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Growth Estimates for m-th ($m \geq 1$) Derivatives of Algebraic Polynomials in Domains with Piecewise Quasismooth Boundaries

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Abstract

In this paper, we investigate the growth behavior of the m-th derivatives ($m \geq 1$) of arbitrary algebraic polynomials in bounded and unbounded domains whose boundaries are piecewise quasismooth curve and possess interior and exterior zero angles of power type, within weighted Lebesgue spaces. We derive estimates for the growth of these derivatives in terms of the characteristics of the weight function, the geometric properties of the boundary, and the tangency exponents of the boundary arcs at their junction points. The obtained estimates are established both in the closure of the bounded domain and at exterior points, while the pointwise estimates in the unbounded domain additionally reflect the location of the point within the domain.

Keywords: algebraic polynomials; quasismooth curve; Bernstein–Walsh–Markoff inequalities

MSC: 30A10; 30C10; 41A17

1. Introduction and Definitions

One of the fundamental problems in the theory of analytic functions of a complex variable is determining the behavior of the modulus of an analytic function as its domain of analyticity expands. This problem becomes particularly relevant in the case of algebraic polynomials. Similar studies arise naturally in various areas of analysis, including inverse problems in polynomial approximation theory. The aim of this work is to investigate growth estimates for the modulus of algebraic polynomials at points lying outside the closure of a given domain. We simultaneously obtain corresponding estimates on the closure of the domain. The combination of these estimates provides a comprehensive description of the growth behavior of the modulus of the polynomials under consideration on the entire complex plane.

Before discussing the main results, we introduce the necessary notation and preliminary definitions.

Let $\mathbb{C}$ be a complex plane, $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, and $G \subset \mathbb{C}$ be a bounded Jordan domain with boundary $L := \partial G$; $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = extL$. For $t \in \mathbb{C}$ and $\delta > 0$, let us set: $\Delta(t, \delta) :=$

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$\{w \in \mathbb{C} : |w-t| > \delta\}$; $\Delta := \Delta(0,1)$. Consider the univalent conformal map $\Phi : \Omega \to \Delta$ normalized at infinity by $\Phi(\infty) = \infty$ and $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$. Let Ψ denote the inverse function of Φ.

For each $\delta \geq 0$, we define the exterior level curves associated with L by

$$L_{1+\delta} := \{z : \ |\Phi(z)| = 1+\delta\}, \ L_1 \equiv L,$$

and we set $G_{1+\delta} := intL_{1+\delta}$, $\Omega_{1+\delta} := extL_{1+\delta}$. Throughout the paper, the notation

$$\wp_n := \left\{ P_n(z) = \sum_{k=0}^{n} a_k z^k : a_k \in \mathbb{C} \right\},$$

will be used for the family of all algebraic polynomials with complex coefficients whose degrees do not exceed $n \in \mathbb{N}$. For a set $M \subset \mathbb{C}$, denote by $d(z,M) := dist(z,M) = \inf\{|\zeta - z| : \ \zeta \in M\}$.

Consider a fixed set of distinct points $\{z_j\}_{j=1}^{l} \subset L$. For a given constant R_0, where $1 < R_0 < \infty$, we define the generalized Jacobi weight function as follows:

$$h(z) := \begin{cases} h_0(z) \prod_{j=1}^{l} |z - z_j|^{\gamma_j}, & z \in \overline{G}_{R_0}, \\ 0, & z \in \mathbb{C}\backslash\overline{G}_{R_0}, \end{cases} \tag{1}$$

where $\gamma_j > -1$, for $j = 1, 2, \ldots, l$, and the function h_0 satisfies $h_0(z) \geq c_0(L) > 0$ for some positive constant $c_0(L) > 0$.

Consider a Jordan domain G whose boundary $L = \partial G$ is rectifiable. For $0 < p \leq \infty$, we introduce

$$\|P_n\|_p := \|P_n\|_{\mathcal{L}_p(h,L)} := \left(\int_L h(z)|P_n(z)|^p |dz| \right)^{1/p} < \infty, \quad 0 < p < \infty;$$

$$\|P_n\|_\infty := \|P_n\|_{\mathcal{L}_\infty(1,L)} := \max_{z\in L} |P_n(z)|, \quad p = \infty; \ \mathcal{L}_p(1,L) =: \mathcal{L}_p(L).$$

A fundamental result from Bernstein and Walsh [1] states that

$$\|P_n\|_{C(\overline{G}_{1+\delta})} \leq (1+\delta)^n \|P_n\|_{C(\overline{G})}, \ \forall P_n \in \wp_n. \tag{2}$$

A $\mathcal{L}_p(L)$-version of the above inequality, valid for $p > 0$, was proved in [2]:

$$\|P_n\|_{\mathcal{L}_p(L_{1+\delta})} \leq (1+\delta)^{n+\frac{1}{p}} \|P_n\|_{\mathcal{L}_p(L)}, \ \forall P_n \in \wp_n. \tag{3}$$

A weighted version of estimate (3) in the space $\mathcal{L}_p(h, L_{1+\delta})$, with h defined (1), was obtained in ([3], Lemma 2.4):

$$\|P_n\|_{\mathcal{L}_p(h,L_{1+\delta})} \leq (1+\delta)^{n+\frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \ \gamma^* = \max\{0; \gamma_j : 1 \leq j \leq \ l\}. \tag{4}$$

Before presenting the analogous inequality (2)–(4) for the two-dimensional analogue of $\|P_n\|_{A_p(h,G)}$, defined for an arbitrary Jordan domain G, we first introduce the following definition:

$$\|P_n\|_{A_p(h,G)} := \left(\iint\limits_G h(z)|P_n(z)|^p d\sigma_z \right)^{1/p}, \quad 0 < p < \infty,$$

$$\|P_n\|_{A_\infty(1,G)} := \max_{z\in\overline{G}} |P_n(z)|, \quad p = \infty;\ A_p(1,G) =: A_p(G),$$

where σ denotes the area measure.

We begin by introducing the following definitions.

A curve L is called quasiconformal if, for arbitrary points $z_1 \in L$ and $z_2 \in L$, the diameter of the shorter arc $l(z_1, z_2)$ of the curve L joining points z_1, z_2 satisfies the inequality ([4], p. 102)

$$\frac{diam\, l(z_1, z_2)}{|z_1 - z_2|} \le c < +\infty. \tag{5}$$

We recall that quasiconformal curves are not necessarily rectifiable (see, for instance [4], p. 104 and [5]).

Two-dimensional analogues of the inequalities (2)–(4) for a domain G bounded by an arbitrary quasiconformal curve and a weight function h defined as (1) for $\gamma_j > -2$, $j = 1, 2, \ldots, l$, are presented in [6] as follows:

$$\|P_n\|_{A_p(h,G_{1+\delta})} \le c_1 \left[1 + c_2\delta\right]^{n+\frac{1}{p}} \|P_n\|_{A_p(h,G)},\ p > 0, \tag{6}$$

where $c_2 > 0$ and $c_1 := c_1(G, p, c_2) > 0$ are constants, independent of n and z. Later, ([7], Theorem 1.1) established the following version of this estimate for an arbitrary Jordan domain G when $h(z) \equiv 1$:

$$\|P_n\|_{A_p(G_{1+\delta})} \le c_3(1+\delta)^{n+\frac{2}{p}} \|P_n\|_{A_p(G_{1+\frac{1}{n}})},\ \delta > \frac{1}{n},\ p > 0,$$

where $c_3 = \left(\frac{2}{e^p - 1}\right)^{\frac{1}{p}} \left[1 + O(\frac{1}{n})\right]$, $n \to \infty$, is asymptotically sharp constant.

In [8], for domains bounded by a rectifiable quasiconformal curve, an inequality analogous to (6) was established, incorporating the location of a point exterior to the given domain, expressed in the form

$$|P_n(z)| \le c(L)\frac{\sqrt{n}}{d(z,L)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1},\ z \in \Omega,$$

where $c(L) > 0$ is a constant depending only on the curve L.

Let $S \subset \mathbb{C}$ be a rectifiable Jordan curve or arc, and consider its natural parametrization: $z = z(s)$, $s \in [0, |S|]$, $|S| := mes\, S$. For two arbitrary points $z_1, z_2 \in S$, let $l(z_1, z_2)$ denote the subarc of S connecting these points and having the shorter diameter. The notation $|l(z_1, z_2)|$ will be used for the linear measure (length) of this subarc. According to ([9], p. 163), a bounded Jordan curve S is called a λ-quasismooth (in the sense of Lavrentiev) curve if there exists a constant $\lambda = \lambda(S) \ge 1$ such that

$$|l(z_1, z_2)| \le \lambda |z_1 - z_2|, \qquad z_1, z_2 \in S.$$

The domain G is said to be a λ-quasismooth domain if its boundary $L = \partial G$ is a λ-quasismooth curve. The family of all such curves and their subarcs is denoted by $QS(\lambda)$. In particular, a Jordan domain G belongs to $QS(\lambda)$ whenever $\partial G \in QS(\lambda)$, where $\lambda \ge 1$. Moreover, we use the notation $L \in QS$ (or $G \in QS$) whenever $L \in QS(\lambda)$ (or $G \in QS(\lambda)$) for some $\lambda \ge 1$.

In [10] (for L is $\lambda-$quasismooth curve) the authors investigated the problems on pointwise and uniform estimates for the $|P_n(z)|$ in unbounded domain Ω and on the closure of the bounded domain G and the estimation was obtained for the following type:

$$|P_n(z)| \le c_4 \|P_n\|_p \begin{cases} \nu_n, & z \in \overline{G}, \\ \eta_n, & z \in \Omega, \end{cases} \tag{7}$$

where $c_4 = c_4(L, p, m, \gamma) > 0$ is a constant independent of n, h, P_n, $\nu_n = \nu_n(L, h, p) > 0$ and $\eta_n = \eta_n(L, h, p, z) \to \infty$, as $n \to \infty$, are constants depending on the properties of L and h.

Note that λ-quasismooth curves do not admit cusps, i.e., their boundaries contain neither interior nor exterior zero angles. To extend this setting, we introduce a class of domains with piecewise quasismooth boundaries that may possess a finite number of interior and exterior zero angles at boundary points.

Let L be a bounded Jordan curve or arc and let $z \in L$. The curve L is said to be locally λ-quasismooth at z provided that there exists a closed subarc $\ell \subset L$ with $z \in \ell$ such that each open subarc of ℓ containing z is a λ-quasismooth arc. By the "three-point" criterion ([4], p. 100) and [11], every quasismooth curve is quasiconformal.

