of order 2 belonging to ${}_{(2)}S^{(2)\wedge}$. The α-norm of the tensor denoted by ${}_{34}^{12}f$, and representing X_{34}^{12}, is given by

$$\|{}_{34}^{12}f\|_{\alpha}^{2} = \begin{vmatrix} \|_{12}f\|_{\alpha}^{2} & {}_{12}f \odot {}_{34}f \\ {}_{34}f \odot {}_{12}f & \|_{34}f\|_{\alpha}^{2} \end{vmatrix}, \tag{46}$$

where ${}_{12}f$ and ${}_{34}f$ are two antisymmetric tensors of order 2 representing X_{12} and X_{34}, respectively. A frequency distribution of X_{34}^{12} operationally identifies a square matrix of order 2 given by

$$\begin{bmatrix} \|_{12}f\|_{\alpha}^{2} & {}_{12}f \odot {}_{34}f \\ {}_{34}f \odot {}_{12}f & \|_{34}f\|_{\alpha}^{2} \end{bmatrix}, \tag{47}$$

where each entry of (47) is made up of four elements, each of which depends on an association tensor.

3.4. Empirical Data: Time Series of a Finite Length

We focus on annual returns on the same D stock listed on four stock exchanges: the Milan Stock Exchange, the London Stock Exchange, the New York Stock Exchange, and the Toronto Stock Exchange. Four time series of a finite length are observed. Each time series is a frequency distribution defined with respect to a simple statistical variable. The four marginal frequency distributions identify a multiple frequency distribution that characterizes a multiple statistical variable of order 4. With regard to the Milan Stock Exchange, the London Stock Exchange, the New York Stock Exchange, and the Toronto Stock Exchange, time series of a finite length are first put in the form of the following tables:

Year	Annual Return on the *D* Stock: The Milan Stock Exchange	Frequency
2010	0.12	1/16
2011	0.085	1/16
…	…	…
2018	0.11	1/16
2019	0.096	1/16
2020	0.087	1/16
2021	0.074	1/16
2022	0.13	1/16
2023	0.092	1/16
2024	0.14	1/16
2025	0.088	1/16

Year	Annual Return on the *D* Stock: The London Stock Exchange	Frequency
2010	0.116	1/16
2011	0.073	1/16
…	…	…
2018	0.081	1/16
2019	0.083	1/16
2020	0.072	1/16
2021	0.091	1/16
2022	0.094	1/16
2023	0.061	1/16
2024	0.063	1/16
2025	0.084	1/16

Year	Annual Return on the *D* Stock: The New York Stock Exchange	Frequency
2010	0.123	1/16
2011	0.092	1/16
…	…	…
2018	0.082	1/16
2019	0.085	1/16
2020	0.079	1/16
2021	0.093	1/16
2022	0.081	1/16
2023	0.073	1/16
2024	0.076	1/16
2025	0.069	1/16

and

Year	Annual Return on the D Stock: The Toronto Stock Exchange	Frequency
2010	0.13	1/16
2011	0.084	1/16
...	...	...
2018	0.059	1/16
2019	0.066	1/16
2020	0.072	1/16
2021	0.065	1/16
2022	0.062	1/16
2023	0.123	1/16
2024	0.134	1/16
2025	0.113	1/16

We write

$$\left\{\begin{pmatrix}2010\\2011\\\vdots\\2018\\2019\\2020\\2021\\2022\\2023\\2024\\2025\end{pmatrix};\left[\begin{pmatrix}0.12\\0.085\\\vdots\\0.11\\0.096\\0.087\\0.074\\0.13\\0.092\\0.14\\0.088\end{pmatrix},\begin{pmatrix}0.116\\0.073\\\vdots\\0.081\\0.083\\0.072\\0.091\\0.094\\0.061\\0.063\\0.084\end{pmatrix},\begin{pmatrix}0.123\\0.092\\\vdots\\0.082\\0.085\\0.079\\0.093\\0.081\\0.073\\0.076\\0.069\end{pmatrix},\begin{pmatrix}0.13\\0.084\\\vdots\\0.059\\0.066\\0.072\\0.065\\0.062\\0.123\\0.134\\0.113\end{pmatrix}\right],\begin{pmatrix}1/16\\1/16\\\vdots\\1/16\\1/16\\1/16\\1/16\\1/16\\1/16\\1/16\\1/16\end{pmatrix}\right\} \tag{48}$$

to denote a multiple frequency distribution. The items within P are given by

$$\begin{pmatrix}2010\\2011\\\vdots\\2018\\2019\\2020\\2021\\2022\\2023\\2024\\2025\end{pmatrix}.$$

In particular, a divided quadruple statistical variable is studied, so it is possible to consider the following multiple frequency distributions:

$$\left\{\begin{pmatrix}2010\\2011\\\vdots\\2018\\2019\\2020\\2021\\2022\\2023\\2024\\2025\end{pmatrix};\left[\begin{pmatrix}0.12\\0.085\\\vdots\\0.11\\0.096\\0.087\\0.074\\0.13\\0.092\\0.14\\0.088\end{pmatrix},\begin{pmatrix}0.116\\0.073\\\vdots\\0.081\\0.083\\0.072\\0.091\\0.094\\0.061\\0.063\\0.084\end{pmatrix}\right],\begin{pmatrix}1/16\\1/16\\\vdots\\1/16\\1/16\\1/16\\1/16\\1/16\\1/16\\1/16\\1/16\end{pmatrix}\right\}, \tag{49}$$

and

$$\left\{\begin{pmatrix}2010\\2011\\\vdots\\2018\\2019\\2020\\2021\\2022\\2023\\2024\\2025\end{pmatrix};\left[\begin{pmatrix}0.123\\0.092\\\vdots\\0.082\\0.085\\0.079\\0.093\\0.081\\0.073\\0.076\\0.069\end{pmatrix},\begin{pmatrix}0.13\\0.084\\\vdots\\0.059\\0.066\\0.072\\0.065\\0.062\\0.123\\0.134\\0.113\end{pmatrix}\right],\begin{pmatrix}1/16\\1/16\\\vdots\\1/16\\1/16\\1/16\\1/16\\1/16\\1/16\\1/16\\1/16\end{pmatrix}\right\}. \tag{50}$$