Throughout the paper, $c, c_0, c_1, c_2, \dots$ denote positive constants, while $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$ stand for sufficiently small positive constants. These constants may vary from one relation to another and, in general, depend on L and on other parameters that are inessential for the arguments. Any explicit dependence of a constant on specific parameters will be stated separately. For integers $k \ge 0$ and $m > k$, the notation $i = \overline{k, m}$ is used to mean $i = k, k+1, \dots, m$. Moreover, we assume that the points $\{z_i\}_{i=1}^{l} \subset L$ introduced in (1) and Definition 1 coincide. Without loss of generality, the points $\{z_i\}_{i=0}^{l}$ are arranged along the curve L according to its positive orientation. With this arrangement, G has interior zero angles at the points $\{z_i\}_{i=1}^{l_1}$ for $l_1 \ge 1$, and exterior zero angles at the points $\{z_i\}_{i=l_1+1}^{l}$ for $l \ge l_1 + 1$.

For each $j = 1, 2, \dots$ and for sufficiently small $\varepsilon_1 > 0$, let $f_j, g_j : [0, \varepsilon_1] \to \mathbb{R}$ be twice continuously differentiable functions satisfying $f_j(0) = g_j(0) = 0$ and $f_j^{(k)}(x) > 0$, $g_j^{(k)}(x) > 0$ for $x > 0$ and $k = 0, 1, 2$.

Definition 1 ([10]). *We say that a Jordan domain $G \in PQS\left(\lambda; f_i, g_j\right)$, $\lambda \ge 1$, $f_i = f_i(x)$, $i = \overline{1, l_1}$, $g_j = g_j(x)$, $j = \overline{l_1+1, l}$, if $L := \partial G = \bigcup_{j=0}^{l} L_j$ is the union of the finite number of λ-quasismooth arcs L_j, connecting at the points $\{z_j\}_{j=0}^{l} \in L$, and such that L is a locally λ-quasismooth arc at the $z_0 \in L \setminus \{z_j\}_{j=1}^{l}$ and, in the (x, y) local coordinate system with the origin at the z_j, $1 \le j \le l$, and axes parallel to the main coordinate system XOY, the following conditions are satisfied:*

(a) For every $z_j \in L$, $j = \overline{1, l_1}$, $l_1 \le l$,

$$\{z = x + iy : |z| \le \varepsilon_1, c_1 f_i(x) \le y \le c_2 f_i(x)\} \subset \overline{G},$$

$$\{z = x + iy : |z| \le \varepsilon_1, |y| \ge \varepsilon_2 x\} \subset \overline{\Omega};$$

(b) For every $z_j \in L$, $j = \overline{l_1+1, l}$,

$$\{z = x + iy : |z| < \varepsilon_3, \quad c_3 g_j(x) \le y \le c_4 g_j(x), \quad 0 \le x \le \varepsilon_3\} \subset \overline{\Omega},$$

$$\{z = x + iy : |z| < \varepsilon_3, \quad |y| \ge \varepsilon_4 x, \quad 0 \le x \le \varepsilon_3\} \subset \overline{G},$$

For some constants $-\infty < c_1 < c_2 < \infty$, $-\infty < c_3 < c_4 < \infty$, $\varepsilon_i > 0$, $i = \overline{1, 4}$.

According to Definition 1, a domain $G \in PQS(\lambda; f_i, g_j)$ can contain l_1 interior zero angles and $l - l_1$ exterior zero angles with respect to $\overline{G}$. In the absence of interior zero angles, i.e., when $l_1 = 0$, we use the notation $G \in PQS(\lambda; 0, g_j)$; similarly, if there are no exterior zero angles, i.e., $l_1 = l$, we write $G \in PQS(\lambda; f_i, 0)$. Finally, when G has no zero angles ($l = 0$), its boundary is a λ-quasismooth circle, and we identify this case with $PQS(\lambda, 0, 0) \equiv QS(\lambda)$.

In the work [12], for each $m \geq 1$ and for $p > 1$, the authors used a recurrence formula to investigate a similar problem for domains bounded by a piecewise $\lambda-$quasismooth curve having interior and exterior zero angles and obtained an estimate of the following type:

$$\left|P_n^{(m)}(z)\right| \leq \eta_n \|P_n\|_p,\ z \in \Omega,\ \eta_n = \eta_n(L, h, p, z) \to \infty, n \to \infty. \tag{8}$$

However, estimates obtained by means of the recurrence formula require lengthy and labor-intensive calculations. Moreover, this approach leads to a significant increase in the order of magnitude of the polynomial derivative estimate. In this work, bypassing the recurrence formula and using a different approach, sharper estimates of the type (7) are obtained for domains of the class $G \in PQS\left(\lambda\,; f_i,\ g_j\right)$ with some given functions f and g of power type.

Analogous results of the type (8) for various norms and classes of unbounded domains were obtained by N.A. Lebedev, P.M. Tamrazov, V.K. Dzjadyk (see, for example, ([13], pp. 418–428, [14], p. 383)), F. G. Abdullayev et al. in [15] (for domains with piecewise smooth boundary with interior angles, $m = 0$, $p > 0$), [16] ($L - \partial G$ rectifiable asymptotically conformal curve, $m = 0$, $p > 0$), $m = 0$, $p > 0$), [3] ($L - \partial G$ rectifiable piecewise Dini-smooth curve, $m = 0$, $p > 0$), [10] ($L - \partial G$ piecewise quasismooth curve with cusps, $m = 0$, $p > 0$); see also the references cited therein.

It should be noted that most of these studies rely on a recurrence formula, in which the estimate for each derivative is obtained from the estimate of the preceding one. In the present paper, we avoid the use of the recurrence formula and derive estimates of the type (8) for each $m \geq 1$, independently of the estimates for lower-order derivatives.

To obtain global estimates in the whole complex plane, one also requires Bernstein–Markoff–Nikol'skii-type inequalities for the derivatives of algebraic polynomials $P_n^{(m)}(z)$, $z \in \overline{G}$. Specifically, we consider inequalities of the form

$$\|P_n^{(m)}\|_\infty \leq \nu_n \|P_n\|_p,\ m = 1, 2, \ldots, \tag{9}$$

where the value $\nu_n := \nu_n(L, h, p, m) > 0$ ($\nu_n \to \infty$, $n \to \infty$), depends on the properties of the boundary L and the weight function h.

It should be noted that inequalities of type (9) trace back to the classical works of [17–19] and have subsequently been developed and generalized in numerous directions. In particular, Bernstein–Markov–Nikol'skii-type inequalities for $m \geq 0$ and various function spaces have been investigated by many authors: see, for example, [12], ([13], pp. 418–428), [20–22], ([23], pp. 122–133), [24–28], as well as the references cited therein.

Combining the interior estimates (9) with the exterior estimate (8), we obtain the following global growth estimate for the m-th derivative of an algebraic polynomial on the whole complex plane:

$$\left|P_n^{(m)}(z)\right| \leq c_4 \|P_n\|_p \begin{cases} \nu_n, & z \in \overline{G}_R, \\ \eta_n, & z \in \Omega_R, \end{cases} \tag{10}$$

where $c_4 = c_4(L, h, p, m) > 0$ is a constant independent of n, P_n, and $\nu_n \to \infty$, $\eta_n \to \infty$ as $n \to \infty$, depending on the properties of L and h.

2. Main Results

Before stating the main results, we introduce some notation that will be used throughout the paper. To simplify the presentation of the theorems and their proofs and to avoid cumbersome formulas, we restrict our attention to domains whose boundary contains at most two zero-angle points. These zero angles may be of the same type (both interior or both exterior) or of different types (one interior and one exterior). The extension of the results obtained below to domains with an arbitrary finite number of zero-angle points is straightforward and can be carried out by analogous arguments. For the sake of simplicity, we assume that $i = 1,2$; $l_1 = 1$ and $l = 2$. Thus, the domain G is assumed to possess one interior zero angle at a point z_1, characterized by an " f_1-touching" condition with $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$, and one exterior zero angle at a point z_2 characterized by a "g_2-touching" condition with $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$. Here C_1 and C_2 are finite constants satisfying $-\infty < C_1 < +\infty$, $-\infty < C_2 < +\infty$, where $C_1 := C_1(c^1_{11}, c^1_{12})$ and $C_2 := C_2(c^2_{21}, c^2_{22})$ and the parameters $c^i_{j,k}$, $i,j,k = 1,2$, are those introduced in Definition 1.

Throughout this paper, we assume that $m \geq 1$ and that $\gamma_i > -1$, $\alpha_i \geq 0$, and $\beta_i > 0$ for $i = 1,2$. We introduce the following quantities:

$$\gamma^* := \max\{0; \gamma_k,\ k = 1,2\}; \qquad \mu := 2\left(1 - \frac{1}{\pi}\arcsin\frac{1}{\lambda}\right), \quad 1 < \mu < 2,$$

$$\widetilde{\mu} := \begin{cases} \mu, & \text{if } \alpha_i = 0, \\ 2, & \text{if } \alpha_i \neq 0, \end{cases} \qquad i = 1,2.$$

2.1. Estimates for $\left|P_n^{(m)}(z)\right|$ in Unbounded Domains

In this subsection, we consider the case where the boundary of the domain contains both an interior and exterior zero angle, and we derive estimates for $\left|P_n^{(m)}(z)\right|$ in such unbounded domains Ω. The following statement holds:

Theorem 1. *Let $p \geq 1$; $G \in PQS(\lambda; f_1, g_2)$, for some $\lambda \geq 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$, $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$ and let $P_n \in \wp_n$, $n \in \mathbb{N}$. Assume that the weight function h is defined by (1) with $\gamma_j > -1$, $j = 1,2$. Then, the following holds for every $m = 0,1,\ldots,n$:*

$$\left|P_n^{(m)}(z)\right| \leq c_1 \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L)} \|P_n\|_p \cdot A_{n,1}(m),\ z \in \Omega, \tag{11}$$

where the constant $c_1 = c_1(L, \gamma_i, \beta, m, p) > 0$ is independent of bothn and z; $A_{n,1}(m) := \left(A^1_{n,1} + A^2_{n,1}\right) \cdot D_{n,1}(m)$,

$$A^1_{n,1} := \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)\widetilde{\mu}}, & 1 \leq p < \gamma_1 + 1, \quad \gamma_1 > 0, \\ (\ln n)^{1-\frac{1}{p}}, & p = \gamma_1 + 1, \quad \gamma_1 > 0, \\ 1, & p > \gamma_1 + 1, \quad \gamma_1 > 0, \\ 1, & p \geq 1, \quad -1 < \gamma_1 \leq 0; \end{cases}$$

$$A^2_{n,1} := \begin{cases} n^{\left(\frac{\gamma_2+1}{p}-1\right)\frac{\mu}{1+\beta_2}}, & 1 \leq p < \gamma_2 + 1, \quad \gamma_2 > 0, \\ (\ln n)^{1-\frac{1}{p}}, & p = \gamma_2 + 1, \quad \gamma_2 > 0, \\ 1, & p > \gamma_2 + 1, \quad \gamma_2 > 0, \\ 1, & p \geq 1, \quad -1 < \gamma_2 \leq 0, \end{cases}$$

$$D_{n,1}(m) := n^{\widetilde{\mu}m} + n^{\frac{\mu m}{1+\beta_2}}.$$

As established in Theorem 1, the estimate for $A_{n,1}(m)$ explicitly describes the dependence of the growth of the polynomial and its m-th derivatives in the unbounded domain Ω on the geometric features of the boundary and the behavior of the weight function. Consequently, it provides a quantitative characterization of the influence exerted by the domain geometry and the weight.

We next consider domains whose boundary contains only zero angles of the same type. Assume that the domain G has two interior zero angles at the boundary points z_1 and z_2, with corresponding f_i-touching

$$f_i(x) = C_i x^{1+\alpha_i},\ \alpha_i \geq 0,\ i = 1,2,$$

where C_i $(-\infty < C_i < +\infty, i = 1,2)$, are finite constants. Collecting the terms corresponding to these two interior angles yields the following result:

Corollary 1. *Let $p \geq 1$; $G \in PQS(\lambda; f_1, f_2)$, for some $\lambda \geq 1$, $f_i(x) = C_i x^{1+\alpha_i}$, $\alpha_i \geq 0$, $i = 1,2$, and let $P_n \in \wp_n$, $n \in \mathbb{N}$. Assume that the weight function h is defined by (1) with $\gamma_j > -1$, $j = 1,2$. Then, the following holds for every $m = 0,1,...,n$:*

$$\left|P_n^{(m)}(z)\right| \leq c_2 \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L)} \|P_n\|_p \cdot A_{n,2}(m),\ z \in \Omega, \tag{12}$$

where the constant $c_2 = c_2(L, \gamma_i, m, p) > 0$ is independent of both n and z;

$$A_{n,2}(m) := \begin{cases} n^{\left(\frac{\max\{\gamma_1;\gamma_2\}+1}{p}+m-1\right)\widetilde{\mu}}, & 1 \leq p < \max\{\gamma_1;\gamma_2\}+1, & \gamma_1,\gamma_2 > 0, \\ n^{\widetilde{\mu}m}(\ln n)^{1-\frac{1}{p}}, & \begin{array}{l} p = \gamma_1+1, p \geq \gamma_2+1 \\ \text{or } p \geq \gamma_1+1, p = \gamma_2+1, \end{array} & \gamma_1,\gamma_2 > 0, \\ n^{\widetilde{\mu}m}, & p > \max\{\gamma_1;\gamma_2\}+1, & \gamma_1,\gamma_2 > 0, \\ n^{\widetilde{\mu}m}, & p \geq 1, & -1 < \gamma_1,\gamma_2 \leq 0. \end{cases}$$

Analogously, suppose that the domain G has two exterior zero angles at the boundary points z_1 and z_2, with corresponding g_i-touching of the form

$$g_i(x) = C_i x^{1+\beta_i},\ \beta_i > 0,\ i = 1,2.$$

Then the following estimate holds:

Corollary 2. *Let $p \geq 1$; $G \in PQS(\lambda; g_1, g_2)$, for some $\lambda \geq 1$, $g_i(x) = C_i x^{1+\beta_i}$, $\beta_i > 0$, $i = 1,2$, and let $P_n \in \wp_n$, $n \in \mathbb{N}$. Assume that the weight function h is defined by (1) with $\gamma_j > -1$, $j = 1,2$. Then, the following holds for every $m = 0,1,...,n$:*

$$\left|P_n^{(m)}(z)\right| \leq c_3 \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L)} \|P_n\|_p \cdot A_{n,3}(m),\ z \in \Omega, \tag{13}$$

where the constant $c_3 = c_3(L, \gamma_i, \beta, m, p) > 0$ is independent of both n and z;

$$A_{n,3}(m) := \begin{cases} n^{\left(\frac{\max\{\gamma_1;\gamma_2\}+1}{p}+m-1\right)\frac{\mu}{1+\min\{\beta_1;\beta_2\}}}, & 1 \leq p < \max\{\gamma_1;\gamma_2\}+1, & \gamma_1,\gamma_2 > 0, \\ n^{\frac{\mu m}{1+\min\{\beta_1;\beta_2\}}}(\ln n)^{1-\frac{1}{p}}, & \begin{array}{l} p = \gamma_1+1,\ p \geq \gamma_2+1 \\ \text{or } p \geq \gamma_1+1,\ p = \gamma_2+1, \end{array} & \gamma_1,\gamma_2 > 0, \\ n^{\frac{\mu m}{1+\min\{\beta_1;\beta_2\}}}, & p > \max\{\gamma_1;\gamma_2\}+1, & \gamma_1,\gamma_2 > 0, \\ n^{\frac{\mu m}{1+\min\{\beta_1;\beta_2\}}}, & p \geq 1, & -1 < \gamma_1,\gamma_2 \leq 0. \end{cases}$$

2.2. Estimates for $\left|P_n^{(m)}(z)\right|$ in Bounded Domains

To obtain estimates for $\left|P_n^{(m)}(z)\right|$, $m \geq 0$, in the whole complex plane, we first derive the corresponding estimates in bounded domains $G \in PQS(\lambda; f_i, g_j)$. Therefore, in this subsection we present estimates for $\left|P_n^{(m)}(z)\right|$, $z \in \overline{G}$, in the domains specified above. The following results hold:

Theorem 2. *Let $p \geq 1$; $G \in PQS(\lambda; f_1, g_2)$, for some $\lambda \geq 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$, $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$, and let $P_n \in \wp_n$, $n \in \mathbb{N}$. Assume that the weight function h is defined by (1) with $\gamma_j > -1$, $j = 1, 2$. Then, the following holds for every $m = 0, 1, \ldots, n$:*

$$\left\|P_n^{(m)}\right\|_\infty \leq c_4 E_{n,1}(m) \|P_n\|_p, \tag{14}$$

where the constant $c_4 = c_4(L, \gamma_i, \beta, p) > 0$ is independent of both n and z; $E_{n,1}(m) := E_{n,1}^1(m) + E_{n,1}^2(m)$ and

$$E_{n,1}^1(m) := n^{\left(\frac{\gamma_1^*+1}{p}+m\right)\tilde{\mu}};$$

$$E_{n,1}^2(m) := \begin{cases} n^{\left(\frac{\gamma_2^*+1}{p}\right)\frac{\mu}{1+\beta_2}}, & 1 \leq p < 1 + \frac{(\gamma_2^*+1)\mu}{1+\beta_2}, \quad \beta_2 > 0, \ m = 0, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{(\gamma_2^*+1)\mu}{1+\beta_2}, \quad \beta_2 > 0, \ m = 0, \\ n^{1-\frac{1}{p}}, & p > 1 + \frac{(\gamma_2^*+1)\mu}{1+\beta_2}, \quad \beta_2 > 0, \ m = 0, \\ n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{\mu}{1+\beta_2}}, & p \geq 1, \quad \beta_2 < m\mu - 1, \ m \geq 1, \\ n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{\mu}{1+\beta_2}}, & 1 \leq p < \frac{(\gamma_2^*+1)\mu+1+\beta_2}{1+\beta_2-m\mu}, \quad \beta_2 \geq m\mu - 1, \ m \geq 1, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{(\gamma_2^*+1)\mu+1+\beta_2}{1+\beta_2-m\mu}, \quad \beta_2 \geq m\mu - 1, \ m \geq 1, \\ n^{1-\frac{1}{p}}, & p > \frac{(\gamma_2^*+1)\mu+1+\beta_2}{1+\beta_2-m\mu}, \quad \beta_2 \geq m\mu - 1, \ m \geq 1. \end{cases}$$

The following corollaries describe the special cases of Theorem 2 corresponding to different values of m. Specifically, Corollary 3 is obtained for $m = 0$, whereas Corollary 4 is obtained for $m = 1, 2, \ldots, n$.

Corollary 3. *Let $p \geq 1$; $G \in PQS(\lambda; f_1, g_2)$, for some $\lambda \geq 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$, $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$, and let $P_n \in \wp_n$, $n \in \mathbb{N}$. Assume that the weight function h is defined by (1) with $\gamma_j > -1$, $j = 1, 2$. Then, the following holds:*

$$\|P_n\|_\infty \leq c_5 E_{n,2}(0) \|P_n\|_p, \tag{15}$$

where the constant $c_5 = c_5(L, \gamma_i, \beta, p) > 0$ is independent of both n and z; $E_{n,2}(0)$ is defined for arbitrary small $\varepsilon > 0$ as follows:

$$E_{n,2}(0) := \begin{cases} n^{\left(\frac{\gamma_1^*+1}{p}\right)\tilde{\mu}}, & 1 \leq p < 1 + \frac{(\gamma_2^*+1)\mu}{1+\beta_2}, \ \gamma_1 \geq \frac{(\gamma_2+1)\mu}{\tilde{\mu}(1+\beta_2)} - 1, \ \gamma_2 > -1, \\ n^{\left(\frac{\gamma_2^*+1}{p}\right)\frac{\mu}{1+\beta_2}}, & 1 \leq p < 1 + \frac{(\gamma_2^*+1)\mu}{1+\beta_2}, \ -1 < \gamma_1 < \frac{(\gamma_2+1)\mu}{\tilde{\mu}(1+\beta_2)} - 1, \ \gamma_2 > -1, \\ n^{\left(\frac{\gamma_1^*+1}{p}\right)\tilde{\mu}}, & p = 1 + \frac{(\gamma_2^*+1)\mu}{1+\beta_2}, \ \gamma_1 \geq \frac{(\gamma_2+1)\mu}{\tilde{\mu}(1+\beta_2)} - 1 - \varepsilon, \ \gamma_2 > -1, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1 + \frac{(\gamma_2^*+1)\mu}{1+\beta_2}, \ -1 < \gamma_1 < \frac{(\gamma_2+1)\mu}{\tilde{\mu}(1+\beta_2)} - 1 - \varepsilon, \ \gamma_2 > -1, \\ n^{\left(\frac{\gamma_1^*+1}{p}\right)\tilde{\mu}}, & p > 1 + \frac{(\gamma_2^*+1)\mu}{1+\beta_2}, \ \gamma_1 \geq \frac{(\gamma_2+1)\mu}{\tilde{\mu}(1+\beta_2)} - 1, \ \gamma_2 > -1, \\ n^{1-\frac{1}{p}}, & p > 1 + \frac{(\gamma_2^*+1)\mu}{1+\beta_2}, \ -1 < \gamma_1 < \frac{(\gamma_2+1)\mu}{\tilde{\mu}(1+\beta_2)} - 1, \ \gamma_2 > -1. \end{cases}$$