The vectors

$$\left\{\left[{}_{(1)}\dot{x}=\begin{pmatrix}0.12\\0.085\\\vdots\\0.11\\0.096\\0.087\\0.074\\0.13\\0.092\\0.14\\0.088\end{pmatrix},{}_{(2)}\dot{x}=\begin{pmatrix}0.116\\0.073\\\vdots\\0.081\\0.083\\0.072\\0.091\\0.094\\0.061\\0.063\\0.084\end{pmatrix},{}_{(3)}\dot{x}=\begin{pmatrix}0.123\\0.092\\\vdots\\0.082\\0.085\\0.079\\0.093\\0.081\\0.073\\0.076\\0.069\end{pmatrix},{}_{(4)}\dot{x}=\begin{pmatrix}0.13\\0.084\\\vdots\\0.059\\0.066\\0.072\\0.065\\0.062\\0.123\\0.134\\0.113\end{pmatrix}\right]\right\} \tag{51}$$

are invariant, in the sense that their contravariant components remain unchanged with respect to the vectors contained in any orthonormal basis whatsoever of E_{16}. Since we write

$$0 \cdot \begin{pmatrix} 0.12 \\ 0.085 \\ \vdots \\ 0.11 \\ 0.096 \\ 0.087 \\ 0.074 \\ 0.13 \\ 0.092 \\ 0.14 \\ 0.088 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0.116 \\ 0.073 \\ \vdots \\ 0.081 \\ 0.083 \\ 0.072 \\ 0.091 \\ 0.094 \\ 0.061 \\ 0.063 \\ 0.084 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0.123 \\ 0.092 \\ \vdots \\ 0.082 \\ 0.085 \\ 0.079 \\ 0.093 \\ 0.081 \\ 0.073 \\ 0.076 \\ 0.069 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0.13 \\ 0.084 \\ \vdots \\ 0.059 \\ 0.066 \\ 0.072 \\ 0.065 \\ 0.062 \\ 0.123 \\ 0.134 \\ 0.113 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{52}$$

${}_{(1)}\dot{x}$, ${}_{(2)}\dot{x}$, ${}_{(3)}\dot{x}$, and ${}_{(4)}\dot{x}$ are linearly independent. Moreover, this implies that ${}_{(1)}\dot{x}$, ${}_{(2)}\dot{x}$, ${}_{(3)}\dot{x}$, and ${}_{(4)}\dot{x}$ are also logically independent. It is possible to consider a reduction in the dimension of the vector space over $\mathbb{R}$ within which the four vectors denoted by ${}_{(1)}\dot{x}$, ${}_{(2)}\dot{x}$, ${}_{(3)}\dot{x}$, and ${}_{(4)}\dot{x}$ are initially studied. In this way, four axes are obtained. They are perpendicular in pairs. Each axis of them is not a linear subspace of E_{16}, but it is a one-dimensional vector space over $\mathbb{R}$ coinciding with the real line. The point where the four axes meet is called the origin of the coordinate system and it has $(0,0,0,0)$ as coordinates. On each axis, it is also possible to add a point that coincides with zero. That point is associated with a frequency that is equal to zero. The center of a multiple frequency distribution is the mean of it. The center of a multiple frequency distribution can geometrically be divided. Such a center is therefore a point whose coordinates are the centers of marginal frequency distributions. In particular, from

$$X_{12\ldots4} \equiv \{{}_1X, {}_2X, \ldots, {}_4X\},$$

it follows that one has

$$\overline{X}_{12\ldots4} \equiv ({}_1\overline{X}, {}_2\overline{X}, \ldots, {}_4\overline{X}). \tag{53}$$

Marginal frequency distributions are the components of the multiple frequency distribution under consideration. Each center of a marginal frequency distribution belongs to one of the four axes. Given the centers of marginal frequency distributions, even statistical weights identifying the center of a multiple frequency distribution do not change. Since marginal weights do not change whenever they have been made explicit, not even the centers of marginal frequency distributions change. Such centers are therefore said to be invariant. If a reduction in the dimension of the initial vector space over $\mathbb{R}$ denoted by E_{16} is not considered, then the center of the frequency distribution defined with respect to X_{34}^{12} is given by

$$\overline{X}_{34}^{12} = \begin{vmatrix} \|{}_{12}f\|_{\alpha}^2 & {}_{12}f \odot {}_{34}f \\ {}_{34}f \odot {}_{12}f & \|{}_{34}f\|_{\alpha}^2 \end{vmatrix}, \tag{54}$$

where one has

$$\|{}_{12}f\|_{\alpha}^2 = \begin{vmatrix} \|{}_{(1)}\dot{x}\|_{\alpha}^2 & {}_{(1)}\dot{x} \odot {}_{(2)}\dot{x} \\ {}_{(2)}\dot{x} \odot {}_{(1)}\dot{x} & \|{}_{(2)}\dot{x}\|_{\alpha}^2 \end{vmatrix}, \tag{55}$$

$${}_{12}f \odot {}_{34}f = \begin{vmatrix} {}_{(1)}\dot{x} \odot {}_{(3)}\dot{x} & {}_{(1)}\dot{x} \odot {}_{(4)}\dot{x} \\ {}_{(2)}\dot{x} \odot {}_{(3)}\dot{x} & {}_{(2)}\dot{x} \odot {}_{(4)}\dot{x} \end{vmatrix}, \tag{56}$$

$${}_{34}f \odot {}_{12}f = \begin{vmatrix} {}_{(3)}\dot{x} \odot {}_{(1)}\dot{x} & {}_{(3)}\dot{x} \odot {}_{(2)}\dot{x} \\ {}_{(4)}\dot{x} \odot {}_{(1)}\dot{x} & {}_{(4)}\dot{x} \odot {}_{(2)}\dot{x} \end{vmatrix}, \tag{57}$$

and

$$\|_{34}f\|_{\alpha}^{2} = \begin{vmatrix} \|_{(3)}\dot{x}\|_{\alpha}^{2} & {}_{(3)}\dot{x} \odot {}_{(4)}\dot{x} \\ {}_{(4)}\dot{x} \odot {}_{(3)}\dot{x} & \|_{(4)}\dot{x}\|_{\alpha}^{2} \end{vmatrix}. \tag{58}$$

That center denoted by $\overline{X}_{34}^{12}$ shows the value of the square of the area of a specific geometric shape, whose edges are given by two multiple statistical variables of order 2, X_{12} and X_{34}, represented by two antisymmetric tensors of order 2. The geometric shape under consideration is a 2-parallelepiped. It is embedded into a linear subspace of E_{16}. The α-norm of the tensor of order 2 denoted by ${}_{12}f$ is written as $\|_{12}f\|_{\alpha}^{2}$. The α-product of the tensors ${}_{12}f$ and ${}_{34}f$ is written as ${}_{12}f \odot {}_{34}f$. The Bravais-Pearson correlation coefficient can be extended. Hence, the Bravais-Pearson correlation coefficient associated with a divided quadruple statistical variable is given by

$$R_{12,34} = \frac{{}_{12}d \odot {}_{34}d}{\|_{12}d\|_{\alpha}\|_{34}d\|_{\alpha}}, \tag{59}$$

where ${}_{12}d$ and ${}_{34}d$ are two tensors of order 2 based on deviations. On the other hand, the following index given by

$$r_{12} = \frac{{}_{(1)}\dot{d} \odot {}_{(2)}\dot{d}}{\|_{(1)}\dot{d}\|_{\alpha}\|_{(2)}\dot{d}\|_{\alpha}} \tag{60}$$

expresses the Bravais-Pearson correlation coefficient associated with X_{12}. Thus, one has

$$r_{12} = \cos\gamma = \frac{{}_{(1)}\dot{d} \odot {}_{(2)}\dot{d}}{\|_{(1)}\dot{d}\|_{\alpha}\|_{(2)}\dot{d}\|_{\alpha}}. \tag{61}$$

The Bravais-Pearson correlation coefficient associated with X_{12} can also be written as

$$r_{12}^{2} = 1 - \frac{\|_{12}d\|_{\alpha}^{2}}{\|_{12}\hat{d}\|_{\alpha}^{2}}, \tag{62}$$

with

$$\|_{12}\hat{d}\|_{\alpha}^{2} = \begin{vmatrix} \|_{(1)}\dot{d}\|_{\alpha}^{2} & 0 \\ 0 & \|_{(2)}\dot{d}\|_{\alpha}^{2} \end{vmatrix}. \tag{63}$$

The expression given by (62) shows the Bravais-Pearson correlation coefficient associated with X_{12} to be a measure which is established on the surface of a 2-parallelepiped, whose edges are two vectors. The numerator of the subtrahend of the right-hand side of (62) is therefore assumed to be as the foundation of the notion of superficial metric. This latter is a quadratic metric of a nonlinear nature. Hence, one of the practical advantages of the framework which is proposed in this research work is that further statistical indices, obtained using a quadratic metric of a nonlinear nature, can enlarge the range of estimation problems to be addressed. Even the Bravais-Pearson correlation coefficient associated with a divided quadruple statistical variable can be shown using a quadratic metric of a nonlinear nature: it can be denoted by $R_{12,34}^{2}$. Usual statistical models do not use a nonlinear metric. Antisymmetric tensors of order 2 are essential tools to be used inside a nonlinear metric. For this reason, those tools are treated in this subsection and in the two previous subsections. Note the following:

Remark 9. *Given a multiple statistical variable of order* 2 *or greater, it is only useful to obtain a measure which is referred to it. Such a measure intrinsically depends on two or more marginal distributions which are summarized. To know the exact distribution associated with a multiple statistical variable of order* 2 *or greater is not of interest to us. For instance, if a* 2*-parallelepiped has its two touching edges which are known, then its area is a measure also obtained from the product of three factors: the lengths of two touching edges and the sine of the angle between them. The square of that measure can be considered. It can be compared with the squares of other measures characterizing other* 2*-parallelepipeds.*

4. Results

4.1. From Frequency Distributions to Random Variables

We focused on a finite statistical population denoted by P. We also focused on a simple statistical variable denoted by X: it is intrinsically characterized "a priori". Given X, a simple frequency distribution has been denoted by (X_i, P_i) as i varies from 1 to m. A simple frequency distribution is characterized "a posteriori". Note the following:

Remark 10. *We first study the distinct values of X, where X is a simple statistical variable. We, secondly, pass on to define the notion of random variable. The distinct values of X are now all possible outcomes of a random experiment. Such an experiment is about determining the mathematical expectation or prevision of a random variable. This latter is again denoted by X. Thus, the distinct values of a random variable denoted by X belong to the sample space S, where S is a finite set. They are expressed by* x^i*, with* $i = 1, \ldots, m$*. In general, a random variable X, defined on S, is a function from S into the set* $\mathbb{R}$ *of real numbers such that the pre-image of any interval of* $\mathbb{R}$ *is an event in S.*

Here, it is possible to establish the following:

Definition 2. *The sample space denoted by S consists of exactly m elements, where m is a positive integer. A marginal random variable denoted by X, and defined on S, is a function from S into* $\mathbb{R}$*, where the pre-image of a specific interval of* $\mathbb{R}$ *denoted by* $[x^i, x^i] = \{x^i\}$*, with* $i = 1, \ldots, m$*, is an event in S.*

Since we follow a probability approach for which an event is not intrinsically a repeatable fact, but is a single case, X is also said to be a marginal random quantity. The possible values for X are the elements of the following set

$$I(X) = \{x^1, x^2, \ldots, x^m\}, \tag{64}$$

where we put $x^1 < x^2 < \ldots < x^m$ for convenience, so X is a bounded quantity from above and below. The possible values for X given by $I(X)$ are the image of X. They are the contravariant components of a vector denoted by $\dot{x}$ belonging to an m-dimensional vector space E_m over $\mathbb{R}$ having a Euclidean nature. An orthonormal basis of E_m denoted by $\{\dot{e}_i\}$, with $i = 1, \ldots, m$, is considered, so from the linear combination expressed by

$$\dot{x} = x^i \dot{e}_i,$$

it follows that one writes

$$\dot{x} = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^m \end{pmatrix}. \tag{65}$$

A multiple statistical variable of order k, where k is a positive integer, is expressed by

$$X_{12\ldots k} \equiv \{{}_1X, {}_2X, \ldots, {}_kX\}. \tag{66}$$

Each simple statistical variable ${}_hX$, with $h \in I_k = \{1, 2, \ldots, k\}$, is a component of $X_{12\ldots k}$. Each simple statistical variable is a marginal variable. The generic value of $X_{12\ldots k}$ is given by

$$X_{i_1 i_2 \ldots i_k} \equiv ({}_1X_{i_1}, {}_2X_{i_2}, \ldots, {}_kX_{i_k}). \tag{67}$$

Given a statistical population P and a multiple statistical variable of order k denoted by $X_{12\ldots k}$, an item within P with respect to $X_{12\ldots k}$ is written as

$$U_{i_1, i_2, \ldots, i_k}^{{}_1X, {}_2X, \ldots, {}_kX}. \tag{68}$$

It is an element of P where the value $X_{i_1 i_2 \ldots i_k}$ of $X_{12\ldots k}$ appears. A multiple frequency distribution of order k is the set of values of $X_{12\ldots k}$ appearing within P. Since it is not necessary that the values of $X_{12\ldots k}$ are all distinct, a multiple frequency distribution of order k is expressed by

$$(X_{i_1 i_2 \ldots i_k}, P_{i_1 i_2 \ldots i_k}). \tag{69}$$

High-dimensional data are caught by multiple frequency distributions. A multiple frequency distribution of order k defined with respect to a multiple statistical variable of the same order identifies k marginal random variables that are the components of another variable. Such variables are defined on the same sample space denoted by S. It is possible to establish the following:

Definition 3. *Given k marginal random variables being the components of another variable, if such variables are studied along with k marginal probability distributions of a nonparametric nature, where each marginal probability distribution of a nonparametric nature uniquely identifies a marginal random variable, then k^2 bivariate probability distributions of the same nature take place.*

Check the following:

Example 3. *If $k = 2$, then ${}_1X$ and ${}_2X$ are two marginal random variables being the components of X_{12}: ${}_1X$ and ${}_2X$ are defined on the same sample space S. Let $I({}_1X) = \{{}_1x^1, {}_1x^2, \ldots, {}_1x^m\}$ be the image of ${}_1X$. Let $I({}_2X) = \{{}_2x^1, {}_2x^2, \ldots, {}_2x^m\}$ be the image of ${}_2X$. The Cartesian product of $I({}_1X)$ and $I({}_2X)$ is not commutative: for this reason, $2^2 = 4$ joint distributions are studied. The function h on $I({}_1X) \times I({}_2X)$ is the joint distribution of ${}_1X$ and ${}_2X$. The marginal distributions of ${}_1X$ and ${}_2X$ are uniquely determined. Conversely, the masses of the joint distribution of ${}_1X$ and ${}_2X$ can change in infinite ways. Thus, the joint distribution of ${}_1X$ and ${}_2X$ is not uniquely determined. Anyway, a joint distribution of ${}_1X$ and ${}_2X$ is chosen. The function h on $I({}_2X) \times I({}_1X)$ is the joint distribution of ${}_2X$ and ${}_1X$. This latter coincides with the joint distribution of ${}_1X$ and ${}_2X$. The function h' on $I({}_1X) \times I({}_1X)$ is the marginal distribution of ${}_1X$ seen as a joint probability function. The function h'' on $I({}_2X) \times I({}_2X)$ is the marginal distribution of ${}_2X$ seen as a joint probability function. Probability spaces on $I({}_1X) \times I({}_2X)$, $I({}_2X) \times I({}_1X)$, $I({}_1X) \times I({}_1X)$, and $I({}_2X) \times I({}_2X)$ are defined by h, h', and h''. Similarly, if $k = 3$, then ${}_1X$, ${}_2X$, and ${}_3X$ are three marginal random variables being the components of X_{123}: ${}_1X$, ${}_2X$, and ${}_3X$ are defined on the same sample space S. Since the Cartesian product of $I({}_1X)$, $I({}_2X)$, and $I({}_3X)$ is not commutative, $3^2 = 9$ joint distributions are studied. Nevertheless, only the masses of $9 - 3 = 6$ joint distributions can change in infinite ways. After choosing them, such distributions are equal in pairs. In general, if $k > 3$, then only the masses of $k^2 - k$ joint distributions can change in infinite ways. After choosing them, such distributions are equal in pairs.*

4.2. Previsions of Random Variables

Given X, the prevision of X is based on the probability distribution of X. Such a distribution is of a nonparametric nature. We write

$$\dot{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{pmatrix}, \tag{70}$$

where the covariant components of the vector denoted by $\dot{p}$ are probabilities. Here, there are ∞^{m-1} probabilities which can be determined at a first stage. Law of total probability holds at a first stage every time, so the sum of m probabilities is equal to 1. Hence, there are ∞^{m-1} probability laws that can formally be admitted at a first stage. Every probability law reflects a specific subjective opinion associated with a given individual regarding m probabilities of m single events. It is therefore possible to establish the following:

Definition 4. *The degree of belief in the occurrence of a single event, attributed by an individual having a given state of information and knowledge at a given instant, is the subjective probability of that event.*

Definition 5. *After considering all the possible values, of an objective nature, for a random variable, if an individual distributes among them his expectations and sensations of probability, then a prevision of a random variable takes place.*

Within this context, the prevision of X denoted by $\mathbf{P}(X)$ is geometrically the scalar product of two ordered sequences of real numbers [39–41]. We write

$$\mathbf{P}(X) = \langle \dot{x}, \dot{p} \rangle = \langle \dot{p}, \dot{x} \rangle, \tag{71}$$

where $\dot{x}$ and $\dot{p}$ belong to the same finite dimensional vector space over $\mathbb{R}$ denoted by E_m. If we put $x^1, \ldots, x^m$ on the real line, where this line is a one-dimensional vector space over $\mathbb{R}$, then all probability laws that are formally admissible at a first stage identify a closed line segment. Its endpoints are the two extreme points, expressed by x^1 and x^m, respectively, of a one-dimensional closed convex set given by $[x^1, x^m]$. We first focus on that bounded interval denoted by $[x^1, x^m]$ containing all real numbers that can be accepted as estimates of the mean value of X. The multiplication theorem for conditional probability is used at a second stage for choosing $\mathbf{P}(X)$ as a point of that one-dimensional closed convex set. This implies that the state of information and knowledge associated with a given individual is intrinsically variable [42,43].

Given X_{34}^{12}, the prevision of X_{34}^{12} is expressed by

$$\mathbf{P}(X_{34}^{12}) = \begin{vmatrix} \|_{12}f\|_{\alpha}^2 & {}_{12}f \odot {}_{34}f \\ {}_{34}f \odot {}_{12}f & \|_{34}f\|_{\alpha}^2 \end{vmatrix}, \tag{72}$$

where each entry of the square matrix whose determinant, given by (72), is computed is characterized by four elements, each of which identifies a two-dimensional closed convex set. Thus, the prevision of X_{34}^{12} denoted by $\mathbf{P}(X_{34}^{12})$ is appropriately decomposed. The prevision of X_{34}^{12} denoted by $\mathbf{P}(X_{34}^{12})$ also shows the value of the square of the area of a specific geometric shape, whose edges are given by two variables, X_{12} and X_{34}, respectively, where one has $X_{12} = \{{}_1X, {}_2X\}$ and $X_{34} = \{{}_3X, {}_4X\}$.