Corollary 4. *Let $p \geq 1$; $G \in PQS(\lambda; f_1, g_2)$, for some $\lambda \geq 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$, $g_2(x) = C_2 x^{1+\beta_2}, \beta_2 > 0$, and let $P_n \in \wp_n$, $n \in \mathbb{N}$. Assume that the weight function h be given by (1) with $\gamma_j > -1$, $j = 1, 2$. Then, the following holds for every $m = 1, \ldots, n$:*

$$\left\| P_n^{(m)} \right\|_\infty \leq c_6 E_{n,3}(m) \|P_n\|_p, \tag{16}$$

where the constant $c_6 = c_6(L, \gamma_i, \beta, p) > 0$ is independent of both n and z; $E_{n,3}(m)$ is defined as

$$E_{n,3}(m) := \begin{cases} n^{\left(\frac{\gamma_1^*+1}{p}+m\right)\widetilde{\mu}}, & p \geq 1,\ \gamma_1 \geq \widetilde{\gamma}_1,\ \gamma_2 > \widetilde{\gamma}_2,\ \beta_2 < m\mu - 1, \\ n^{\left(\frac{\gamma_1^*+1}{p}+m\right)\widetilde{\mu}}, & p \geq 1,\ \gamma_1 > -1,\ -1 < \gamma_2 \leq \widetilde{\gamma}_2,\ \beta_2 < m\mu - 1, \\ n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{\mu}{1+\beta_2}}, & p \geq 1,\ -1 < \gamma_1 < \widetilde{\gamma}_1,\ \gamma_2 > \widetilde{\gamma}_2,\ \beta_2 < m\mu - 1, \\ n^{\left(\frac{\gamma_1^*+1}{p}+m\right)\widetilde{\mu}}, & 1 \leq p < \frac{(\gamma_2+1)\mu+1+\beta_2}{1+\beta_2-m\mu},\ \gamma_1 \geq \widetilde{\gamma}_1,\ \gamma_2 > \widetilde{\gamma}_2,\ \beta_2 \geq m\mu - 1, \\ n^{\left(\frac{\gamma_1^*+1}{p}+m\right)\widetilde{\mu}}, & 1 \leq p < \frac{(\gamma_2+1)\mu+1+\beta_2}{1+\beta-m\mu},\ \gamma_1 > -1,\ -1 < \gamma_2 \leq \widetilde{\gamma}_2,\ \beta_2 \geq m\mu - 1, \\ n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{\mu}{1+\beta_2}}, & 1 \leq p < \frac{(\gamma_2+1)\mu+1+\beta_2}{1+\beta_2-m\mu},\ 0 < \gamma_1 < \widetilde{\gamma}_1,\ \gamma_2 > \widetilde{\gamma}_2,\ \beta_2 \geq m\mu - 1, \\ n^{\left(\frac{\gamma_1^*+1}{p}+m\right)\widetilde{\mu}}, & p \geq \frac{(\gamma_2^*+1)\mu+1+\beta_2}{1+\beta_2-m\mu},\ \gamma_1 > -1,\ \gamma_2 > -1,\ \beta_2 \geq m\mu - 1, \end{cases}$$

and $\widetilde{\gamma}_1 := \frac{(\gamma_2+pm+1)\mu}{\widetilde{\mu}(1+\beta_2)} - (pm+1)$; $\widetilde{\gamma}_2 := \left[\frac{\widetilde{\mu}(1+\beta_2)}{\mu} - 1\right] pm - 1$.

When the domain G has singular boundary points of the same type, namely, two interior zero angles or two exterior zero angles, respectively, Corollaries 5 and 6 are obtained as special cases of Theorem 2.

Corollary 5. *Let $p \geq 1$; $G \in PQS(\lambda; f_1, f_2)$, for some $\lambda \geq 1$, $f_i(x) = C_i x^{1+\alpha_i}$, $\alpha_i \geq 0$, $i = 1, 2$, and let $P_n \in \wp_n$, $n \in \mathbb{N}$. Assume that the weight function h is defined by (1) with $\gamma_j > -1$, $j = 1, 2$. Then, the following holds for every $m = 0, 1, \ldots, n$:*

$$\left\| P_n^{(m)} \right\|_\infty \leq c_7 E_{n,4}(m) \|P_n\|_p, \tag{17}$$

where the constant $c_7 = c_7(L, \gamma_i, \beta, p) > 0$ is independent of both n and z;

$$E_{n,4}(m) := n^{\left(\frac{\max\left\{\gamma_1^*;\gamma_1^*\right\}+1}{p}+m\right)\widetilde{\mu}}.$$

Corollary 6. *Let $p \geq 1$; $G \in PQS(\lambda; g_1, g_2)$, for some $\lambda \geq 1$, $g_i(x) = C_i x^{1+\beta_i}$, $\beta_i > 0$, $i = 1, 2$, and let $P_n \in \wp_n$, $n \in \mathbb{N}$. Assume that the weight function h is defined by (1) with $\gamma_j > -1$, $j = 1, 2$. Then, following holds for every $m = 0, 1, \ldots, n$:*

$$\left\| P_n^{(m)} \right\|_\infty \leq c_8 E_{n,5}(m) \|P_n\|_p, \tag{18}$$

where the constant $c_8 = c_8(L, \gamma_i, \beta, p) > 0$ is independent of both n and z;

$$E_{n,5}(m) := n^{\left(\frac{\max\left\{\gamma_1^*;\gamma_1^*\right\}+1}{p}\right)\frac{\mu}{1+\min\left\{\beta_1;\beta_2\right\}}}.$$

Theorem 3. *Let $p \geq 1$; $G \in PQS(\lambda; f_1, g_2)$, for some $\lambda \geq 1$, $f_1(x) = c_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$, $g_2(x) = c_2 x^{1+\beta_2}$, $\beta_2 > 0$, and let $P_n \in \wp_n$, $n \in \mathbb{N}$. Assume that the weight function h is defined by (1) with $\gamma_j > -1$, $j = 1, 2$. Then, the following holds for every $m = 0, 1, \ldots, n$:*

$$\left|P_n^{(m)}(z_j)\right| \leq c_9 n^{\left(\frac{\gamma_j+1}{p}+m\right)\omega_j} \|P_n\|_p, \ j = 1, 2, \tag{19}$$

where the constant $c_9 = c_9(L, \gamma, m, p) > 0$ is independent of both n and z; γ^*; ν_j is defined as follows:

$$\omega_j := \begin{cases} \widetilde{\mu}, & j = 1, \\ \frac{\mu}{1+\beta_2}, & j = 2. \end{cases}$$

Sharpness of the Estimates

Remark 1 ([12], Remark 2.16). *Estimates (14)–(19) are sharp in the order sense for some specially chosen domain and weight function.*

2.3. Estimates for $\left|P_n^{(m)}(z)\right|$ in Whole Complex Plane

In this part, we combine the results obtained in the preceding sections for infinite and finite domains, respectively, and gain insight into the behavior of derivatives of polynomials in the entire complex plane. Hence, combining estimations (14)–(18) with the results of Theorem 1, we obtain growth estimate for $|P_n^{(m)}(z)|$ in the whole complex plane.

Theorem 4. *Let $p \geq 1$; $G \in PQS(\lambda; f_1, g_2)$, for some $\lambda \geq 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$, $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$, and let $P_n \in \wp_n$, $n \in \mathbb{N}$. Assume that the weight function h is defined by (1) with $\gamma_j > -1$, $j = 1, 2$. Then, the following holds for every $m = 0, 1, \ldots, n$:*

$$\left|P_n^{(m)}(z)\right| \leq c_7 \|P_n\|_p \begin{cases} E_{n,1}(m), & z \in \overline{G}, \\ \frac{|\Phi^{n-m+1}(z)|}{d(z,L)} \|P_n\|_p \cdot A_{n,1}(m), & z \in \Omega, \end{cases}$$

where the constant $c_7 = c_7(L, \gamma_i, \beta, m, p) > 0$ is independent of both n and z; $E_{n,1}(m)$ and $\widetilde{A}_{n,1}(m)$ are defined as in Theorem 2 for all $z \in \overline{G}$ and in Theorem 1 for all $z \in \Omega$, respectively.

In the special case where the domain G has identical zero angles at both boundary points z_1 and z_2 namely, $f_i(x) = C_i x^{1+\alpha_i}$, or $g_i(x) = C_i x^{1+\beta_i}$, $i = 1, 2$, results similar to Corollaries 3 and 4 follow directly from the above estimates together with Corollaries 1 and 2, respectively.

As immediate consequences of Theorem 4, we obtain Corollaries 7 and 8 by combining Theorem 1 with Corollary 3 for the case $m = 0$, and Theorem 1 with Corollary 4 for the cases $m = 1, 2, \ldots, n$, respectively:

Corollary 7. *Under the assumptions of Theorem 4, the following holds:*

$$|P_n(z)| \leq c_8 \|P_n\|_p \begin{cases} E_{n,2}(0), & z \in \overline{G}, \\ \frac{|\Phi^{n+1}(z)|}{d(z,L)} \|P_n\|_p \cdot A_{n,1}(0), & z \in \Omega, \end{cases}$$

where the constant $c_8 = c_8(L, \gamma_i, \beta, p) > 0$ is independent of both n and z; $E_{n,2}(0)$ and $A_{n,1}(0)$ are defined as in (15) and Theorem 1, respectively.

Corollary 8. *Under the assumptions of Theorem 4, the following holds for every $m = 1, 2, \ldots, n$:*

$$\left|P_n^{(m)}(z)\right| \le c_8 \|P_n\|_p \begin{cases} E_{n,3}(m), & z \in \overline{G}, \\ \frac{|\Phi^{n+1-m}(z)|}{d(z,L)} \|P_n\|_p \cdot A_{n,1}(m), & z \in \Omega, \end{cases}$$

where the constant $c_8 = c_8(L, \gamma_i, \beta, p) > 0$ is independent of both n and z; $E_{n,2}(m)$ and $A_{n,1}(m)$ are defined as in (16) and Theorem 1, respectively.

Similarly, by combining Corollary 1 with Corollary 5, and Corollary 2 with Corollary 6, respectively, we obtain the following two corollaries:

Corollary 9. *Under the assumptions of Corollary 1, the following holds for every $m = 0, 1, 2, \ldots, n$:*

$$\left|P_n^{(m)}(z)\right| \le c_8 \|P_n\|_p \begin{cases} E_{n,4}(m), & z \in \overline{G}, \\ \frac{|\Phi^{n+1-m}(z)|}{d(z,L)} \|P_n\|_p \cdot A_{n,2}(m), & z \in \Omega, \end{cases}$$

where the constant $c_8 = c_8(L, \gamma_i, p) > 0$ is independent of both n and z; $E_{n,4}(m)$ and $A_{n,2}(m)$ are defined as in Corollary 5 and Corollary 1, respectively.