4.3. Qualitative Axioms

We focus on ${}_{(1)}\dot{x} \odot {}_{(2)}\dot{x}$. Two marginal random variables denoted by ${}_1X$ and ${}_2X$, respectively, are involved. The possible values for ${}_1X$ and ${}_2X$ are the contravariant components of ${}_{(1)}\dot{x}$ and ${}_{(2)}\dot{x}$, respectively. We put the possible values for ${}_1X$ on the real line. In addition, we put the possible values for ${}_2X$ on another real line. Those two lines are perpendicular: ${}_1X$ and ${}_2X$ are assumed to be logically independent. A reduction in the dimension of the algebraic structure where ${}_{(1)}\dot{x}$ and ${}_{(2)}\dot{x}$ are initially studied is observed. Conversely, the covariant components of ${}_{(1)}\dot{p}$ and ${}_{(2)}\dot{p}$ are probabilities associated with the possible values for ${}_1X$ and ${}_2X$, respectively. The covariant components of ${}_{(1)}\dot{p}$ express a nonparametric probability distribution of ${}_1X$ every time. Here, there are infinitely many nonparametric distributions of ${}_1X$ at a first stage, each of which expresses the opinion of a given individual having a given state of information and knowledge at a certain instant. Such distributions identify a one-dimensional closed convex set. The covariant components of ${}_{(2)}\dot{p}$ express a nonparametric probability distribution of ${}_2X$ every time. Here, there are infinitely many nonparametric distributions of ${}_2X$ at a first stage, each of which expresses the opinion of the same individual having a given state of information and knowledge at a certain instant. Such distributions identify another one-dimensional closed convex set. Thus, there are two one-dimensional closed convex sets on two perpendicular real lines. Those two one-dimensional closed convex sets identify a two-dimensional closed convex set. This latter can be thought as a square or a rectangle embedded into a Cartesian coordinate plane. One of their vertices could also be the origin of a Cartesian coordinate plane. It is possible to establish the following:

Definition 6. *Every point of a two-dimensional closed convex set is a prevision bundle. It is denoted by* $(\mathbf{P}({}_1X), \mathbf{P}({}_2X))$*, where* $\mathbf{P}({}_1X)$ *is the prevision or mathematical expectation of* ${}_1X$ *and* $\mathbf{P}({}_2X)$ *is the prevision or mathematical expectation of* ${}_2X$*.*

Note the following:

Remark 11. *After reducing the dimension of the algebraic structure where* ${}_{(1)}\dot{x}$ *and* ${}_{(2)}\dot{x}$ *are initially studied,* ${}_{(1)}\dot{x} \odot {}_{(2)}\dot{x}$ *is a point of a two-dimensional closed convex set. It is always decomposed into* $\mathbf{P}({}_1X)$ *and* $\mathbf{P}({}_2X)$*, respectively, where* $\mathbf{P}({}_1X)$ *and* $\mathbf{P}({}_2X)$ *are chosen at a second stage whenever the state of information and knowledge associated with an individual becomes more precise. Thus,* $\mathbf{P}({}_1X)$ *and* $\mathbf{P}({}_2X)$ *are the coordinates of a point of a two-dimensional closed convex set. This latter is embedded into a Cartesian coordinate plane.*

A fundamental invariance property is established in the following:

Definition 7. *Whenever* $\mathbf{P}({}_1X)$ *and* $\mathbf{P}({}_2X)$ *are made explicit, marginal probabilities used to obtain them are fixed. They are said to be invariant. Conversely, there are infinitely many bivariate probabilities compatible with the marginal ones through which* $\mathbf{P}({}_1X)$ *and* $\mathbf{P}({}_2X)$ *are obtained. While a specific set of bivariate probabilities is chosen, only* $\mathbf{P}({}_1X)$ *and* $\mathbf{P}({}_2X)$ *are directly observable.*

Although bivariate masses can vary, they do not come into play whenever we write $(\mathbf{P}({}_1X), \mathbf{P}({}_2X))$ to identify a prevision bundle belonging to the "budget set" containing all bundles that are affordable [44–46]. Bivariate masses characterizing any bivariate distribution of a nonparametric nature whatsoever are not directly observable, unlike $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$. Conversely, with regard to $\|{}_{(1)}\dot{x}\|_\alpha^2$ and $\|{}_{(2)}\dot{x}\|_\alpha^2$, marginal and bivariate masses are the same. Hence, bivariate masses do not vary. Note the following:

Remark 12. *If a two-dimensional closed convex set is considered whenever two logically independent marginal random variables are studied, then a uniquely determined bivariate distribution of a nonparametric nature could be missing. Given* $\mathbf{P}({}_1X)$ *and* $\mathbf{P}({}_2X)$*, bivariate masses can be chosen based on a specific working hypothesis which is made explicit by an individual. Bivariate masses could also be chosen in such a way that one of three extreme hypotheses takes place. Thus, bivariate masses can be equal to the product of the marginal ones. They can be chosen in such a way that the possible values for* ${}_2X$ *increase when the possible values for* ${}_1X$ *increase and elsewhere such masses are equal to zero, or the possible values for* ${}_2X$ *decrease when the possible values for* ${}_1X$ *increase and elsewhere they are equal to zero.*

If a two-dimensional closed convex set is managed whenever two logically independent marginal random variables are studied, given any two prevision bundles, denoted by $(\mathbf{P}'({}_1X), \mathbf{P}'({}_2X))$ and $(\mathbf{P}''({}_1X), \mathbf{P}''({}_2X))$, respectively, then it is possible to study how a given individual can rank them as to their desirability. It is therefore possible to study how a given individual can establish that one of the prevision bundles is strictly better than the other, or determine that he is indifferent between the two bundles. If we write

$$(\mathbf{P}''({}_1X), \mathbf{P}''({}_2X)) \succ (\mathbf{P}'({}_1X), \mathbf{P}'({}_2X)), \tag{73}$$

then a given individual strictly prefers $(\mathbf{P}''({}_1X), \mathbf{P}''({}_2X))$ to $(\mathbf{P}'({}_1X), \mathbf{P}'({}_2X))$. More explicitly, a given individual strictly prefers $(\mathbf{P}''({}_1X), \mathbf{P}''({}_2X))$ to $(\mathbf{P}'({}_1X), \mathbf{P}'({}_2X))$ if and only if we observe

$$\begin{cases} \mathbf{P}''({}_1X) > \mathbf{P}'({}_1X) \\ \mathbf{P}''({}_2X) > \mathbf{P}'({}_2X) \end{cases} \tag{74}$$

on two perpendicular real lines separately. If a given individual is indifferent between two prevision bundles, then it is possible to write

$$(\mathbf{P}''({}_1X), \mathbf{P}''({}_2X)) \sim (\mathbf{P}'({}_1X), \mathbf{P}'({}_2X)), \tag{75}$$

That is, one has

$$\begin{cases} \mathbf{P}''({}_1X) = \mathbf{P}'({}_1X) \\ \mathbf{P}''({}_2X) = \mathbf{P}'({}_2X) \end{cases} \tag{76}$$

on two perpendicular real lines separately. If a given individual prefers or is indifferent between two prevision bundles, then it is possible to say that he weakly prefers $(\mathbf{P}''({}_1X), \mathbf{P}''({}_2X))$ to $(\mathbf{P}'({}_1X), \mathbf{P}'({}_2X))$. We write

$$(\mathbf{P}''({}_1X), \mathbf{P}''({}_2X)) \succeq (\mathbf{P}'({}_1X), \mathbf{P}'({}_2X)), \tag{77}$$

so one has

$$\begin{cases} \mathbf{P}''({}_1X) \geq \mathbf{P}'({}_1X) \\ \mathbf{P}''({}_2X) \geq \mathbf{P}'({}_2X) \end{cases} \tag{78}$$

on two perpendicular real lines separately.

It is possible to make some assumptions about the coherence of an individual's choices. The first assumption is called completeness assumption, so we assume that any two prevision bundles can be compared. That is, given $(\mathbf{P}''({}_1X), \mathbf{P}''({}_2X))$ and $(\mathbf{P}'({}_1X), \mathbf{P}'({}_2X))$, it is possible to assume that $(\mathbf{P}''({}_1X), \mathbf{P}''({}_2X)) \succeq (\mathbf{P}'({}_1X), \mathbf{P}'({}_2X))$, or $(\mathbf{P}'({}_1X), \mathbf{P}'({}_2X)) \succeq (\mathbf{P}''({}_1X), \mathbf{P}''({}_2X))$, or both, in which case a given individual is indifferent between the two prevision bundles belonging to a specific two-dimensional closed convex set or "budget set".

The second assumption is called reflexivity assumption, so it is possible to assume that any prevision bundle is at least as good as itself. We therefore write both $(\mathbf{P}''({}_1X), \mathbf{P}''({}_2X)) \succeq (\mathbf{P}''({}_1X), \mathbf{P}''({}_2X))$ and $(\mathbf{P}'({}_1X), \mathbf{P}'({}_2X)) \succeq (\mathbf{P}'({}_1X), \mathbf{P}'({}_2X))$.

The third axiom is called transitivity assumption, so if

$$(\mathbf{P}''({}_1X), \mathbf{P}''({}_2X)) \succeq (\mathbf{P}'({}_1X), \mathbf{P}'({}_2X)),$$

and

$$(\mathbf{P}'({}_1X), \mathbf{P}'({}_2X)) \succeq (\mathbf{P}'''({}_1X), \mathbf{P}'''({}_2X)),$$

then we assume that

$$(\mathbf{P}''({}_1X), \mathbf{P}''({}_2X)) \succeq (\mathbf{P}'''({}_1X), \mathbf{P}'''({}_2X)).$$

In other words, if a given individual thinks that $\mathbf{P}''$ is at least as good as $\mathbf{P}'$ and that $\mathbf{P}'$ is at least as good as $\mathbf{P}'''$, then the same individual coherently thinks that $\mathbf{P}''$ is at least as good as $\mathbf{P}'''$.

4.4. Qualitative Axioms and Their Psychological Value

The possible values, of an objective nature, for ${}_1X$ are given by

$$I({}_1X) = \{{}_1x^1, {}_1x^2, \ldots, {}_1x^m\}, \tag{79}$$

where we put ${}_1x^1 < {}_1x^2 < \ldots < {}_1x^m$ for convenience, so ${}_1X$ is a bounded quantity from above and below. We write

$$\inf I({}_1X) = {}_1x^1, \tag{80}$$

and

$$\sup I({}_1X) = {}_1x^m, \tag{81}$$

so one has

$$\inf I({}_1X) \leq \mathbf{P}({}_1X) \leq \sup I({}_1X). \tag{82}$$

Since $\mathbf{P}({}_1X)$ cannot be greater than $\sup I({}_1X)$ nor less than $\inf I({}_1X)$, no individual is willing to make greater sacrifices, where sacrifices are denoted by O, than he would make to secure the advantage A, of whatever nature it may be, conditional on the occurrence of $\sup I({}_1X)$. Similarly, no individual is willing to make less sacrifices, where sacrifices are denoted by O, than he would make to secure the advantage A, of whatever nature it may be, conditional on the occurrence of $\inf I({}_1X)$. The possible values, of an objective nature, for ${}_2X$ are expressed by

$$I({}_2X) = \{{}_2x^1, {}_2x^2, \ldots, {}_2x^m\}, \tag{83}$$

where we put ${}_2x^1 < {}_2x^2 < \ldots < {}_2x^m$ for convenience, so even ${}_2X$ is a bounded quantity from above and below. We write

$$\inf I({}_2X) = {}_2x^1, \tag{84}$$

and

$$\sup I({}_2X) = {}_2x^m, \tag{85}$$

so one has

$$\inf I({}_2X) \leq \mathbf{P}({}_2X) \leq \sup I({}_2X). \tag{86}$$

Since $\mathbf{P}({}_2X)$ cannot be greater than $\sup I({}_2X)$ nor less than $\inf I({}_2X)$, no individual is willing to make greater sacrifices, where sacrifices are denoted by O, than he would

make to secure the advantage A conditional on the occurrence of sup $I({}_2X)$. Similarly, no individual is willing to make less sacrifices, where sacrifices are denoted by O, than he would make to secure the advantage A conditional on the occurrence of inf $I({}_2X)$.

We focus on the principle expressed as follows:

Definition 8. *In order to be able to enjoy an advantage A, of whatever nature it may be, in the case that one of the events of ${}_1X$ expressed by ${}_1x^1, {}_1x^2, \ldots, {}_1x^m$ occurs, a given individual is willing to make sacrifices, where sacrifices are denoted by O, which are all the more serious the greater the mathematical expectation or prevision of ${}_1X$.*

Similarly, it is possible to establish the following:

Definition 9. *In order to be able to enjoy an advantage A, of whatever nature it may be, in the case that one of the events of ${}_2X$ expressed by ${}_2x^1, {}_2x^2, \ldots, {}_2x^m$ occurs, the same individual is willing to make sacrifices, where sacrifices are denoted by O, which are all the more serious the greater the mathematical expectation or prevision of ${}_2X$.*

Given $\mathbf{P}'$ and $\mathbf{P}''$, one has

$$\mathbf{P}'({}_1X) > \mathbf{P}''({}_1X), \tag{87}$$

or

$$\mathbf{P}'({}_1X) < \mathbf{P}''({}_1X), \tag{88}$$

or

$$\mathbf{P}'({}_1X) = \mathbf{P}''({}_1X) \tag{89}$$

on the real line. To enjoy an advantage A conditionally to the occurrence of one of the events of ${}_1X$, to which a certain set of probabilities is associated, an individual is willing to sustain sacrifices, where sacrifices are denoted by O, greater than, less than, or equal to the sacrifices O that the same individual is willing to sustain to enjoy A conditionally to the occurrence of one of the events of ${}_1X$, to which another set of probabilities is associated. Given $\mathbf{P}'$ and $\mathbf{P}''$, one writes

$$\mathbf{P}'({}_2X) > \mathbf{P}''({}_2X), \tag{90}$$

or

$$\mathbf{P}'({}_2X) < \mathbf{P}''({}_2X), \tag{91}$$

or

$$\mathbf{P}'({}_2X) = \mathbf{P}''({}_2X) \tag{92}$$

on another real line. To enjoy an advantage A conditionally to the occurrence of one of the events of ${}_2X$, to which a certain set of probabilities is associated, an individual is willing to sustain sacrifices O greater than, less than, or equal to the sacrifices O that the same individual is willing to sustain to enjoy A conditionally to the occurrence of one of the events of ${}_2X$, to which another set of probabilities is associated.