Corollary 10. *Under the assumptions of Corollary 2, the following holds for every $m = 0, 1, 2, \ldots, n$:*

$$\left|P_n^{(m)}(z)\right| \le c_8 \|P_n\|_p \begin{cases} E_{n,5}(m), & z \in \overline{G}, \\ \frac{|\Phi^{n+1-m}(z)|}{d(z,L)} \|P_n\|_p \cdot A_{n,3}(m), & z \in \Omega, \end{cases}$$

where the constant $c_8 = c_8(L, \gamma_i, \beta, p) > 0$ is independent of both n and z; $E_{n,5}(m)$ and $A_{n,3}(m)$ are defined as in Corollary 6 and Corollary 2, respectively.

3. Some Auxiliary Results

For $a > 0$ and $b > 0$, the notation "$a \preceq b$" means that $a \le cb$ for some constant $c > 0$, while "$a \asymp b$" means that $c_1 a \le b \le c_2 a$ for some constants $c_1, c_2 > 0$.

Lemma 1 ([6]). *Let G be a quasidisk, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \preceq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then*

(a) The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent. Therefore, $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$ also are equivalent.

(b) If $|z_1 - z_2| \preceq |z_1 - z_3|$, then

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c_1} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c_2},$$

where $0 < r_0 < 1$ is a constant, depending on G.

Corollary 11. *Under the conditions of Lemma 1, we have*

$$|w_1 - w_2|^{c_1} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{\varepsilon},$$

where $\varepsilon = \varepsilon(G) < 1$.

Lemma 2 ([29,30]). *Let $L \in QS(\lambda)$ for some $\lambda \ge 1$. Then*

$$|\Psi(w_1) - \Psi(w_2)| \succeq |w_1 - w_2|^{\mu},$$

for all $w_1, w_2 \in \overline{\Delta}$, *where* $\mu := 2(1 - \frac{1}{\pi}\arcsin\frac{1}{\lambda})$.

This follows from a suitable result for mappings $f \in \Sigma(\kappa)$ established in ([31], p. 287), together with an estimate for Ψ' obtained in ([32], Th. 2.8):

$$d(\Psi(\tau), L) \asymp \left|\Psi'(\tau)\right|(|\tau| - 1). \tag{20}$$

Let $\{z_j\}_{j=1}^{l} \in L$ be fixed system, and let h be the weight function defined by (1) with $\gamma_j > -1$, $j = \overline{1,l}$.

Lemma 3 ([3]). *Let $L = \partial G$ be a rectifiable Jordan curve, and let $P_n(z)$, $\deg P_n \le n$, $n = 1, 2, \ldots$, be an arbitrary polynomial. Assume that the weight function $h(z)$ satisfies the condition (1). Then for any $R > 1$, $p > 0$ and $n = 1, 2, \ldots$*

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \le R^{n+\frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h,L)},\ \gamma^* = \max\{0; \gamma_j : 1 \le j \le l\}.$$

4. Proofs of Theorems

Proof of Theorem 1. Let $G \in PQS(\lambda; f_1, g_2)$, for some $\lambda \ge 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \ge 0$, and $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$. Let $R = 1 + \frac{\varepsilon_0}{n}$, $0 < \varepsilon_0 < \frac{1}{2}$. For each $n \in \mathbb{N}$ and m with $1 \le m < n$, define

$$H_{n,m}(z) := \frac{P_n^{(m)}(z)}{\Phi^{n-m+1}(z)},\ z \in \Omega.$$

The function $H_{n,m}(z)$ is analytic in Ω, continuous on $\overline{\Omega}$ and satisfies $H_{n,m}(\infty) = 0$, then the Cauchy integral formula for an unbounded domain gives

$$H_{n,m}(z) = -\frac{1}{2\pi i}\int_L H_{n,m}(\zeta)\frac{d\zeta}{\zeta - z},\ z \in \Omega.$$

Then,

$$\left|\frac{P_n^{(m)}(z)}{\Phi^{n-m+1}(z)}\right| \le \frac{1}{2\pi}\int_L \left|\frac{P_n^{(m)}(\zeta)}{\Phi^{n-m+1}(\zeta)}\right|\frac{|d\zeta|}{|\zeta - z|}.$$

Since $\left|\Phi^{n-m+1}(\zeta)\right| = 1$, for all $\zeta \in L$, it follows that

$$\left|P_n^{(m)}(z)\right| \preceq \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L)}\int_L \left|P_n^{(m)}(\zeta)\right||d\zeta|. \tag{21}$$

Cauchy's integral representation for $P_n^{(m)}(\zeta)$ gives

$$P_n^{(m)}(\zeta) = \frac{1}{2\pi i}\int_{L_R}\frac{P_n(t)}{(t-\zeta)^{m+1}}dt,\ \zeta \in G_R.$$

For $\zeta \in L$, substituting $P_n^{(m)}(\zeta)$ into (21) yields

$$\begin{aligned}\left|P_n^{(m)}(z)\right| &\preceq \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L)}\cdot\int_L\left|\frac{1}{2\pi i}\int_{L_R}\frac{P_n(t)}{(t-\zeta)^{m+1}}dt\right||d\zeta| \\ &\preceq \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L)}\cdot\int_L\left(\int_{L_R}\frac{|P_n(t)|}{|t-\zeta|^{m+1}}|dt|\right)|d\zeta|.\end{aligned} \tag{22}$$

$$\left|P_n^{(m)}(z)\right| \preceq \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L)} \cdot \left[\int\limits_{L_R} |P_n(t)||dt|\right] \cdot \sup_{t\in L_R}\left(\int\limits_{L} \frac{|d\zeta|}{|t-\zeta|^{m+1}}\right). \tag{23}$$

Denote by

$$A_n := \int\limits_{L_R} |P_n(t)||dt|; \quad B_{n,m}(t) := \int\limits_{L} \frac{|d\zeta|}{|t-\zeta|^{m+1}}; \tag{24}$$

and estimate these integrals separately.

We begin with an estimate of the integral A_n. Assuming that $p > 1$, we rewrite the integral by multiplying and dividing the integrand by $h^{\frac{1}{p}}(\zeta)$ and then apply Hölder's inequality to obtain

$$\begin{aligned} A_n &= \int\limits_{L_R} |P_n(t)||dt| = \int\limits_{L_R} \frac{h^{\frac{1}{p}}(t)|P_n(t)|}{h^{\frac{1}{p}}(t)}|dt| \\ &\leq \left(\int\limits_{L_R} h(t)|P_n(t)|^p|dt|\right)^{\frac{1}{p}} \times \left(\int\limits_{L_R} \frac{|dt|}{h^{\frac{q}{p}}(t)}\right)^{\frac{1}{q}} \\ &\preceq \|P_n\|_{\mathcal{L}_p(h,L_R)} \left(\int\limits_{L_R} \frac{|dt|}{h^{q-1}(t)}\right)^{\frac{1}{q}}, \quad \frac{1}{p}+\frac{1}{q}=1. \end{aligned} \tag{25}$$

According to Lemma 3, we get

$$\begin{aligned} A_n &\preceq \|P_n\|_p \left(\int\limits_{L_R} \frac{|dt|}{h^{q-1}(t)}\right)^{\frac{1}{q}} \\ &\preceq \|P_n\|_p \left[\left(\int\limits_{L_R^1} \frac{|d\zeta|}{|\zeta-z_1|^{(q-1)\gamma_1}}\right)^{\frac{1}{q}} + \left(\int\limits_{L_R^2} \frac{|d\zeta|}{|\zeta-z_2|^{(q-1)\gamma_2}}\right)^{\frac{1}{q}}\right] =: \|P_n\|_p \left(J_R^1 + J_R^2\right), \end{aligned} \tag{26}$$

since the points $z_1, z_2 \in L$ are distinct, and we begin to estimate this integrals.

To simplify the calculations, we choose the points $z_1 = -1$ and $z_2 = 1$ such that $(-1,1) \subset G$. We further assume that the local coordinate axes introduced in Definition 1 are parallel to the coordinate axes OX and OY of the system XOY. Let $L_R = L_R^+ \cup L_R^-$, where $L_R^+ := \{z \in L_R : \mathrm{Im} z \geq 0\}$, $L_R^- := \{z \in L_R : \mathrm{Im} z < 0\}$. Furthermore, consider $z_R^{\pm} \in \Psi(w_R^{\pm})$, where $w_R^{\pm} := \left\{w = R \cdot e^{i\theta} : \theta = \frac{\varphi_1 \pm \varphi_2}{2}\right\}$, and let $z_{R,i} \in \Psi(R\Phi(z_i))$. Denote by $L_{R,i}^{\pm}(z_{R,i}, z_R^{\pm})$ the arcs connecting the points $z_{R,i}$ and $z_R^{\pm}$, respectively. We write $\left|L_{R,i}^{\pm}\right| := mes\, L_{R,i}^{\pm}(z_{R,i}, z_R^{\pm})$, $i = 1,2$. Then, from (26), we have

$$A_n \preceq \|P_n\|_p \left\{\left[J_R^1\right]^{\frac{1}{q}} + \left[J_R^2\right]^{\frac{1}{q}}\right\}. \tag{27}$$

To estimate the integrals J_R^1 and J_R^2, we first introduce the following notation. For any $\rho \geq 1$, we define

$$d_{i,R} := d(z_i, L_R); \tag{28}$$

$$E_{\rho,1}^{1,\pm} := \left\{\zeta \in L_\rho^1 : |\zeta - z_1| < c_1 d_{1,R}\right\},$$
$$E_{\rho,2}^{1,\pm} := \left\{\zeta \in L_\rho^1 : c_1 d_{1,R} \leq |\zeta - z_1| \leq \left|L_{\rho,1}^{\pm}\right|\right\},$$
$$E_{\rho,1}^{2,\pm} := \left\{\zeta \in L_\rho^2 : |\zeta - z_2| < c_2 d_{2,R}\right\},$$
$$E_{\rho,2}^{2,\pm} := \left\{\zeta \in L_\rho^2 : c_2 d_{2,R} \leq |\zeta - z_2| \leq \left|L_{\rho,2}^{\pm}\right|\right\};$$
$$\Im_{\rho,k}^{i,\pm} := \int\limits_{E_{\rho,k}^{i,\pm}} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i(q-1)}}, \quad i,k = 1,2.$$

Then, using these notations and for $\rho = R$, from (27) we get

$$A_n \preceq \|P_n\|_p \sum_{i=1}^{2} \left[\Im_{R,1}^{i,\pm} + \Im_{R,2}^{i,\pm}\right]^{\frac{1}{q}}, \quad i = 1,2. \tag{29}$$

Therefore, we proceed with the estimation of the integrals $\Im_{R,k}^{i,\pm}$ with $i,k = 1,2$.

1.1. Let $\gamma_1, \gamma_2 > 0$. Then, we get

$$\Im_{R,1}^{1,\pm} \preceq \int\limits_0^{c_1 d_{1,R}} \frac{ds}{s^{\gamma_1(q-1)}} \preceq \left(\frac{1}{d_{1,R}}\right)^{\gamma_1(q-1)} \cdot \operatorname{mes}\left(E_{R,1}^{1,\pm}\right) \preceq \left(\frac{1}{d_{1,R}}\right)^{\gamma_1(q-1)-1}; \tag{30}$$

$$\Im_{R,2}^{1,\pm} \preceq \int\limits_{c_1 d_{1,R}}^{|L_{R,1}^{\pm}|} \frac{ds}{s^{\gamma_1(q-1)}} \preceq \begin{cases} \left(\frac{1}{d_{1,R}}\right)^{\gamma_1(q-1)-1}, & \gamma_1(q-1) > 1, \\ \ln\left(\frac{1}{d_{1,R}}\right), & \gamma_1(q-1) = 1, \\ 1, & \gamma_1(q-1) < 1; \end{cases}$$

$$\Im_{R,1}^{1,\pm} + \Im_{R,2}^{1,\pm} \preceq \left(\frac{1}{d_{1,R}}\right)^{\gamma_1(q-1)-1} + \begin{cases} \left(\frac{1}{d_{1,R}}\right)^{\gamma_1(q-1)-1}, & \gamma_1(q-1) > 1, \\ \ln\left(\frac{1}{d_{1,R}}\right), & \gamma_1(q-1) = 1, \\ 1, & \gamma_1(q-1) < 1, \end{cases}$$

$$\preceq \begin{cases} \left(\frac{1}{d_{1,R}}\right)^{\gamma_1(q-1)-1}, & \gamma_1(q-1) > 1, \\ \ln\left(\frac{1}{d_{1,R}}\right), & \gamma_1(q-1) = 1, \\ 1, & \gamma_1(q-1) < 1. \end{cases}$$

Similarly, for the J_R^2 in the neighborhood of the point z_2, we have

$$\Im_{R,1}^{2,\pm} \preceq \int\limits_0^{c_2 d_{2,R}} \frac{ds}{s^{\gamma_2(q-1)}} \preceq \left(\frac{1}{d_{2,R}}\right)^{\gamma_2(q-1)} \cdot \operatorname{mes}\left(E_{R,1}^{2,\pm}\right) \preceq \left(\frac{1}{d_{2,R}}\right)^{\gamma_2(q-1)-1}; \tag{31}$$

$$\Im_{R,2}^{2,\pm} \preceq \int\limits_{c_2 d_{2,R}}^{|L_{R,2}^{\pm}|} \frac{ds}{s^{\gamma_2(q-1)}} \preceq \begin{cases} \left(\frac{1}{d_{2,R}}\right)^{\gamma_2(q-1)-1}, & \gamma_2(q-1) > 1, \\ \ln\left(\frac{1}{d_{2,R}}\right), & \gamma_2(q-1) = 1, \\ 1, & \gamma_2(q-1) < 1; \end{cases}$$

$$\Im_{R,1}^{2,\pm}+\Im_{R,1}^{2,\pm}\preceq\left(\frac{1}{d_{2,R}}\right)^{\gamma_2(q-1)-1}+\begin{cases}\left(\frac{1}{d_{2,R}}\right)^{\gamma_2(q-1)-1}, & \gamma_2(q-1)>1,\\ \ln\left(\frac{1}{d_{2,R}}\right), & \gamma_2(q-1)=1,\\ 1, & \gamma_2(q-1)<1,\end{cases}$$

$$\preceq\begin{cases}\left(\frac{1}{d_{2,R}}\right)^{\gamma_2(q-1)-1}, & \gamma_2(q-1)>1,\\ \ln\left(\frac{1}{d_{2,R}}\right), & \gamma_2(q-1)=1,\\ 1, & \gamma_2(q-1)<1.\end{cases}$$

1.2. Let $\gamma_1,\gamma_2\leq 0$. Then, analogously to the (30) and (31), we obtain

$$\Im_{R,1}^{1,\pm}=\int\limits_{E_{R,1}^{1,\pm}}|\zeta-z_1|^{-\gamma_1(q-1)}|d\zeta|\preceq(\operatorname{diam}L_R)^{-\gamma_1(q-1)}\operatorname{mes}E_{R,1}^{1,\pm}\preceq 1; \tag{32}$$

$$\Im_{R,2}^{1,\pm}\preceq\int\limits_{E_{R,2}^{1,\pm}}|\zeta-z_1|^{-\gamma_1(q-1)}|d\zeta|\preceq\int\limits_{c_1d_{1,R}}^{|L_1^{\pm}|}s^{-\gamma_1(q-1)}\,ds\preceq 1;$$

and

$$\Im_{R,1}^{2,\pm}\preceq\int\limits_{E_{R,1}^{2,\pm}}|\zeta-z_2|^{-\gamma_2(q-1)}|d\zeta|\preceq(\operatorname{diam}L_R)^{(-\gamma_2)(q-1)}\operatorname{mes}E_{R,1}^{2,\pm}\preceq 1;$$

$$\Im_{R,2}^{2,\pm}\preceq\int\limits_{E_{R,2}^{2,\pm}}|\zeta-z_2|^{-\gamma_2(q-1)}|d\zeta|\preceq\int\limits_{c_2d_{2,R}}^{|L_2^{\pm}|}ds\preceq 1.$$

In this case, combining (29)–(32), we get

$$A_n\preceq\|P_n\|_p\begin{cases}\left(\frac{1}{d_{1,R}}\right)^{\frac{(\gamma_1+1)}{p}-1}+\left(\frac{1}{d_{2,R}}\right)^{\frac{(\gamma_2+1)}{p}-1}, & \gamma_1>p-1,\gamma_2>p-1,\\ \left(\ln\left(\frac{1}{d_{1,R}}\right)\right)^{1-\frac{1}{p}}+\left(\ln\left(\frac{1}{d_{2,R}}\right)\right)^{1-\frac{1}{p}}, & \gamma_1=p-1,\gamma_2=p-1,\\ 1, & -1<\gamma_1,\gamma_2<p-1.\end{cases} \tag{33}$$

According to Lemma 2, ([32], p. 61) and ([33], p. 10), we obtain

$$d_{1,R}\succeq\begin{cases}n^{-\mu}, & \text{if } \alpha_1=0;\\ n^{-2}, & \text{if } \alpha_1\neq 0.\end{cases} \tag{34}$$

Now, we define $z_R\in L_R: d_{2,R}=|z_2-z_R|$; $\zeta^{\pm}\in L^{\pm}: d(z_R,L^2\cap L^{\pm}):=d(z_R,L^{+})$; $z_2^{\pm}:=\zeta\in L^2:\ |\zeta-z_2|=c_2d_{2,R}$ for the estimate $d_{2,R}$. Then, we have

$$d_R^{\pm}:=d(z_R,L^2\cap L^{\pm})\asymp|z_R-z_2^{\pm}|\asymp d_{2,R}^{1+\beta_2},$$

from Lemma 1. Hence, $d_{2,R}=\left(d_R^{\pm}\right)^{\frac{1}{1+\beta_2}}$. Furthermore, it follows from Lemma 2 and ([33], Corollary 2) that $d_R^{\pm}\succeq n^{-\mu}$ and

$$d_{2,R}\succeq n^{\frac{-\mu}{1+\beta_2}}. \tag{35}$$

Therefore, taking into account these estimates in (33), we obtain

$$A_n \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)\tilde{\mu}} + n^{\left(\frac{\gamma_2+1}{p}-1\right)\frac{\mu}{1+\beta_2}}, & \gamma_1,\gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \begin{cases} \gamma_1 = p-1,\ -1 < \gamma_2 \le p-1, \\ \text{or } -1 < \gamma_1 \le p-1,\ \gamma_2 = p-1, \end{cases} \\ 1, & -1 < \gamma_1,\gamma_2 < p-1. \end{cases} \tag{36}$$

We now estimate $B_{n,m}(\rho)$. Using notations (28) and setting $\rho = 1$, we get

$$B_{n,m}(t) = \int\limits_L \frac{|d\zeta|}{|t-\zeta|^{m+1}} =: \sum_{i=1}^{2} \left[\Im_{1,1}^{i,\pm} + \Im_{1,2}^{i,\pm}\right], \tag{37}$$

where

$$\Im_{1,k}^{i,\pm} := \int\limits_{E_{1,k}^{i,\pm}} \frac{|d\zeta|}{|\zeta - z_i|^{m+1}}\,; i,k = 1,2.$$

We have

$$\Im_{1,1}^{1,\pm} = \int\limits_{E_{1,1}^{1,\pm}} \frac{|d\zeta|}{|\zeta - z_1|^{m+1}} \preceq \int\limits_0^{c_1 d_{1,R}} \frac{ds}{s^{m+1}} \preceq \left(\frac{1}{d_{1,R}}\right)^{m+1} \cdot \text{mes}\left(E_{1,1}^{1,\pm}\right) \preceq \left(\frac{1}{d_{1,R}}\right)^{m}, \tag{38}$$

$$\Im_{1,2}^{1,\pm} = \int\limits_{E_{1,2}^{1,\pm}} \frac{|d\zeta|}{|\zeta - z_1|^{m+1}} \preceq \int\limits_{c_1 d_{1,R}}^{|L_{1,1}^{\pm}|} \frac{ds}{s^{m+1}} \preceq \left(\frac{1}{d_{1,R}}\right)^{m},$$

$$\Im_{1,1}^{1,\pm} + \Im_{1,2}^{1,\pm} \preceq \left(\frac{1}{d_{1,R}}\right)^{m},$$

and

$$\Im_{1,1}^{2,\pm} = \int\limits_{E_{1,1}^{2,\pm}} \frac{|d\zeta|}{|\zeta - z_2|^{m+1}} \preceq \int\limits_0^{c_2 d_{2,R}} \frac{ds}{s^{m+1}} \preceq \left(\frac{1}{d_{2,R}}\right)^{m+1} \cdot \text{mes}\left(E_{1,1}^{2,\pm}\right) \preceq \left(\frac{1}{d_{2,R}}\right)^{m}, \tag{39}$$

$$\Im_{1,2}^{2,\pm} = \int\limits_{E_{1,2}^{2,\pm}} \frac{|d\zeta|}{|\zeta - z_2|^{m+1}} \preceq \int\limits_{c_2 d_{2,R}}^{|L_{1,2}^{\pm}|} \frac{ds}{s^{m+1}} \preceq \left(\frac{1}{d_{2,R}}\right)^{m},$$

$$\Im_{1,1}^{2,\pm} + \Im_{1,2}^{2,\pm} \preceq \left(\frac{1}{d_{2,R}}\right)^{m}.$$

From (37)–(39), we get

$$B_{n,m}(t) \preceq \left(\frac{1}{d_{1,R}}\right)^{m} + \left(\frac{1}{d_{2,R}}\right)^{m}.$$

Therefore, using estimates (34) and (35), we obtain

$$B_{n,m}(t) \preceq n^{\tilde{\mu}m} + n^{\frac{-\mu}{1+\beta_2}m}. \tag{40}$$

Combining estimates (23), (36) and (40), we have

$$\left|P_n^{(m)}(z)\right| \preceq \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L)}\|P_n\|_p\left[n^{\widetilde{\mu}m}+n^{\frac{-\mu}{1+\beta_2}m}\right]$$
$$\times\left[\begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)\widetilde{\mu}}+n^{\left(\frac{\gamma_2+1}{p}-1\right)\frac{\mu}{1+\beta_2}}, & \gamma_1,\gamma_2>p-1,\\ (\ln n)^{1-\frac{1}{p}}, & \begin{cases}\gamma_1=p-1,\ -1<\gamma_2\le p-1\\ \text{or } -1<\gamma_1\le p-1,\ \gamma_2=p-1,\end{cases}\\ 1, & -1<\gamma_1,\gamma_2<p-1.\end{cases}\right]$$

Now suppose that $p=1$.