Given $\mathbf{P}'$, $\mathbf{P}''$, and $\mathbf{P}'''$, an individual cannot judge $\mathbf{P}'({}_1X)$ greater than $\mathbf{P}''({}_1X)$, $\mathbf{P}''({}_1X)$ greater than $\mathbf{P}'''({}_1X)$, and $\mathbf{P}'''({}_1X)$ greater than $\mathbf{P}'({}_1X)$. Similarly, the same individual cannot judge $\mathbf{P}'({}_2X)$ greater than $\mathbf{P}''({}_2X)$, $\mathbf{P}''({}_2X)$ greater than $\mathbf{P}'''({}_2X)$, and $\mathbf{P}'''({}_2X)$ greater than $\mathbf{P}'({}_2X)$.

Given $I({}_1X)$, before reducing the dimension of the algebraic structure where the possible values, of an objective nature, for ${}_1X$ are initially studied, we write

$$ {}_1\dot{x} = \begin{pmatrix} {}_1x^1 \\ {}_1x^2 \\ \vdots \\ {}_1x^m \end{pmatrix}. \tag{93} $$

It is always possible to decompose the possible values for ${}_1X$, so one has

$$ {}_1X = {}_1Y + {}_1Z, \tag{94} $$

that is,

$$ {}_1\dot{x} = {}_1\dot{y} + {}_1\dot{z}. \tag{95} $$

If one writes

$$ \mathbf{P}''({}_1Y + {}_1Z) > \mathbf{P}'({}_1Y + {}_1Z), \tag{96} $$

then we say that in order to enjoy the advantage A subject to the occurrence of one of the events of ${}_1Y + {}_1Z$ treated by a set of probabilities giving rise to $\mathbf{P}'$, a given individual is willing to sustain a group of sacrifices denoted by $O + O'$. It follows that in order to enjoy the advantage A subject to the occurrence of one of the events of ${}_1Y + {}_1Z$ treated by a set of probabilities giving rise to $\mathbf{P}''$, the same individual is willing to sustain the same group of sacrifices denoted by $O + O'$. Given $I({}_2X)$, before reducing the dimension of the algebraic structure where the possible values, of an objective nature, for ${}_2X$ are initially studied, one has

$$ {}_2\dot{x} = \begin{pmatrix} {}_2x^1 \\ {}_2x^2 \\ \vdots \\ {}_2x^m \end{pmatrix}. \tag{97} $$

It is always possible to decompose the possible values for ${}_2X$ into ${}_2Y + {}_2Z$, so one writes

$$ {}_2X = {}_2Y + {}_2Z, \tag{98} $$

that is,

$$ {}_2\dot{x} = {}_2\dot{y} + {}_2\dot{z}. \tag{99} $$

If one writes

$$ \mathbf{P}''({}_2Y + {}_2Z) > \mathbf{P}'({}_2Y + {}_2Z), \tag{100} $$

then we say that in order to enjoy the advantage A subject to the occurrence of one of the events of ${}_2Y + {}_2Z$ treated by a set of probabilities giving rise to $\mathbf{P}'$, a given individual is willing to sustain a group of sacrifices denoted by $O + O'$. It follows that in order to enjoy the advantage A subject to the occurrence of one of the events of ${}_2Y + {}_2Z$ treated by a set of probabilities giving rise to $\mathbf{P}''$, the same individual is willing to sustain the same group of sacrifices denoted by $O + O'$.

When considering two or more marginal variables which are logically independent, the study of the coherence of an individual's choices must be appropriately decomposed [47–49]. If we focus on the other α-products characterizing (72), where (72) extends the notion of mean value or mathematical expectation of a random variable, then the same considerations we have developed in this subsection and in the previous one, with respect to a specific α-product, hold. The coherence of an individual's choices can therefore be defined based solely on qualitative considerations. It is possible to establish the following:

Definition 10. *An individual is coherent if he no longer evaluates the exact values of different mathematical expectations or previsions of a random variable, but only the inequalities between those values in order to respect certain qualitative axioms of an objective nature.*

Those axioms have been made explicit in this research work. The arguments developed with regard to qualitative axioms are not precise and mathematically rigorous, and they are not even intended to be precise and mathematically rigorous. Within this context, the most mathematically precise reasoning deserves a greater suspicion, as its greater or lesser adherence to a specific psychological process is hard to establish. Such a process is intrinsically of an imprecise and vague nature.

4.5. Two-Dimensional Closed Convex Sets and σ-Algebras on Perpendicular Real Lines

Given a random variable having two or more marginal random variables as its components, each two-dimensional closed convex set incorporates bivariate distributions of a nonparametric nature identifying the possible values, of an objective nature, for two marginal random variables that are logically independent. For instance, let ${}_1X$ and ${}_2X$ be two marginal random variables. After reducing the dimension of the algebraic structure where the possible values for ${}_1X$ and ${}_2X$ are initially studied, the support of each marginal variable is the real line. Hence, we focus on a larger and manageable space. Note the following:

Remark 13. *In this paper, each marginal random variable is assumed to have m possible values of an objective nature. Nevertheless, it is possible that ${}_1X$ has m different values, whereas n different values identify ${}_2X$, with $m \neq n$. Anyway, if this happens, then nothing changes.*

The conditional prevision of ${}_1X$ given H, or the prevision of ${}_1X$ under the condition or hypothesis H, is expressed by the following multiplication theorem for conditional probability

$$\mathbf{P}({}_1X \wedge H) = \mathbf{P}({}_1X \cdot H) = \mathbf{P}(H)\mathbf{P}({}_1X|H), \tag{101}$$

where the logical product $\wedge$ is also denoted by $\cdot$. That theorem is always used at a second stage [50]. A point of a closed line segment is accordingly chosen at a second stage, so a Bayesian interpretation of the concept of probability is accepted [51]. On the other hand, if a closed line segment is denoted by A, then the following set

$$\Sigma = \{A,\ A^c, \emptyset,\ U\} \tag{102}$$

is a σ-algebra of sets. The real line of which A is a portion is denoted by U, where U stands for universal set. Here, A^c is the complement of A. Similarly, the conditional prevision of ${}_2X$ given H is expressed by

$$\mathbf{P}({}_2X \wedge H) = \mathbf{P}({}_2X \cdot H) = \mathbf{P}(H)\mathbf{P}({}_2X|H). \tag{103}$$

Regarding another real line that is not the support of ${}_1X$, but is the support of ${}_2X$, if a closed line segment is denoted by B, then the following set

$$\Sigma' = \{B, B^c, \emptyset, U\} \tag{104}$$

is a σ-algebra of sets. The real line of which B is a portion is again denoted by U, where U stands for universal set. Here, B^c is the complement of B. Note the following:

Remark 14. *If two marginal random variables are the elements of a bivariate distribution, then they are logically independent. They come into play along with bivariate probabilities. The support of each marginal variable is the real line. Even if logical independence is a condition more restrictive than linear independence, we always think of two perpendicular lines.*

5. Discussion

If the psychological sensations of a given individual, along with his state of information and knowledge that is intrinsically variable, are at the basis of the notion of probability, then it is possible to make considerations of a qualitative nature. Since the logical foundation of probability calculus is here extended, such qualitative considerations are accordingly extended. The logical foundation of probability calculus is given by linearity when considering two or more random variables of which their linear combination is studied. Hence, a random variable is linearly dependent on other variables. Conversely, the logical foundation of probability calculus is given by multilinearity when studying two or more logically independent random variables. They are the components of another variable. To neglect marginal distributions to focus on an only multivariate distribution is what is usually made in the literature. On the contrary, in this paper, given different marginal probability distributions identifying different marginal random variables being the components of another variable, it is possible to study different bivariate distributions. A basic reason associated with the mathematical idea of a quadratic metric is therefore considered. The mean of a multivariate distribution can be decomposed using basic ideas related to a tensor decomposition [52,53]. Thus, even the mean values of different bivariate distributions can be decomposed observing symmetry relationships holding in pairs [54]. Each bivariate distribution is intrinsically made up of two marginal distributions. Since marginal distributions characterizing each marginal variable are invariant whenever their marginal probabilities are empirically made explicit by a given individual at a second stage, there is no loss of information. The real comparison is therefore made by a given individual among the mean values of marginal variables. They are directly observable. In particular, if two or more logically independent random variables are handled, then the real comparison is always studied on each real line of two perpendicular real lines, after reducing the dimension of the algebraic structure where specific geometric entities are initially considered. A specific algebraic structure for which linearity and multilinearity properties hold is here studied. Thus, we focus on finite dimensional vector spaces over $\mathbb{R}$ for which their dimensions can decrease or increase.

All marginal variables that are the components of another variable can be treated inside the same scale of preference. We focus on their mean values or previsions [55–58]. Hence, if the scale of utility associated with a given individual is the 45-degree line, then his best choice or estimate is operationally that leading to the marginal variable with the highest prevision or mean value. It is always possible to handle prices denoted by $\mathbf{P}$. For instance, given ${}_1X$, ${}_2X$, ${}_3X$, and ${}_4X$, whose values are pure numbers such as annual rates of return on the same stock listed on four stock exchanges, since we write

$$\mathbf{P}[a({}_hX)] = a\mathbf{P}({}_hX), \tag{105}$$

with $h = 1, 2, 3, 4$, where a is a real number, it is possible to define the following expression

$$\mathbf{P}({}_hX) = \frac{1}{a}\mathbf{P}[a({}_hX)], \tag{106}$$

with $h = 1, 2, 3, 4$. Here, we choose a, where a is a coefficient, in such a way that $a({}_hX)$ is a monetary value. For instance, we can choose $a = \$$, so (106) is an identity expressed by

$$\mathbf{P}({}_hX) = \frac{1}{a}a\mathbf{P}({}_hX), \tag{107}$$

with $h = 1, 2, 3, 4$, which is invariant regarding the choice of a. Since the price is defined only in terms of the subjective preference that it measures, if the scale of utility associated with a given individual is the 45-degree line, then his best choice or estimate is operationally that leading to the marginal variable ${}_hX$, $h = 1, 2, 3, 4$, with the highest price $\mathbf{P}$. Note the following:

Remark 15. *If the scale of utility associated with a given individual is the* 45*-degree line, then there is the identity of monetary value and utility. The higher the monetary value, the higher the utility.*

6. Conclusions

It is possible to leave the notion of probability undefined, much the same as point and line are undefined in geometry. Nevertheless, in this paper, the concept of probability is not undefined, but it has a psychological value and meaning. It is possible to extend axioms of a qualitative nature that are not in conflict with those characterizing the development of modern probability theory. Here, probabilistic evaluations that are made over a specific set of events, and interpretable as the opinions of a given individual, are always in agreement with the fundamental theorems of probability calculus. We focus on law of total probability at a first stage and the multiplication theorem for conditional probability at a second stage. In this paper, it is shown that the mean value of a bivariate distribution is a point of a two-dimensional closed convex set. This latter is embedded into a Cartesian coordinate plane. That point intrinsically depends on two real numbers that are its coordinates. They are the mean values, directly observable, of two marginal variables. Such mean values are compatible with infinitely many bivariate probabilities. It follows that a bivariate distribution does not need to be uniquely determined. Here, different probability spaces can be defined. Two probability spaces expressed by $(S, \Sigma, \mathbf{P})$ and $(S, \Sigma', \mathbf{P})$ are explicitly considered. Each probability space consists of three elements, two of which are always the same: S, the sample space which is a finite set within this context, and $\mathbf{P}$, the function of prevision which is meant to be an expression of the subjective opinion of a given individual about the probabilities of single events. The third element is intrinsically different: it is a σ-algebra of sets containing a closed line segment as its element, whose endpoints are two possible values for a specific marginal variable that is the component of another variable. We are interested in assigning probabilities only to all possible values for each marginal variable. We use tensors to obtain multilinear indices. The logical foundation of probability calculus is extended to multilinear issues using tensors. Accordingly, a quadratic metric of a nonlinear nature can be handled. It is therefore possible to compare not only numbers on perpendicular real lines, where each set of numbers to compare is on a specific real line, but also the squares of the areas of 2-parallelepipeds, the squares of the volumes of 3-parallelepipeds, and the squares of the hypervolumes of l-parallelepipeds, with $l > 3$. Each geometric shape of those is embedded into a specific geometric structure. In addition, since tensors used to obtain multilinear indices also work along with continuous distributions of a parametric nature such as normal distributions, it is not necessary to consider probability mass functions only.

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