After multiplying the integrand of the second integral on the right-hand side of inequality (22) by $h(t)$ and applying Lemma 3, the following estimate is obtained:

$$\left|P_n^{(m)}(z)\right| \preceq \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L)}\cdot\left[\int_{L_R}h(t)|P_n(t)||dt|\right]\cdot\sup_{t\in L_R}\left(\int_L\frac{|d\zeta|}{h(t)|\zeta-t|^{m+1}}\right) \tag{41}$$
$$\preceq \frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,L)}\cdot\|P_n\|_1\cdot\sup_{t\in L_R}\left(\int_L\frac{|d\zeta|}{h(t)|\zeta-t|^{m+1}}\right).$$

Denote by

$$\widetilde{B}_{n,m}(t):=\frac{1}{h(t)}\int_L\frac{|d\zeta|}{|\zeta-t|^{m+1}}=\frac{1}{h(t)}B_{n,m}(t),$$

and estimate this integral. We have

$$\frac{1}{h(t)}\preceq\frac{1}{|t-z_1|^{\gamma_1}|t-z_2|^{\gamma_2}}\preceq\begin{cases} n^{\widetilde{\mu}\gamma_1^*}, & t\in E_{1,R}^{1,\pm}\cup E_{2,R}^{1,\pm},\\ n^{\mu}, & t\in L_R\setminus\bigcup\limits_{i,j=1}^{2}E_{j,R}^{i,\pm},\\ n^{\frac{\mu\gamma_2^*}{1+\beta_2}}, & t\in E_{1,R}^{2,\pm}\cup E_{2,R}^{2,\pm},\end{cases}$$

and, so, from (40), we get

$$\widetilde{B}_{n,m}(t)\preceq\left[n^{\widetilde{\mu}m}+n^{\frac{-\mu}{1+\beta_2}m}\right]\begin{cases} n^{\widetilde{\mu}\gamma_1^*}, & t\in E_{1,R}^{1,\pm}\cup E_{2,R}^{1,\pm},\\ n^{\mu}, & t\in L_R\setminus\bigcup\limits_{i,j=1}^{2}E_{j,R}^{i,\pm},\\ n^{\frac{\gamma_2^*}{1+\beta_2}\mu}, & t\in E_{1,R}^{2,\pm}\cup E_{2,R}^{2,\pm},\end{cases}$$
$$\preceq\left[n^{\widetilde{\mu}m}+n^{\frac{m}{1+\beta_2}\mu}\right]\left[n^{\widetilde{\mu}\gamma_1^*}+n^{\frac{\gamma_2^*}{1+\beta_2}\mu}\right]$$
$$=n^{(\gamma_1^*+m)\widetilde{\mu}}+n^{\frac{\gamma_2^*+m}{1+\beta_2}\mu}.$$

Therefore, we have completed the proof of Theorem 1. □

Proof of Theorem 2. The proof for $p>1$ was proved in ([12], Theorem 2.7). We note that in the conditions of Theorem 2.7 [12] $p>0$ was incorrectly written and should have been written as $p>1$, although in subsequent corollaries and in the proof of the theorem it is correctly indicated that $p>1$. Also, on the left-hand side of the estimate is written $\|P_n\|_\infty$, while it should have been written as $\left\|P_n^{(m)}\right\|_\infty$.

We now prove that this is also true for $p = 1$. Let the domain $G \in PQS(\lambda; f_i, g_i)$, for some $\lambda \geq 1$, $f_1(x) = c_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$, and $g_2(x) = c_2 x^{1+\beta_2}, \beta_2 > 0$. Assume that the weight function h is defined by (1), where $\gamma_j > -1$, for $j = 1, 2$. The Cauchy integral representation for m-th derivatives on G_R yields

$$P_n^{(m)}(z) = \frac{1}{2\pi i} \int_{L_R} P_n(\zeta) \frac{d\zeta}{(\zeta - z)^{m+1}}, \; z \in G_R.$$

Then,

$$\left| P_n^{(m)}(z) \right| \leq \frac{1}{2\pi} \int_{L_R} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|^{m+1}}, \; z \in G_R.$$

Let $z \in L$ be arbitrary fixed. According to Lemma 3, we get

$$\left| P_n^{(m)}(z) \right| \leq \frac{1}{2\pi} \int_{L_R} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|^{m+1}} \leq \sup_{z \in L} \left(\frac{1}{h(\zeta)|\zeta - z|^{m+1}} \right) \cdot \int_{L_R} h(\zeta) |P_n(\zeta)| |d\zeta| \tag{42}$$

$$= \|P_n\|_{\mathcal{L}_1(h, L_R)} \cdot \sup_{z \in L} \left(\frac{1}{h(\zeta)|\zeta - z|^{m+1}} \right) \preceq \|P_n\|_1 \cdot \sup_{z \in L} \widetilde{D}_m(\zeta),$$

where

$$\widetilde{D}_m(\zeta) := \frac{1}{h(\zeta)|\zeta - z|^{m+1}}, \; z \in L, \; \zeta \in L_R. \tag{43}$$

It remains for us to estimate $\widetilde{D}_m(\zeta)$. According to (1) for $j = 1, 2$, we have

$$\widetilde{D}_m(\zeta) \preceq \frac{1}{|\zeta - z_1|^{\gamma_1} |\zeta - z_2|^{\gamma_2} |\zeta - z|^{m+1}} =: \bigcup_{k=0}^{2} \widetilde{D}_{n,m}^k, \tag{44}$$

where $\widetilde{D}_{n,m}^k(\zeta) := \left\{ \zeta : \zeta \in L_R^k, \; k = 0, 1, 2 \right\}$.
(a) Let $\zeta \in L_R^0$. Then,

$$\widetilde{D}_{n,m}^0 = \frac{1}{|\zeta - z_1|^{\gamma_1} |\zeta - z_2|^{\gamma_2} |\zeta - z|^{m+1}} \preceq \frac{1}{|\zeta - z|^{m+1}} \preceq \frac{1}{d^{m+1}(z, L_R)} \preceq n^{(m+1)\mu}; \tag{45}$$

(b) For the $\zeta \in L_R^1$, we get

$$\widetilde{D}_{n,m}^1 = \frac{1}{|\zeta - z_1|^{\gamma_1} |\zeta - z_2|^{\gamma_2} |\zeta - z|^{m+1}} \preceq \frac{1}{|\zeta - z_1|^{\gamma_1} |\zeta - z|^{m+1}} \tag{46}$$

$$\preceq \frac{1}{d_{1,R}^{\gamma_1 + m}} \preceq n^{(\gamma_1 + m + 1)\widetilde{\mu}}, \quad \text{if } \gamma_1 > 0;$$

$$\widetilde{D}_{n,m}^1 = \frac{|\zeta - z_1|^{-\gamma_1}}{|\zeta - z_2|^{\gamma_2} |\zeta - z|^{m+1}} \preceq \frac{1}{|\zeta - z|^{m+1}} \preceq \frac{1}{d_{1,R}^{m+1}} \preceq n^{(m+1)\widetilde{\mu}}, \quad \text{if } \gamma_1 \leq 0.$$

(c) Let us set: $\left(E_{2,R}^{1,\pm}\right)_1 := \{\zeta \in L_R^2 : |\zeta - z| \geq |\zeta - z_2|\}$, $\left(E_{2,R}^{1,\pm}\right)_2 := E_{2;R}^{1,\pm} \setminus \left(E_{2,R}^{1,\pm}\right)_1$. Then,

$$\widetilde{D}_{n,m}^2 \asymp \frac{1}{|\zeta - z_2|^{\gamma_2+m+1}} \asymp \frac{1}{d_{2,R}^{\gamma_2+m+1}} \asymp n^{(\gamma_2+m+1)\frac{\mu}{1+\alpha_2}}, \quad z \in \left(E_{2,R}^{1,\pm}\right)_1, \text{ if } \gamma_2 > 0; \tag{47}$$

$$\widetilde{D}_{n,m}^2 \asymp \frac{1}{|\zeta - z|^{\gamma_2+m+1}} \asymp \frac{1}{d_{2,R}^{\gamma_2+m+1}} \asymp n^{(\gamma_2+m+1)\frac{\mu}{1+\alpha_2}}, \quad z \in \left(E_{2,R}^{1,\pm}\right)_2, \text{ if } \gamma_2 > 0;$$

$$\widetilde{D}_{n,m}^2 = \frac{|\zeta - z_2|^{-\gamma_2}}{|\zeta - z_2|^{m+1}} \asymp \frac{1}{d_{2,R}^{m+1}} \asymp n^{\frac{m+1}{1+\alpha_2}\mu}, \quad z \in \left(E_{2,R}^{1,\pm}\right)_1, \text{ if } \gamma_2 \leq 0;$$

$$\widetilde{D}_{n,m}^2 = \frac{|\zeta - z_2|^{-\gamma_2}}{|\zeta - z|^{m+1}} \asymp \frac{1}{d_{2,R}^{m+1}} \asymp n^{\frac{m+1}{1+\alpha_2}\mu}, \quad z \in \left(E_{2,R}^{1,\pm}\right)_2, \text{ if } \gamma_2 \leq 0;$$

Combining estimates (43)–(47), we get

$$\widetilde{D}_m(\zeta) \asymp n^{(\gamma_1^*+m+1)\widetilde{\mu}} + n^{\left(\gamma_2^*+m+1\right)\frac{\mu}{1+\alpha_2}}.$$

Therefore, we complete the proof for any $p = 1$. □

Proof of Theorem 3. Let $G \in PQS(\lambda; f_1, g_2)$, for some $\lambda \geq 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$, and $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$. Let $R = 1 + \frac{\varepsilon_0}{n}$, $0 < \varepsilon_0 < \frac{1}{2}$. For $0 \leq m < n$ and $z \in G_R$, by Cauchy integral formulas for derivatives, we have

$$P_n^{(m)}(z_j) = \frac{m!}{2\pi i} \int\limits_{L_R} \frac{P_n(\zeta)}{(\zeta - z_j)^{m+1}} d\zeta, \; j = 1, 2. \tag{48}$$

First, consider the case $p > 1$. By multiplying and dividing the integrand by $h^{\frac{1}{p}}$, and applying Hölder's inequality, we obtain

$$\left|P_n^{(m)}(z_j)\right| \leq \frac{m!}{2\pi} \left(\int\limits_{L_R} h(\zeta) |P_n(\zeta)|^p |d\zeta| \right)^{1/p} \cdot \left(\int\limits_{L_R} \frac{|d\zeta|}{h^{q/p}(\zeta) |\zeta - z_j|^{q(m+1)}} \right)^{1/q} \tag{49}$$

$$=: M_{n,1} \cdot M_{n,2}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Applying Lemma 3 to the integral $M_{n,1}$, we obtain the following estimate:

$$M_{n,1} \asymp \|P_n\|_p. \tag{50}$$

Using (20), we obtain the following estimate for the integral $M_{n,2}$:

$$(M_{n,2})^q = \int\limits_{L_R} \frac{|d\zeta|}{h^{q-1}(\zeta) |\zeta - z_j|^{q(m+1)}} \tag{51}$$

$$\asymp \int\limits_{L_R} \frac{|d\zeta|}{|\zeta - z_1|^{(q-1)\gamma_1} |\zeta - z_2|^{(q-1)\gamma_2} |\zeta - z_j|^{q(m+1)}}$$

$$\asymp \int\limits_{L_R^1} \frac{|d\zeta|}{|\zeta - z_1|^{(q-1)\gamma_1 + q(m+1)}} + \int\limits_{L_R^2} \frac{|d\zeta|}{|\zeta - z_2|^{(q-1)\gamma_2 + q(m+1)}} =: \widetilde{J}_R^1 + \widetilde{J}_R^2.$$

Now, for estimates of the integrals $\widetilde{J}_R^1$ and $\widetilde{J}_R^2$, let us use the notation (28) for $\rho = R$ and denote

$$\widetilde{J}_{R,k}^{i,\pm} := \int\limits_{E_{R,k}^{i,\pm}} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i(q-1)+q(m+1)}} \; ; \; i,k = 1,2.$$

Then, using these notations from (51), we get

$$(M_{n,2})^q \preceq \sum_{i=1}^{2} \left[\widetilde{J}_{R,1}^{i,\pm} + \widetilde{J}_{R,2}^{i,\pm} \right], \quad i = 1,2. \tag{52}$$

Thus, it remains for us to estimate the integrals $\widetilde{J}_{R,k}^{i,\pm}$ for each $i,k = 1,2$. For arbitrary $\gamma_1, \gamma_2 > -1$, we get

$$\begin{aligned} \widetilde{J}_{R,1}^{1,\pm} &\preceq \int\limits_0^{c_1 d_{1,R}} \frac{ds}{s^{\gamma_1(q-1)+q(m+1)}} \\ &\preceq \left(\frac{1}{d_{1,R}}\right)^{\gamma_1(q-1)+q(m+1)} \cdot \operatorname{mes}\left(E_{R,1}^{1,\pm}\right) \preceq \left(\frac{1}{d_{1,R}}\right)^{\gamma_1(q-1)+q(m+1)-1}; \end{aligned} \tag{53}$$

$$\begin{aligned} \widetilde{J}_{R,2}^{1,\pm} &\preceq \int\limits_{c_1 d_{1,R}}^{|L_{R,1}^{\pm}|} \frac{ds}{s^{\gamma_1(q-1)+q(m+1)}} \\ &\preceq \left(\frac{1}{d_{1,R}}\right)^{\gamma_1(q-1)+q(m+1)-1}; \end{aligned}$$

$$\widetilde{J}_{R,1}^{1,\pm} + \widetilde{J}_{R,2}^{1,\pm} \preceq \left(\frac{1}{d_{1,R}}\right)^{\gamma_1(q-1)+q(m+1)-1}.$$

Similarly, for the $\widetilde{J}_R^2$ in the neighborhood of the point z_2, we have

$$\begin{aligned} \widetilde{J}_{R,1}^{2,\pm} &\preceq \int\limits_0^{c_2 d_{2,R}} \frac{ds}{s^{\gamma_2(q-1)+q(m+1)}} \\ &\preceq \left(\frac{1}{d_{2,R}}\right)^{\gamma_2(q-1)+q(m+1)} \cdot \operatorname{mes}\left(E_{R,1}^{2,\pm}\right) \preceq \left(\frac{1}{d_{2,R}}\right)^{\gamma_2(q-1)+q(m+1)-1}; \end{aligned} \tag{54}$$

$$\begin{aligned} \widetilde{J}_{R,2}^{2,\pm} &\preceq \int\limits_{c_2 d_{2,R}}^{|L_{R,2}^{\pm}|} \frac{ds}{s^{\gamma_2(q-1)+q(m+1)}} \\ &\preceq \left(\frac{1}{d_{2,R}}\right)^{\gamma_2(q-1)+q(m+1)-1}; \end{aligned}$$

$$\widetilde{J}_{R,1}^{2,\pm} + \widetilde{J}_{R,1}^{2,\pm} \preceq \left(\frac{1}{d_{2,R}}\right)^{\gamma_2(q-1)+q(m+1)-1}.$$

In this case, combining (49)–(54), we get

$$\left|P_n^{(m)}(z_j)\right| \preceq \|P_n\|_p \left(\frac{1}{d_{j,R}}\right)^{\frac{\gamma_j+1}{p}+m}.$$

To complete the proof of Theorem 3 in the case $p > 1$, it remains to use estimates (34) and (35).

The proof argument for $p = 1$ is identical to that used in the proof of Theorem 2 for $p = 1$, upon setting $z = z_j$, $j = 1, 2$. □

Proof of Remark 1. The sharpness of estimates (14)–(19) follows from the fact that they combine the classical sharp Markov inequalities $\left\|P_n^{(m)}\right\|_\infty \asymp n^m \|P_n\|_\infty$, $m \geq 1$, with weighted norm inequalities of the form $\|P_n\|_\infty \asymp F_n(\gamma, p)\|P_n\|_{\mathcal{L}(h^*, L)}$, $p > 1$. The sharpness of the above inequalities may be verified by means of the following examples. Let $T_n(z) = 1 + z + \cdots + z^n$ and consider the weights $h^*(z) = h_0(z)$ and $h^{**}(z) = |z-1|^\gamma$, $\gamma > 0$, on the unit circle $L := \{z : |z| = 1\}$. Then, for every $n \in \mathbb{N}$, there exist constants $c_3 = c_3(h^*, p) > 0$ and $c_3' = c_3'(h^{**}, p) > 0$, satisfying

$$(a) \quad \|T_n\|_\infty \geq c_3 n^{\frac{1}{p}} \|T_n\|_{\mathcal{L}_p(h^*, L)}, \quad p > 1;$$

$$(b) \quad \|T_n\|_\infty \geq c_3' n^{\frac{\gamma+1}{p}} \|T_n\|_{\mathcal{L}_p(h^{**}, L)}, \quad p > \gamma + 1.$$

Indeed, if $L := \{z : |z| = 1\}$, then L is quasismooth, and hence, $L \in PQS(1; 0, 0) = QS(1)$.

(a) $h^*(z) \equiv 1$; (b) $h^{**}(z) = |z-1|^\gamma$, $\gamma > 0$.

Obviously,

$$|T_n(z)| \leq \sum_{j=0}^{n-1} \left|z^j\right| = n, \ |z| = 1; \ |T_n(1)| = n.$$

So

$$\|T_n\|_\infty = n.$$

On the other hand, it follows from ([19], p. 236) that

$$\|T_n\|_{\mathcal{L}_p(h^*, L)} \asymp n^{1-\frac{1}{p}}, \ p > 1,$$

and

$$\|T_n\|_{\mathcal{L}_p(h^{**}, L)} \asymp n^{1-\frac{\gamma+1}{p}}, \quad p > \gamma + 1.$$

Therefore,

$$(a) \quad \|T_n\|_\infty = n \asymp n^{\frac{1}{p}} \|T_n\|_{\mathcal{L}_p(h^*, L)}, \quad p > 1;$$

$$(b) \quad \|T_n\|_\infty = n = n \cdot n^{1-\frac{\gamma+1}{p}} \cdot n^{\frac{\gamma+1}{p}-1} \asymp n^{\frac{\gamma+1}{p}} \|T_n\|_{\mathcal{L}_p(h^{**}, L)}, \quad p > \gamma + 1.$$

□

5. Discussion

In this paper, we continue our study of the growth properties of the m-th ($m \geq 1$) derivatives of algebraic polynomials in weighted Lebesgue spaces on both bounded and unbounded domains. In our previous works and related studies (see the references in the introduction), this problem has been investigated for various classes of curves. It should be noted that in [12] a similar problem was treated using a recursive method; namely, the estimate for the m-th derivative was derived from the corresponding estimate for the $(m-1)$-th derivative. In contrast, in the present paper we avoid recursive arguments and treat all $m \geq 1$ simultaneously, obtaining stronger results than those in [12]. We explicitly demonstrate the influence of the order of tangency of boundary arcs at junction points, as well as the effect of zeros and poles of the weight function on the growth of the polynomial (see Theorem 1 and the corresponding corollaries).

Furthermore, we extend the range of the exponent p in the space $\mathcal{L}_p(h, L)$ to $p \geq 1$ in the estimates for the m-th $(m \geq 1)$ derivatives. Here, we reveal the influence of the type of zero angles on the growth behavior of polynomials and demonstrate the sharpness of the obtained estimates in several special cases. By combining the results of Theorems 1, 2 and the corresponding corollaries, we derive global estimates for the growth of polynomials in the entire complex plane. Moreover, we also establish growth estimates at the boundary points z_j, $j = 1, 2$. The obtained estimates clearly demonstrate the influence of the nature of the zero angles of tangency of the boundary arcs and the "zeros" and "poles" of the weight function on the orders of growth of the modulus of polynomial.

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